Optimal patent licensing: from three to two part tariffs

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Abstract

We study the licensing of a cost-reducing innovation in a Cournot oligopoly where an outside innovator uses three part tariffs that are combinations of upfront fees, per unit royalties and ad valorem royalties. Under general demand, the maximum possible licensing revenue under three part tariffs can be always attained by a policy that uses at most two of the three components. For relatively significant innovations, there exists an optimal policy consisting of a per unit royalty and upfront fee and a continuum of other optimal policies that are three part tariffs whose all components are positive. Completely characterizing optimal policies under linear demand, we show that for oligopolies with four or more firms: (i) pure upfront fees are optimal for insignificant innovations; (ii) for intermediate and significant innovations: (a) there is a continuum of optimal policies which always includes a two part tariff with unit royalty and upfront fee and (b) a two part tariff with an ad valorem royalty and fee or a two part royalty can be optimal for some, but not all parametric configurations.

Keywords: patent licensing; per unit royalties; ad valorem royalties; three part tariffs; acceptability and feasibility constraints

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1 Introduction

A patent grants an innovator monopoly rights over its innovation for a given period of time. This paper studies the problem of patent licensing of a process innovation in a Cournot oligopoly with an outside innovator. The licensing policies we study are general three part tariffs where a licensee pays an upfront fee, a per unit royalty and an ad valorem royalty that is a fraction of its revenue.

Patent licensing by means of upfront fees, royalties or combinations of fees and different kinds of royalties are commonly observed in practice. The survey of Rostoker (1983) on licensing agreements of 37 corporations in the United States reports the prevalence of these policies and differences in their relative uses across industries. For instance, in the electrical industry, "The favored license type was the straight royalties method (55%) with decreasing dependence given to downpayment and royalties (30%), paid-up licenses (14%)..." (p.66) while in the transportation industry, "...the preference clearly was for the down payment and royalties method (58%), followed by straight royalties (35%) and paid-up licenses (7%)..." (p.71).

In their study of 241 licensing contracts of Spanish firms, Macho-Stadler et al. (1996) find that most contracts have simple, linear structures, involving either fixed fees, or "variable payments" (royalties based on volumes of outputs or sales), or two part tariffs that combine fees and royalties. The survey of Aulakh et al. (1998) on international licensing agreements of U.S. firms reports that out of 110 agreements, 29 use only lump-sum fees, 49 are based on royalties and the remaining 32 use combinations of multiple policies or other methods.

In a dataset of 1458 patent licenses filed with the U.S. Securities and Exchange Commission (SEC), Varner (2011) finds that 94% of all licenses involve one of the three modes: fixed fees, ad valorem royalties based on sales percentages or unit royalties. This study also finds variation in relative uses of different policies. For instance, only 3% agreements in the life sciences industry include unit royalties, while they are more frequently used in high-tech (15%) and "other" industries that include chemicals, automotive and financial services (21%) (Table 4, p.234).

A natural question is whether three part tariffs are observed in practice. The study of Vishwasrao (2007) on foreign technology licenses by manufacturing firms in India is illuminating in this regard. Technology transfer by a foreign licensor may involve an "equity purchase," which is an ad valorem profit royalty where the licensor obtains a fraction of the licensee's profit. A contract having an equity purchase together with a unit royalty and a fee is a three part tariff. Out of 4025 contracts in this study, 11% are three part tariffs, while 28% are "arms' length" (that is, licensing without equity purchase) two part tariffs with unit royalties and fees. Other arms' length contracts are unit royalties (12%) and fees (36%). The remaining 7% are two part tariffs with equities and fees and 6% are two part royalties with equities and unit royalties.[1] There is variation in relative

[1] These percentages are computed from the numbers given in Table 2 (p.748) of Vishwasrao (2007). We study three part tariffs that are combinations of upfront fees, unit royalties and *ad valorem revenue* royalties rather than equities or *ad valorem profit* royalties that are observed in Vishwasrao (2007). If the rate of equity is relatively small, any equity-based three part tariff is equivalent to a revenue-based three part tariff in our theoretical framework (see Remark 3). It should be mentioned that for international technology transfer agreements, sometimes the rate of equity may be small due to an institutionally fixed upper bound on foreign equity ownerships (see, e.g., Eun and

uses of equity-based and arms' length contracts across industries (Table 4, p.749).

The broad pattern that emerges from these empirical studies is that fixed fees, per unit and ad valorem royalties are commonly used either separately or as two part tariffs. Three part tariffs that are combinations of all of these three components are also observed, but their use is less frequent. To fully understand the interrelation of these licensing devices and their effects on market outcomes, it is useful to see if incorporating both kinds of royalties in licensing contracts would increase the licensing revenue despite the potential complexity it may add to the transaction.

Another stylized fact is that relative uses of different policies vary across industries. In a related context of licensing of mining rights, Hogan and Goldsworthy (2010) state: "The specific or unit-based royalty is still utilized in most countries for low value, high volume minerals (for example, industrial minerals)... The main advantage of the specific royalty is its relative administrative simplicity..." (p.142). On the other hand, Gajigo et al. (2012) find an ad valorem or sales royalty to be the most commonly used policy for gold mining in African countries, noting that: "...it is less onerous on mining companies relative to unit royalty when commodity prices fall..." (p.2). Thus transaction costs and external factors could be plausible reasons behind the variation of licensing policies across industries. However, in situations where different contracts can work seamlessly, it is of interest to know if a specific royalty format is preferable over others.

Seeking to better understand these issues, we present a theoretical analysis of licensing with three part tariffs in a parsimonious model of a general oligopoly that has no uncertainty, risk or informational asymmetry. We show that the maximum possible licensing revenue under three part tariffs can be always attained by a policy that uses at most two of the three components. For relatively significant innovations: (i) there always exists an optimal policy that is a two part tariff consisting of a per unit royalty and an upfront fee; (ii) there is a continuum of other optimal policies that are three part tariffs in which all three components are positive and (iii) a two part tariff with an ad valorem royalty and upfront fee or a two part royalty with ad valorem and unit royalties can achieve the maximum possible licensing revenue for some, but not all parametric configurations (Propositions 3,4).

Our results show that even in the absence of transaction costs of writing more complicated contracts, at most two components are sufficient to attain the maximum possible licensing revenue under three part tariffs. The presence of transaction costs will further tilt the scale in favor of two part tariffs. This provides a clear explanation of why licensing contracts in practice more frequently use one or two components. We also show that whether a two part tariff with a fee and an ad valorem royalty is optimal or not may depend on intercepts of the demand curve (Propositions 4,5). This indicates that aspects such as market size and elasticity can determine the use of different kinds of royalties, providing a plausible explanation of variation of licensing policies across industries.

The theoretical literature of patent licensing can be traced back to Arrow (1962), who argued that a perfectly competitive industry provides a higher incentive to innovate compared to a monopoly. The early literature found upfront fees to be superior to unit royalties for outside

Janakiramanan 1986, Hoang and Mateus 2023).

innovators (e.g., Kamien and Tauman 1986; Kamien et al. 1992), although royalties can be optimal when the innovator is an incumbent firm in the industry (e.g., Shapiro 1985, Wang 1998, Kamien and Tauman 2002), or due to integer constraints on the number of licenses (Sen 2005a).

The literature has looked at several aspects of licensing in oligopolies such as cost asymmetry (e.g., Katz and Shapiro 1985, Marjit 1990), differentiated products (e.g., Muto 1993, Faulí-Oller and Sandonís 2002, Poddar and Sinha 2004, Stamatopoulos and Tauman 2008, Colombo 2012), leader-follower structures (e.g., Kabiraj 2004, Filippini 2005), returns to scale (e.g., Sen and Stamatopoulos 2009a, 2016, 2019, Faulí-Oller and Sandonís 2022), the role of licensing or patent agreements in facilitating or deterring entry (e.g., Eswaran 1994, Rodriguez 2002, Che and Facchini 2009, Duchêne and Serfes 2012, Duchêne et al. 2015) and informational asymmetry (e.g., Gallini and Wright 1990, Macho-Stadler and Pérez-Castrillo 1991, Beggs 1992, Sen 2005b, Fan et al. 2018a).[2] Two part tariffs with upfront fees and per unit royalties are studied in Sen and Tauman (2007), but they do not consider ad valorem royalties.

This is the first work to study three part tariffs in a general oligopoly. The literature on three part tariffs is sparse. Bousquet et al. (1998) consider these policies for a product innovation with an outside innovator and one risk averse firm. Savva and Taneri (2015) examine three part tariffs consisting of an equity (an ad valorem profit royalty[3] as in Vishwasrao 2007), a per unit royalty and a fixed fee. Recently Banerjee et al. (2023) study three part tariffs in a Hotelling duopoly where the innovator is one of the two competing firms.

In all of these papers on three part tariffs, the opportunity cost for a license does not depend on licensing policies, because there is only one potential licensee. This is not the case in an oligopoly where the profit a firm obtains without a license depends on the licensing policy as well as the number of other firms having licenses. Our analysis of three part tariffs in a general oligopoly contributes to the existing literature by fully exploring the strategic interaction among potential licensees in the downstream market.

As a starting point, a key question is: to what extent are the two kinds of royalties substitutable? In other words, can the same market outcome be sustained by slightly lowering one kind of royalty and raising the other? Attempting to answer this question, we show that the same *effective magnitude* of the innovation (denoted by δ) which determines the market outcomes (price,

[2]Other related issues include sequential innovations and patent pools (e.g., Brenner 2009, Llanes and Trento 2012, Choi and Stefanadis 2018), R&D spillovers (e.g., Suzumara 1992, Amir et al. 2000), capacities (e.g., Mukherjee 2001, Colombo et al. 2023) and asymptotic properties for large sizes of oligopolies (e.g., Tauman and Watanabe 2007, Hiroaki and Kawamori 2009). The literature is large and growing, and we do not attempt to summarize it here (for surveys, see, Kamien 1992; Bhattacharya et al. 2014). For two part tariffs with fees and only one kind of royalties, whether ad valorem or per unit royalty is superior can depend on factors such as the nature of product differentiation (Colombo and Filippini 2015, San Martín and Saracho 2015), demand uncertainty (Ma and Tauman 2020), informational asymmetry (Heywood et al. 2014) or scale economies (Colombo and Filippini 2016).

[3]In a cost asymmetric differentiated duopoly with an incumbent licensor, Mukhopadhyay et al. (1999) show that ad valorem profit royalties can induce profitable licensing regardless of the extent of product differentiation. Colombo et al. (2021) show that ad valorem profit royalties are generally equivalent to upfront fees for an outside innovator. For an incumbent innovator that has a *partial passive ownership* (PPO) of a rival firm (which is an ad valorem profit royalty with the royalty fraction not exceeding half), Leonardos et al. (2021) show that consumer surplus and social welfare are improved whenever licensing by lump-sum fees is profitable under PPO.

quantities) can generally be supported by a continuum of per unit and ad valorem royalties. Offering a licensing policy with any δ, the innovator faces two constraints: (i) an acceptability constraint to ensure that the net profit from having a license is no less than the profit without a license and (ii) a feasibility constraint to ensure that the royalties are within their permissible bounds.

The licensing revenue of the innovator under a three part tariff is completely determined by the effective magnitude,[4] so the same licensing revenue can be attained through multiple combinations of the two kinds of royalties. In the case of a monopoly, we show that among all three part tariffs, it is optimal to set a pure upfront fee (Proposition 1). For a general oligopoly we show that there always exists an optimal three part tariff that uses at most two of the three components. Moreover for relatively significant innovations, there always exists an optimal policy that is a two part tariff with a per unit royalty and an upfront fee and there is a continuum of other optimal policies that are three part tariffs in which all three components are positive (Proposition 3).

Completely characterizing the optimal policies under linear demand,[5] we show that for oligopolies with at least four firms: (i) pure upfront fees are optimal for insignificant innovations, (ii) for intermediate and significant innovations, there is a continuum of optimal licensing policies which always includes a two part tariff consisting of a unit royalty and upfront fee, (iii) for significant innovations, there also exists an optimal policy that is a two part tariff with ad valorem royalty and fee, (iv) for intermediate innovations, a two part tariff with ad valorem royalty and fee or a two part royalty can be optimal depending on the demand intercept and the magnitude of the innovation (Proposition 4(I)-(IV)). Similar qualitative conclusions for insignificant and certain intermediate innovations are obtained for smaller sizes of oligopolies with two or three firms (Proposition 5).

Furthermore, under linear demand we obtain the limiting result that for any fixed demand intercept, marginal cost and magnitude of the innovation, for all sufficiently large sizes of oligopolies: (i) there is a continuum of optimal licensing policies, (ii) there exists an optimal policy that is a two part tariff with a per unit royalty and upfront fee and (iii) there is a continuum of other optimal policies that are three part tariffs with all three components positive (Proposition 4(V)).

The paper is organized as follows. The model is presented in Section 2. The acceptability and feasibility constraints are discussed in Section 3. Optimal licensing policies are presented in Section 4. Some general properties of optimal policies are studied in Section 4.2 and the case of linear demand is considered in Section 5.7. Section 4.4 discusses optimal policies for an incumbent innovator in a duopoly. We conclude in Section 5.

2 The model

Consider a Cournot oligopoly with $n \geq 2$ firms where the set of competing firms is $N = \{1, \ldots, n\}$. For $j \in N$, let q_j be the quantity produced by firm j and $Q = \sum_{j \in N} q_j$. Initially each firm $j \in N$

[4] When the innovator is a competing firm, the market outcomes separately depend on ad valorem and per unit royalties. This is illustrated in Section 4.4, where the innovator is an incumbent firm in a duopoly.

[5] Table 1 presents the complete description of optimal three part tariffs for an outside innovator under linear demand.

produces under constant marginal cost $c > 0$. An outside innovator I has a patent for a process innovation that reduces the per unit cost from c to $c - \varepsilon$ ($0 < \varepsilon < c$), so ε is the magnitude of the innovation. I can license the innovation to some or all firms in N.

For the licensing policies that we study in this paper, under any licensing configuration firms without a license have the initial marginal cost c, while firms with a license have marginal cost $c - \delta$ for some $\delta \in [0, \varepsilon]$. Below we list two alternative sets of sufficient conditions that ensure existence and uniqueness of Cournot equilibrium under any licensing configuration.

(A1) The price function or the inverse demand function $p(Q)$: $\mathrm{R}_{++} \to \mathrm{R}_{+}$ is non-increasing and $\exists\ \overline{Q} > 0$ such that $p(Q)$ is decreasing and twice continuously differentiable for $Q \in (0, \overline{Q})$.

(A2) $\overline{p} \equiv \lim_{Q \downarrow 0} p(Q) > c$, $\lim_{Q \downarrow 0} p'(Q)Q + \overline{p} - c > 0$ and $\exists\ 0 < Q^c < Q^{c-\varepsilon} < \overline{Q}$ such that $p(Q^c) = c > p(Q^{c-\varepsilon}) = c - \varepsilon > p(\overline{Q})$.

(A3) $p(Q)$ is log-concave for $Q \in (0, \overline{Q})$.

Noting that cost functions of all firms are convex (since firms have constant marginal costs), applying the results of Amir (1996) and Amir and Lambson (2000),[6] it follows that when [A1-A3] hold, under any licensing configuration, the resulting oligopoly has a unique Cournot equilibrium.

An alternative set of sufficient conditions involves dropping (A3) and adding conditions (A4), (A5):

(A4) For $p \in (0, \overline{p})$, the price elasticity $\eta(p) := -pQ'(p)/Q(p)$ is non-decreasing.

(A5) For $Q \in (0, \overline{Q})$, the revenue function $\gamma(Q) := p(Q)Q$ is strictly concave, that is, $\gamma''(Q) = 2p'(Q) + Qp''(Q) < 0$.

By Kamien et al. (1992), the uniqueness of Cournot equilibrium is also ensured by the alternative set of assumptions [A1-A2, A4-A5]. Using $p' < 0$, it can be seen that (A3) implies (A4). Regarding (A5), recall the condition of Novshek (1985): $p'(Q) + Qp''(Q) \leq 0$ (this means each firm's marginal revenue is decreasing in the aggregate quantity of other firms). Since $p' < 0$, (A5) is a weaker requirement than the Novshek condition.

We assume either [A1-A3] or [A1-A2, A4-A5] hold.[7] This ensures: (a) the Cournot equilibrium is unique for any licensing configuration and (b) the general conclusions of our analysis are applicable to a wide class of demand functions that satisfy either of these two alternative sets of sufficient conditions. As examples: (i) the inverse demand function $p(Q) = \max\{(a - Q)^t, 0\}$ (where $a, t > 0$, $t \leq 1$ and $c < a^t$), satisfies all conditions (A1)-(A5); (ii) for $s, t > 0$, the constant elasticity inverse demand function $p(Q) = s/Q^t$ satisfies (A4), but does not satisfy (A3); for $0 < t < 1$, it satisfies [A1-A2, A4-A5], but for $t \geq 1$ it fails to satisfy (A2) or (A5).

Drastic and nondrastic innovations The notion of *drastic* innovations (Arrow, 1962) is useful for the analysis of licensing. A cost-reducing innovation is drastic if the monopoly price under

[6] See the discussion after Theorem 2.3 (p.240), Amir and Lambson (2000). For other sufficient conditions on the existence and uniqueness of Cournot equilibrium, see, e.g., Novshek (1985), Gaudet and Salant (1991), Badia et al. (2014). Also see Dixit (1986).

[7] There are two comparative statics results (Lemma 1(vi)(c) and Lemma 2(ii)(b)) that specifically require (A5). These results hold under [A1-A2, A4-A5], but may not hold under [A1-A3]. Both of these involve the comparison of the Cournot profit of a licensee when all firms have licenses with that of a non-licensee when all but one have licenses.

the new technology does not exceed the old marginal cost c; otherwise it is nondrastic. If only one firm in an oligopoly has a drastic innovation, it becomes a monopolist with the new technology and drives all other firms out of the market. To classify drastic and nondrastic innovations, define

$$\theta \equiv c/\eta(c) = -Q(c)/Q'(c) \tag{1}$$

For $k = 1, \ldots, n$, define the function $H^k : (0, \overline{p}) \to R$ as

$$H^k(p) := p[1 - 1/k\eta(p)] \tag{2}$$

Note that $H^k(p)$ is the marginal revenue of a firm in a k-firm oligopoly when each firm produces $Q(p)/k$. The following property is immediate from condition (A4) that[8] $\eta(p)$ is non-decreasing.

Observation 1 *Let $p, \widetilde{p} \in (0, \overline{p})$ and suppose $H^k(\widetilde{p}) > 0$. Then $H^k(p) > H^k(\widetilde{p})$ for $p > \widetilde{p}$ and $H^k(p) < H^k(\widetilde{p})$ for $p < \widetilde{p}$.*

To characterize drastic innovations, consider a monopolist who has unit cost $c - \varepsilon$. The profit of this monopolist at price p is

$$G(p) := [p - (c - \varepsilon)]Q(p) \tag{3}$$

If either [A1-A3] or [A1-A2, A4-A5] hold, there exists a unique $p_M(\varepsilon)$ (the monopoly price) that maximizes $G(p)$. The monopoly price satisfies $p_M(\varepsilon) \in (c - \varepsilon, \overline{p})$ and $H^1(p_M(\varepsilon)) = c - \varepsilon$, where H^k is given in (2) (see the Appendix for the proof of these properties). The monopoly profit under cost $c - \varepsilon$ is denoted by $\phi_M(\varepsilon)$, that is, $\phi_M(\varepsilon) = G(p_M(\varepsilon))$. Note that $\phi_M(\varepsilon) > 0$.

An innovation of magnitude ε is drastic if and only if $p_M(\varepsilon) \le c$. Since $c - \varepsilon > 0$, by Observation 1, $p_M(\varepsilon) \le c \Leftrightarrow H^1(p_M(\varepsilon)) \le H^1(c) \Leftrightarrow \varepsilon \ge c/\eta(c) \equiv \theta$. Thus, an innovation of magnitude ε is drastic if $\varepsilon \ge \theta$ and nondrastic if $\varepsilon < \theta$.

The industry profit at price p is at most $G(p)$. Since $G(p) \le \phi_M(\varepsilon) = G(p_M(\varepsilon))$, under any licensing configuration the industry profit is bounded above by the monopoly profit $\phi_M(\varepsilon)$. So the maximum licensing revenue that the innovator can obtain is $\phi_M(\varepsilon)$. If I has a drastic innovation, it can sell only one license and collect the entire monopoly profit $\phi_M(\varepsilon)$ from the sole license through an upfront fee.[9] Henceforth we consider nondrastic innovations.

2.1 Licensing policies

The set of licensing policies for I is the set of all three part tariffs consisting of a unit royalty, an ad valorem revenue royalty and an upfront fee. Under a three part tariff (k, r, v), I offers $k \in \{1, \ldots, n\}$ licenses (committing to sell no more than k) at a unit royalty $r \ge 0$, an ad valorem royalty $v \in [0, 1]$ and charges a non-negative upfront fee from each licensee. Under this policy any licensee has to

[8] Since (A3) implies (A4), the condition (A4) holds under both sets of alternative conditions [A1-A3] and [A1-A2, A4-A5].

[9] It should be mentioned that this pure upfront fee policy is not the unique optimal policy for an outside innovator of a drastic innovation. There are other optimal policies involving positive per unit royalties and upfront fees that also enable the innovator to earn the monopoly profit with a drastic innovation. See Sen and Stamatopoulos (2009b).

pay: (i) an upfront fee, (ii) r for every unit it produces and (iii) fraction v of its revenue to I. When $v = 0$, we have a *two part tariff FR* (a policy consisting of an upfront fee and unit royalty, but no ad valorem royalty) and when $r = 0$, we have a *two part tariff FV* (a policy consisting of an upfront fee and ad valorem royalty, but no unit royalty).

The upfront fee is collected through an auction (possibly with a minimum bid).[10] Firms are asked to simultaneously place non-negative bids. Suppose I offers to sell k licenses. If $m \leq k$ firms place bids, each of the bidding firms wins a license. If $m > k$ firms place bids, bids are arranged in descending order as $f_1 \geq \ldots \geq f_k \geq \ldots \geq f_m$. If $f_k > f_{k+1}$, firms with k highest bids win licenses. If $f_k = f_{k+1}$: (a) firms with bids strictly higher than f_k win licenses and (b) a random tie breaking process is run among the firms who placed bid f_k to determine who get the remaining licenses.[11] Any firm that wins a license pays its bid as upfront fee to I.

Remark 1 If I offers n licenses (that is, $k = n$), each firm is guaranteed a license, so no one will place a positive bid. To ensure positive upfront fees, I announces to auction n licenses together with a minimum bid $\hat{f} > 0$. It can be shown that under this modified process, for suitably chosen $\hat{f}$, there is an equilibrium in which each firm will have a license with the minimum bid, so each will pay upfront fee $\hat{f}$ (see Lemma 3).

2.2 The Licensing game Γ

The strategic interaction between I and the firms in N is modeled as the *licensing game* Γ that has the following stages. In stage 1, I announces a licensing policy. In stage 2, firms simultaneously decide whether to purchase a license or not. Any firm willing to purchase a license places its bid. Following the rules described before, the set of licensees is determined and commonly known. In stage 3, firms in N compete in quantities. Any licensee produces under the reduced marginal cost and pays I according to the licensing policy. Any non-licensee produces under the initial marginal cost. Denote by $\Gamma(k, r, v)$ the subgame of Γ that follows the announcement of the policy (k, r, v).

Consider a licensing policy with unit royalty r and ad valorem royalty v. Let $L \subseteq N$ be the set of licensees, so $\bar{L} = N \setminus L$ is the set of non-licensees. Any licensee $j \in L$ has marginal cost $c - \varepsilon$ and its payment for the license has the following components: (i) it pays r to I for every unit, (ii) it pays fraction v of its revenue to I, so it keeps the remaining fraction $1 - v$ and (iii) it pays fee $f_j \geq 0$ to I. A non-licensee has marginal cost c and makes no payment. So the payoffs of firms are

$$\pi_j = \begin{cases} (1-v)p(Q)q_j - (c-\varepsilon)q_j - rq_j - f_j & \text{if } j \in L \\ [p(Q) - c]q_j & \text{if } j \in \bar{L} \end{cases} \tag{4}$$

The payoff of I is the sum of three components: (i) revenue from unit royalty, (ii) revenue from ad

[10] The innovator can alternatively set the fee as a posted price. However, compared to a posted price, an auction generates more competition among firms that raises the willingness to pay for a license (see, e.g., Kamien and Tauman 1986, Katz and Shapiro 1986).

[11] The only constraint we impose on the random tie breaking process is that every firm with bid f_k has a positive probability of winning a license.

valorem royalty and (iii) upfront fees. This payoff is

$$\pi_I = \sum_{j\in L} r q_j + \sum_{j\in L} v p(Q) q_j + \sum_{j\in L} f_j \tag{5}$$

We confine to Subgame Perfect Nash Equilibrium (SPNE) outcomes of Γ.

Remark 2 Observe from (4) that if $v = 1$, a licensee obtains negative or zero payoff. Clearly a negative payoff will not be acceptable to a firm. With $v = 1$, the payoff of a licensee can be zero only if both $q_j = 0$ and $f_j = 0$, but in that case I has zero licensing revenue, so such a licensing policy is redundant. Ruling out unacceptable or redundant policies, henceworth we only consider policies with $v < 1$.

Remark 3 As mentioned in the introduction, a three part tariff in Vishwasrao (2007) consists of: (i) an equity purchase, which is an *ad valorem profit* royalty $\alpha \in (0,1)$ (the licensor keeps fraction α of the licensee's *profit*), (ii) a per unit royalty $\hat{r} \in (0,\varepsilon)$ and (iii) a fixed fee. Under this policy the payoff of any licensee firm $j \in L$ that pays fixed fee $\hat{f}_j$ is $\hat{\pi}_j = (1-\alpha)[p(Q)q_j - (c-\varepsilon)q_j - \hat{r}q_j] - \hat{f}_j$ and the payoff of the innovator is $\hat{\pi}_I = \sum_{j\in L} \hat{r}q_j + \sum_{j\in L} \alpha[p(Q)q_j - (c-\varepsilon)q_j - \hat{r}q_j] + \sum_{j\in L} \hat{f}_j$. Taking $v = \alpha$, $r = (1-\alpha)\hat{r} - \alpha(c-\varepsilon)$ and $f_j = \hat{f}_j$, we have $\hat{\pi}_j = \pi_j$ (given in (4)) and $\hat{\pi}_I = \pi_I$ (given in (5)). Since $r < \hat{r}$ and $\hat{r} < \varepsilon$, we have $r < \varepsilon$. Noting that $r > 0$ iff $\alpha < \hat{r}/(\hat{r} + c - \varepsilon)$, we conclude that provided the rate of equity α is relatively small, an equity-based three part tariff is equivalent to a revenue-based three part tariff.

