Comments on B. Hansen's Reply to "A Comment on: 'A Modern Gauss-Markov Theorem'", and Some Related Discussion

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Abstract

In Pötscher and Preinerstorfer (2022) and in the abridged version Pötscher and Preinerstorfer (2024, published in Econometrica) we have tried to clear up the confusion introduced in Hansen (2022a) and in the earlier versions Hansen (2021a,b). Unfortunatelly, Hansen's (2024) reply to Pötscher and Preinerstorfer (2024) further adds to the confusion. While we are already somewhat tired of the matter, for the sake of the econometrics community we feel compelled to provide clarification. We also add a comment on Portnoy (2023), a "correction" to Portnoy (2022), as well as on Lei and Wooldridge (2022).

1 Comments on Hansen (2024)

We organize our comments by section in Hansen (2024). The more important comments relate to Sections 2 and 3 in Hansen (2024).

1.1 Comments on Section 1 in Hansen (2024)

- Hansen wants to see Theorem 3.4 in Pötscher and Preinerstorfer (2022, 2024) as providing an alternative proof of Theorem 4 in Hansen (2022a), his 'modern Aitken Theorem'. We rather see it as a result showing that his 'modern Aitken Theorem', i.e., Theorem 4 in Hansen (2022a), is nothing else than the classical Aitken Theorem in disguise.
- The reference to Portnoy (2022) is not entirely to the point: As pointed out in Remark 3.6 in Pötscher and Preinerstorfer (2022) as well as in Remark 3.6 in Pötscher and Preinerstorfer (2024), Portnoy (2022) does **not** establish linearity of the estimators, but only Lebesgue

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a.e. linearity. Why this difference matters is explained in the before mentioned remarks. [This difference actually seems to have inspired Portnoy to write the "correction" Portnoy (2023), see the discussion in Section 2 further below.]

1.2 Comments on Section 2 in Hansen (2024)

- The statement that Pötscher and Preinerstorfer (2024) does not examine the case of independent sampling is misleading to say the least: Hansen's (2022) results on independent sampling are discussed in the abstract, in the introduction, and in the conclusion of Pötscher and Preinerstorfer (2024), and the reader is referred to the more extensive discussion in Pötscher and Preinerstorfer (2022), a paper Hansen is well aware of. [This more extensive discussion was removed from Pötscher and Preinerstorfer (2024) on the request of the editor of Econometrica.]
- The discussion up to and including Theorem 5 (and the paragraph following this theorem) is just a **repetition** of material in Hansen (2022), and our criticism of this material given in Section 5 of Pötscher and Preinerstorfer (2022) still stands.
- Theorem 11.1 is attributed to the book Hansen (2022b). The discussion in this book claims that it is taken from Hansen (2022a). However, this result is **nowhere** to be found in Hansen (2022a) (nor is it contained in any of the earlier versions Hansen (2021a,b)): While Hansen (2022a) indeed has a result for the location case (Theorem 7), this is a **different** result. [Hansen (2021a,b) also has results for the location case (Theorem 6 in both versions), but again these are results **different** from Theorem 11.1 in Hansen (2022b). As discussed in Pötscher and Preinerstorfer (2022), one of these two results is correct, the other one is incorrect.]
- A sketch of a proof of Theorem 11.1 is given in the book Hansen (2022b). This sketch is modelled after proofs in Hansen (2022a). Unfortunately, this sketch of a proof rests on Theorem 10.6 in Hansen (2022b), his rendition of the Cramér-Rao lower bound, a rendition that is lacking mathematical rigor and likely is not correct as given, because well-known regularity conditions (e.g., conditions for interchange of integration and differentiation) are omitted from Theorem 10.6 in Hansen (2022b). As a consequence of using Theorem 10.6 in Hansen (2022b), such regularity conditions are then **not** checked in the proof of Theorem 11.1. While the proof of Theorem 11.1 can perhaps be repaired, what is **given** in Hansen (2022b) certainly does **not** constitute a proof of Theorem 11.1.[1]
- Fortunately, there is actually **no need** to come up with a proof of Theorem 11.1 as it is an old result already proved in Halmos (1946). This is well-known, and is also discussed in

[1] There is a further problem here: From the discussion surrounding Theorem 10.6 in Hansen (2022b) it appears that this theorem is given for parametric models defined through densities w.r.t. Lebesgue measure. However, in a rigorous proof of Theorem 11.1 one needs to be able to apply the Cramér-Rao bound also in models that are not necessarily described by Lebesgue densities, but by densities w.r.t. another base measure. This problem can certainly be fixed, but Theorem 10.6 **as given** is not applicable.

Section 6 of Pötscher and Preinerstorfer (2022), a discussion that can hardly have escaped Hansen's attention.[2] Nevertheless the book Hansen (2022b) is silent on this issue and does not make any mention of Halmos (1946).

- The claim that Theorem 11.1 would be a strict improvement over the Cramér-Rao Theorem is obviously nonsense as the Cramér-Rao Theorem applies to unbiased estimators in general parametric models (the estimators and the model satisfying certain regularity conditions). Maybe Hansen wanted to say that Theorem 11.1 is a strict improvement over the result that in the context of i.i.d. normal data the mean is best unbiased (which can be obtained from the Cramér-Rao Theorem). But this again is nonsense, as the classes of unbiased estimators in these two statements are not guaranteed to be the same.