2.3 Effective magnitude of the innovation

To determine SPNE of Γ, consider stage 3 (Cournot stage) of this game where firms in N choose quantities. By (4), in stage 3 any licensee firm $j \in L$ has payoff

$$\pi_j = (1-v)p(Q)q_j - (c-\varepsilon)q_j - rq_j - f_j = (1-v)[p(Q) - (c-\varepsilon+r)/(1-v)]q_j - f_j \tag{6}$$

For $r \geq 0$ and $0 \leq v < 1$, denote

$$\delta(r,v) := [\varepsilon - (r + cv)]/(1-v) \tag{7}$$

Using $\delta(r,v)$ from (7) in (6), we have

$$\pi_j = (1-v)[p(Q) - (c - \delta(r,v))]q_j - f_j \tag{8}$$

Since the fee f_j is paid upfront, it does not affect the choice of quantities at the Cournot stage. Because $1 - v > 0$, by (8), at the Cournot stage any licensee solves the same problem as a firm that has marginal cost $c - \delta(r,v)$. This means for a licensing policy that has unit royalty r and ad valorem royalty v, the *effective magnitude* of the innovation is $\delta(r,v)$. Note from (7) that $\delta(r,v)$ is decreasing in each of r, v, so $\delta(r,v) \leq \delta(0,0) = \varepsilon$.

Any firm has marginal cost c without a license. Assuming no firm will accept a policy that raises its marginal cost, we only consider policies for which the effective magnitude of the innovation is non-negative, so we restrict r, v such that $\delta(r, v) \geq 0$. Thus $0 \leq \delta(r, v) \leq \varepsilon$.

Note that $\delta(r, v) \geq 0$ if and only if $r + cv \leq \varepsilon$. The line AB in Figure 1 has equation $r + cv = \varepsilon$, so $\delta(r, v) \geq 0$ if and only if (r, v) is in ΔOAB. In general, for any $\delta \in [0, \varepsilon]$, $\delta(r, v) = \delta$ if and only if $r + (c - \delta)v = \varepsilon - \delta$. This is demonstrated by the line CD in Figure 1, which has equation $r + (c - \delta)v = \varepsilon - \delta$ for some $\delta \in (0, \varepsilon)$.

Remark 4 For $r \geq 0$ and $0 \leq v < 1$, the unique (r, v) that gives $\delta(r, v) = \varepsilon$ is $(r = 0, v = 0)$. On the other hand, for any $\delta \in [0, \varepsilon)$, there is a continuum of (r, v) for which $\delta(r, v) = \delta$, as shown by lines AB and CD in Figure 1. For any δ, the maximum v that can support δ, denoted by $\bar{v}(\delta)$, is found by taking $r = 0$ in the equation $r + (c - \delta)v = \varepsilon - \delta$, so $\bar{v}(\delta) = (\varepsilon - \delta)/(c - \delta)$, as shown in Figure 1.

2.4 The auxiliary Cournot oligopoly game $\mathcal{C}^n(k, \delta)$

Let $n \geq 2$ and $0 < \varepsilon < c$. For $k \in \{0, 1, \ldots, n\}$ and $\delta \in [0, \varepsilon]$, denote by $\mathcal{C}^n(k, \delta)$ the Cournot oligopoly game with n firms in which k firms (licensees) have marginal cost $c - \delta$ and the remaining $n - k$ firms (non-licensees) have marginal cost c. From the discussion of Section 2.3, we know that when there are k licensees under a licensing policy with per unit royalty r and ad valorem royalty v, the game $\mathcal{C}^n(k, \delta(r, v))$ is played at the Cournot stage. For our analysis it is useful to look at the equilibrium properties of $\mathcal{C}^n(k, \delta)$ for all k, δ.

To determine (Cournot-Nash) equilibrium of $\mathcal{C}^n(k, \delta)$ for $k \geq 1$, the threshold θ/k will be useful (recall $\theta \equiv c/\eta(c)$). Lemma 1 shows that if $\delta < \theta/k$, all firms are active and if $\delta \geq \theta/k$, all non-licensees drop out of the market and a k-firm natural oligopoly is created with k licensees.

Lemma 1 *Let $n \geq 2$ and $0 < \varepsilon < c$. Suppose either* [A1-A3] *or* [A1-A2, A4-A5] *hold. For $k \in \{0, 1, \ldots, n\}$ and $\delta \in [0, \varepsilon]$, the game $\mathcal{C}^n(k, \delta)$ has a unique equilibrium. Let $\overline{q}^n(k, \delta), \underline{q}^n(k, \delta)$ be the Cournot quantities of a licensee, non-licensee. Let $\overline{\phi}^n(k, \delta)$, $\underline{\phi}^n(k, \delta)$ be the corresponding Cournot profits and $p^n(k, \delta)$ be the Cournot price. Any licensee always obtains a positive Cournot profit. Properties* (i)-(v) *and* (vi)(a)-(b) *hold under either* [A1-A3] *or* [A1-A2, A4-A5] *and* (vi)(c) *holds under* [A1-A2, A4-A5].

(i) *Let $k = 0$ (no firm has a license). Then $c < p^n(0, \delta) < \overline{p}$ and $p^n(0, \delta)$ is the unique solution of $H^n(p) = c$ over $p \in (0, \overline{p})$. All firms obtain positive Cournot profits.*

(ii) *Let $1 \leq k \leq n - 1$ (there is at least one licensee and at least one non-licensee).*

  (a) *If $\delta < \theta/k$, then $c < p^n(k, \delta) < \overline{p}$ and $p^n(k, \delta)$ is the unique solution of $H^n(p) = c - k\delta/n$ over $p \in (0, \overline{p})$. All firms obtain positive Cournot profits.*

  (b) *If $\delta \geq \theta/k$, then $c - \delta < p^n(k, \delta) \leq c$ [equality iff $\delta = \theta/k$] and $p^n(k, \delta)$ is the unique solution of $H^k(p) = c - \delta$ over $p \in (0, \overline{p})$. A k-firm natural oligopoly is created, k licensees obtain positive Cournot profit and the $n - k$ non-licensees drop out of the market.*

(iii) *Let $k = n$ (all firms have licenses). Then $c - \delta < p^n(n, \delta) < \overline{p}$ and $p^n(n, \delta)$ is the unique solution of $H^n(p) = c - \delta$ over $p \in (0, \overline{p})$, where $p^n(n, \delta) > c$ if $\delta < \theta/n$ and $p^n(n, \delta) \leq c$ if $\delta \geq \theta/n$ [equality iff $\delta = \theta/n$]. All firms obtain positive Cournot profits.*

(iv) *For $k \geq 1$ and $\delta \leq \theta/k$, the Cournot price depends on k and δ only through the term $k\delta$. Specifically, for any $1 \leq k \leq n - 1$ and $\delta \leq \theta/k$, if $(k+1)\tilde{\delta} = k\delta$, then $p^n(k+1, \tilde{\delta}) = p^n(k, \delta)$.*

(v) *For $1 \leq k \leq n$, the Cournot price $p^n(k, \delta)$ is decreasing in δ; for $1 \leq k \leq n - 1$, the Cournot profit of a non-licensee $\underline{\phi}^n(k, \delta)$ is decreasing for $\delta \leq \theta/k$ and equals zero for $\delta \geq \theta/k$.*

(vi) *For $\delta = 0$, all firms obtain the same Cournot profit; specifically $\overline{\phi}^n(n, 0) = \overline{\phi}^n(k, 0) = \underline{\phi}^n(k, 0)$ for all $1 \leq k \leq n - 1$. For any $\delta > 0$:*

   (a) *$p^n(k-1, \delta) > p^n(k, \delta)$ for $1 \leq k \leq n$.*

   (b) *$\underline{\phi}^n(k-1, \delta) \geq \underline{\phi}^n(k, \delta)$ and $\overline{\phi}^n(k, \delta) > \underline{\phi}^n(k, \delta)$ for $1 \leq k \leq n - 1$.*

   (c) *$\overline{\phi}^n(n, \delta) > \underline{\phi}^n(n-1, \delta)$.*

Proof When either [A1-A3] or [A1-A2, A4-A5] hold, by applying the first and second order conditions of profit functions, it follows that $\mathcal{C}^n(k, \delta)$ has a unique equilibrium and parts (i)-(iii) hold. Part (iv) follows from parts (ii) and (iii) by applying Observation 1. See the Appendix for the proofs of parts (v)-(vi). ■

2.5 The monopoly case

Before studying licensing policies for a general oligopoly, let us first consider the case of a monopoly, where the innovator I interacts with a monopolist M who faces the inverse demand described before in Section 2. Without a license, M has constant marginal cost c. By (6) and (8), under a licensing policy with unit royalty $r \geq 0$ and ad valorem royalty $v \in [0, 1)$, the effective marginal cost of M is $c - \delta$, where $\delta = \delta(r, v)$ given in (7). When either [A1-A3] or [A1-A2, A4-A5] holds, the monopoly problem under marginal cost $c - \delta$ has a unique solution for any $\delta \in [0, \varepsilon]$. The monopoly price, quantity, profit under marginal cost $c - \delta$ are denoted by $p_M(\delta)$, $Q_M(\delta)$, $\phi_M(\delta)$. As functions of δ, $p_M(\delta)$ is decreasing and $Q_M(\delta), \phi_M(\delta)$ are increasing (see the Appendix for the proof of these properties).

Proposition 1 *When an outside innovator interacts with a monopolist, the unique optimal three part tariff is a pure upfront fee policy (that is, it has no ad valorem or per unit royalty) with positive fee $\phi_M(\varepsilon) - \phi_M(0)$. Under this policy, the innovator obtains $\phi_M(\varepsilon) - \phi_M(0)$ and the monopolist earns net payoff $\phi_M(0)$.*

Proof Consider a three part tariff with per unit royalty r, ad valorem royalty v and upfront fee f. If the monopolist M has a license under this policy, its operational monopoly profit is $\phi_M(\delta)$, where $\delta = \delta(r, v)$ given by (7). The per unit royalty r is already included in δ. By (8), taking into account the ad valorem royalty payment, M keeps fraction $1 - v$ of its operational profit and in addition pays upfront fee f, so its payoff is $(1-v)\phi_M(\delta) - f$. Without a license M obtains $\phi_M(0)$. Thus, for a three part tariff with r, v, the maximum upfront fee f that can be set is where the payoff of M with a license equals its payoff without a license:[12]

$$(1-v)\phi_M(\delta) - f = \phi_M(0) \Rightarrow f = (1-v)\phi_M(\delta) - \phi_M(0) \equiv \hat{f}(v, \delta)$$

The payoff of the innovator I at this policy is the sum of its revenues from unit royalty, ad valorem royalty and the upfront fee $\hat{f}(v, \delta)$, which is

$$\Pi_{FRV}(r, v, \delta) = rQ_M(\delta) + vp_M(\delta)Q_M(\delta) + \hat{f}(v, \delta)$$

$$= rQ_M(\delta) + vp_M(\delta)Q_M(\delta) + (1-v)[p_M(\delta) - (c-\delta)]Q_M(\delta) - \phi_M(0)$$

Noting that $(1-v)(c-\delta) = c - \varepsilon + r$ (by (7)) and using the function $G(p)$ from (3), we have $\Pi_{FRV}(r, v, \delta) = \Pi_{FRV}(\delta) = G(p_M(\delta)) - \phi_M(0)$. Since $p_M(\delta)$ is decreasing for $\delta \in [0, \varepsilon]$ and $p_M(\varepsilon)$ is the unique maximizer of $G(p)$, we have $G(p_M(\delta)) \leq G(p_M(\varepsilon))$, with equality iff $\delta = \varepsilon$. This shows that the unique maximum of $\Pi_{FRV}(\delta)$ is attained at $\delta = \varepsilon$, which implies $r = 0$, $v = 0$ (see Figure 1), proving that the unique optimal three part tariff is the pure upfront fee policy with fee $\hat{f}(0, \varepsilon) = \phi_M(\varepsilon) - \phi_M(0)$. Since $\phi_M(\varepsilon) > \phi_M(0)$, this fee is positive. The payoffs are immediate. ■

3 Acceptability and feasibility constraints

To study licensing policies for a general oligopoly, we need to consider two kinds of constraints: (i) acceptability constraints that ensure that the payoff of a licensee net of its licensing payments is no less than its payoff without a license and (ii) feasibility constraints that ensure that per unit and ad valorem royalties are within their permissible bounds.

3.1 Acceptability constraint: relative gain in operational profit from a license

When there are k licensees under a licensing policy with unit royalty r and ad valorem royalty v, the Cournot oligopoly game $\mathcal{C}^n(k, \delta)$ (see Section 2.4) is played in stage 3 of Γ where $\delta = \delta(r, v)$ given in (7). In this game any non-licensee has marginal cost c and its payoff is simply its operational Cournot profit $\underline{\phi}^n(k, \delta)$. Any licensee, having marginal cost $c - \delta$, has operational Cournot profit $\overline{\phi}^n(k, \delta)$. Note that the unit royalty is already included in δ. By (8), taking into account its ad valorem royalty payment, a licensee keeps fraction $1 - v$ of its operational Cournot profit and in addition pays its upfront fee (say f). So the payoff of a licensee is $(1-v)\overline{\phi}^n(k, \delta) - f$.

Surplus from acquiring a license To determine the surplus of a firm from acquiring a license, first suppose there are $k \leq n - 1$ licensees. If a licensee chooses to not have a license, it

[12]Note that v has to be constrained so that the fee $\hat{f}(v, \delta)$ is non-negative; however, we solve the unconstrained problem to show that the fee is positive under the unique optimal policy.

would obtain[13] either $\underline{\phi}^n(k-1,\delta)$ or $\underline{\phi}^n(k,\delta)$. Since $\underline{\phi}^n(k-1,\delta) \geq \underline{\phi}^n(k,\delta)$ (Lemma 1(vi)(b)), the maximum upfront fee f that can be obtained from a licensee in this case is where the payoff with a license equals the payoff without a license:

$$(1-v)\overline{\phi}^n(k,\delta) - f = \underline{\phi}^n(k,\delta) \Rightarrow f = (1-v)\overline{\phi}^n(k,\delta) - \underline{\phi}^n(k,\delta) \tag{9}$$

When all firms have licenses (that is, $k = n$), by refusing to have a license, a firm lowers the number of licensees by one and obtains $\underline{\phi}^n(n-1,\delta)$. So for $k = n$, the maximum upfront fee f that can be obtained from a licensee is where

$$(1-v)\overline{\phi}^n(n,\delta) - f = \underline{\phi}^n(n-1,\delta) \Rightarrow f = (1-v)\overline{\phi}^n(n,\delta) - \underline{\phi}^n(n-1,\delta) \tag{10}$$

Using (9) and (10), denote

$$f^n(k,v,\delta) := \begin{cases} (1-v)\overline{\phi}^n(k,\delta) - \underline{\phi}^n(k,\delta) & \text{if } k = 1,\ldots,n-1 \\ (1-v)\overline{\phi}^n(n,\delta) - \underline{\phi}^n(n-1,\delta) & \text{if } k = n \end{cases} \tag{11}$$

When there are k licensees with unit royalty r and ad valorem royalty v, $f^n(k,v,\delta)$ in (11) presents the surplus from acquiring a license (where $\delta = \delta(r,v)$ given in (7)). There are two possibilities.

(i) $f^n(k,v,\delta) \geq 0$: In this case, there is non-negative surplus from acquiring a license, so it is worthwhile to have a license. The innovator can extract this surplus from each licensee through an upfront fee that is collected using an auction, possibly with a minimum bid (see Lemma 3).

(ii) $f^n(k,v,\delta) < 0$: In this case, the surplus from acquiring a license is negative, so it is not worthwhile for a firm to have a license, rendering the corresponding licensing policy unacceptable.

The conclusions of (i)-(ii) above lead us to the *acceptability constraint* for a licensing policy, which is simply the requirement that the surplus from acquiring a license is non-negative: $f^n(k,v,\delta) \geq 0$.

Relative gains in operational profits Consider the function $\gamma^n(k,\delta)$ below, which presents the *relative gains in operational Cournot profits from a license.*

$$\gamma^n(k,\delta) := \begin{cases} [\overline{\phi}^n(k,\delta) - \underline{\phi}^n(k,\delta)]/\overline{\phi}^n(k,\delta) & \text{if } k = 1,\ldots,n-1 \\ [\overline{\phi}^n(n,\delta) - \underline{\phi}^n(n-1,\delta)]/\overline{\phi}^n(n,\delta) & \text{if } k = n \end{cases} \tag{12}$$

Observe from (11) and (12) that

$$f^n(k,v,\delta) \geq 0 \Leftrightarrow v \leq \gamma^n(k,\delta) \text{ and } f^n(k,v,\delta) = 0 \Leftrightarrow v = \gamma^n(k,\delta) \tag{13}$$

[13] If the number of firms willing to buy licenses does not exceed the number of licenses offered, any firm that has a license can lower the number of licensees by one if it chooses to not have a license. On the other hand, if the number of firms willing to buy licenses is more than the number of licenses offered, the number of licensees does not alter when one firm refuses to have a license.

This means a license is acceptable if and only if the ad valorem royalty fraction v does not exceed the relative gain in the operational profit from a license. The next lemma summarizes the basic properties of these functions.

Lemma 2 *Properties* (i)(a)-(b) *and* (ii)(a) *hold under either* [A1-A3] *or* [A1-A2, A4-A5], *while property* (ii)(b) *holds only under* [A1-A2, A4-A5].

(i) *For* $k = 1, \ldots, n-1$:

(a) $\gamma^n(k, 0) = 0$ *and* $\gamma^n(k, \delta) = 1$ *for* $\delta \geq \theta/k$.

(b) *For* $0 < \delta < \theta/k$, $\gamma^n(k, \delta)$ *is increasing in* δ *and* $0 < \gamma^n(k, \delta) < 1$.

(ii) *For* $k = n$:

(a) $\gamma^n(n, 0) = 0$ *and* $\gamma^n(n, \delta) = 1$ *for* $\delta \geq \theta/(n-1)$.

(b) *For* $0 < \delta < \theta/(n-1)$, $\gamma^n(n, \delta)$ *is increasing in* δ *and* $0 < \gamma^n(n, \delta) < 1$.

Proof Parts (i)(a), (ii)(a) are immediate from (12) by noting that $\overline{\phi}^n(k, 0) = \underline{\phi}^n(k, 0)$ for $1 \leq k \leq n-1$, $\overline{\phi}^n(n, 0) = \underline{\phi}^n(n-1, 0)$ (Lemma 1(vi)) and for $1 \leq k \leq n-1$, $\delta \geq \theta/k$: $\underline{\phi}^n(k, \delta) = 0$ (Lemma 1(ii)(b)). See the Appendix for the proofs of (i)(b), (ii)(b). ■

Figures 2(a), 2(b) depict $\gamma^n(k, \delta)$, $\gamma^n(n, \delta)$ as functions of δ. The monotonicity of these functions follow by Lemma 2. For linear demand, these are also concave functions of δ, as shown in Figures 2(a), 2(b).

3.2 Acceptability versus feasibility constraints

To determine optimal licensing policies, together with acceptability constraints we need to look at *feasibility* constraints of ad valorem and unit royalties. The feasibility constraints are determined by the bounds of r and v: both are non-negative, r is bounded above by ε and v by 1. By Remark 4, for any δ, the maximum ad valorem royalty v for which $\delta(r, v)$ given in (7) equals δ is

$$\bar{v}(\delta) = (\varepsilon - \delta)/(c - \delta) \tag{14}$$

As shown in Figure 1, when $\delta(r, v) = \delta$, then $v = \bar{v}(\delta) \Leftrightarrow r = 0$ (the ad valorem royalty is maximum if and only if the per unit royalty is at its minimum level zero). Note that (i) $\bar{v}(\delta)$ is decreasing and concave in δ, (ii) $\bar{v}(\varepsilon) = 0$, $\bar{v}(0) = \varepsilon/c < 1$, so $0 < \bar{v}(\delta) < 1$ for $\delta \in [0, \varepsilon)$.

For $k = 1, \ldots, n$ and $\delta \in [0, \varepsilon]$, the acceptability constraint is $v \leq \gamma^n(k, \delta)$ and the feasibility constraint is $v \leq \bar{v}(\delta)$. Thus, for any k, δ, the ad valorem royalty v is both acceptable and feasible if and only if:

$$v \leq \min\{\gamma^n(k, \delta), \bar{v}(\delta)\} \tag{15}$$

Let $\hat{\theta}(k) = \min\{\varepsilon, \theta/k\}$ for $k = 1, \ldots, n-1$ and $\hat{\theta}(n) = \min\{\varepsilon, \theta/(n-1)\}$. If $\hat{\theta}(k) = \varepsilon$, then $\bar{v}(\hat{\theta}(k)) = \bar{v}(\varepsilon) = 0 < \gamma^n(k, \varepsilon) = \gamma^n(k, \hat{\theta}(k))$; otherwise $\bar{v}(\hat{\theta}(k)) < 1 = \gamma^n(k, \hat{\theta}(k))$. In either case

$\bar{v}(\hat{\theta}(k)) < \gamma^n(k, \hat{\theta}(k))$. Also note that $\bar{v}(0) = \varepsilon/c > 0 = \gamma^n(k, 0)$. Since $\bar{v}(\delta)$ is decreasing and $\gamma^n(k,\delta)$ is increasing for $\delta \in [0, \hat{\theta}(k)]$, $\exists$ a unique $\hat{\delta}^n(k) \in (0, \hat{\theta}(k))$ such that

$$\gamma^n(k,\delta) \gtreqless \bar{v}(\delta) \Leftrightarrow \delta \gtreqless \hat{\delta}^n(k) \tag{16}$$

The threshold $\hat{\delta}^n(k)$ is identified in Figure 3(a)-(d).

In Figures 3(a)-3(d), given any k, the set of all acceptable and feasible combinations of (δ, v) is given by the region OAB. Any (δ, v) on the horizontal line OB has $v = 0$ (no ad valorem royalty). For any (δ, v) on OA, the acceptability constraint binds: $v = \gamma^n(k,\delta)$ (the maximum acceptable v for that δ), so a licensee is left with no surplus, implying that in this case the combination of per unit and ad valorem and royalties collects the entire surplus and upfront fees are not needed. For any (δ, v) on AB, the feasibility constraint binds: $v = \bar{v}(\delta)$ (the maximum feasible v for that δ), so the per unit royalty r is zero. Any point in the interior of OAB has: (i) $v > 0$ (positive ad valorem royalty), (ii) $v < \bar{v}(\delta)$ (positive unit royalty) and (iii) $v < \gamma^n(k,\delta)$ (positive surplus from acquiring a licensee, which can be collected through upfront fees, see Lemma 3).

Note that a policy (k, r, v) is both acceptable and feasible if and only if (15) holds, where $\delta = \delta(r, v)$ given in (7). The next lemma shows that for any such (k, r, v), there exists an SPNE of the subsequent subgame in which: (i) the desired number k of licenses are sold and (ii) I collects the maximum possible upfront fee $f^n(k, v, \delta)$ (given in (11)) from each licensee, thus extracting the entire surplus of acquiring a license.

Lemma 3 *Let $k \in \{1, \ldots, n\}$, $r \geq 0$, $v \in [0, 1)$ and $\delta = \delta(r, v)$ given in (7). Consider a three part tariff (k, r, v), in which I announces to auction off k licenses with unit royalty r and ad valorem royalty v (additionaly, for $k = n$, specifying the minimum bid $\hat{f} = f^n(n, v, \delta)$ given in (11)). Suppose this policy is both acceptable and feasible, that is, (15) holds. The subgame $\Gamma(k, r, v)$ that follows this announcement has the following properties.*

(i) *For $1 \leq k \leq n-1$, there exists an SPNE of $\Gamma(k, r, v)$ in which the highest bid is $f^n(k, v, \delta)$, the highest bid is placed by at least $k+1$ firms and k of them are chosen at random to be licensees.*

(ii) *For $k = n$, there exists an SPNE of $\Gamma(n, r, v)$ in which all firms place the specified minimum bid $f^n(n, v, \delta)$ and all of them become licensees.*

Proof Note from (13) that when (15) holds: $f^n(k, v, \delta) \geq 0$. Thus, there is non-negative surplus from acquiring a license. Using this, the result follows by noting that at the outcomes described in (i)-(ii), there is no gainful unilateral deviation for any firm. ■

Remark 5 Lemma 3 shows that for any acceptable and feasible (k, r, v), when k licenses are offered with unit royalty r and ad valorem royalty v, there is an SPNE of $\Gamma(k, r, v)$ with k licensees where I can collect the maximum possible upfront fee. It should be noted that this SPNE is not necessarily unique. Denote $\underline{\gamma}^n = [\overline{\phi}^n(1,\delta) - \underline{\phi}^n(0,\delta)]/\overline{\phi}^n(1,\delta)$. For a linear demand, $\underline{\gamma}^n < \gamma^n(k,\delta)$

for $1 \leq k \leq n-1$ and $\delta > 0$. Unless the ad valorem royalty is sufficiently smaller than the acceptable threshold (specifically, $v < \underline{\gamma}^n$), $\Gamma(k, r, v)$ has an SPNE in which no firm buys a license.

4 Optimal licensing policies

Denote by $\mathbb{T}$ the set of all three part tariffs (k, r, v) where:

(i) $k \in \{1, \ldots, n\}$, $r \geq 0$, $v \in [0, 1)$;

(ii) (k, r, v) is both acceptable and feasible;

(iii) There exists an SPNE of $\Gamma(k, r, v)$ in which the number of licensees is k and each licensee pays the maximum possible upfront fee $f^n(k, v, \delta)$ given in (11) as its winning bid to acquire its license (where $\delta = \delta(r, v)$ given in (7)).

Note by Lemma 3 that $\mathbb{T}$ is non-empty.

Definition 1 Suppose for every $(k, r, v) \in \mathbb{T}$, the SPNE of $\Gamma(k, r, v)$ described in Lemma 3 is played. We say (k^*, r^*, v^*) is an *optimal three part tariff* if (a) $(k^*, r^*, v^*) \in \mathbb{T}$ and (b) over all $(k, r, v) \in \mathbb{T}$, the payoff of the innovator is maximized at (k^*, r^*, v^*).

To determine optimal licensing policies, first let $1 \leq k \leq n-1$. Consider any $(k, r, v) \in \mathbb{T}$, where as before $\delta = \delta(r, v)$ given in (7). Under the SPNE outcome of Lemma 3, there are k licensees, each paying the upfront fee $f^n(k, v, \delta)$. From each licensee, I obtains:[14] (i) per unit royalty payments $r\overline{q}^n(k, \delta)$, (ii) ad valorem fraction v of its revenue $p^n(k, \delta)\overline{q}^n(k, \delta)$ and (iii) upfront fee $f^n(k, v, \delta)$ given in (11). Hence the payoff of I is

$$\Pi^n(k, r, v, \delta)_{FRV} = kr\overline{q}^n(k, \delta) + kvp^n(k, \delta)\overline{q}^n(k, \delta) + kf^n(k, \delta)$$

$$= kr\overline{q}^n(k, \delta) + kvp^n(k, \delta)\overline{q}^n(k, \delta) + k(1 - v)[p^n(k, \delta) - (c - \delta)]\overline{q}^n(k, \delta) - k\underline{\phi}^n(k, \delta)$$

Noting that $(1 - v)(c - \delta) = c - \varepsilon + r$ (by (7)), using the industry quantity $Q^n(k, \delta) = k\overline{q}^n(k, \delta) + (n - k)\underline{q}^n(k, \delta)$ and the function $G(p)$ from (3), we have

$$\Pi^n(k, r, v, \delta)_{FRV} = \Pi^n(k, \delta)_{FRV} = G(p^n(k, \delta)) - n\underline{\phi}^n(k, \delta) - \varepsilon(n - k)\underline{q}^n(k, \delta) \tag{17}$$

Observe that the payoff above is a function of δ and does not separately depend on r, v.

Next consider $k = n$ and $(n, r, v) \in \mathbb{T}$, where $\delta = \delta(r, v)$ given in (7). Under the SPNE outcome of Lemma 3, there are n licensees, each paying the upfront fee $f^n(n, v, \delta)$ given in (11). Hence the payoff of I is

$$\Pi^n(n, r, v, \delta)_{FRV} = nr\overline{q}^n(n, \delta) + nvp^n(n, \delta)\overline{q}^n(n, \delta) + nf^n(n, v, \delta)$$

[14] Recall from Lemma 1 that $p^n(k, \delta)$ denote the Cournot price and $\overline{q}^n(k, \delta)$, $\underline{q}^n(k, \delta)$ the Cournot quantity of a licensee, non-licensee for the Cournot oligopoly game $\mathcal{C}^n(k, \delta)$.

$$= nr\overline{q}^n(n,\delta) + nvp^n(n,\delta)\overline{q}^n(n,\delta) + n(1-v)[p^n(n,\delta) - (c-\delta)]\overline{q}^n(n,\delta) - n\underline{\phi}^n(n-1,\delta)$$

As before, using $(1-v)(c-\delta) = c - \varepsilon + r$ (by (7)), the industry quantity $Q^n(n,\delta) = n\overline{q}^n(n,\delta)$ and the function $G(p)$ from (3), we have

$$\Pi^n(n,r,v,\delta)_{FRV} = \Pi^n(n,\delta)_{FRV} = G(p^n(n,\delta)) - n\underline{\phi}^n(n-1,\delta) \tag{18}$$

Like (17), the payoff above is a function of only δ.

The problem of finding optimal policies for I reduces to[15] choosing $k \in \{1,\ldots,n\}$ and $\delta \in [0,\varepsilon]$ to maximize $\Pi^n(k,\delta)_{FRV}$. Since for any k, the functions in (17)-(18) are continuous and bounded in δ, this problem has a solution.

4.1 From three to two part tariffs

For any $\delta \in [0,\varepsilon]$, we say a combination of per unit and ad valorem royalties (r,v) *supports* δ if $\delta(r,v)$ given in (7) equals δ, that is, if $[\varepsilon - (r+cv)]/(1-v) = \delta$. Denote by $\overline{\mathbb{S}}(\delta)$ the set of all (r,v) that supports δ:

$$\overline{\mathbb{S}}(\delta) := \{(r,v) | r \geq 0, v \in [0,1), \delta(r,v) = \delta\} \tag{19}$$

Note that any $\delta \in [0,\varepsilon)$ can be supported by a continuum of combinations of (r,v) (see Figure 1). For $k \in \{1,\ldots,n\}$ and $\delta \in [0,\varepsilon]$, denote by $\mathbb{S}^n(k,\delta)$ the subset of $\overline{\mathbb{S}}(\delta)$ that contains only those (r,v) for which the three part tariff (k,r,v) is both acceptable and feasible. Using (15):

$$\mathbb{S}^n(k,\delta) := \{(r,v) | (r,v) \in \overline{\mathbb{S}}(\delta), v \leq \min\{\overline{v}(\delta), \gamma^n(k,\delta)\}\} \tag{20}$$

To identify the set $\mathbb{S}^n(k,\delta)$, consider any $\delta \in (0,\varepsilon)$. As an illustration, one such δ is presented as point D in Figure 3(a). As shown in Figure 3(a), the vertical line from D meets the curve OAB at point C (for any point outside OAB, either acceptability or feasibility will be violated). The line CD presents the set $\mathbb{S}^n(k,\delta)$. Observe that δ stays the same across all points on the line CD. For two different points (δ,v_1), (δ,v_2) on CD, setting $\delta(r_1,v_1) = \delta$, $\delta(r_2,v_2) = \delta$ gives us the corresponding (r_1,v_1), $(r_2,v_2) \in \mathbb{S}^n(k,\delta)$.

For any k, the payoffs in (17)-(18) are functions of only δ and do not separately depend on r,v. This means for any $\delta \in (0,\varepsilon)$, all $(r,v) \in \mathbb{S}^n(k,\delta)$ are payoff-equivalent for the innovator. Thus, for the δ located at point D of Figure 3(a), all policies on the line CD give the same payoff to I.

To better understand the payoff-equivalence, observe that all points on the line CD are below the curve $\gamma^n(k,\delta)$, but different points have different distances from the curve $\gamma^n(k,\delta)$. This means all policies on CD result in positive surplus from acquiring a license, but different policies on CD, having different v, give different surpluses $f^n(k,v,\delta)$. The policies are payoff-equivalent because for

[15] Note that an outsider innovator obtains zero payoff by not offering any license (that is, by choosing $k=0$). Offering licenses to one or more firms with a sufficiently small positive unit royalty or upfront fee makes the licensing offer acceptable and feasible, yielding positive payoff for the innovator. This shows that choosing $k=0$ cannot be optimal.

each such policy, the innovator is able to collect the corresponding surplus through upfront fees using an auction (see Lemma 3).

Also by Figure 3(a), the line CD always contains its boundary point D, where $v = 0$. Thus, the set $\mathbb{S}^n(k, \delta)$ of (20) always contains a point $(r, 0)$ (note from (15) that both acceptability and feasibility constraints always hold for $v = 0$). Setting $\delta(r, 0) = \delta$ gives us $r = \varepsilon - \delta$. Therefore for any $\delta \in (0, \varepsilon)$, there exists $r = \varepsilon - \delta > 0$ such that: (i) $(r, 0)$ supports δ and (ii) the three part tariff $(k, r, 0)$ is both acceptable and feasible. This means any $\delta \in (0, \varepsilon)$ can be supported by an acceptable and feasible $(k, r, 0)$, which is effectively a two part tariff consisting of a per unit royalty and upfront fee. These observations are more specifically spelled out in the next proposition.

Proposition 2 (Relation between three part and two part tariffs) *For any $k \in \{1, \ldots, n\}$, the set $\mathbb{S}^n(k, \delta)$ is non-empty for every $\delta \in [0, \varepsilon]$ and it has the following properties.*

(I) $\mathbb{S}^n(k, 0)$ *is the singleton set* $\{(\varepsilon, 0)\}$ *and* $\mathbb{S}^n(k, \varepsilon)$ *is the singleton set* $\{(0, 0)\}$. *That is, the only acceptable and feasible three part tariff that supports $\delta = 0$ is the pure per unit royalty $r = \varepsilon$ and the only acceptable and feasible three part tariff that supports $\delta = \varepsilon$ is the pure upfront fee policy with fee $f^n(k, 0, \varepsilon)$ (given in* (11)*).*

(II) *For every $0 < \delta < \varepsilon$, the set $\mathbb{S}^n(k, \delta)$ is a continuum and it always contains a point (r, v) where $r = \varepsilon - \delta > 0$ and $v = 0$ (such a point lies on the line OB in Figures* 3(a)-(d)*). That is, there is a continuum of acceptable and feasible combinations of (r, v) that support δ and there always exists a two part tariff consisting of a positive per unit royalty and upfront fee but no ad valorem royalties that supports δ.*

(III) *For every $0 < \delta < \varepsilon$, the set $\mathbb{S}^n(k, \delta)$ also contains other points (r, v) where $r > 0$, $v > 0$ and the acceptability constraint is not binding (such a point lies in the interior of OAB in Figures* 3(a)-(d)*). That is, there is a continuum of acceptable and feasible combinations of (r, v) that support δ for which both kinds of royalties as well as the upfront fee is positive.*

(IV) *If $0 < \delta < \hat{\delta}^n(k)$, then $\mathbb{S}^n(k, \delta)$ contains a point (r, v) where $r > 0$ and $v = \gamma^n(k, \delta) > 0$ (such a point lies on curve OA in Figures* 3(a)-(d)*). That is, $\mathbb{S}^n(k, \delta)$ contains a two part royalty policy with positive per unit and ad valorem royalties (but no upfront fees) that binds the acceptability constraint.*

(V) *If $\hat{\delta}^n(k) < \delta < \varepsilon$, then $\mathbb{S}^n(k, \delta)$ contains a point (r, v) where $r = 0$ and $v = \bar{v}(\delta) > 0$ (such a point lies on curve AB in Figures* 3(a)-(d)*). That is, $\mathbb{S}^n(k, \delta)$ contains a two part tariff with a positive ad valorem royalty and upfront fee (but no per unit royalties) that binds the feasibility constraint.*

Proof By using (20), the results follow from Figures 3(a)-(d) and the preceeding discussion. ■

4.2 Optimal policies under general demand

Using Proposition 2, we are now in a position to state some general properties of optimal licensing policies.

Proposition 3 *Consider an outside innovator of a nondrastic innovation of magnitude ε who licenses its innovation using three part tariffs to firms in a Cournot oligopoly described in Section 2 and suppose the number of firms is at least three. The optimal licensing policies have the following properties.*

(I) *The maximum possible licensing revenue can be always attained by a policy that uses at most two of the three components and there always exists an optimal policy that uses no ad valorem royalty.*

(II) *There always exists an optimal policy in which the number of licenses offered is either $n-1$ or n. Moreover, for $1 \leq k \leq n-2$, if (k, δ) is part of an optimal policy in which k licenses are offered, then $\delta = \theta/k$, the post-innovation Cournot price equals c and the innovator obtains $G(c) = \varepsilon Q(c)$, where $G(p)$ is given in (3).*

(III) *The following hold if $\varepsilon > \theta/(n-1)$ (relatively significant innovations):*

   (i) *Under any optimal policy, the innovator obtains at least $G(c)$.*

   (ii) *Under any optimal policy, the post-innovation Cournot price is at least c and the post-innovation price falls below its pre-innovation level.*

   (iii) *It is not optimal to set a pure per unit royalty or a pure upfront fee.*

   (iv) *There always exists an optimal policy that is a two part tariff consisting of a positive per unit royalty and upfront fee, but no ad valorem royalty. There also exist other optimal policies that are three part tariffs whose all components (upfront fees, per unit and ad valorem royalties) are positive.*

Proof (I) By (17)-(18), the payoff function of the innovator is continuous and bounded, so there exists a policy that gives the maximum possible licensing revenue under three part tariffs. Consider an optimal policy and suppose the effective magnitude of the innovation under that policy is $\delta \in [0, \varepsilon]$. By Proposition 2(I), the unique acceptable and feasible policy supporting $\delta = 0$ is a pure per unit royalty and that supporting $\delta = \varepsilon$ is a pure upfront fee, so for each of these two polar cases the corresponding policy uses only one of the three components. For any $\delta \in (0, \varepsilon)$, by Proposition 2(II), there always exists an acceptable and feasible two part tariff consisting of a positive per unit royalty and upfront fee that supports δ, completing the proof of (I).

See the Appendix for the proofs of (II)-(III). ■

Some of the conclusions of Proposition 3 (e.g., existence of optimal policies with $n-1$ or n licensees, the lower bound $G(c)$) were obtained in Sen and Tauman (2018) for two part tariffs with per unit royalties and upfront fees.

Proposition 3 shows that there always exists an optimal policy that uses no ad valorem royalty. This does not imply that there never exists an optimal policy with an ad valorem royalty. As we demonstrate in Proposition 4 with linear demand, for significant innovations, a two part tariff with ad valorem royalty and upfront fee is optimal. Such a policy can be also optimal for intermediate innovations under certain parameric configurations.

The literature has sometimes suggested to impose an "antitrust constraint" on licensing policies that requires the post-licensing prices to not exceed the price without licensing (see, e.g., Niu 2013, Fan et al. 2018b; see Niu 2014 for other policy interventions). Such a constraint ensures that licensing does not lower consumer surplus. For an outside innovator, in the absence of licensing the price is simply the pre-innovation Cournot price. Proposition 3 shows that for relatively signficant innovations, the post-licensing Cournot price is lower than its pre-innovation level, so for these innovations, optimal three part tariffs improve consumer surplus without any external constraint.

4.3 Optimal policies under linear demand

To completely characterize optimal three part tariffs, we consider a linear demand curve given by

$$p(Q) = \max\{a - Q, 0\} \text{ where } a > c > 0 \tag{21}$$

Note that in this case $\theta = c/\eta(c) = a - c$. So a cost-reducing innovation of magnitude ε is drastic if $\varepsilon \geq a - c$ and nondrastic if $\varepsilon < a - c$. Also note from (3) that in this case $G(c) = \varepsilon(a - c)$.

In a Cournot oligopoly under demand (21), optimal two part tariffs with combinations of per unit royalties and upfront fees are characterized in Sen and Tauman (2007). In terms of our framework, a two part tariff with a per unit royalty and an upfront fee is simply a three part tariff with $v = 0$. Recall that any such two part tariff corresponds to a point on the horizontal line OB in Figure 3(a)-(d). Taking $v = 0$ in (7), $\delta(r, 0) = \delta \Leftrightarrow \delta = \varepsilon - r$. This shows that payoffs of I under two part tariffs with per unit royalties and upfront fees are again functions of δ given by (17)-(18), but in this case there is a one-to-one relation between δ and r (specifically, $\delta = \varepsilon - r$), so the multiplicity we face for three part tariffs disappears.

Because as functions of δ, the payoffs that I seeks to maximize are same, we can apply the results of Sen and Tauman (2007) as follows: identify optimal (k, δ) from their results and then determine the set of all acceptable and feasible combinations of r, v that supports that (k, δ). This set is simply $\mathbb{S}^n(k, \delta)$ given in (20). By Proposition 2(II), whenever $0 < \delta < \varepsilon$, the set $\mathbb{S}^n(k, \delta)$ is a continuum.

Consider again the illustration in Figure 3(a), where a certain $\delta \in (0, \varepsilon)$ is identified as point D. The line CD presents the set $\mathbb{S}^n(k, \delta)$. While this general conclusion holds for any $\delta \in (0, \varepsilon)$, to precisely identify $\mathbb{S}^n(k, \delta)$, we need to know the location of δ in relation to the threshold $\hat{\delta}^n(k)$ (see Figures 3(a)-(d)). If $\delta < \hat{\delta}^n(k)$, the line CD meets the curve OA at C (in that case C would correspond to a two part royalty policy) while if $\delta > \hat{\delta}^n(k)$, the line CD meets the curve AB at point C (so C would correspond to a two part tariff with an ad valorem royalty and an upfront fee). The next proposition makes this comparison of optimal δ with $\hat{\delta}^n(k)$ to precisely identify the set of all optimal three part tariffs.

Remark 6 It is assumed in Sen and Tauman (2007, p.167) that "...whenever the innovator is indifferent between two licensing policies, it chooses the one where the number of licensees is higher". Under this simplifying assumption, the number of licenses sold is at least $n-1$, because there always exists an optimal policy with $n-1$ or n licenses (Proposition 3(II)). We relax this assumption and determine all optimal three part tariffs by noting that for $n \geq 3$: (i) if there is any optimal policy in which less than $n-1$ licenses are offered, I must obtain $G(c)$ under that policy (Proposition 3(II)) and (ii) under linear demand, the payoff of I at any optimal policy is always at least $G(c)$ (see Proposition 3(a), ibid., p.172). By (i) and (ii), under any optimal policy with $n-1$ or n licenses, if I obtains more than $G(c)$, there is no optimal policy with less than $n-1$ licenses, but if I obtains exactly $G(c)$, it is possible to have other optimal policies with less than $n-1$ licenses (see Proposition 4(II)).

For $n \geq 2$, define the function $\delta_n^* : \{n-1, n\} \to R_+$ as

$$\delta_n^*(k) := \begin{cases} (\theta + 2\varepsilon)/2(n-1) & \text{if } k = n-1 \\ [(n-1)\theta + (n+1)\varepsilon]/2(n^2 - n + 1) & \text{if } k = n \end{cases} \tag{22}$$

Define the functions

$$z(n) := [n^3 - n + \sqrt{(n+1)(n^2-n+1)(n^3 - 6n^2 + 5n - 4)}]/(2n^2 - n + 1),$$

$$u(n) := (n+1)(1 + \sqrt{n^2 - n + 1})^2/n(n-1)^2 \tag{23}$$

Proposition 4 *In a Cournot oligopoly under linear demand (21) with an outside innovator of a nondrastic innovation of magnitude ε who uses three part tariffs, for any $n \geq 4$, the optimal licensing policies have the following general properties.*

(A) *Under any optimal policy, the innovator obtains at least $G(c) = \varepsilon(a - c)$ and the post-licensing Cournot price is lower than the price without licensing.*

(B) *A pure per unit royalty is never optimal. A pure ad valorem royalty is never optimal for generic magnitudes of a, c, ε.*

(C) *For insignificant innovations, it is optimal to use a pure upfront fee.*

(D) *For significant or intermediate innovations:*

  (i) *There always exists an optimal policy that is a two part tariff with a positive per unit royalty and upfront fee;*

  (ii) *There is a continuum of other optimal policies whose all three components (per unit royalty, ad valorem royalty, upfront fee) are positive;*

(iii) *A two part tariff with ad valorem royalty and upfront fee or a two part royalty with per unit and ad valorem royalties may be optimal depending on the magnitude of the innovation and the demand intercept.*

Specific properties of the optimal policies are as follows, where the functions $z(n)$, $u(n)$ *are given in* (33), *with* $\theta/(2n-4) < \theta/z(n) < \theta/5$ *and* $\theta/2 < \theta/u(n) < \theta$ *for* $n \geq 7$.

(I) (**Insignificant innovations**) *If* $\varepsilon < \theta/(2n-4)$, *the unique optimal policy is to sell* $n-1$ *or* n *licenses using a pure upfront fee, the Cournot price is higher than* c *and the innovator obtains more than* $G(c)$.

(II) (**Significant innovations**) *If* $[n = 4, 5, 6$ *and* $\varepsilon > \theta/2]$ *or* $[n \geq 7$ *and* $\varepsilon > \theta/u(n)]$, *there is a continuum of optimal licensing policies. Under any optimal policy the Cournot price equals* c *and the innovator obtains* $G(c)$. *The set of all optimal policies is* $\cup_{k=2}^{n-1}\mathbb{S}^n(k, \theta/k)$. *For* $k \in \{2, \ldots, n-1\}$, *under any optimal policy in* $\mathbb{S}^n(k, \theta/k)$ (*presented by the line* DE *in Figure* 4(a)):

- (i) *The license is sold to* k *firms, the effective magnitude of the innovation is* θ/k *and a* k*-firm natural oligopoly is created with* k *licensees, with all non-licensees dropping out of the market. Each licensee pays* $(1-v)\overline{\phi}^n(k, \theta/k)$ *as upfront fee and obtains zero net payoff.*
- (ii) *There exists an optimal policy that is a two part tariff consisting of a positive per unit royalty and upfront fee* (*the point* E *in Figure* 4(a)).
- (iii) *There also exists an optimal policy that is a two part tariff consisting of a positive ad valorem royalty and upfront fee* (*the point* D *in Figure* 4(a)).
- (iv) *There is a continuum of other optimal policies that are three part tariffs with all components positive* (*points on the line* DE *excluding points* D, E *in Figure* 4(a)).

(III) (**Intermediate innovations**) *If* $n \geq 7$ *and* $\theta/z(n) < \varepsilon < \theta/u(n)$, *there is a continuum of optimal licensing policies. Under any optimal policy, the license is sold to all* n *firms and the effective magnitude of the innovation is* $\delta_n^*(n)$ *given in* (22). *The Cournot price is higher than* c *and the innovator obtains more than* $G(c)$. *Specifically:*

- (i) *There exists an optimal policy that is a two part tariff consisting of a positive per unit royalty and upfront fee* (*the points* E_1 *in Figure* 4(b), E_2 *in Figure* 4(c) *with* $k = n$).
- (ii) *There are also optimal policies that are three part tariffs with all components positive* (*points on the lines* D_1E_1, D_2E_2 *excluding points* D_1, E_1, D_2, E_2 *in Figures* 4(b), 4(c) *with* $k = n$).
- (iii) *There are functions* $2 < \underline{\kappa}(n) < 3$ *and* $\overline{\kappa}(n) > 5$ *such that*

(a) *If* $a < \underline{\kappa}(n)c$ *(the demand intercept is relatively small), then* $\delta_n^*(n) > \hat{\delta}^n(n)$ *and the set of all optimal policies is* $\mathbb{S}^n(n, \delta_n^*(n))$*, given by* D_1E_1 *in Figure* 4(b) *with* $k = n$*. In this case there exists an optimal policy that is a two part tariff consisting of a positive ad valorem royalty and upfront fee (the point* D_1 *in Figure* 4(b)*).*

(b) *If* $a > \overline{\kappa}(n)c$ *(the demand intercept is relatively large), then* $\delta_n^*(n) < \hat{\delta}^n(n)$ *and the set of all optimal policies is* $\mathbb{S}^n(n, \delta_n^*(n))$*, given by* D_2E_2 *in Figure* 4(c) *with* $k = n$*. In this case there exists an optimal policy that is a two part royalty having positive ad valorem and unit royalties but no upfront fees (the point* D_2 *in Figure* 4(c)*).*

(c) *If* $\underline{\kappa}(n)c < a < \overline{\kappa}(n)c$ *(the demand intercept is of intermediate size),* $\exists\ \overline{\varepsilon}(n) \in (\theta/z(n), \theta/u(n))$ *such that if* $\varepsilon < \overline{\varepsilon}(n)$*, the conclusion is the same as* (a) *and if* $\varepsilon > \overline{\varepsilon}(n)$*, the conclusion is the same as* (b).

(IV) (**Intermediate innovations**) *If* $[n = 4, 5$ *and* $\theta/(2n-4) < \varepsilon < \theta/2]$ *or* $[n \geq 6$ *and* $\theta/(2n-4) < \varepsilon < \theta/z(n)]$ *there is a continuum of optimal licensing policies. Under any optimal policy, the license is sold to* $n-1$ *firms and the effective magnitude of the innovation is* $\delta_n^*(n-1)$ *given in* (22)*. The Cournot price is higher than* c *and the innovator obtains more than* $G(c)$*. Specifically:*

(i) *There exists an optimal policy that is a two part tariff consisting of a positive per unit royalty and upfront fee (the points* E_1 *in Figure* 4(b)*,* E_2 *in Figure* 4(c) *with* $k = n-1$*).*

(ii) *There are also optimal policies that are three part tariffs with all components positive (points on the lines* D_1E_1*,* D_2E_2 *excluding points* D_1, E_1*,* D_2, E_2 *in Figures* 4(b)*,* 4(c) *with* $k = n-1$*).*

(iii) *There is a function* $\kappa(n) \geq 3$ *such that*

(a) *If* $a < \kappa(n)c$ *(the demand intercept is relatively small), then* $\delta_n^*(n-1) > \hat{\delta}^n(n-1)$ *and the set of all optimal policies is* $\mathbb{S}^n(n-1, \delta_n^*(n-1))$*, given by* D_1E_1 *in Figure* 4(b) *with* $k = n-1$*. In this case there exists an optimal policy that is a two part tariff consisting of a positive ad valorem royalty and upfront fee (the point* D_1 *in Figure* 4(b)*).*

(b) *If* $a > \kappa(n)c$ *(the demand intercept is relatively large),* $\exists\ \underline{\varepsilon}(n)$ *in the relevant domain of* ε *such that if* $\varepsilon < \underline{\varepsilon}(n)$*, the conclusion is the same as* (a)*. If* $\varepsilon > \underline{\varepsilon}(n)$*, then* $\delta_n^*(n-1) < \hat{\delta}^n(n-1)$*, the set of all optimal policies is* $\mathbb{S}^n(n-1, \delta_n^*(n-1))$ *(given by* D_2E_2 *in Figure* 4(c) *with* $k = n-1$*) and there exists an optimal policy that is a two part royalty having positive ad valorem and unit royalties but no upfront fees (the point* D_2 *in Figure* 4(c)*).*

(V) (**Limiting results**) *For* $n \geq 7$*,* $\theta/(2n-4)$*,* $\theta/z(n)$ *are decreasing and* $\theta/u(n)$ *is increasing in* n*, with* $lim_{n\to\infty}\ \theta/(2n-4) = lim_{n\to\infty}\ \theta/z(n) = 0$*,* $lim_{n\to\infty}\theta/u(n) = \theta$*. Moreover* $\underline{\kappa}(n)$ *is decreasing and* $\overline{\kappa}(n)$ *is increasing in* n*, with* $lim_{n\to\infty}\underline{\kappa}(n)c = 2c$*,* $lim_{n\to\infty}\overline{\kappa}(n)c = \infty$*.*

Consequently for every a, c, ε where $a > c > \varepsilon$ and $0 < \varepsilon < \theta$, $\exists\ \bar{n}(a, c, \varepsilon)$ such that for all $n > \bar{n}(a, c, \varepsilon)$:

(i) *There is a continuum of optimal licensing policies. Under any optimal policy, the license is sold to all n firms, the effective magnitude of the innovation is $\delta_n^*(n)$ given in (22), the Cournot price is higher than c and the innovator obtains more than $G(c)$.*

(ii) *There exists an optimal policy that is a two part tariff consisting of a positive per unit royalty and upfront fee. There also exists a continuum of other optimal policies that are three part tariffs with all components positive.*

(iii) *If $a < 2c$, there exists an optimal policy that is a two part tariff consisting of a positive ad valorem royalty and upfront fee.*

(iv) *If $a > 2c$, then $c < \theta$ and $\exists\ \hat{\varepsilon} \in (0, c)$ (specifically, $\hat{\varepsilon} \equiv [2\sqrt{a^2 - 3ac + 3c^2} - (2a - 3c)]\theta/c$) such that if $\varepsilon < \hat{\varepsilon}$, there exists an optimal policy that is a two part tariff consisting of a positive ad valorem royalty and upfront fee and if $\varepsilon > \hat{\varepsilon}$, there exists an optimal policy that is a two part royalty having positive ad valorem and unit royalties but no upfront fees.*

Proof We present the proofs of general properties (A)-(D) and specific properties (I)-(II). Proofs of the remaining parts are given in the Appendix. Note from (7) that under two part tariffs with per unit royalties and fees, when the per unit royalty is r, the effective magnitude of the innovation is $\delta = \delta(r, 0) = \varepsilon - r$.