- It is interesting to note that the same problems that plague the proof of Theorem 11.1 in Hansen (2022b) actually also plague the proofs of Theorems 4, 5, 6, and 7 in Hansen (2022a) as they are again based on the non-rigorous rendition of the Cramér-Rao lower bound given in Theorem 10.6 in Hansen (2022b). While these proofs can possibly be repaired, we have not bothered to check this in any detail.

- We note that the set $\boldsymbol{F}_2^0$ defined in Hansen (2024) is **different** from the set $\boldsymbol{F}_2^0$ defined in Hansen (2022a), and both are different from the set $\boldsymbol{F}_2^0$ defined in Hansen (2021a,b). This is unfortunate as it creates unnecessary confusion.

- An earlier version (November 2023) of Hansen (2024) claimed that our counterexample given in Appendix A of Pötscher and Preinerstorfer (2024) (also given in Appendix A of Pötscher and Preinerstorfer (2022)) is mislabeled as a "counterexample". This is nonsense, and fortunately is not repeated in Hansen (2024).

1.3 Comments on Section 3 in Hansen (2024)

- Here Theorems 5 and 11.1 are repeated as Theorems 5' and 11.1' in a less formal notation. This lends itself to an increased chance of misunderstanding. For example, Theorem 5' in this section is not a precise mathematical statement; it can easily be understood as a statement where the σ_i^2 are arbitrary but fixed, which then gives an incorrect statement. We doubt that the two sentences following that theorem will help much to avoid the possibility of misunderstanding. Especially so, when this statement is specialized to the case of OLS. Here the typical reader will believe that one can set $\sigma_i^2 = \sigma^2$ in that theorem. We thus warn against using the imprecise formulation of the results given in that section. That section should best be ignored.

[2] The reference to Halmos (1946) is also mentioned in Appendix A of Pötscher and Preinerstorfer (2024).

2 Comments on Portnoy (2023)

In the "correction" Portnoy (2023) and the accompanying supplementary material, Portnoy claims that the theorem in Portnoy (2022) would state linearity of the estimators and that the proof would be in error. This is not correct as claimed, since the theorem in Portnoy (2022) **only** states Lebesgue a.e. linearity of the estimators considered, and **not** linearity. [The proof in Portnoy (2022) rests on Lusin's Theorem and inspection shows that it only delivers Lebesgue a.e. linearity. Whether this proof is actually correct, we have not checked.] It seems that in the "correction" Portnoy (2023) now wants to give a proof of a full linearity result in hindsight after such a result has appeared in Theorem 3.4 in Pötscher and Preinerstorfer (2022) (which does establish linearity by exploiting unbiasedness under certain discrete distributions). It is welcomed that Portnoy now provides an alternative proof of the full linearity result, but this is an **extension** of Portnoy (2022) rather than a **correction**.[3] Why this is advertised as a "correction" to Portnoy (2022) puzzles us. Also note that, for some reason, Portnoy (2023), and the supplementary material, do **not** acknowledge the full linearity result (Theorem 3.4) in Pötscher and Preinerstorfer (2022) at all.[4]

3 Comments on Lei and Wooldridge (2022)

Lei and Wooldridge (2022) incorrectly claim to have given a proof of the linearity result on Twitter, and they provide Twitter links in their paper. Inspection of the links shows that what is discussed there is a conjecture that the estimators in question must be linear, but there is no proof given there at all for at least two reasons: (i) The purported proof claims that the set of $n \times n$ symmetric matrices spans $\mathbb{R}^{n\times n}$, a blatantly false claim. (ii) It also rests on Theorem 4.3 in Koopmann (1982), for which no complete proof is given in that reference, see the discussion on p.9 of Pötscher and Preinerstorfer (2022) (see also Remark 3.7 and Appendix B in Pötscher and Preinerstorfer (2024)). Hence, while Lei and Wooldridge correctly guessed linearity (like probably many others), they did **not** provide a proof in their Twitter discussion. [Lei and Wooldridge (2022), released almost a year after their Twitter discussion, now claims to have finally managed to come up with a proof of Theorem 4.3 of Koopmann (1982), which – if correct – would void objection (ii), but not (i).]

Pötscher and Preinerstorfer (2022) provided a proof of the linearity result (Theorem 3.4) that does **not** rest on Theorem 4.3 in Koopmann (1982). Additionally, Pötscher and Preinerstorfer (2022) also provided (another) proof **conditional** on Theorem 4.3 in Koopmann (1982), explicitly stressing the conditionality. Lei and Wooldridge (2022) further claim that this conditional proof in Pötscher and Preinerstorfer (2022) would be identical to their own purported proof on Twitter. This is **incorrect**, as the conditional proof in Pötscher and Preinerstorfer (2022) does **not** rely on the incorrect claim (i) appearing in Lei's and Wooldridge's Twitter discussion;

[3] Again, we have not checked the details of the proof given in Portnoy (2023) (and the supplementary material).
[4] This full linearity result is also given in the abridged version Pötscher and Preinerstorfer (2024).

furthermore, – contrary to the purported Twitter-proof – the conditional proof in Pötscher and Preinerstorfer (2022) is correct.

We abstain from providing a list of all the errors and inconsistencies that can be found in Lei and Wooldridge (2022), and that make it difficult to gauge the actual contribution of that paper.

4 References

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