Property (A) follows from Proposition 3(a) (p.172) of Sen and Tauman (2007). From Table A.5 (ibid., p.183) we note that for $n \geq 4$, the effective magnitude δ of the innovation is positive under any optimal policy and so by Proposition 2(I), a pure unit royalty is never optimal. Note from Figures (3)(a)-(d) that the only point in the acceptable and feasible region OAB that corresponds to a pure ad valorem royalty is the point A, so such a policy can be optimal if and only if optimal δ exactly equals $\hat{\delta}^n(k)$, which is never the case for generic magnitudes of a, c, ε, which proves (B).

Again from Table A.5 (ibid., p.183), for insignificant innovations, it is optimal to set a pure upfront fee, which proves (C). For intermediate and significant innovations, it is optimal to set $\delta \in (0, \varepsilon)$ and (D) follows by Proposition 2(II)-(V).

For (I), note from Table A.5 (ibid., p.183) that for $n \geq 4$ and $\varepsilon < \theta/(2n-4)$, it is optimal to sell $n-1$ or n licenses with unit royalty $r = 0$, so the optimal δ is $\delta = \varepsilon$. The unique acceptable and feasible three part tariff that supports $\delta = \varepsilon$ is a pure upfront fee (Proposition 2(I)). Moreover the innovator obtains more than $G(c)$ under such a policy, so there is no optimal policy with less than $n-1$ licenses (see Remark 6).

For (II), note from Table A.5 (ibid., p.183) that for $\varepsilon > \theta/u(n)$, it is optimal to sell $n-1$ licenses with $\delta = \theta/(n-1)$ and I obtains $G(c)$. Noting that $1 < u(n) < 2$ for $n \geq 7$, when $\varepsilon > \theta/u(n)$, we have $\varepsilon > \theta/2$. Thus $\varepsilon > \theta/k$ for $k \geq 2$. Since $\delta \in [0, \varepsilon]$, for $2 \leq k \leq n-1$, it is possible to offer three part tariffs with k licenses and $\delta = \theta/k$ (the innovation being nondrastic, we have $\varepsilon < \theta$, so it is not possible to offer $k = 1$ and $\delta = \theta$). By Lemma 1(ii)(b), for any such policy, the post-innovation Cournot price equals c and by (17), I obtains $G(c)$. Thus, in addition to $(k = n-1, \delta = \theta/(n-1))$, (k, δ) is also part of an optimal three part tariff for $2 \le k \le n-2$ and $\delta = \theta/k$. Since the set of all acceptable and feasible combinations of r, v that supports $(k, \theta/k)$ is $\mathbb{S}^n(k, \theta/k)$ given in (20), the set of all optimal three part tariffs is $\cup_{k=2}^{n-1}\mathbb{S}^n(k, \theta/k)$. Part (II)(i) follows from (11) by noting that $\underline{\phi}^n(k, \theta/k) = 0$. Noting that $\varepsilon > \theta/k > \hat{\delta}^n(k)$, parts (II)(ii)-(iii) follow by Proposition 2((II),(III),(V)). ■

The next proposition shows that for relatively small sizes of industry ($n = 2, 3$), the conclusions for insignificant and intermediate innovations are qualitatively similar to Proposition 4. See Table 1 for the complete presentation of optimal three part tariffs for an outside innovator under linear demand for all $n \ge 2$.

Proposition 5 *In a Cournot oligopoly under linear demand (21) with an outside innovator of a nondrastic innovation of magnitude ε who uses three part tariffs, for $n = 2, 3$, the optimal licensing policies have the following properties, where $n - 1 < t(n) < 2n - 1$.*

(I) *Under any optimal policy, the innovator obtains at least $G(c)$ for $n = 3$ and lower than $G(c)$ for $n = 2$.*

(II) *If $\varepsilon < \theta/(2n-1)$, the unique optimal policy is to sell n licenses using a pure upfront fee. If $[n = 2$ and $\varepsilon > t(2)]$ or $[n = 3$ and $\varepsilon \in (\theta/t(3), \theta/2)]$, the unique optimal policy is to sell $n - 1$ licenses using a pure upfront fee.*

(III) *If $n = 3$ and $\varepsilon > \theta/2$, there is a continuum of optimal licensing policies. Under any optimal policy, the license is sold to 2 firms, the effective magnitude of the innovation equals $\theta/2$, the Cournot price equals c and a natural duopoly is created with the sole non-licensee dropping out of the market. The set of all optimal policies is $\mathbb{S}^n(2, \theta/2)$ and the same conclusion as* (II)(i)-(iv) *of Proposition 4 holds with $k = 2$.*

(IV) *If $\theta/(2n-1) < \varepsilon < \theta/t(n)$, there is a continuum of optimal policies. For any optimal policy, the innovation is licensed to all n firms and the effective magnitude of the innovation is $\delta_n^*(n)$ given in (22) and the same conclusion as* (III)(i)-(ii) *of Proposition 4 holds. There exists $\hat{\kappa}(n) > 2$ such that:*

   (i) *If $a < \hat{\kappa}(n)c$, the conclusion is the same as in Proposition 4*(III)(iii)(a).

   (ii) *If $a > \hat{\kappa}(n)c$, $\exists\ \underline{\varepsilon}(n) \in (\theta/(2n-1), \theta/t(n))$ such that if $\varepsilon < \underline{\varepsilon}(n)$, the conclusion is the same as in Proposition 4*(III)(iii)(a) *and if $\varepsilon > \underline{\varepsilon}(n)$, the conclusion is the same as in Proposition 4*(III)(iii)(b).

Proof See the Appendix. ■

A general finding of Propositions 4,5 is that for intermediate innovations, a two part tariff with an ad valorem royalty and upfront fee is optimal when the demand intercept a is relatively small.

To see the intuition of this result, we note that for fixed n, c, ε and $k = n-1, n$, the relative gain in profit $\gamma^n(k,\delta)$ is decreasing in demand intercept a. Since $\bar{v}(\delta)$ does not involve a, by (16) it follows that the threshold $\hat{\delta}^n(k)$ is increasing in a. This is illustrated in Figure 5, where a fall in the demand intercept from a to $a_0 < a$ results in a shift of $\gamma^n(k,\delta)$ to $\gamma^n_0(k,\delta) > \gamma^n(k,\delta)$. The curve $\gamma^n_0(k,\delta)$ meets $\bar{v}(\delta)$ at $\hat{\delta}^n_0(k) < \hat{\delta}^n(k)$. As a result a larger interval $(\hat{\delta}^n_0(k), \varepsilon)$ of δ can support two part tariffs with ad valorem royalties and upfront fees. As shown in Figure 5, such two part tariffs are given by AB under demand intercept a and A_0B under demand intercept $a_0 < a$.

4.4 An incumbent innovator in a Cournot duopoly

The main driving force for the analysis of three part tariffs with an outside innovator is that for any r, v, the outcome at the Cournot stage is completely determined by $\delta(r,v)$ given in (7) and the Cournot price and quantities do not separately depend on r, v. This is not the case when the innovator is one of the incumbent firms, because the quantity of an incumbent innovator affects the price, which in turn affects the revenue from ad valorem royalty.

To illustrate this point, consider a Cournot duopoly with two firms 1, 2 under demand $p(Q) = \max\{a - Q, 0\}$, where $Q = q_1 + q_2$. Initially both firms have the same constant marginal cost c, where $0 < c < a$. Firm 1 has a cost-reducing innovation of magnitude ε that reduces the marginal cost from c to $c - \varepsilon$ $(0 < \varepsilon < c)$. The innovation is nondrastic: $\varepsilon < a - c$.

If firm 1 licenses the innovation[16] to firm 2 using a three part tariff with unit royalty $r \in [0, \varepsilon]$, ad valorem royalty $v \in [0, 1)$ and[17] upfront fee $f \geq 0$, the payoff of firm 2 at the Cournot stage is

$$\pi_2 = (1-v)p(Q)q_2 - (c-\varepsilon)q_2 - rq_2 - f = (1-v)[p(Q) - (c-\delta)]q_2 - f \tag{24}$$

where $\delta = \delta(r,v)$ given in (7). Thus, at the Cournot stage firm 2 solves the same problem as a firm that has marginal cost $c - \delta$. The payoff of firm 1 at the Cournot stage is the sum of: (i) its operating profit, (ii) the revenue from ad valorem royalty, (iii) the revenue from unit royalty and (iv) the fee f, which is

$$\pi_1 = [p(Q)q_1 - (c-\varepsilon)q_1] + vp(Q)q_2 + rq_2 + f \tag{25}$$

The revenue from unit royalty rq_2 is not affected by q_1. However, q_1 affects the price $p(Q)$, which in turn affects the ad valorem royalty revenue $vp(Q)q_2$. For this reason, unlike the case of an outside innovator, in this case the (unique) Cournot price and quantities are functions of both v and δ.

Let $p(v,\delta)$ be the Cournot price, $q_i(v,\delta)$ the Cournot quantity of firm i and $Q(v,\delta) = q_1(v,\delta) + q_2(v,\delta)$ (see Lemma L3 of the Appendix for their expressions). Let $\phi_i(v,\delta)$ be the operating Cournot

[16] If firm 1 licenses a nondrastic innovation with $r = \varepsilon$, $v = 0$, $f = 0$ (a pure per unit royalty policy), the effective marginal cost of firm 2 is $c - \varepsilon + r = c - \varepsilon + \varepsilon = c$, which is the same as not licensing. This ensures that firm 1 has the same operating Cournot profit as not licensing, but in addition it obtains a positive royalty revenue. This shows that for a nondrastic innovation, licensing is superior to not licensing for firm 1.

[17] Since the innovation is nondrastic, firm 2 earns a positive profit without a license. Any licensing policy with $v = 1$ gives a zero or negative payoff to firm 2, so such a policy will not be acceptable. For this reason we restrict $v < 1$. See also Remark 2.

profit of firm i, that is,

$$\phi_1(v,\delta) = [p(v,\delta) - (c-\varepsilon)]q_1(v,\delta) \text{ and } \phi_2(v,\delta) = [p(v,\delta) - (c-\delta)]q_2(v,\delta)$$

Also denote $\psi(v,\delta) := (1-v)\phi_2(v,\delta)$. Using these in (59)-(60), the payoffs of firms are

$$\pi_2 = (1-v)\phi_2(v,\delta) - f = \psi(v,\delta) - f \text{ and } \pi_1 = \phi_1(v,\delta) + vp(v,\delta)q_2(v,\delta) + rq_2(v,\delta) + f \tag{26}$$

Since the innovation is nondrastic, without a license firm 2 obtains a positive profit $\underline{\phi}$, so a policy is acceptable if and only if $\pi_2 = \psi(v,\delta) - f \geq \underline{\phi}$. Thus, for a policy (v,δ), the maximum upfront fee that can be obtained from firm 2 is $\bar{f}(v,\delta) = \psi(v,\delta) - \underline{\phi}$ which presents the surplus from a license. A policy is acceptable to firm 2 if and only if this surplus is non-negative.

The problem of firm 1 For any acceptable and feasible (v,δ), taking $f = \bar{f}(v,\delta)$ in (61), the payoff of firm 1 is $\pi_1 = \phi_1(v,\delta) + [p(v,\delta) - (1-v)(c-\delta) + r]q_2(v,\delta) - \underline{\phi}$. As $\delta = \delta(r,v)$ (given in (7)), we have $(1-v)(c-\delta) = c - \varepsilon + r$, so that $\pi_1 = G(p(v,\delta)) - \underline{\phi}$ where $G(p) = [p-(c-\varepsilon)]Q(p)$ is the monopolist's profit at price p under marginal cost $c-\varepsilon$.

Note that $p_M(\varepsilon) = (a+c-\varepsilon)/2$ is the monopoly price under marginal cost $c-\varepsilon$. For $\delta \in [0,\varepsilon]$ and $v \in [0,1)$, the Cournot price $p(v,\delta)$ is increasing in v, decreasing in δ and $\lim_{v\uparrow 1} p(v,0) = p_M(\varepsilon)$ (see Lemma L3, Appendix), so $p(v,\delta) < p_M(\varepsilon)$. As $G(p)$ is increasing for $p < p_M(\varepsilon)$, the payoff of firm 1 is increasing in $p(v,\delta)$, so the payoff is maximum at policies where $p(v,\delta)$ is maximum.

Acceptable and feasible policies Note that $\psi(v,\delta)$ is decreasing in v and increasing in δ (see Lemma L3, Appendix). As $\psi(0,0) = \underline{\phi}$, $\psi(0,\delta) > \underline{\phi}$ for $\delta > 0$ and $\lim_{v\uparrow 1} \psi(v,\delta) = 0 < \underline{\phi}$ for all δ, for every $\delta \in [0,\varepsilon]$, $\exists$ a unique $\gamma(\delta)$ such that a policy is acceptable (that is, $\psi(v,\delta) \geq \underline{\phi}$) if and only if $v \leq \gamma(\delta)$. Note that $\gamma(0) = 0$ and $\gamma(\delta)$ is increasing.

Recall that for any $\delta \in [0,\varepsilon]$, the maximum feasible v for that δ is given by $\bar{v}(\delta) = (\varepsilon-\delta)/(c-\delta)$ (see Figure 1). Noting that $\bar{v}(\delta)$ is decreasing in δ with $\bar{v}(0) = \varepsilon/c > \gamma(0) = 0$ and $\bar{v}(\varepsilon) = 0 < \gamma(\varepsilon)$, $\exists\ \delta_A \in (0,\varepsilon)$ such that $\min\{\gamma(\delta), \bar{v}(\delta)\} = \gamma(\delta)$ if $\delta \leq \hat{\delta}$ (given by the curve OA in Figure 6) and $\min\{\gamma(\delta), \bar{v}(\delta)\} = \bar{v}(\delta)$ if $\delta > \hat{\delta}$ (given by the curve AB in Figure 6). The set of all acceptable and feasible (v,δ) is $\{(v,\delta) | 0 \leq \delta \leq \varepsilon, 0 \leq v \leq \min\{\gamma(\delta), \bar{v}(\delta)\}\}$, which is given by the region OAB in Figure 6. The problem of firm 1 is to find (v,δ) in OAB where the Cournot price $p(v,\delta)$ is maximum.

The modified problem under an antitrust constraint Following the *upward pricing pressure* (UPP) principle of merger policy (see, e.g., Farrell and Shapiro, 2010), it is sometimes suggested in the literature that licensing agreements should be subject to an "antitrust constraint" which requires that the post-licensing price must not exceed the price without licensing (see, e.g., Niu 2013; Fan et al. 2018b). The modified problem of firm 1 under this constraint will be to find those (v,δ) that give the maximum Cournot price in the subset of OAB where the post-licensing Cournot price does not exceed the price without licensing. The next propostion identifies optimal three part tariffs for both problems: one without the antitrust constraint and the modified problem with the constraint.

Proposition 6 *Consider a Cournot duopoly with two firms* $1, 2$ *under linear demand* (21) *where firm* 1 *has a nondrastic innovation.*

(I) *When there is no constraint on post-licensing prices, the unique optimal three part tariff is the pure ad valorem royalty policy with* $v = \bar{v}(\delta_A)$*, given by the point* A *in Figure* 6*. Among all points in* OAB (*the set of all acceptable and feasible policies*)*, the point* A *results in the maximum possible Cournot price.*

(II) *When there is an antitrust constraint that the post-licensing price must not exceed the price without licensing, there is a continumm of payoff-equivalent optimal three part tariffs, represented by the curve* OE *in Figure* 6*. Under all of these policies, the Cournot price equals the price without licensing. In particular:*

  (i) *The point* O *corresponds to the pure per unit royalty policy* $r = \varepsilon$.

  (ii) *The point* E *corresponds to a two part tariff that has positive ad valorem royalty* $v = \bar{v}(\delta_E)$ *and positive upfront fee, but no per unit royalty.*

  (iii) *All other optimal policies* (*given by points on* OE *excluding* O *and* E) *are three part tariffs whose all three components* (*per unit, ad valorem royalties and upfront fees*) *are positive.*

Proof (I) The payoff of firm 1, $\pi_1 = G(p(v, \delta)) - \underline{\phi}$, is increasing in the Cournot price $p(v, \delta)$, so the payoff is maximum at policies where the Cournot price is maximum. Since $p(v, \delta)$ is increasing in v, for any δ, the payoff of firm 1 is maximum when v is maximum. Thus any optimal (v, δ) must be on the curve OA for $\delta \leq \delta_A$ and on the curve AB for $\delta > \delta_A$ in Figure 6.

Since the curve AB (which represents $v = \bar{v}(\delta)$) is decreasing, for any point on AB other than A, δ is higher and v is lower compared to A. Because $p(v, \delta)$ is increasing in v and decreasing in δ, the price at A is higher than the price at any other point on AB. This shows that any optimal (v, δ) must be on the curve OA.

For any (v, δ) on OA: $v = \gamma(\delta)$. As $\gamma(\delta)$ is increasing, there is an increasing function $h(v)$ such that $v = \gamma(\delta) \Leftrightarrow \delta = h(v)$ (see the Appendix for details). For any v on the line OD in Figure 6, $h(v)$ is given by OA and the corresponding Cournot price is $p(v, \delta) = p(v, h(v))$, which is increasing in v (see the Appendix for the proof). This shows that over all points on OA, the Cournot price is maximum at point A, so the unique optimal three part tariff is given by A.

At point A: $v = \gamma(\delta_A)$ (zero surplus from the license, so the upfront fee is zero) as well as $v = \bar{v}(\delta_A)$ (maximum feasible ad valorem royalty, so the unit royalty is zero). This shows that the unique optimal three part tariff A is a pure ad valorem royalty policy.

(II) Noting that $(v = 0, \delta = 0)$ results in the same Cournot price as in the case of no licensing, when there is no licensing, the price is $p(0, 0)$ (the price at the point O in Figure 6). The antitrust constraint requires to choose acceptable and feasible policies for which $p(v, \delta) \leq p(0, 0)$.

Recall that any point on OA other than O has a price higher than $p(0, 0)$. So the price at A is higher than $p(0, 0)$. The price at B, $p(0, \varepsilon)$, is lower than $p(0, 0)$. As prices on the curve AB increase as we move from B to A, there exists a point E on AB at which the price equals $p(0,0)$, so that prices are higher than $p(0,0)$ on the part AE and lower than $p(0,0)$ on the part EB. The point E is identified in Figure 6 and the corresponding δ is denoted by δ_E. For any point in the region FEB other than E, the price is lower than $p(0,0)$.

For any $\delta \in (0, \delta_E)$, the maximum v that makes (v, δ) both acceptable and feasible lies on OAE; it lies on OA if $\delta \leq \delta_A$ and on AE if $\delta_A < \delta < \delta_E$. For any v on OAE, we have $(v, \delta) = (m(\delta), \delta)$ where $m(\delta) = \min\{\bar{v}(\delta), \gamma(\delta)\}$. As the price at any point on OAE (excluding O) is higher than $p(0,0)$, for any $\delta \in (0, \delta_E)$: $p(0,\delta) < p(0,0) < p(m(\delta), \delta)$. So there is a unique $\hat{v}(\delta) \in (0, m(\delta))$ such that $p(v,\delta) \gtreqless p(0,0) \Leftrightarrow v \gtreqless \hat{v}(\delta)$. As $p(v,\delta)$ is increasing in v and decreasing in δ, $\hat{v}(\delta)$ is increasing in δ. The function $\hat{v}(\delta)$ is presented by the curve OE in Figure 6 (with $\hat{v}(0) = 0$ and $\hat{v}(\delta_E)$ coinciding with point E).

The region OEB is the set of all acceptable and feasible (v, δ) for which the post-licensing Cournot price $p(v,\delta)$ does not exceed $p(0,0)$. For any point on OE, $p(v,\delta) = p(0,0)$, whereas for any other point in the region OEB, $p(v,\delta) < p(0,0)$. Since the payoff of firm 1 is maximum when $p(v,\delta)$ is maximum, the set of all optimal policies is the continuum given by the curve OE where the price exactly equals $p(0,0)$. For any such policy, firm 2 obtains $\underline{\phi}$ and firm 1 obtains $G(p(0,0)) - \underline{\phi}$.

At O: $v = 0 = \gamma(0)$ (no ad valorem royalty and zero surplus from the license, so no upfront fee as well). As $\delta = 0$ at O, by (7), setting $\delta(r,v) = 0$ and $v = 0$ gives $r = \varepsilon$, proving (i). At E: $v > 0$ (positive ad valorem royalty), $v = \bar{v}(\delta_E)$ (maximum feasible v, so the unit royalty is zero) and $v < \gamma(\delta_E)$ (positive surplus from the license, so positive upfront fee), which is a two part tariff with ad valorem royalty and upfront fee, proving (ii).

For any point on OE excluding O and E: $v > 0$ (positive ad valorem royalty), $v < \bar{v}(\delta)$ (less than maximum feasible v, so the unit royalty is positive) and $v < \gamma(\delta)$ (positive surplus from the license, so positive upfront fee), so all three components are positive, proving (iii). ■

In the setting of this section, that is, in a Cournot duopoly with linear demand, symmetric initial constant marginal costs and an incumbent innovator who has a nondrastic cost-reducing innovation that lowers the marginal costs of both firms by the same magnitude, pure per unit royalties are superior to pure upfront fees (Wang, 1998) and they are also optimal among all combinations of unit royalties and upfront fees (Sen and Tauman, 2007), where the unique optimal policy is to set per unit royalty $r = \varepsilon$, implying that the post-licensing Cournot price equals the price without licensing.

In this setting, San Martín and Saracho (2010) find pure ad valorem royalties to be superior to pure per unit royalties. However, in the presence of the antitrust constraint that the post-licensing Cournot price must not exceed the price without licensing, under two part tariffs with fees and one kind of royalties, Fan et al. (2018b) show that offering per unit royalties with fees are equally profitable to ad valorem royalties plus fees. Since the unique optimal policy among combinations of unit royalties and fees is the pure unit royalty $r = \varepsilon$, it follows that this policy gives the same payoff as any optimal combination of ad valorem royalty and fee.

Proposition 6 qualifies the findings above by showing that when there is no constraint on post-licensing prices, a pure ad valorem royalty is the unique optimal policy among all three part tariffs. When post-licensing prices are constrained to not exceed the price without licensing, there is a continumm of optimal three part tariffs, all of which give the same price as the no licensing level. In particular, the pure unit royalty $r = \varepsilon$ is an optimal policy, there is another optimal policy that is a two part tariff consisting of an ad valorem royalty and fee and the rest of the optimal policies are three part tariffs whose all three components are positive.

5 Concluding remarks

This paper studies patent licensing for an outside innovator in a Cournot oligopoly who uses three part tariffs consisting of upfront fees, per unit and ad valorem royalties. Our analysis shows that the maximum possible licensing revenue under three part tariffs can be always attained by policies that use at most two components. This provides an explanation of why licensing contracts in practice more frequently use one or at most two components.

Our analysis is restricted to linear contracts and non-negative royalties. Many licensing contracts in real life are indeed simple and linear in nature (see, e.g., Macho-Stadler et al. 1996). However, it should be noted that more general royalties, non-linear or dynamic contracts can increase the licensing revenue and in some cases may even enable an innovator to extract the maximum industry profit from the licensees (see, e.g., Sen 2002, Liao and Sen 2005, Erutku and Richelle 2007, Fan et al. 2018c).

The analysis of three part tariffs for an incumbent innovator in a duopoly establishes two contrasts with the outsider case: (i) the market outcomes separately depend on per unit and ad valorem royalties and (ii) the constraint that the post-licensing price must not exceed the price without licensing can alter licensing policies and lower the price.

This paper presents a parsimonious model to theoretically study the implications of incorporating both kinds of royalties together with fees. Our analysis is carried out under constant unit costs of production where the innovation reduces the unit cost of any firm by the same magnitude. Our results provide a useful foundation to study alternative configurations such as scale economies, asymmetric reduction of costs across firms and informational asymmetry.

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Appendix

The Appendix is organized as follows. We begin with some intermediary lemmas in Section 5.1. The proof of Observation 1 presented in Section 5.2. The proof of the results for the monopoly is presented in Section 5.3. The proof of parts (v)-(vi) of Lemma 1 is presented in Section 5.4. The proof of Lemma 2 is presented in Section 5.5. Section 5.6 presents the proof of parts (II) and (III) of Proposition 3. Section 5.7 presents the proof of parts (III)-(V) of Proposition 4, with Section 5.7.1 presenting the proof of part (III), Section 5.7.2 presenting part (IV) and Section 5.7.3 part (V). Section 5.7.4 presents the proof of results for the regions (in particular, specific regions when $n = 6$) not covered in Proposition 4. Section 5.8 presents the proof of Proposition 5 (the results for $n = 2, 3$). Section 5.9 presents the results for the case of an incumbent innovator in a Cournot duopoly and in particular, presents the detailed proof of Proposition 6 in Section 5.9.1.

5.1 Some intermediary lemmas

It is useful to recall the set of alternative sufficient conditions for the Cournot oligopoly. It is assumed that either [A1-A3] or [A1-A2, A4-A5] hold, where:

(A1) The price function or the inverse demand function $p(Q)$: $\mathrm{R}_{++} \to \mathrm{R}_{+}$ is non-increasing and $\exists\, \overline{Q} > 0$ such that $p(Q)$ is decreasing and twice continuously differentiable for $Q \in (0, \overline{Q})$.

(A2) $\overline{p} \equiv \lim_{Q\downarrow 0} p(Q) > c$, $\lim_{Q\downarrow 0} p'(Q)Q + \overline{p} - c > 0$ and $\exists\, 0 < Q^c < Q^{c-\varepsilon} < \overline{Q}$ such that $p(Q^c) = c > p(Q^{c-\varepsilon}) = c - \varepsilon > p(\overline{Q})$.

(A3) $p(Q)$ is log-concave for $Q \in (0, \overline{Q})$.

(A4) For $p \in (0, \overline{p})$, the price elasticity $\eta(p) := -pQ'(p)/Q(p)$ is non-decreasing.

(A5) For $Q \in (0, \overline{Q})$, the revenue function $\gamma(Q) := p(Q)Q$ is strictly concave, that is, $\gamma''(Q) = 2p'(Q) + Qp''(Q) < 0$.

Also note that (A3) implies (A4).

For $\delta \in [0, c)$, let

$$\psi_\delta(Q) := [p(Q) - (c - \delta)]/[-p'(Q)] \tag{27}$$

Lemma L1 *If assumptions* [A1-A3] *hold and* $p(Q) > c - \delta$, *then the function* $\psi_\delta(Q)$ *given in* (27) *is decreasing in* Q.

Proof Note from (27) that

$$\psi'_\delta(Q) = -1 + [p(Q) - (c - \delta)]p''(Q)/[p'(Q)]^2$$

Since $p(Q) > c - \delta$, we note that $\psi'_\delta(Q) < 0$ if $p''(Q) \leq 0$. So let $p''(Q) > 0$. Since $c - \delta > 0$, in that case $\psi'_\delta(Q) < -1 + p(Q)p''(Q)/[p'(Q)]^2$. Since $p(Q)$ is log concave (Assumption A3), we have $p(Q)p''(Q) \leq [p'(Q)]^2$, implying that $-1 + p(Q)p''(Q)/[p'(Q)]^2 \leq 0$ and hence $\psi'_\delta(Q) < 0$, proving the assertion. ■

The next result does not depend on the assumptions above. This is a general result that holds under either [A1-A3] or [A1-A2, A4-A5].

Lemma L2 *Let*

$$\beta_\delta(p) := 1 - (p-c)^2/[p-(c-\delta)]^2 \tag{28}$$

If $p > c$, then $\beta_\delta(p)$ is decreasing in p for any $\delta > 0$.

Proof Immediate by taking derivative with respect to p. ∎

5.2 Proof of Observation 1

For $k = 1, \ldots, n$, recall that the function $H^k : (0, \overline{p}) \to R$ given in (2): $H^k(p) := p[1 - 1/k\eta(p)]$. Since Assumption (A3) implies Assumption (A4), under either [A1-A3] or [A1-A2, A4-A5], (A4) holds, that is, the function $\eta(p)$ is non-decreasing for $p \in (0, \overline{p})$. Thus, $\eta(p) \geq \eta(\widetilde{p})$ for $p > \widetilde{p}$ and $\eta(p) \leq \eta(\widetilde{p})$ for $p < \widetilde{p}$, so that $[1-1/k\eta(p)] \geq [1-1/k\eta(\widetilde{p})]$ for $p > \widetilde{p}$ and $[1-1/k\eta(p)] \leq [1-1/k\eta(\widetilde{p})]$ for $p < \widetilde{p}$. Then by (2): $H^k(p)/p \geq H^k(\widetilde{p})/\widetilde{p}$ for $p > \widetilde{p}$ and $H^k(p)/p \leq H^k(\widetilde{p})/\widetilde{p}$ for $p < \widetilde{p}$.

For $p, \widetilde{p} \in (0, \overline{p})$, $p/\widetilde{p} > 1$ for $p > \widetilde{p}$ and $p/\widetilde{p} < 1$ for $p < \widetilde{p}$. Thus, if $H^k(\widetilde{p}) > 0$, we have $pH^k(\widetilde{p})/\widetilde{p} > H^k(\widetilde{p})$ for $p > \widetilde{p}$ and $pH^k(\widetilde{p})/\widetilde{p} < H^k(\widetilde{p})$ for $p < \widetilde{p}$. Using these inequalities with the inequalities of the last paragraph, for $p > \widetilde{p}$: $H^k(p) = pH^k(p)/p \geq pH^k(\widetilde{p})/\widetilde{p} > H^k(\widetilde{p})$ and for $p < \widetilde{p}$: $H^k(p) = pH^k(p)/p \leq pH^k(\widetilde{p})/\widetilde{p} < H^k(\widetilde{p})$. This completes the proof of the observation. ∎

5.3 Proof of the results for monopoly

The monopoly problem Let $\varepsilon \in (0, c)$ and $\delta \in [0, \varepsilon]$. Consider a monopolist M who has marginal cost $c - \delta$. The profit function of M as function of Q is $\pi_\delta(Q) = [p(Q) - (c-\delta)]Q$. By assumption (A2), $\lim_{Q \downarrow 0} p(Q) \equiv \overline{p} > c$. Thus, there is a small positive Q for which $p(Q) > c \geq c - \delta$ and $\pi_\delta(Q) > 0$. This shows that the maximized value of the monopolist's profit must be positive and any Q that gives a negative or zero profit cannot be optimal. In particular, $Q = 0$ (for demand functions where $p(0)$ is well defined) is not optimal.

By assumptions (A1) and (A2), $p(Q)$ is non-increasing and $\exists\, Q^{c-\varepsilon} \in (0, \overline{Q})$ such that $p(Q^{c-\varepsilon}) = c - \varepsilon \leq c - \delta$. Thus $\pi_\delta(Q) \leq 0$ for $Q \geq Q^{c-\varepsilon}$. So it is sufficient to consider $Q < Q^{c-\varepsilon}$. By assumption (A1), $p(Q)$ is decreasing for $Q \in (0, Q^{c-\varepsilon})$. Noting that $\lim_{Q\downarrow 0} p(Q) \equiv \overline{p} > c \geq c - \delta$ and $p(Q^{c-\varepsilon}) = c - \varepsilon < c - \delta$, it follows that $\exists\, Q^{c-\delta} \in (0, Q^{c-\varepsilon})$ such that $p(Q) \gtreqless c - \delta \Leftrightarrow Q \lesseqgtr Q^{c-\delta}$ and so $\pi_\delta(Q) \leq 0$ for $Q \geq Q^{c-\delta}$. Thus, for all $\delta \in [0, \varepsilon]$, it is sufficient to consider $Q < Q^{c-\delta}$.

By assumption (A1), for $Q \in (0, Q^{c-\delta}) \subseteq (0, Q^{c-\varepsilon})$, $p(Q)$ is decreasing and twice continuously differentiable. Note that $\pi'_\delta(Q) = p'(Q)Q + p(Q) - (c-\delta)$. By assumption (A2): $\overline{p} \equiv \lim_{Q\downarrow 0} p(Q) > c$ and $\lim_{Q\downarrow 0} p'(Q)Q + \overline{p} - c > 0$, implying that $\lim_{Q\downarrow 0} \pi'_\delta(Q) = \lim_{Q\downarrow 0} p'(Q)Q + \overline{p} - (c-\delta) > 0$. Thus, $\pi_\delta(Q)$ is increasing for small positive values of Q. Next observe that $\pi'_\delta(Q^{c-\delta}) = p'(Q^{c-\delta})Q^{c-\delta} < 0$. This shows that $\pi_\delta(Q)$ is decreasing at $Q = Q^{c-\delta}$.

First suppose assumptions [A1-A2,A4-A5] hold. Then for $Q \in (0, \overline{Q})$, the revenue function $p(Q)Q$ is strictly concave and so is the profit function $\pi_\delta(Q)$. As $\pi_\delta(Q)$ is increasing for small positive values of Q and decreasing at $Q = Q^{c-\delta}$, it follows that $\pi_\delta(Q)$ attains a unique maximum over $Q \in (0, Q^{c-\delta})$, denoted by $Q_M(\delta)$ (the monopoly quantity under marginal cost $c-\delta$) and it satisfies the first order condition $\pi'_\delta(Q_M(\delta)) = 0$. Also note that $\pi_\delta(Q)$ is increasing for $Q \in (0, Q_M(\delta))$, decreasing for $Q \in (Q_M(\delta), Q^{c-\delta})$ and $\pi_\delta(Q) \leq 0$ for $Q \geq Q^{c-\delta}$.

Note that the profit of the monopolist as function of p is $G_\delta(p) = [p-(c-\delta)]Q(p)$ (for $\delta = \varepsilon$, this function is $G(p)$ given in (3)). Denoting $p_M(\delta) \equiv p(Q_M(\delta))$, it follows that the unique maximum of $G_\delta(p)$ is attained at $p = p_M(\delta)$. As $Q_M(\delta) \in (0, Q^{c-\delta})$, we have $p_M(\delta) \in (c-\delta, \overline{p})$. As $p(Q)$ is decreasing for $Q \in (0, \overline{Q})$, it follows that $G_\delta(p)$ is increasing for $p \in (c-\delta, p_M(\delta))$, decreasing for $p \in (p_M(\delta), \overline{p})$, $G_\delta(p) \leq 0$ for $p \in [0, c-\delta]$ and if $\overline{p}$ is finite, then $G_\delta(p) = 0$ for $p \geq \overline{p}$.

Next suppose assumptions [A1-A3] hold. Then $p(Q)$ is log-concave. Noting that $p > c-\delta$ for $Q \in (0, Q^{c-\delta})$, we can apply the conclusion of Lemma L1. We note that for $Q \in (0, Q^{c-\delta})$: $\pi'_\delta(Q) \gtreqless 0 \Leftrightarrow Q \lesseqgtr \psi_\delta(Q)$ where $\psi_\delta(Q)$ is given in (27). Note that as function of Q, $f(Q) = Q$ is increasing and $\psi_\delta(Q)$ is decreasing (by Lemma L1). As $\lim_{Q\downarrow 0} \psi_\delta(Q) = [\overline{p} - (c-\delta)]/[-\lim_{Q\downarrow 0} p'(Q)] > 0$ and $\lim_{Q\uparrow Q^{c-\delta}} \psi_\delta(Q) = 0 < Q^{c-\delta}$, it follows that $\exists$ a unique $Q \in (0, Q^{c-\delta})$ where $Q = \psi_\delta(Q)$. Denote this Q by $Q_M(\delta)$ (monopoly quantity under marginal cost $c-\delta$). This shows that $\pi'_\delta(Q_M(\delta)) = 0$, $\pi_\delta(Q)$ is increasing for $Q \in (0, Q_M(\delta))$, decreasing for $Q \in (Q_M(\delta), Q^{c-\delta})$ and its unique maximum is attained at $Q = Q_M(\delta)$. Denoting $p_M(\delta) = p(Q_M(\delta))$, similar conclusions hold when when we look at the profit of the monopolist as a function of p.

Recall that the elasticity at price p is $\eta(p) = -pQ'(p)/Q(p)$. Note that

$$G'_\delta(p) = [p-(c-\delta)]Q'(p) + Q(p) = Q'(p)[\{p-(c-\delta)\} - p/\eta(p)]$$

$$= Q'(p)[p\{1 - 1/\eta(p)\} - (c-\delta)] = Q'(p)[H^1(p) - (c-\delta)]$$

where $H^k(p)$ is given in (2). As $Q'(p) < 0$: $G'_\delta(p) = 0 \Leftrightarrow H^1(p) = c-\delta$. Since $p_M(\delta)$ satisfies $G'_\delta(p_M(\delta)) = 0$, we have $H^1(p_M(\delta)) = c-\delta$. In particular, when $\delta = \varepsilon$, we have $H^1(p_M(\varepsilon)) = c-\varepsilon$.

Comparative statics Let $\delta_1, \delta_2 \in [0, \varepsilon]$ such that $\delta_1 < \delta_2$. The quantity $Q_M(\delta_1)$ satisfies the first order condition $\pi'_{\delta_1}(Q_M(\delta_1)) = 0$, so that $p'(Q_M(\delta_1)) + p(Q_M(\delta_1)) - (c-\delta_1) = 0$. As $\delta_1 < \delta_2$, we have $Q^{c-\delta_1} < Q^{c-\delta_2}$. Since $Q_M(\delta_1) \in (0, Q^{c-\delta_1})$, we have $Q_M(\delta_1) \in (0, Q^{c-\delta_2})$.

Note that $\pi'_{\delta_2}(Q_M(\delta_1)) = p'(Q_M(\delta_1)) + p(Q_M(\delta_1)) - (c-\delta_2) > p'(Q_M(\delta_1)) + p(Q_M(\delta_1)) - (c-\delta_1) = 0$. Thus, $\pi'_{\delta_2}(Q_M(\delta_1)) > 0$. This shows that $\pi_{\delta_2}(Q)$ is increasing at $Q = Q_M(\delta_1)$, implying that $Q_M(\delta_1) < Q_M(\delta_2)$. Thus $Q_M(\delta)$ is increasing in δ and so $p_M(\delta)$ is decreasing in δ.

Denote by $\phi_M(\delta)$ the monopoly profit (the maximized value of the monopolist's profit) under marginal cost $c-\delta$, that is, $\phi_M(\delta) := \pi_\delta(Q_M(\delta))$. Let $\delta_1, \delta_2 \in [0, \varepsilon]$ such that $\delta_1 < \delta_2$. Noting that for any $Q > 0$, $\pi_{\delta_1}(Q) = [p(Q) - (c-\delta_1)]Q < \pi_{\delta_2}(Q) = [p(Q) - (c-\delta_2)]Q$, we have $\phi_M(\delta_1) = \pi_{\delta_1}(Q_M(\delta_1)) < \pi_{\delta_2}(Q_M(\delta_1))$. As $Q_M(\delta_2)$ is the unique maximizer of $\pi_{\delta_2}(Q)$ and $Q_M(\delta_1) < Q_M(\delta_2)$, we have $\pi_{\delta_2}(Q_M(\delta_1)) < \pi_{\delta_2}(Q_M(\delta_2)) = \phi_M(\delta_2)$. Thus $\phi_M(\delta_1) < \phi_M(\delta_2)$, proving that $\phi_M(\delta)$ is increasing in δ.

5.4 Proof of parts (v)-(vi) of Lemma 1

As mentioned in the main text, when either [A1-A3] or [A1-A2, A4-A5] hold, by applying the first and second order conditions of profit functions, it follows that $\mathcal{C}^n(k,\delta)$ has a unique equilibrium and parts (i)-(iii) hold. Part (iv) follows from parts (ii) and (iii) by applying Observation 1. In what follows, we show that (v) and (vi)(a)-(b) hold under either [A1-A3] or [A1-A2, A4-A5] and (vi)(c) holds under [A1-A2, A4-A5].

Proof of part (v): Suppose either [A1-A3] or [A1-A2, A4-A5] hold.

Case 1 $1 \le k \le n-1$: Let $\delta, \hat{\delta} \in [0,\varepsilon]$ and $\hat{\delta} > \delta$.

If $\delta < \theta/k \le \hat{\delta}$, then by parts (ii)(a)-(b) of Lemma 1, $p^n(k,\hat{\delta}) \le c$ and $c < p^n(k,\delta)$. Hence $p^n(k,\hat{\delta}) < p^n(k,\delta)$ in this case.

If $\delta < \hat{\delta} < \theta/k$, by part (ii)(a) of Lemma 1: $H^n(p^n(k,\delta)) = c - k\delta/n$, $H^n(p^n(k,\hat{\delta})) = c - k\hat{\delta}/n$ and $p^n(k,\delta), p^n(k,\hat{\delta}) \in (c,\overline{p}) \subset (0,\overline{p})$, where $H^n(p)$ is given in (2). Noting that $c - k\delta/n > c - k\hat{\delta}/n > c - \varepsilon > 0$, by Observation 1, we conclude that $p^n(k,\hat{\delta}) < p^n(k,\delta)$.

Finally if $\theta/k \le \delta < \hat{\delta}$, by part (ii)(b) of Lemma 1: $H^k(p^n(k,\delta)) = c - \delta$, $H^k(p^n(k,\hat{\delta})) = c - \hat{\delta}$ and $p^n(k,\delta), p^n(k,\hat{\delta}) \in (0,\overline{p})$. Noting that $c - \delta > c - \hat{\delta} \ge c - \varepsilon > 0$, by Observation 1, we conclude that $p^n(k,\hat{\delta}) < p^n(k,\delta)$. This proves the first statement of part (v) for $1 \le k \le n-1$.

Case 2 $k = n$: Let $\delta, \hat{\delta} \in [0,\varepsilon]$ and $\hat{\delta} > \delta$. By part (iii) of Lemma 1: $H^n(p^n(n,\delta)) = c - \delta$, $H^n(p^n(n,\hat{\delta})) = c - \hat{\delta}$ and $p^n(n,\delta), p^n(n,\hat{\delta}) \in (c-\delta,\overline{p}) \subset (0,\overline{p})$, where $H^n(p)$ is given in (2). Noting that $c - \delta > c - \hat{\delta} \ge c - \varepsilon > 0$, by Observation 1, we conclude that $p^n(n,\hat{\delta}) < p^n(n,\delta)$. This proves the first statement of part (v) for $k = n$.

To prove the second statement of part (v), let $1 \le k \le n-1$. First note from part (ii)(b) of Lemma 1 that $\underline{\phi}^n(k,\delta) = 0$ for $\delta \ge \theta/k$. Also note from part (ii)(a) of Lemma 1 that for any $\delta < \theta/k$, $\underline{\phi}^n(k,\delta) > 0 = \underline{\phi}^n(k,\theta/k)$. To complete the proof, it remains to show that $\underline{\phi}^n(k,\delta)$ is decreasing for $\delta < \theta/k$.

If [A1-A2, A4-A5] hold, then along the same lines as the proof of Lemma A.2 of Sen and Tauman (2018, p.44),[18] it follows that $\underline{\phi}^n(k,\delta)$ is decreasing for $\delta < \theta/k$.

Next suppose [A1-A3] hold. By part (ii)(a) of Lemma 1, for $\delta < \theta/k$, any non-licensee firm produces a positive Cournot quantity. The profit function of any non-licensee firm j is $p(Q)q_j - cq_j$. Thus, the quantity q_j of any non-licensee firm j satisfies the following first order condition: $q_j = [p(Q)-c]/[-p'(Q)]$, so that $q_j = \psi_0(Q)$ (where $\psi_\delta(Q)$ is given in (27)). Thus $\underline{q}^n(k,\delta) = \psi_0(Q^n(k,\delta))$ (where $Q^n(k,\delta) = k\overline{q}^n(k,\delta) + (n-k)\underline{q}^n(k,\delta)$ is the equilibrium industry quantity of $\mathcal{C}^n(k,\delta)$).

Let $\delta, \hat{\delta} \in [0,\theta/k)$ and $\hat{\delta} > \delta$. Then by part (ii)(a) and the first statement of part (v) of Lemma 1: $c < p^n(k,\hat{\delta}) < p^n(k,\delta)$. Thus $Q^n(k,\hat{\delta}) > Q^n(k,\delta)$ and by Lemma L1, $\psi_0(Q^n(k,\hat{\delta})) < \psi_0(Q^n(k,\delta))$, implying that $\underline{q}^n(k,\hat{\delta}) < \underline{q}^n(k,\delta)$. Thus $\underline{\phi}^n(k,\hat{\delta}) = [p^n(k,\hat{\delta}) - c]\underline{q}^n(k,\hat{\delta}) < \underline{\phi}^n(k,\delta) = [p^n(k,\delta) - c]\underline{q}^n(k,\delta)$, completing the proof of part (v).

Proof of part (vi): The first statement of (vi) is immediate, so consider $\delta \in (0,\varepsilon]$. We show that

[18] Sen, D., Tauman, Y. (2018) Patent licensing in a Cournot oligopoly: General results. Mathematical Social Sciences 96, 37-48.

(vi)(a)-(b) hold under both [A1-A3] and [A1-A2, A4-A5], while (vi)(c) holds under [A1-A2, A4-A5].

Proof of part (vi)(a) Suppose either [A1-A3] or [A1-A2, A4-A5] hold.

Case 1 $k = 1$: By part (i) of Lemma 1, $p^n(k-1, \delta) = p^n(0, \delta) > c$.

If $\delta \geq \theta$, then by part (ii)(b) of Lemma 1, $p^n(k, \delta) = p^n(1, \delta) \leq c$. Hence $p^n(1, \delta) < p^n(0, \delta)$ in this case.

If $0 < \delta < \theta$, then by parts (i) and (ii)(a) of Lemma 1: $H^n(p^n(0, \delta)) = c$, $H^n(p^n(1, \delta)) = c - \delta/n$ and $p^n(0, \delta), p^n(1, \delta) \in (c, \overline{p}) \subset (0, p)$, where $H^n(p)$ is given in (2). Since $c > c - \delta/n > c - \varepsilon > 0$, by Observation 1, it follows that $p^n(1, \delta) < p^n(0, \delta)$.

Case 2 $2 \leq k \leq n$: First suppose $0 < \delta < \theta/k$, so $\delta < \theta/(k-1)$. Then by part (ii)(a) of Lemma 1 (for $k \leq n-1$) and part (iii) of Lemma 1 (for $k = n$): $H^n(p^n(k, \delta)) = c - k\delta/n$. By part (ii)(a) of Lemma 1: $H^n(p^n(k-1, \delta)) = c - (k-1)\delta/n$. Moreover, $p^n(k, \delta), p^n(k-1, \delta) \in (c, \overline{p}) \subset (0, \overline{p})$. As $c - (k-1)\delta/n > c - k\delta/n \geq c - \varepsilon > 0$, by Observation 1, we have $p^n(k, \delta) < p^n(k-1, \delta)$.

Next suppose $\theta/k \leq \delta < \theta/(k-1)$. In this case by part (ii)(a) of Lemma 1: $p^n(k-1, \delta) > c$. By part (ii)(b) of Lemma 1 (for $k \leq n-1$) and part (iii) of Lemma 1 (for $k = n$): $p^n(k, \delta) \leq c$. Hence $p^n(k, \delta) < p^n(k-1, \delta)$.

Finally suppose $\delta \geq \theta/(k-1)$, so that $\delta > \theta(k)$. In this case by part (ii)(b) of Lemma 1 (for $k \leq n-1$) and part (iii) of Lemma 1 (for $k = n$): $H^k(p^n(k, \delta)) = c - \delta$. By part (ii)(b) of Lemma 1: $H^{k-1}(p^n(k-1, \delta)) = c - \delta$. Moreover, $p^n(k, \delta), p^n(k-1, \delta) \in (0, \overline{p})$. Note from (2) that for $p \in (0, \overline{p})$, the elasticity function $\eta(p)$ is positive and $H^k(p) > H^{k-1}(p)$ for any $k \geq 2$. Hence $H^k(p^n(k-1, \delta)) > H^{k-1}(p^n(k-1, \delta))$. Since $H^{k-1}(p^n(k-1, \delta)) = H^k(p^n(k, \delta))$ (both equal $c - \delta$), we conclude that $H^k(p^n(k-1, \delta)) > H^k(p^n(k, \delta)) = c - \delta > 0$. Then by Observation 1, it follows that $p^n(k-1, \delta) > p^n(k, \delta)$. This completes the proof of part (vi)(a).

Proof of part (vi)(b)

First we prove that for any $\delta > 0$**:** $\underline{\phi}^n(k-1, \delta) \geq \underline{\phi}^n(k, \delta)$ **for** $1 \leq k \leq n-1$. Under [A1-A2, A4-A5], the result follows from Lemma 2, p.488, Kamien et al. (1992).[19] In what follows, we prove the result under [A1-A3].

Case 1 $k = 1$: In this case, $k - 1 = 0$. Note from part (i) of Lemma 1 that when the number of licensees is zero, all firms are non-licensess and all of them obtain positive Cournot profit so that $\underline{\phi}^n(0, \delta) > 0$.

If $\delta \geq \theta$, then by part (ii)(b) of Lemma 1, $\underline{\phi}^n(1, \delta) = 0$. So $\underline{\phi}^n(1, \delta) < \underline{\phi}^n(0, \delta)$ in this case.

If $\delta < \theta$, then by part (ii)(a) of Lemma 1, all $n-1$ non-licensee firms produce positive Cournot quantities. From the first order condition of any non-licensee firm, we have $\underline{q}^n(1, \delta) = \psi_0((Q^n(1, \delta))$ (where $\psi_\delta(Q)$ is given by (27)). When the number of licensees is zero, all firms are non-licensees and the first order condition of any firm gives $\underline{q}^n(0, \delta) = \psi_0((Q^n(0, \delta))$. By part (vi)(a) of Lemma 1: $p^n(0, \delta) > p^n(1, \delta)$, implying $Q^n(0, \delta) < Q^n(1, \delta)$, so by Lemma L1: $\underline{q}^n(0, \delta) > \underline{q}^n(1, \delta)$. Since $p^n(0, \delta) > p^n(1, \delta) > c$ and $\underline{q}^n(0, \delta) > \underline{q}^n(1, \delta)$, we have $\underline{\phi}^n(0, \delta) = [p^n(0, \delta) - c]\underline{q}^n(0, \delta) > \underline{\phi}^n(1, \delta) =$

[19] Kamien, M.I., Oren, S.S., Tauman, Y. (1992) Optimal licensing of cost-reducing innovation. Journal of Mathematical Economics 21, 483-508.

$[p^n(1,\delta) - c]\underline{q}^n(1,\delta)$.

Case 2 $2 \le k \le n-1$: First suppose $\delta \ge \theta/(k-1)$. Then $\delta > \theta/k$ and by part (ii)(b) of Lemma 1: $\underline{\phi}^n(k-1,\delta) = \underline{\phi}^n(k,\delta) = 0$.

Next suppose $\theta/k \le \delta < \theta/(k-1)$. Then by parts (ii)(a)-(b) of Lemma 1: $\underline{\phi}^n(k,\delta) = 0$ and $\underline{\phi}^n(k-1,\delta) > 0$. Hence $\underline{\phi}^n(k-1,\delta) < \underline{\phi}^n(k,\delta)$.

Finally suppose $0 < \delta < \theta/k$. Then $\delta < \theta/(k-1)$. In this case by part (ii)(a) of Lemma 1: when there are k or $k-1$ licensees, all non-licensee firms produce positive Cournot quantities. From the first order condition of any non-licensee firm, we have $\underline{q}^n(m,\delta) = \psi_0((Q^n(m,\delta))$ for $m = k-1, k$ (where $\psi_\delta(Q)$ is given by (27)). By part (vi)(a) of Lemma 1: $p^n(k-1,\delta) > p^n(k,\delta)$, implying $Q^n(k-1,\delta) < Q^n(k,\delta)$, so by Lemma L1, $\underline{q}^n(k-1,\delta) > \underline{q}^n(k,\delta)$. Since $p^n(k-1,\delta) > p^n(k,\delta) > c$ and $\underline{q}^n(k-1,\delta) > \underline{q}^n(k,\delta)$, we have $\underline{\phi}^n(k-1,\delta) = [p^n(k-1,\delta) - c]\underline{q}^n(k-1,\delta) > \underline{\phi}^n(k,\delta) = [p^n(k,\delta) - c]\underline{q}^n(k,\delta)$.

Now we prove that for any $\delta > 0$**:** $\overline{\phi}^n(k,\delta) > \underline{\phi}^n(k,\delta)$ **for** $1 \le k \le n-1$. In what follows, it shown that this result holds under either [A1-A3] or [A1-A2, A4-A5].

First consider $\delta \ge \theta/k$. Then by part (ii)(b) of Lemma 1: $\overline{\phi}^n(k,\delta) > 0 = \underline{\phi}^n(k,\delta)$.

Next consider $0 < \delta < \theta/k$. Then all firms (licensees as well as non-licensees) produce positive Cournot quantities. The profit function of any licensee firm i is $p(Q)q_i - (c-\delta)q_i$ and its quantity q_i satisfies the first order condition: $q_i = [p(Q) - (c-\delta)]/[-p'(Q)]$. The profit function of any non-licensee firm j is $p(Q)q_j - cq_j$ and its quantity q_j satisfies the first order condition: $q_j = [p(Q) - c]/[-p'(Q)]$. Using the function $\psi_\delta(Q)$ from (27), we have $q_i = \psi_\delta(Q)$ and $q_j = \psi_0(Q)$. Thus, $\overline{q}^n(k,\delta) = \psi_\delta(Q^n(k,\delta))$ and $\underline{q}^n(k,\delta) = \psi_0(Q^n(k,\delta))$.

As $p' < 0$ and $\delta > 0$, by (27), we have $\psi_\delta(Q) > \psi_0(Q)$, so that $\psi_\delta(Q^n(k,\delta)) > \psi_0(Q^n(k,\delta))$. This implies $\overline{q}^n(k,\delta) > \underline{q}^n(k,\delta)$. Thus $\overline{\phi}^n(k,\delta) = [p^n(k,\delta) - (c-\delta)]\overline{q}^n(k,\delta) > [p^n(k,\delta) - c]\overline{q}^n(k,\delta) > \underline{\phi}^n(k,\delta) = [p^n(k,\delta) - c]\underline{q}^n(k,\delta)$.

Proof of part (vi)(c) We prove $\overline{\phi}^n(n,\delta) > \underline{\phi}^n(n-1,\delta)$ for $\delta > 0$ when [A1-A2, A4-A5] hold.

First note that if $\delta \ge \theta/(n-1)$, then by part (ii)(b) of Lemma 1, $\underline{\phi}^n(n-1,\delta) = 0$ and the inequality clearly holds.

Next consider $0 < \delta < \theta/(n-1)$ and let $\tilde{\delta}$ be such that $n\tilde{\delta} = (n-1)\delta$ (since $\delta < \theta/(n-1)$, we have $\tilde{\delta} = (n-1)\delta/n < \theta/n$; also note that $\tilde{\delta} < \delta$). Then applying the result of part (iv) of Lemma 1, we have $p^n(n-1,\delta) = p^n(n,\tilde{\delta})$ and so $Q^n(n-1,\delta) = Q^n(n,\tilde{\delta})$. Hence

$$\overline{\phi}^n(n,\tilde{\delta}) = [p^n(n,\tilde{\delta}) - (c-\tilde{\delta})]Q^n(n,\tilde{\delta})/n = [p^n(n-1,\delta) - (c-\tilde{\delta})]Q^n(n-1,\delta)/n \tag{29}$$

Note that $Q^n(n-1,\delta) = (n-1)\overline{q}^n(n-1,\delta) + \underline{q}^n(n-1,\delta)$. In the proof of the last paragraph of part (vi)(b), we have shown that under either [A1-A3] or [A1-A2, A4-A5], $\overline{q}^n(k,\delta) > \underline{q}^n(k,\delta)$ for $1 \le k \le n-1$, so in particular $\overline{q}^n(n-1,\delta) > \underline{q}^n(n-1,\delta)$. Therefore $Q^n(n-1,\delta)/n > \underline{q}^n(n-1,\delta)$. As $\tilde{\delta} > 0$, from (29) we have

$$\overline{\phi}^n(n,\tilde{\delta}) > [p^n(n-1,\delta) - c]\underline{q}^n(n-1,\delta) = \underline{\phi}^n(n-1,\delta)$$

When [A1-A2, A4-A5] hold, for any $1 \leq k \leq n$, the Cournot profit of a licensee $\overline{\phi}^n(k,\delta)$ is increasing in δ (see, Lemma A.2, Sen and Tauman 2018, p.44). Using this result, noting that $\delta > \tilde{\delta}$, we have $\overline{\phi}^n(n,\delta) > \overline{\phi}^n(n,\tilde{\delta})$. Because $\overline{\phi}^n(n,\tilde{\delta}) > \underline{\phi}^n(n-1,\delta)$, it follows that $\overline{\phi}^n(n,\delta) > \underline{\phi}^n(n-1,\delta)$. ■

5.5 Proof Lemma 2

As mentioned in the main text, parts (i)(a), (ii)(a) are immediate from (12) by noting that $\overline{\phi}^n(k,0) = \underline{\phi}^n(k,0)$ for $1 \leq k \leq n-1$, $\overline{\phi}^n(n,0) = \underline{\phi}^n(n-1,0)$ (part (vi), Lemma 1) and for $1 \leq k \leq n-1$, $\delta \geq \theta/k$: $\underline{\phi}^n(k,\delta) = 0$ (part (ii)(b), Lemma 1). In what follows, we show that property (i)(b) holds under either [A1-A3] or [A1-A2, A4-A5] and property (ii)(b) holds under [A1-A2, A4-A5].

Proof of (i)(b) Suppose either [A1-A3] or [A1-A2, A4-A5] hold. Let $1 \leq k \leq n-1$. Note that $\overline{\phi}^n(k,\delta) > \underline{\phi}^n(k,\delta) > 0$ for $0 < \delta < \theta/k$ (parts (ii)(a), (iv)(b) of Lemma 1). Then from (12), it follows that $0 < \gamma^n(k,\delta) < 1$ for $0 < \delta < \theta/k$.

To complete the proof of (i)(b), it remains to show that when $1 \leq k \leq n-1$, $\gamma^n(k,\delta)$ is increasing for $0 < \delta < \theta/k$. Let $1 \leq k \leq n-1$ and $\delta \in (0,\theta/k)$. Then by part (ii)(a) of Lemma 1, all firms produce positive Cournot quantities and the quantities of both a licensee and a non-licensee are obtained from first order conditions. In this case the quantity q_i of any licensee firm i is given by $q_i = [p(Q)-(c-\delta)]/[-p'(Q)]$ and the quantity q_j of any non-licensee firm j is given by $q_j = [p(Q)-c]/[-p'(Q)]$. Thus the equilibrium profit of any licensee firm i is $\pi_i = [p(Q)-(c-\delta)]q_i = [p(Q)-(c-\delta)]^2/[-p'(Q)]$ and that of any non-licensee firm j is $\pi_j = [p(Q)-c]q_j = [p(Q)-c]^2/[-p'(Q)]$. So the ratio π_j/π_i equals $[p(Q)-c]^2/[p(Q)-(c-\delta)]^2$. Therefore

$$\underline{\phi}^n(k,\delta)/\overline{\phi}^n(k,\delta) = [p^n(k,\delta)-c]^2/[p^n(k,\delta)-(c-\delta)]^2$$

Hence

$$\gamma^n(k,\delta) = [\overline{\phi}^n(k,\delta) - \underline{\phi}^n(k,\delta)]/\overline{\phi}^n(k,\delta) = \beta_\delta(p^n(k,\delta))$$

where the function $\beta_\delta(p)$ is given in (28). Note that

$$\mathrm{d}\beta_\delta(p^n(k,\delta))/\mathrm{d}\delta = [\partial\beta_\delta(p)/\partial p][\partial p^n(k,\delta)/\partial\delta]_{p=p^n(k,\delta)} + [\partial\beta_\delta(p)/\partial\delta]_{p=p^n(k,\delta)} \tag{30}$$

Since $\beta_\delta(p)$ is decreasing in p (Lemma L2) and $p^n(k,\delta)$ is decreasing in δ (part (v), Lemma 1), the first term on the right side of (30) is positive. Note from (28) that $\beta_\delta(p)$ is increasing in δ for any $p > c-\delta$, so the second term of (30) is also positive. This shows that $\beta_\delta(p^n(k,\delta))$ is increasing in δ and therefore $\gamma^n(k,\delta)$ is increasing in δ. This completes the proof of (i)(b).

Proof of (ii)(b) Suppose [A1-A2, A4-A5] hold and let $k = n$. Observe that under [A1-A2, A4-A5], $\overline{\phi}^n(n,\delta) > \underline{\phi}^n(n-1,\delta) > 0$ for $0 < \delta < \theta/(n-1)$ (parts (ii)(a), (vi)(c) of Lemma 1). Then from (12), it follows that $0 < \gamma^n(n,\delta) < 1$ for $0 < \delta < \theta/(n-1)$.

It remains to show that $\gamma^n(n,\delta)$ is increasing in δ for $0 < \delta < \theta/(n-1)$. Under [A1-A2, A4-A5], $\underline{\phi}^n(n-1,\delta)$ is decreasing in δ for $\delta \in (0,\theta/(n-1))$ (by part (v) of Lemma 1). By Lemma A.2, Sen and Tauman (2018, p.44), under [A1-A2, A4-A5]: $\overline{\phi}^n(n,\delta)$ is increasing in δ. From these results, using (12) it follows that $\gamma^n(n,\delta)$ is increasing for $0<\delta<\theta/(n-1)$. ∎

5.6 Proof of parts (II)-(III) of Proposition 3

The conclusions of Proposition 2 will be useful for the proof of Proposition 3. The proof of part (I) is given in the main text. Here we prove parts (II) and (III). First note from Proposition 2(I)-(II) that for any $k \in \{1,\ldots,n\}$ and $\delta \in [0,\varepsilon]$, the set $\mathbb{S}^n(k,\delta)$ given in (20) is non-empty (more specifically, $\mathbb{S}^n(k,0)$, $\mathbb{S}^n(k,\varepsilon)$ are both singleton sets and $\mathbb{S}^n(k,\delta)$ is a continuum for $0<\delta<\varepsilon$). This implies that for every $k \in \{1,\ldots,n\}$ and $\delta \in [0,\varepsilon]$, there is always at least one acceptable and feasible policy that supports (k,δ).

To determine optimal licensing policies for I, we choose $k \in \{1,\ldots,n\}$ and $\delta \in [0,\varepsilon]$ to maximize $\Pi^n(k,\delta)_{FRV}$ given in (17),(18). For any solution (k^*,δ^*) to this maximization problem, the set of all optimal licensing policies corresponding to this solution is the set of all acceptable and feasible (k^*,r,v) that supports (k^*,δ^*), which is the set $\{(k^*,r,v)|(r,v)\in \mathbb{S}^n(k^*,\delta^*)\}$, where $\mathbb{S}^n(k,\delta)$ is given in (20).

To prove the results, first we show that it is never optimal for I to choose either (i) (k,δ) with $\delta>\theta/k$ and $1\le k\le n-1$ or (ii) (n,δ) with $\delta>\theta/(n-1)$.

Note that $\delta\in[0,\varepsilon]$. For $1\le k\le n-1$, if $\varepsilon\le\theta/k$, it is not even feasible to choose (k,δ) with $\delta>\theta/k$. Similarly, if $\varepsilon\le\theta/(n-1)$, it is not even feasible to choose (n,δ) with $\delta>\theta/(n-1)$. So clearly policies satisfying (i) or (ii) above are not optimal in these cases. In what follows, we show that such policies are not optimal in cases where it is feasible to offer these policies, so for $1\le k\le n-1$, we show that any (k,δ) with $\delta>\theta/k$ is not optimal when $\varepsilon>\theta/k$ and any (n,δ) with $\delta>\theta/(n-1)$ is not optimal when $\varepsilon>\theta/(n-1)$.

By part (ii)(b) of Lemma 1, when there are k licensees with $1\le k\le n-1$ and $\delta\ge\theta/k$, the Cournot quantity and profit of any non-licensee is zero. Using this in (17)-(18), the payoff of I is $G(p^n(k,\delta))$ if either (i) $1\le k\le n-1$ and $\delta>\theta/k$ or (ii) $k=n$ and $\delta>\theta/(n-1)$ (note that if $\delta>\theta/(n-1)$, we also have $\delta>\theta/n$).

Recall that $p_M(\varepsilon)$ denotes the monopoly price under marginal cost $c-\varepsilon$. Since the innovation is nondrastic, we have $c<p_M(\varepsilon)$. Note by (3) that $G(p)$ (the monopolist's profit at price p under marginal cost $c-\varepsilon$) is increasing for $p\le c$. As $p^n(k,\delta)<c$ for $\delta>\theta/k$ (parts (ii)(b),(iii), Lemma 1), for either (i) $1\le k\le n-1$ and $\delta>\theta/k$ or (ii) $k=n$ and $\delta>\theta/(n-1)$ (and thus $\delta>\theta/n$), we have $G(p^n(k,\delta))<G(c)=G(p^n(k,\theta/k))$, showing that $(k,\theta/k)$ gives a strictly higher payoff[20] than (k,δ). This proves the assertion that it is never optimal for I to choose either (i) (k,δ) with $\delta>\theta/k$ and $1\le k\le n-1$ or (ii) (n,δ) with $\delta>\theta/(n-1)$.

In view of the assertion in the last paragraph, it is sufficient to consider (k,δ) with $\delta\le\theta/k$ for $1\le k\le n-1$ and (n,δ) with $\delta\le\theta/(n-1)$.

[20]Note that when $\varepsilon>\theta/k$, it is feasible to offer a policy $(k,\theta/k)$, because for such a policy, $\delta=\theta/k<\varepsilon$. As $0<\theta/k<\varepsilon$, by Proposition 2(II), $\mathbb{S}^n(k,\theta/k)$ is a continuum, so in particular there is at least one acceptable and feasible policy that supports $(k,\theta/k)$.

Proof of part (II) Since $n \geq 3$, we have $n-2 \geq 1$. To prove part (II) of Proposition 3, consider any (k, δ) with $1 \leq k \leq n-2$ and $\delta \leq \theta/k$. Consider another policy $(n-1, \tilde{\delta})$ such that $k\delta = (n-1)\tilde{\delta}$. By part (iv) of Lemma 1, $p^n(k, \delta) = p^n(n-1, \tilde{\delta})$.

If $\delta = \theta/k$, then $\tilde{\delta} = \theta/(n-1)$ and by Lemma 1(ii)(b): $p^n(k, \delta) = p^n(n-1, \tilde{\delta}) = c$, any non-licensee produces zero and obtains zero profit under both (k, δ) and $(n-1, \tilde{\delta})$.

If $\delta < \theta/k$, then $\tilde{\delta} < \theta/(n-1)$, so by Lemma 1(ii)(a): $p^n(k, \delta) = p^n(n-1, \tilde{\delta}) > c$. Under either (k, δ) or $(n-1, \tilde{\delta})$, the Cournot quantity of any non-licensee firm j is positive and it satisfies the first order condition: $q_j = [p(Q) - c]/[-p'(Q)]$. Since $p^n(k, \delta) = p^n(n-1, \tilde{\delta})$ and $Q^n(k, \delta) = Q^n(n-1, \tilde{\delta})$, we conclude that the Cournot quantity and profit of any non-licensee stay the same under (k, δ) and $(n-1, \tilde{\delta})$.

Thus for any $\delta \leq \theta/k$, we have $\underline{q}^n(k, \delta) = \underline{q}^n(n-1, \tilde{\delta})$ and $\underline{\phi}^n(k, \delta) = \underline{\phi}^n(n-1, \tilde{\delta})$. Using this in (17): $\Pi_{FRV}(n-1, \tilde{\delta}) - \Pi_{FRV}(k, \delta) = (n-1-k)\underline{q}^n(k, \delta) \geq 0$. This shows that for any policy with $1 \leq k \leq n-2$, there exists a policy with $k = n-1$ which gives the same or higher payoff. Hence there always exists an optimal policy with $k = n-1$ or $k = n$. This completes the proof of the first statement of part (II).

To prove the second statement of part (II), if (k, δ) is part of an optimal policy for $1 \leq k \leq n-2$, then we must have $\delta \leq \theta/k$ and $\Pi_{FRV}(n-1, \tilde{\delta}) = \Pi_{FRV}(k, \delta)$ where $k\delta = (n-1)\tilde{\delta}$. Note from the last paragraph that $\Pi_{FRV}(n-1, \tilde{\delta}) = \Pi_{FRV}(k, \delta) \Leftrightarrow (n-1-k)\underline{q}^n(k, \delta) = 0$. Since $k \leq n-2 < n-1$, $(n-1-k)\underline{q}^n(k, \delta) = 0 \Leftrightarrow \underline{q}^n(k, \delta) = 0$. As $\delta \leq \theta/k$, we note that $\underline{q}^n(k, \delta) = 0 \Leftrightarrow \delta = \theta/k$. When $\delta = \theta/k$, by Lemma 1(ii)(b): $p^n(k, \delta) = c$, $\underline{q}^n(k, \delta) = 0$, $\underline{\phi}^n(k, \delta) = 0$ and by (17), we have $\Pi_{FRV}(k, \delta) = G(c)$. Taking $p = c$ in (3), we have $G(c) = \varepsilon Q(c)$.

Proof of part (III) Let $\varepsilon > \theta/(n-1)$.

Proof of (III)(i) Since $\varepsilon > \theta/(n-1)$, for any policy with $\delta = \theta/(n-1)$, we have $0 < \delta < \varepsilon$. Taking $k = n-1$ and $\delta = \theta/(n-1)$ in Proposition 2(II), the set $\mathbb{S}^n(n-1, \theta/(n-1))$ is a continuum (where $S(k, \delta)$ is given in (20)). Thus, in particular, there is at least one acceptable and feasible policy with $k = n-1$ and $\delta = \theta/(n-1)$.

Consider any policy with $k = n-1$, $\delta = \theta/(n-1)$. By Lemma 1(ii)(b), $p^n(n-1, \theta/(n-1)) = c$, $\underline{q}^n(n-1, \theta/(n-1)) = 0$. Using this in (17), we have $\Pi_{FRV}(n-1, \theta/(n-1)) = G(c)$. This shows that there exist policies at which I can obtain $G(c)$. Taking $p = c$ in (3), we note that $G(c) = \varepsilon Q(c)$. Because I obtains $G(c)$ under some policies, it must obtain at least $G(c)$ under any optimal policy. This completes the proof of (III)(i).

Proof of the first statement of (III)(ii) that under any optimal policy, the post-innovation Cournot price is at least c: Since the innovation is nondrastic, we have $c < p_M(\varepsilon)$. Note that the function $G(p)$ (given in (3)) is increasing for $p < p_M(\varepsilon)$. Hence $G(p) < G(c)$ for $p < c$. Note from (17) and (18) that for any (k, δ), the payoff of I is bounded above by $G(p^n(k, \delta))$. If $p^n(k, \delta) < c$, I would obtain $G(p^n(k, \delta)) < G(c)$. Since I obtains at least $G(c)$ under any optimal policy (by part (III)(i)), under any optimal policy the resulting Cournot price must be at least c.

Proof of (III)(iii) Note that for a pure unit royalty policy: $v = 0$. Using this in (7), we have $\delta(r, 0) = \varepsilon - r$. Thus a pure unit royalty policy has $\delta = \varepsilon - r$, so that $r = \varepsilon - \delta$.

First consider a pure unit royalty policy with (k, δ) such that $\delta \geq \theta/k$. In this case, by parts (ii)(b) and (iii) of Lemma 1, $p^n(k, \delta) \leq c$, all non-licensees drop out of the market and the payoff of I is its revenue from unit royalty $rk\overline{q}^n(k, \delta) = (\varepsilon - \delta)k\overline{q}^n(k, \delta) = (\varepsilon - \delta)Q^n(k, \delta)$ (where $Q^n(k, \delta)$ is the equilibrium industry quantity of $\mathcal{C}^n(k, \delta)$). Note that

$$(\varepsilon - \delta)Q^n(k, \delta) < (\varepsilon - \delta)Q^n(k, \delta) + k\overline{\phi}^n(k, \delta)$$

$$= (\varepsilon - \delta)Q^n(k, \delta) + [p^n(k, \delta) - (c - \delta)]Q^n(k, \delta) = G(p^n(k, \delta))$$

where $G(p)$ is given in (3). Since $p^n(k, \delta) \leq c$, we have $G(p^n(k, \delta)) \leq G(c)$, so the revenue from unit royalty in this case is less than $G(c)$. By part (III)(i) this proposition, such a policy cannot be optimal.

Next consider a pure unit royalty policy with (k, δ) such that $\delta < \theta/k$. In this case, by parts (ii)(a) and (iii) of Lemma 1, $p^n(k, \delta) > c$, so the equilibrium industry output $Q^n(k, \delta)$ is less than $Q(c)$. The payoff of I is its revenue from unit royalty $rk\overline{q}^n(k, \delta) = (\varepsilon - \delta)k\overline{q}^n(k, \delta) \leq (\varepsilon - \delta)Q^n(k, \delta)$ $< (\varepsilon - \delta)Q(c) \leq \varepsilon Q(c) = G(c)$ and again by part (III)(i), such a policy cannot be optimal.

To show that a pure upfront policy is not optimal, first observe that for any pure upfront fee policy, we have both $r = 0$ and $v = 0$. Using this in (7), we have $\delta(0, 0) = \varepsilon$. Thus a pure upfront fee policy has $\delta = \varepsilon$. It is given that $n \geq 3$ and $\varepsilon > \theta/(n-1)$. Since the innovation is nondrastic, we have $\varepsilon < \theta$. Thus $\theta/(n-1) < \varepsilon < \theta$. For any $\varepsilon \in (\theta/(n-1), \theta)$, $\exists\ 2 \leq m \leq n-1$ such that $\theta/m \leq \varepsilon < \theta/(m-1)$.

For $k \geq m$ and $\delta = \varepsilon$, we have $\delta = \varepsilon \geq \theta/m \geq \theta/k$. Then by Lemma 1(ii)(b) and (iii), the resulting Cournot price $p^n(k, \varepsilon)$ is lower than c, all non-licensees drop out of the market and by (17), I obtains $G(p^n(k, \varepsilon)) < G(c)$. By part (III)(i), such a policy cannot be optimal.

For any policy with $k \leq m-1$ (so that $k < n-1$) and $\delta = \varepsilon$, we have $\delta = \varepsilon < \theta/(m-1) \leq \theta/k$. Consider $(n-1, \tilde{\delta})$ such that $(n-1)\tilde{\delta} = k\delta = k\varepsilon$. By Lemma 1(iv), $p^n(k, \varepsilon) = p^n(n-1, \tilde{\delta})$. Moreover (as shown in the proof of part (II)), $\underline{q}^n(k, \varepsilon) = \underline{q}^n(n-1, \tilde{\delta})$ and $\underline{\phi}^n(k, \varepsilon) = \underline{\phi}^n(n-1, \tilde{\delta})$. Using these in (17), we have $\Pi_{FRV}(n-1, \tilde{\delta}) - \Pi_{FRV}(k, \varepsilon) = (n-1-k)\underline{q}^n(k, \varepsilon) > 0$ (since $k < n-1$, we have $n-1-k > 0$ and since $\varepsilon < \theta/(m-1) \leq \theta/k$, by Lemma 1(ii)(a), we have $\underline{q}^n(k, \varepsilon) > 0$). This shows that any policy with $k \leq m-1$ and $\delta = \varepsilon$ also cannot be optimal. This proves that a pure upfront fee policy is not optimal.

Proof of the second statement of (III)(ii) that under any optimal policy, the post-innovation Cournot price falls below its pre-innovation level: Note from part (i) of Proposition 2 that the only feasible and acceptable three part tariff that supports $\delta = 0$ is the pure per unit royalty $r = \varepsilon$. We have shown in part (III)(iii) that it is not optimal to set a pure per unit royalty. Thus, if (k, δ) is part of an optimal policy, we must have $\delta > 0$. We have shown before in the proof of the first statement of (III)(ii) that for any such optimal policy, $p^n(k, \delta) \geq c$. Then by Lemma 1(ii)(a), $H^n(p^n(k, \delta)) = c - k\delta/n > 0$, where $H^n(p)$ is given in[21] (2). All firms have

[21] By Lemma 1(ii), $p^n(k, \delta) < c$ if and only if $\delta < \theta/k$ and in that case $H^n(p^n(k, \delta)) = c - k\delta/n$. To see that this holds also when $p^n(k, \delta) = c$, note that $p^n(k, \delta) = c$ if and only if $\delta = \theta/k$. In that case $c - k\delta/n = c - \theta/n$. Since

marginal cost $c = c - 0$ in the pre-innovation case. Taking $\delta = 0$, the Cournot oligopoly $\mathcal{C}^n(n, 0)$ corresponds to the pre-innovation scenario. By Lemma 1(iii), the pre-innovation Cournot price satisfies $H^n(p^n(n, 0)) = c > 0$. Since we must have $\delta > 0$ for any optimal policy, $c - k\delta/n < c$. Thus, $H^n(p^n(k, \delta)) < H^n(p^n(n, 0))$. Then by Observation 1, it follows that $p^n(k, \delta) < p^n(n, 0)$. This completes the proof of the assertion.

Proof of (III)(iv) Note from Proposition 2(I) that the only acceptable and feasible policy that supports $\delta = 0$ is a pure royalty policy $r = \varepsilon$ and the only acceptable and feasible policy that supports $\delta = \varepsilon$ is a pure upfront fee. We have shown that a pure royalty or a pure upfront fee is not optimal. Thus, any optimal policy must have $0 < \delta < \varepsilon$.

Consider any (k^*, δ^*) that constitutes an optimal policy. Then we must have $0 < \delta^* < \varepsilon$. Then by Proposition 2(II), $\mathbb{S}^n(k^*, \delta^*)$ (given in (20)) always contains a two part tariff consisting of a positive per unit royalty and upfront fee but no ad valorem royalty. Moreover, by Proposition 2(III), there always exist points in $\mathbb{S}^n(k^*, \delta^*)$ (there is a continuum of such points) that correspond to three part tariffs whose all components (unit royalty, ad valorem royalty and upfront fee) are positive. This completes the proof of part (III)(iv). ■

5.7 Proof of parts (III)-(V) of Proposition 4

5.7.1 Proof of part (III) of Proposition 4

Recall that for linear demand $p(Q) = \max\{a - Q, 0\}$, we have $\theta = a - c$. By Table A.5 (p.183) of Sen and Tauman (2007),[22] for $n \geq 7$, $\exists\ u(n) < v(n)$ such that when $\varepsilon \in (\theta/v(n), \theta/u(n))$, the unique optimal two part tariff with per unit royalty and upfront fee has $k = n$ and $r = \bar{\rho}_0(n)$ where

$$\bar{\rho}_0(n) := (n-1)[(2n-1)\varepsilon - \theta]/2(n^2 - n + 1) \tag{31}$$

Since we denote ad valorem royalties by v, to avoid notational confusion, we denote the function $v(n)$ of Sen and Tauman by $z(n)$. Thus, by Table A.5 of Sen and Tauman, we have

$$z(n) := [n^3 - n + \sqrt{(n+1)(n^2 - n + 1)(n^3 - 6n^2 + 5n - 4)}]/(2n^2 - n + 1) \tag{32}$$

We also note that (ibid., p.183)

$$u(n) := (n+1)(1 + \sqrt{n^2 - n + 1})^2/n(n-1)^2 \tag{33}$$

For $n \geq 7$ and $\varepsilon \in (\theta/z(n), \theta/u(n))$, the unique optimal two part tariff with per unit royalty and upfront fee has $k = n$ and $r = \bar{\rho}_0(n)$. Note from (22) that

$$\delta_n^*(n) := [(n-1)\theta + (n+1)\varepsilon]/2(n^2 - n + 1) = \varepsilon - \bar{\rho}_0(n) \tag{34}$$

$\theta = c/\eta(c)$, from (2) we have $H^n(c) = c - \theta/n$. Thus $H^n(p^n(k, \delta)) = H^n(c) = c - \theta/n = c - k\delta/n$, so the equality $H^n(p^n(k, \delta)) = c - k\delta/n$ also holds when $p^n(k, \delta) = c$.

[22] Sen, D., Tauman, Y. (2007) General licensing schemes for a cost-reducing innovation. Games and Economic Behavior 59, 163-186.

Therefore in this case, any optimal (k, δ) for a three part tariff has $k = n$ and $\delta = \delta_n^*(n)$. The set of all optimal licensing policies in this case is the set of all (r, v) that supports $\delta_n^*(n)$ for which (n, r, v) is both acceptable and feasible. This set is $\mathbb{S}^n(n, \delta_n^*(n))$, where $\mathbb{S}^n(k, \delta)$ is given in (20).

Proof of (III)(i)-(ii) Noting that $\delta_n^*(n) \in (0, \varepsilon)$, parts (III)(i)-(ii) follows by Proposition 2(II)-(III).

Proof of (III)(iii) Taking $k = n$ in parts (IV)-(V) of Proposition 2, we note that if $\delta_n^*(n) < \hat{\delta}^n(n)$, then $\mathbb{S}^n(n, \delta_n^*(n))$ contains a policy that is a two part royalty with positive per unit and ad valorem royalties but no upfront fee. On the other hand, if $\delta_n^*(n) > \hat{\delta}^n(n)$, then $\mathbb{S}^n(n, \delta_n^*(n))$ contains a policy that is a two part tariff with a positive ad valorem royalty and fee but no per unit royalty.

To prove this part, we need to find out when $\delta_n^*(n) < \hat{\delta}^n(n)$ and when $\delta_n^*(n) > \hat{\delta}^n(n)$. Regarding $\hat{\delta}^n(n)$, recall that (see Figures 3(c), 3(d))

$$\gamma^n(n, \delta) \gtreqless \bar{v}(\delta) \Leftrightarrow \delta \gtreqless \hat{\delta}^n(n) \tag{35}$$

where $\gamma^n(n, \delta)$ is given in (12) and $\bar{v}(\delta)$ is given in (**??**). From (35), we note that

$$\delta_n^*(n) \gtreqless \hat{\delta}^n(n) \Leftrightarrow \gamma^n(n, \delta_n^*(n)) \gtreqless \bar{v}(\delta_n^*(n)) \tag{36}$$

From (36) if $\delta_n^*(n) < \hat{\delta}^n(n)$, then $\gamma^n(n, \delta_n^*(n)) < \bar{v}(\delta_n^*(n))$ and the maximum v that can support $\delta_n^*(n)$ is $v = \gamma^n(n, \delta_n^*(n))$ (as shown in Figure 4(b)). On the other hand, if $\delta_n^*(n) > \hat{\delta}^n(n)$, then $\gamma^n(n, \delta_n^*(n)) > \bar{v}(\delta_n^*(n))$ and the maximum v that can support $\delta_n^*(n)$ is $v = \bar{v}(\delta_n^*(n))$ (as shown in Figure 4(c)).

Recall that $\theta \equiv a - c$. Taking $\delta = \delta_n^*(n)$ from (34) in (**??**), we have

$$\bar{v}(\delta_n^*(n)) = (n-1)[(2n-1)\varepsilon - \theta]/[(n-1)(2nc - a) + (n+1)(c - \varepsilon)] \tag{37}$$

Since $z(n) < 2n - 1$ for $n \geq 7$ and $\varepsilon \in (\theta/z(n), \theta/u(n))$, we have $\varepsilon > \theta/(2n-1)$, so $(2n-1)\varepsilon > \theta$. Thus the numerator of $\bar{v}(\delta_n^*(n))$ is positive. Moreover, since $c > \varepsilon$ and $(2n-1)\varepsilon > \theta = a - c$, we have $(2n-1)c > a - c$ so that $2nc > a$ and the denominator of $\bar{v}(\delta_n^*(n))$ is also positive.

Note that under the linear demand $p(Q) = \max\{a - p, 0\}$, for $\delta = \delta_n^*(n)$, the Cournot profits of a licensee when $k = n$ and that of a non-licensee when $k = n - 1$ are given as follows:

$$\overline{\phi}^n(n, \delta_n^*(n)) = [\theta + \delta_n^*(n)]^2/(n+1)^2 \text{ and } \underline{\phi}^n(n-1, \delta_n^*(n)) = [\theta - (n-1)\delta_n^*(n)]^2/(n+1)^2 \tag{38}$$

Taking $\delta = \delta_n^*(n)$ from (34) in (12) and using (38) we have

$$\gamma^n(n, \delta_n^*(n)) = n[(n-1)\theta + (n+1)\varepsilon][(3n^2 - n + 2)\theta - (n^2 - n - 2)\varepsilon]/[(2n^2 - n + 1)\theta + (n+1)\varepsilon]^2 \tag{39}$$

From (37) and (39), we note that

$$\bar{v}(\delta_n^*(n)) \gtreqqless \gamma^n(\delta_n^*(n)) \Leftrightarrow \tau_n(\varepsilon) \gtreqqless 0 \tag{40}$$

where

$$\tau_n(\varepsilon) := -(n^3-3n^2+n-1)(n+1)^2\varepsilon^3$$
$$+(n+1)[(9n^4-10n^3+16n^2-10n+3)a+(2n^5-13n^4+10n^3-14n^2+6n-3)c]\varepsilon^2$$
$$+\theta[(n-1)(8n^5-7n^4+16n^3-8n^2+6n-3)a-(12n^6-11n^5+23n^4-20n^3+18n^2-9n+3)c]\varepsilon$$
$$-(n-1)\theta^2[(n^2+1)^2a+(2n^2-n+1)(3n^3-3n^2+3n-1)c] \tag{41}$$

From (36) and (40), we note that

$$\delta_n^*(n) \lesseqqgtr \hat{\delta}^n(n) \Leftrightarrow \tau_n(\varepsilon) \gtreqqless 0 \tag{42}$$

Note that for $n \geq 7$, $\tau_n(\varepsilon)$ given in (41) is a cubic function of ε and its coefficient of ε^3 is negative. So the third order derivative of $\tau_n(\varepsilon)$ with respect to ε is negative, which implies that $\tau_n''(\varepsilon)$ (the second order derivative of $\tau_n(\varepsilon)$ with respect to ε) is decreasing in ε.

Denoting $\nu_0(n) := \sqrt{n^2-n+1}$, using (33) and (41), we note that for $n \geq 7$, $\tau_n''(\theta/u(n)) = 2\nu_0(n)[\alpha(n)a+\beta(n)c]/[1+\nu_0(n)]^2$, where

$$\alpha(n) := 9n^4-10n^3+16n^2-10n+3+\nu_0(n)(3n^4+n^3+8n^2-7n+3),$$

$$\beta(n) := 2n^5-13n^4+10n^3-14n^2+6n-3+\nu_0(n)(n^5-5n^4-8n^2+3n-3)$$

Since α_n, β_n are both positive, we have $\tau_n''(\theta/u(n)) > 0$, implying that $\tau_n''(\varepsilon) > 0$ for all $\varepsilon \in (\theta/z(n), \theta/u(n))$. Thus for $n \geq 7$, $\tau_n'(\varepsilon)$ (the derivative of $\tau_n(\varepsilon)$ with respect to ε) is increasing in ε for all $\varepsilon \in (\theta/z(n), \theta/u(n))$.

Since $c > \varepsilon$ and $\varepsilon > \theta/z(n) = (a-c)/z(n)$, we have $c > a/[z(n)+1]$. For $n \geq 7$, we observe that there exist functions $1/[z(n)+1] < \underline{t}(n) < \bar{t}(n) < 1$ such that[23]

(I) If $a/[z(n)+1] < c < \underline{t}(n)a$, then $\tau_n'(\theta/z(n)) > 0$ and hence $\tau_n'(\varepsilon) > 0$ for all $\varepsilon \in (\theta/z(n), \theta/u(n))$. In this case $\tau_n(\varepsilon)$ is increasing in ε for all $\varepsilon \in (\theta/z(n), \theta/u(n))$.

(II) If $\underline{t}(n)a < c < \bar{t}(n)a$, then $\tau_n'(\theta/z(n)) < 0$ and $\tau_n'(\theta/u(n)) > 0$. So $\exists\ \varepsilon_0(n) \in (\theta/z(n), \theta/u(n))$ such that $\tau_n(\varepsilon)$ is decreasing in ε for $\varepsilon \in (\theta/z(n), \varepsilon_0(n))$ and it is increasing in ε for $\varepsilon \in (\varepsilon_0(n), \theta/u(n))$.

[23]For $n \geq 7$, the functions $\underline{t}(n)$, $\bar{t}(n)$ are given by $\underline{t}(n) := \underline{t}_1(n)/\underline{t}_2(n)$ and $\bar{t}(n) := \bar{t}_1(n)/\bar{t}_2(n)$ where

$$\nu_1(n) := \sqrt{(n+1)(n^2-n+1)(n^3-6n^2+5n-4)},$$
$$\underline{t}_1(n) := 2\nu_1(n)(n^2+1)(8n^4-15n^3+25n^2-13n+3)+16n^9-78n^8+196n^7-305n^6+356n^5-271n^4+182n^3-81n^2+42n-9,$$
$$\underline{t}_2(n) := 2\nu_1(n)(12n^6-15n^5+35n^4-28n^3+26n^2-9n+3)+(12n^6-15n^5+23n^4-20n^3+14n^2-9n+3)(2n^3-6n^2+5n-3),$$
$$\bar{t}_1(n) := (n-1)[2\nu_0(n)(n^2+n+1)(4n^3-3n^2+6n-3)+2n^7+14n^5-6n^4+17n^3-6n^2+9n-6],$$
$$\bar{t}_2(n) := \nu_0(n)(10n^6+4n^5+14n^4-8n^3+18n^2-12n+6)+2n^8+2n^7+14n^6-8n^5+27n^4-23n^3+27n^2-15n+6.$$

(III) If $\bar{t}(n)a < c < a$, then $\tau_n'(\theta/u(n)) < 0$. So $\tau_n(\varepsilon)$ is decreasing in ε for all $\varepsilon \in (\theta/z(n), \theta/u(n))$.

We next observe that for $n \geq 7$, there exist functions $1/[z(n)+1] < \underline{\lambda}(n) < \overline{\lambda}(n) < 1$ such that[24]

(A) If $a/[z(n)+1] < c < \underline{\lambda}(n)a$, then both $\tau_n(\theta/z(n))$, $\tau_n(\theta/u(n))$ are positive.

(B) If $\underline{\lambda}(n)a < c < \overline{\lambda}(n)a$, then $\tau_n(\theta/z(n)) < 0$ and $\tau_n(\theta/u(n)) > 0$.

(C) If $\overline{\lambda}(n)a < c < a$, then both $\tau_n(\theta/z(n))$, $\tau_n(\theta/u(n))$ are negative.

Noting that $\overline{\lambda}(n) < \underline{t}(n)$ from (I)-(III) and (A)-(C), we conclude the following for $n \geq 7$.

(1) If $a/[z(n)+1] < c < \underline{t}(n)a$, then $\tau_n(\varepsilon)$ is increasing in ε for all $\varepsilon \in (\theta/z(n), \theta/u(n))$. Noting that $(1/[z(n)+1], \underline{t}(n)) = (1/[z(n)+1], \underline{\lambda}(n)] \cup (\underline{\lambda}(n), \overline{\lambda}(n)] \cup (\overline{\lambda}(n), \underline{t}(n))$, we have the following.

(a) If $a/[z(n)+1] < c < \underline{\lambda}(n)a$, then $\tau_n(\theta/z(n)) > 0$, so $\tau_n(\varepsilon) > 0$ for all $\varepsilon \in (\theta/z(n), \theta/u(n))$. By (42), $\delta_n^*(n) < \hat{\delta}^n(n)$ for all $\varepsilon \in (\theta/z(n), \theta/u(n))$.

(b) If $\underline{\lambda}(n)a < c < \overline{\lambda}(n)a$, then $\tau_n(\theta/z(n)) < 0 < \tau_n(\theta/u(n))$, so $\exists\, \overline{\varepsilon}(n) \in (\theta/z(n), \theta/u(n))$ such that $\tau_n(\varepsilon) < 0$ for $\varepsilon \in (\theta/z(n), \overline{\varepsilon}(n))$ and $\tau_n(\varepsilon) > 0$ for $\varepsilon \in (\underline{\varepsilon}(n), \theta/u(n))$. By (42), $\delta_n^*(n) > \hat{\delta}^n(n)$ for $\varepsilon \in (\theta/z(n), \overline{\varepsilon}(n))$ and $\delta_n^*(n) < \hat{\delta}^n(n)$ for $\varepsilon \in (\underline{\varepsilon}(n), \theta/u(n))$.

(c) If $\overline{\lambda}(n)a < c < \underline{t}(n)a$, then $\tau_n(\theta/u(n)) < 0$, so $\tau_n(\varepsilon) < 0$ for all $\varepsilon \in (\theta/z(n), \theta/u(n))$. By (42), $\delta_n^*(n) > \hat{\delta}^n(n)$ for all $\varepsilon \in (\theta/z(n), \theta/u(n))$.

Again noting that $\overline{\lambda}(n) < \underline{t}(n)$, when $c > \underline{t}(n)a$, we have $c > \overline{\lambda}(n)(a)$. In this case both $\tau_n(\theta/z(n))$, $\tau_n(\theta/u(n))$ are negative and the following hold.

(2) If $\underline{t}(n)a < c < \bar{t}(n)a$, then $\exists\, \varepsilon_0(n) \in (\theta/z(n), \theta/u(n))$ such that $\tau_n(\varepsilon)$ is decreasing in ε for $\varepsilon \in (\theta/z(n), \varepsilon_0(n))$ and it is increasing in ε for $\varepsilon \in (\varepsilon_0(n), \theta/u(n))$. Since both $\tau_n(\theta/z(n)), \tau_n(\theta/u(n))$ are negative, in this case $\tau_n(\varepsilon) < 0$ for all $\varepsilon \in (\theta/z(n), \theta/u(n))$. By (42), $\delta_n^*(n) > \hat{\delta}^n(n)$ for all $\varepsilon \in (\theta/z(n), \theta/u(n))$.

(3) If $\bar{t}(n)a < c < a$, then $\tau_n(\varepsilon)$ is decreasing in ε for all $\varepsilon \in (\theta/z(n), \theta/u(n))$. Since $\tau_n(\theta/z(n)) < 0$, in this case $\tau_n(\varepsilon) < 0$ for all $\varepsilon \in (\theta/z(n), \theta/u(n))$. By (42), $\delta_n^*(n) > \hat{\delta}^n(n)$ for all $\varepsilon \in (\theta/z(n), \theta/u(n))$.

[24] For $n \geq 7$, the functions $\underline{\lambda}(n)$, $\overline{\lambda}(n)$ are given by $\underline{\lambda}(n) := \underline{\lambda}_1(n)/\underline{\lambda}_2(n)$ and $\overline{\lambda}(n) := \overline{\lambda}_1(n)/\overline{\lambda}_2(n)$, where

$$\nu_2(n) := \sqrt{n^3 - 6n^2 + 5n - 4},$$

$$\begin{gathered}\underline{\lambda}_1(n) := \nu_0(n)(28n^9 - 122n^8 + 265n^7 - 393n^6 + 437n^5 - 355n^4 + 251n^3 - 127n^2 + 59n - 11) + \\ \sqrt{n+1}\,\nu_2(n)(n^2 - n + 1)(29n^6 - 44n^5 + 69n^4 - 52n^3 + 43n^2 - 16n + 3) - \\ \sqrt{n+1}[\nu_2(n)]^3(n-1)(n^4 + 2n^2 + 1),\end{gathered}$$

$$\begin{gathered}\underline{\lambda}_2(n) := \nu_0(n)(24n^{10} - 96n^9 + 68n^8 + 67n^7 - 377n^6 + 557n^5 - 581n^4 + 389n^3 - 195n^2 + 59n - 11) + \\ \sqrt{n+1}\,\nu_2(n)[\nu_0(n)]^2(18n^7 + 21n^6 - 46n^5 + 87n^4 - 64n^3 + 41n^2 - 12n + 3) + \\ \sqrt{n+1}[\nu_2(n)]^3(n-1)(6n^5 - 9n^4 + 12n^3 - 8n^2 + 4n - 1),\end{gathered}$$

$$\begin{gathered}\overline{\lambda}_1(n) := 4\nu_0(n)(n^8 - 2n^7 + 8n^6 - 15n^5 + 9n^4 - 18n^3 - 2n^2 - 9n - 4) + \\ 6[\nu_0(n)]^2(n+1)(n^4 - 2n^3 - 2n - 1) + 2(8n^8 - 23n^7 + 38n^6 - 52n^5 + 33n^4 - 33n^3 + 2n^2 - 8n - 5),\end{gathered}$$

$$\begin{gathered}\overline{\lambda}_2(n) := 4\nu_0(n)(2n^8 - 2n^7 + 22n^6 - 15n^5 + 26n^4 - 2n^3 - 2n^2 + 7n - 4) + \\ 2[\nu_0(n)]^2(n+1)(8n^5 - 5n^4 + 12n^3 - 6n^2 + 6n - 3) + 2(12n^8 - 23n^7 + 64n^6 - 58n^5 + 53n^4 - 7n^3 - 8n^2 + 12n - 5).\end{gathered}$$

As noted before, since $\varepsilon > \theta/z(n) = (a-c)/z(n)$ and $c > \varepsilon$, we have $c > a/[z(n)+1]$. Denote $1/\underline{\lambda}(n) \equiv \overline{\kappa}(n)$ and $1/\overline{\lambda}(n) \equiv \underline{\kappa}(n)$. Observe that $\underline{\kappa}(n) < \overline{\kappa}(n)$.

From (1)(c), (2) and (3), for $a < c/\overline{\lambda}(n) = \underline{\kappa}(n)c$, we have $\delta_n^*(n) > \hat{\delta}^n(n)$ (see Figure 4(c) with $k = n$). We have seen by Proposition 2(II) that $\mathbb{S}^n(n, \delta_n^*(n))$ is a continuum (shown by D_1E_1 in Figure 4(b) with $k = n$). By Proposition 2(V), $\mathbb{S}^n(n, \delta_n^*(n))$ contains a policy that is two part tariff with positive ad valorem royalty and upfront fee but no per unit royalty (shown by the point D_1 in Figure 4(b)). This completes the proof of part (III)(iii)(a) of Proposition 4.

From (1)(a), for $a > c/\underline{\lambda}(n) = \overline{\kappa}(n)c$, we have $\delta_n^*(n) < \hat{\delta}^n(n)$ (see Figure 4(c)). We have seen by Proposition 2(II) that $\mathbb{S}^n(n, \delta_n^*(n))$ is a continuum (shown by D_2E_2 in Figure 4(c) with $k = n$). By Proposition 2(IV), $\mathbb{S}^n(n, \delta_n^*(n))$ contains a policy that is two part royalty with positive per unit and ad valorem royalties but no upfront fees (shown by the point D_2 in Figure 4(c)). This completes the proof of part (III)(iii)(b) of Proposition 4.

From (1)(b), for $c/\overline{\lambda}(n) = \underline{\kappa}(n)c < a < c/\underline{\lambda}(n) = \overline{\kappa}(n)c$, $\exists\ \overline{\varepsilon}(n) \in (\theta/z(n), \theta/u(n))$ such that $\delta_n^*(n) > \hat{\delta}^n(n)$ for $\varepsilon \in (\theta/z(n), \overline{\varepsilon}(n))$ and $\delta_n^*(n) < \hat{\delta}^n(n)$ for $\varepsilon \in (\overline{\varepsilon}(n), \theta/u(n))$. Thus, for $\varepsilon < \overline{\varepsilon}(n)$, we have the same conclusion as part (III)(iii)(a) of Proposition 4 and for $\varepsilon > \overline{\varepsilon}(n)$, we have the same conclusion as part (III)(iii)(b) of Proposition 4. This completes the proof of part (III)(iii)(c) of Proposition 4.

5.7.2 Proof of part (IV) of Proposition 4

Recall that for linear demand $p(Q) = \max\{a-Q, 0\}$, we have $\theta \equiv a-c$. By Table A.5 (p.183) of Sen and Tauman (2007) for $[n = 4, 5$ and $\varepsilon \in (\theta/(2n-4), \theta/2)]$ or $[n \geq 6$ and $\varepsilon \in (\theta/(2n-4), \theta/z(n))]$, the unique optimal two part tariff[25] with per unit royalty and upfront fee has $k = n-1$ and $r = \tilde{\rho}_0(n)$ where

$$\tilde{\rho}_0(n) := [2(n-2)\varepsilon - \theta]/2(n-1) \tag{43}$$

Note from (22) that

$$\delta_n^*(n-1) := (\theta + 2\varepsilon)/2(n-1) = \varepsilon - \tilde{\rho}_0(n) \tag{44}$$

Therefore in this case, any optimal (k, δ) for a three part tariff has $k = n-1$ and $\delta = \delta_n^*(n-1)$. The set of all optimal licensing policies in this case is the set of all (r, v) that supports $\delta_n^*(n-1)$ for which $(n-1, r, v)$ is both acceptable and feasible. This set is $\mathbb{S}^n(n-1, \delta_n^*(n-1))$, where $\mathbb{S}^n(k, \delta)$ is given in (20).

Proof of (IV)(i)-(ii) Noting that for $\varepsilon \in (\theta/(2n-4), \theta/z(n))$, we have $\delta_n^*(n-1) \in (0, \varepsilon)$, parts (IV)(i)-(ii) follows by Proposition 2(II)-(III).

Proof of (IV)(iii) Taking $k = n-1$ in parts (IV)-(V) of Proposition 2, we note that if $\delta_n^*(n-1) < \hat{\delta}^n(n-1)$, then $\mathbb{S}^n(n-1, \delta_n^*(n-1))$ contains a policy that is a two part royalty with positive per unit and ad valorem royalties but no upfront fee. On the other hand, if $\delta_n^*(n-1) >$

[25]The function $z(n)$ is given in (32). As mentioned before, $z(n)$ is denoted by $v(n)$ in Table A.5 (p.183) of Sen and Tauman (2007). We use a different notation to avoid confusion with v, which we use for ad valorem royalty. Also note that (ibid., p.183): $f_0(n) = 2n-4$ and $d_1 = (210 + \sqrt{5642})/67 = z(6)$. Thus, for $n = 6$, $(\theta/(2n-4), \theta/z(n)) = (\theta/8, \theta/d_1)$.

$\hat{\delta}^n(n-1)$, then $\mathbb{S}^n(n-1,\delta_n^*(n-1))$ contains a policy that is a two part tariff with a positive ad valorem royalty and fee but no per unit royalty.

To prove this part, we need to find out when $\delta_n^*(n-1) < \hat{\delta}^n(n)$ and when $\delta_n^*(n-1) > \hat{\delta}^n(n)$. Regarding $\hat{\delta}^n(n)$, recall that (see Figures 3(a), 3(b))

$$\gamma^n(n-1,\delta) \gtreqless \bar{v}(\delta) \Leftrightarrow \delta \gtreqless \hat{\delta}^n(n-1) \tag{45}$$

where $\gamma^n(n-1,\delta)$ is given in (12) and $\bar{v}(\delta)$ is given in (**??**). From (35), we note that

$$\delta_n^*(n-1) \gtreqless \hat{\delta}^n(n-1) \Leftrightarrow \gamma^n(n-1,\delta_n^*(n-1)) \gtreqless \bar{v}(\delta_n^*(n-1)) \tag{46}$$

From (46), if $\delta_n^*(n-1) < \hat{\delta}^n(n-1)$, then $\gamma^n(n-1,\delta_n^*(n-1)) < \bar{v}(\delta_n^*(n-1))$ and the maximum v that can support $\delta_n^*(n-1)$ is $v = \gamma^n(n-1,\delta_n^*(n-1))$ (as shown in Figure 4(b)). On the other hand, if $\delta_n^*(n-1) > \hat{\delta}^n(n-1)$, then $\gamma^n(n-1,\delta_n^*(n-1)) < \bar{v}(\delta_n^*(n-1))$ and the maximum v that can support $\delta_n^*(n-1)$ is $v = \bar{v}(\delta_n^*(n-1))$ (as shown in Figure 4(c)).

Recall that $\theta \equiv a-c$. Taking $\delta = \delta_n^*(n-1)$ from (44) in (**??**) we have

$$\bar{v}(\delta_n^*(n-1)) = [(2n-4)\varepsilon - \theta]/[(2n-3)c - a + 2(c-\varepsilon)] \tag{47}$$

Since $\varepsilon > \theta/(2n-4)$, the numerator of $\bar{v}(\delta_n^*(n-1))$ is positive. Sonce $c > \varepsilon$ and $\varepsilon > \theta/(2n-4) = (a-c)/(2n-4)$, we have $c > (a-c)/(2n-4)$, so $(2n-3)c > a$ and the denominator of $\bar{v}(\delta_n^*(n-1))$ is also positive.

Note that under the linear demand $p(Q) = \max\{a-p, 0\}$, for $\delta = \delta_n^*(n-1)$, the Cournot profits of a licensee and a non-licensee when $k = n-1$ are given as follows.

$$\overline{\phi}^n(n-1,\delta_n^*(n-1)) = [\theta + 2\delta_n^*(n-1)]^2/(n+1)^2,$$

$$\underline{\phi}^n(n-1,\delta_n^*(n-1)) = [\theta - (n-1)\delta_n^*(n-1)]^2/(n+1)^2 \tag{48}$$

Taking $\delta = \delta_n^*(n-1)$ from (44) in (12) and using (48) we have

$$\gamma^n(n-1,\delta_n^*(n-1)) = (n+1)(\theta+2\varepsilon)[(3n-1)\theta - 2(n-3)\varepsilon]/4(n\theta+2\varepsilon)^2 \tag{49}$$

From (47) and (49), we note that

$$\bar{v}(\delta_n^*(n-1)) \gtreqless \gamma^n(\delta_n^*(n-1)) \Leftrightarrow h_n(\varepsilon) \gtreqless 0 \tag{50}$$

where

$$h_n(\varepsilon) := -8(n-5)\varepsilon^3 + (8cn^2 + 36an - 52cn - 4a - 20c)\varepsilon^2 +$$

$$2\theta(4an^2 - 8cn^2 + an - 9cn - a - 3c)\varepsilon - \theta^2(6cn^2 + an + 3cn - a - c) \tag{51}$$

From (46) and (50), we note that

$$\delta_n^*(n-1) \lesseqqgtr \hat{\delta}^n(n-1) \Leftrightarrow h_n(\varepsilon) \gtreqqless 0 \tag{52}$$

Observe from (51) that for all $n \geq 4$:

$$h_n(\theta/(2n-4)) = -\theta^2(n+1)(3n-5)(n-1)^2[(2n-3)c-a]/(n-2)^3 < 0 \tag{53}$$

The negative sign of $h_n(\theta/(2n-4))$ follows by noting that $c > \varepsilon$ and $\varepsilon > \theta/(2n-4) = (a-c)/(2n-4)$ and thus $c > a/(2n-3)$. Next observe that $h_n(\theta/2) = 4\theta^2(n+1)^2(a-3c)$. Thus

$$h_n(\theta/2) \gtreqqless 0 \Leftrightarrow c \lesseqqgtr a/3 \tag{54}$$

Proof for $n = 5$: For $n = 5$, we have $2n-4 = 6$, so the relevant region is $\varepsilon \in (\theta/6, \theta/2)$. Taking $n = 5$ in (51), we have

$$h_5(\varepsilon) = 16(11a - 5c)\varepsilon^2 + 16\theta(13a - 31c)\varepsilon - 4\theta^2(a + 41c)$$

Note that $h_5(\varepsilon)$ is a u-shaped quadratic function in ε. Since $h_5(\theta/6) < 0$ (by (53)), from (54) we conclude that for $n = 5$: (1) if $c > a/3$, then $h_5(\varepsilon) < 0$ for all $\varepsilon < (\theta/6, \theta/2)$ and (2) if $c < a/3$, then $\exists\ \underline{\varepsilon}(5) \in (\theta/6, \theta/2)$ such that $h_5(\varepsilon) < 0$ for $\varepsilon \in (\theta/6, \underline{\varepsilon}(5))$ and $h_5(\varepsilon) > 0$ for $\varepsilon \in (\underline{\varepsilon}(5), \theta/2)$. Thus if $a < 3c$, then $\delta_5^*(4) > \hat{\delta}^5(4)$ for all $\varepsilon \in (\theta/6, \theta/2)$ and if $a > 3c$, then $\exists\ \underline{\varepsilon}(5) \in (\theta/6, \theta/2)$ such that $\delta_5^*(4) > \hat{\delta}^5(4)$ for $\varepsilon < \underline{\varepsilon}(5)$ and $\delta_5^*(4) < \hat{\delta}^5(4)$ for $\varepsilon > \underline{\varepsilon}(5)$. Using these conclusions taking $\kappa(5) = 3$, together with the conclusions of Proposition 2(IV)-(V) completes the proof of the result for $n = 5$.

Proof for $n = 4$: For $n = 4$, we have $2n - 4 = 4$, so the region under consideration is $\varepsilon \in (\theta/4, \theta/2)$. Taking $n = 4$ in (51), we have

$$h_4(\varepsilon) = 8\varepsilon^3 + 20(7a - 5c)\varepsilon^2 + 2\theta(67a - 167c)\varepsilon - \theta^2(3a + 107c)$$

Note that $h_4(\varepsilon)$ is a cubic function in ε and its coefficient of ε^3 is positive. Thus, the third derivative of $h_4(\varepsilon)$ is positive, implying that its second derivative $h_4''(\varepsilon)$ is increasing. Noting that $h_4''(\theta/4) = 4(73a - 53c) > 0$ (since $a > c$), we conclude that $h_4''(\varepsilon) > 0$ and so $h_4'(\varepsilon)$ is increasing for all $\varepsilon \in (\theta/4, \theta/2)$.

Since $c > \varepsilon$ and $\varepsilon > \theta/4 = (a-c)/4$, we have $c > a/5$. We note that

$$h_4'(\theta/4) = 3\theta(137a - 257c)/2 \text{ and } h_4'(\theta/2) = 40\theta(7a - 11c)$$

Thus, $\exists\ 1/5 < \underline{w}(4) < \overline{w}(4) < 1$ (specifically, $\underline{w}(4) \equiv 137/257$, $\overline{w}(4) \equiv 7/11$) such that

(I) If $a/5 < c < \underline{w}(4)a$, then $h_4'(\theta/4) > 0$ and hence $h_4'(\varepsilon) > 0$ for all $\varepsilon \in (\theta/4, \theta/2)$. In this case $h_n(\varepsilon)$ is increasing in ε for all $\varepsilon \in (\theta/4, \theta/2)$.

(II) If $\underline{w}(4)a < c < \overline{w}(4)a$, then $h_4'(\theta/4) < 0$ and $h_4'(\theta/2) > 0$. In this case $\exists\ \tilde{\varepsilon}(4) \in (\theta/4, \theta/2)$ such

that $h_4(\varepsilon)$ is decreasing in ε for $\varepsilon \in (\theta/4, \tilde{\varepsilon}(4))$ and it is increasing in ε for $\varepsilon \in (\tilde{\varepsilon}(4), \theta/2)$.
(III) If $\overline{w}(4)a < c < a$, then $h_4'(\theta/2) < 0$ and hence $h_4(\varepsilon)$ is decreasing in ε for all $\varepsilon \in (\theta/4, \theta/2)$.

Noting that $1/3 < \underline{w}(4)$, by (53), (54) and (I)-(III) above, we conclude:

(1) If $a/5 < c < \underline{w}(4)a$, then $h_4(\varepsilon)$ is increasing in ε for all $\varepsilon \in (\theta/4, \theta/2)$. Noting that $(1/5, \underline{w}(4)) = (1/5, 1/3] \cup (1/3, \underline{w}(4))$, we have the following.

(a) If $a/5 < c < a/3$, then $h_4(\theta/4) < 0 < h_4(\theta/2)$, so $\exists\ \underline{\varepsilon}(4) \in (\theta/4, \theta/2)$ such that $h_4(\varepsilon) < 0$ for $\varepsilon \in (\theta/4, \underline{\varepsilon}(4))$ and $h_4(\varepsilon) > 0$ for $\varepsilon \in (\underline{\varepsilon}(4), \theta/2)$. By (52), $\delta_4^*(3) > \hat{\delta}^4(3)$ for $\varepsilon \in (\theta/4, \underline{\varepsilon}(4))$ and $\delta_4^*(3) < \hat{\delta}^4(3)$ for $\varepsilon \in (\underline{\varepsilon}(4), \theta/2)$.

(b) If $a/3 < c < \underline{w}(4)a$, then $h_4(\theta/2) < 0$, so $h_4(\varepsilon) < 0$ for all $\varepsilon \in (\theta/4, \theta/2)$ and so $\delta_4^*(3) > \hat{\delta}^4(3)$ for all $\varepsilon \in (\theta/4, \theta/2)$.

(2) If $\underline{w}(4)a < c < \overline{w}(4)a$, then $\exists\ \tilde{\varepsilon}(4) \in (\theta/4, \theta/2)$ such that $h_4(\varepsilon)$ is decreasing in ε for $\varepsilon \in (\theta/4, \tilde{\varepsilon}(4))$ and it is increasing in ε for $\varepsilon \in (\tilde{\varepsilon}(4), \theta/2)$. Since both $h_4(\theta/4), h_4(\theta/2)$ are negative, in this case $h_n(\varepsilon) < 0$ for all $\varepsilon \in (\theta/4, \theta/2)$ and so $\delta_4^*(3) > \hat{\delta}^4(3)$ for all $\varepsilon \in (\theta/4, \theta/2)$.

(3) If $\overline{w}(4)a < c < a$, then $h_4(\varepsilon)$ is decreasing in ε for all $\varepsilon \in (\theta/4, \theta/2)$. Since $h_4(\theta/4) < 0$, in this case $h_4(\varepsilon) < 0$ for all $\varepsilon \in (\theta/4, \theta/2)$ and so $\delta_4^*(3) > \hat{\delta}^4(3)$ for all $\varepsilon \in (\theta/4, \theta/2)$.

From (1)(a)(b), (2)-(3), we conclude that for $n = 4$, if $a < 3c$, then $\delta_4^*(3) > \hat{\delta}^4(3)$ for all $\varepsilon \in (\theta/4, \theta/2)$ and if $a > 3c$, then $\exists\ \underline{\varepsilon}(4) \in (\theta/4, \theta/2)$ such that $\delta_4^*(3) > \hat{\delta}^4(3)$ for $\varepsilon < \underline{\varepsilon}(4)$ and $\delta_4^*(3) < \hat{\delta}^4(3)$ for $\varepsilon > \underline{\varepsilon}(4)$. Using these conclusions with $\kappa(4) = 3$, together with the conclusions of Proposition 2(IV)-(V) completes the proof of the result for $n = 4$.

Proof for $n \geq 6$: For $n \geq 6$, $h_n(\varepsilon)$ given in (51) is a cubic function of ε and its coefficient of ε^3 is negative. So the third order derivative of $h_n(\varepsilon)$ with respect to ε is negative, which implies that $h_n''(\varepsilon)$ (the second order derivative of $h_n(\varepsilon)$ with respect to ε) is decreasing in ε for $n \geq 6$.

For $n \geq 6$, the region under consideration is $\varepsilon \in (\theta/(2n-4), \theta/z(n))$. Denote $\nu_1(n) := \sqrt{(n+1)(n^2-n+1)(n^3-6n^2+5n-4)}$. Note that $h_n''(\theta/z(n)) = 8[\tilde{\alpha}(n)a + \tilde{\beta}(n)c]/[n^3 - n + \nu_1(n)]$ where

$$\tilde{\alpha}(n) := 9n^4 - 13n^3 + 57n^2 - 35n + 30 + \nu_1(n)(9n-1),$$

$$\tilde{\beta}(n) := 2n^5 - 13n^4 + 5n^3 - 53n^2 + 41n - 30 + \nu_1(n)(2n^2 - 13n - 5)$$

Noting that $\nu_1(n)$, $\tilde{\alpha}(n)$, $\tilde{\beta}(n)$ are all positive for $n \geq 7$, we have $h_n''(\theta/z(n)) > 0$. Therefore $h_n''(\varepsilon) > 0$ for and so $h_n'(\varepsilon)$ is increasing in ε for all $\varepsilon \in (\theta/(2n-4), \theta/z(n))$.

Since $c > \varepsilon$ and $\varepsilon > \theta/(2n-4) = (a-c)/(2n-4)$, we have $c > a/(2n-3)$. We observe that there exist functions $1/(2n-3) < \underline{w}(n) < \overline{w}(n) < 1$ such that[26]

[26]For $n \geq 6$, the functions $\underline{w}(n)$, $\overline{w}(n)$ are as follows.

$$\underline{w}(n) := (4n^3 - 11n^2 + 18n - 15)/(8n^3 - 19n^2 + 14n - 7) \text{ and } \overline{w}(n) := \overline{w}_1(n)/\overline{w}_2(n), \text{ where}$$

$$\overline{w}_1(n) := 8n^8 - 22n^7 + 76n^6 - 95n^5 + 230n^4 - 242n^3 + 278n^2 - 137n + 64 + \nu_1(n)(8n^5 + 2n^4 + 62n^3 - 46n^2 + 42n - 4)$$

(I) If $a/(2n-3) < c < \underline{w}(n)a$, then $h_n'(\theta/(2n-4)) > 0$, so $h_n'(\varepsilon) > 0$ for all $\varepsilon \in (\theta/(2n-4), \theta/z(n))$. In this case $h_n(\varepsilon)$ is increasing in ε for all $\varepsilon \in (\theta/(2n-4), \theta/z(n))$.

(II) If $\underline{w}(n)a < c < \overline{w}(n)a$, then $h_n'(\theta/(2n-4)) < 0 < h_n'(\theta/z(n))$. So $\exists\, \tilde{\varepsilon}(n) \in (\theta/(2n-4), \theta/z(n))$ such that $h_n(\varepsilon)$ is decreasing in ε for $\varepsilon \in (\theta/(2n-4), \tilde{\varepsilon}(n))$ and it is increasing in ε for $\varepsilon \in (\tilde{\varepsilon}(n), \theta/z(n))$.

(III) If $\overline{w}(n)a < c < a$, $h_n'(\theta/z(n)) < 0$. So $h_n(\varepsilon)$ is decreasing in ε for all $\varepsilon \in (\theta/(2n-4), \theta/z(n))$.

By (53), $h_n(\theta/(2n-4)) < 0$. For $h_n(\theta/z(n))$, we note that $\exists\, 1/(2n-3) < \lambda(n) < 1$ such that[27]

(A) If $a/(2n-3) < c < \lambda(n)a$, then $h_n(\theta/z(n)) > 0$.

(B) If $\lambda(n)a < c < a$, then $h_n(\theta/z(n)) < 0$.

Noting that $\lambda(n) < \underline{w}(n)$, from (I)-(III) and (A)-(B), we conclude:

(1) If $a/(2n-3) < c < \underline{w}(n)a$, then $h_n(\varepsilon)$ is increasing in ε for all $\varepsilon \in (\theta/(2n-4), \theta/z(n))$. Noting that $(1/(2n-3), \underline{w}(n)) = (1/(2n-3), \lambda(n)] \cup (\lambda(n), \underline{w}(n))$, we have the following.

(a) If $a/(2n-3) < c < \lambda(n)a$, then $h_n(\theta/(2n-4)) < 0 < h_n(z(n))$, so so $\exists\, \underline{\varepsilon}(n) \in (\theta/(2n-4), \theta/z(n))$ such that $h_n(\varepsilon) < 0$ for $\varepsilon \in (\theta/(2n-4), \underline{\varepsilon}(n))$ and $h_n(\varepsilon) > 0$ for $\varepsilon \in (\underline{\varepsilon}(n), \theta/z(n))$. By (52), $\delta_n^*(n-1) > \hat{\delta}^n(n)$ for $\varepsilon < \underline{\varepsilon}(n)$ and $\delta_n^*(n-1) < \hat{\delta}^n(n)$ for $\varepsilon > \underline{\varepsilon}(n)$.

(b) If $\lambda(n)a < c < \underline{w}(n)(a)$, then $h_n(\theta/z(n)) < 0$, so $h_n(\varepsilon) < 0$ for all $\varepsilon \in (\theta/(2n-4), \theta/z(n))$. By (52), $\delta_n^*(n-1) > \hat{\delta}^n(n)$ for all $\varepsilon \in (\theta/(2n-4), \theta/z(n))$.

(2) If $\underline{w}(n)a < c < \overline{w}(n)a$, then $\exists\, \tilde{\varepsilon}(n) \in (\theta/(2n-4), \theta/z(n))$ such that $h_n(\varepsilon)$ is decreasing in ε for $\varepsilon \in (\theta/z(n), \tilde{\varepsilon}(n))$ and it is increasing in ε for $\varepsilon \in (\tilde{\varepsilon}(n), \theta/z(n))$. Since both $h_n(\theta/(2n-4)), h_n(\theta/z(n))$ are negative, in this case $h_n(\varepsilon) < 0$ for all $\varepsilon \in (\theta/(2n-4), \theta/z(n))$. By (52), $\delta_n^*(n-1) > \hat{\delta}^n(n)$ for all $\varepsilon \in (\theta/(2n-4), \theta/z(n))$.

(3) If $\overline{w}(n)a < c < a$, then $h_n(\varepsilon)$ is decreasing in ε for all $\varepsilon \in (\theta/(2n-4), \theta/z(n))$. As $h_n(\theta/(2n-3)) < 0$, in this case $h_n(\varepsilon) < 0$ for all $\varepsilon \in (\theta/(2n-4), \theta/z(n))$. By (52), $\delta_n^*(n-1) > \hat{\delta}^n(n)$ for all $\varepsilon \in (\theta/(2n-4), \theta/z(n))$.

Denote $\kappa(n) := 1/\lambda(n)$. From (1)(a)(b), (2)-(3), we conclude that for $n \geq 6$, if $a < \kappa(n)c$, then $\delta_n^*(n-1) > \hat{\delta}^n(n-1)$ for all $\varepsilon \in (\theta/(2n-4), \theta/z(n))$ and if $a > \kappa(n)c$, then $\exists\, \underline{\varepsilon}(n) \in (\theta/(2n-4), \theta/z(n))$ such that $\delta_n^*(n-1) > \hat{\delta}^n(n-1)$ for $\varepsilon < \underline{\varepsilon}(n)$ and $\delta_n^*(n-1) < \hat{\delta}^n(n-1)$ for

$$\overline{w}_2(n) := 16n^8 - 46n^7 + 88n^6 - 67n^5 + 150n^4 - 274n^3 + 290n^2 - 173n + 48 +$$
$$2\nu_1(n)(8n^5 + n^4 + 51n^3 - 19n^2 + 13n + 10)$$

[27]For $n \geq 6$, the function $\lambda(n)$ is given by $\lambda(n) := \lambda_1(n)/\lambda_2(n)$ where

$$\lambda_1(n) := (n^2 - n + 2)(28n^8 - 54n^7 + 83n^6 - 97n^5 + 154n^4 - 144n^3 + 127n^2 - 57n + 24) -$$
$$(n-1)[\nu_1(n)]^3 + \nu_1(n)(29n^7 - 5n^6 + 122n^5 - 150n^4 + 185n^3 - 97n^2 + 48n - 4),$$
$$\lambda_2(n) := 24n^{11} - 32n^{10} - 188n^9 + 335n^8 - 434n^7 + 167n^6 - 306n^5 + 445n^4 - 646n^3 + 413n^2 - 178n + 16 +$$
$$[\nu_1(n)]^3(6n^2 + 3n - 1) + \nu_1(n)(18n^8 + 73n^7 - 31n^6 + 178n^5 - 160n^4 + 197n^3 - 39n^2 + 20).$$

$\varepsilon > \underline{\varepsilon}(n)$. Using this together with the conclusions of Proposition 2(IV)-(V) complete the proof of the result for $n \geq 6$. ■

5.7.3 Proof of part (V) of Proposition 4

Consider $n \geq 7$. Note that $(2n-4)$ is increasing in n and $\lim_{n\to\infty}(2n-4) = \infty$. From (32) and (33), we note that $z(n)$ is increasing in n, with $\lim_{n\to\infty} z(n) = \infty$ and $u(n)$ is decreasing in n with $\lim_{n\to\infty} u(n) = 1$. Thus, for any a, c, ε, where $a > c > \varepsilon$ and $0 < \varepsilon < \theta \equiv a - c$, we have:

(i) $\theta/(2n-4)$ is decreasing in n and $\lim_{n\to\infty} \theta/(2n-4) = 0$.

(ii) $\theta/z(n)$ is decreasing in n and $\lim_{n\to\infty} \theta/z(n) = 0$.

(iii) $\theta/u(n)$ is increasing in n and $\lim_{n\to\infty} \theta/u(n) = \theta$.

Thus for any a, c, ε with $0 < \varepsilon < \theta$, $\exists\ \hat{n}(a, c, \varepsilon)$ such that for all $n \geq \hat{n}(a, c, \varepsilon)$, we have $\theta \in (\theta/z(n), \theta/u(n))$, so part (III) of Proposition 4 is applicable. We note that $\underline{\kappa}(n)$ is decreasing in n, with $\lim_{n\to\infty} \underline{\kappa}(n) = 2$ and $\lim_{n\to\infty} \overline{\kappa}(n) = \infty$. See footnote 24. Thus $\lim_{n\to\infty} \underline{\kappa}(n)c = 2c$ and $\lim_{n\to\infty} \overline{\kappa}(n)c = \infty$. This implies that if $a < 2c$, then $a < \underline{\kappa}(n)c$ for all n and part (III)(iii)(a) of Proposition 4 applies.

Consider $a > 2c$. Then $\exists\ \tilde{n}(a, c, \varepsilon) > \hat{n}(a, c, \varepsilon)$ such that for all $n \geq \tilde{n}(a, c, \varepsilon)$, we have $a \in (\underline{\kappa}(n)c, \overline{\kappa}(n)c)$ and part (III)(iii)(c) of Proposition 4 applies. Note that in this case optimal $\delta = \delta_n^*(n)$ given in (34). From (37) and (39), we note that

$$\lim_{n\to\infty} \bar{v}(\delta_n^*(n)) = \varepsilon/c \text{ and } \lim_{n\to\infty} \gamma^n(n, \delta_n^*(n)) = (\theta + \varepsilon)(3\theta - \varepsilon)/4\theta^2$$

We note that $\lim_{n\to\infty}[\bar{v}(\delta_n^*(n)) - \gamma^n(n, \delta_n^*(n))] = \psi(\varepsilon)/4c\theta^2$ where $\psi(\varepsilon) := c\varepsilon^2 + 2\theta(2a-3c)\varepsilon - 3c\theta^2$. Since $a > 2c$, we have $\theta = a - c > c$. As $0 < \varepsilon < \theta$ and also $\varepsilon < c$, the relevant region for ε in this case is $\varepsilon \in (0, c)$. Note that $\psi(\varepsilon)$ is a u-shaped quadratic function in ε with $\psi(0) = -3c\theta^2 < 0$ and $\psi(c) = c(a-2c)^2 > 0$. So $\exists\ \hat{\varepsilon} \in (0, c)$ such that $\psi(\varepsilon) < 0$ if $\varepsilon < \hat{\varepsilon}$ and $\psi(\varepsilon) > 0$ if $\varepsilon > \hat{\varepsilon}$. Specifically, $\hat{\varepsilon} \equiv [2\sqrt{a^2 - 3ac + 3c^2} - (2a - 3c)]\theta/c$.

Thus, for $a > 2c$, if $\varepsilon \in (0, \hat{\varepsilon})$, then we have $\lim_{n\to\infty} \bar{v}(\delta_n^*(n)) < \lim_{n\to\infty} \gamma^n(n, \delta_n^*(n))$. In this case $\exists\ \bar{n}(a, c, \varepsilon) > \tilde{n}(a, c, \varepsilon)$ such that for all $n > \bar{n}(a, c, \varepsilon)$, $\bar{v}(\delta_n^*(n)) < \gamma^n(n, \delta_n^*(n))$, which by (36) implies $\delta_n^*(n) > \hat{\delta}^n(n)$ and the conclusion is the same as in part (III)(iii)(a) of Proposition 4.

On the other hand, for $a > 2c$, if $\varepsilon \in (\hat{\varepsilon}, c)$, then we have $\lim_{n\to\infty} \bar{v}(\delta_n^*(n)) > \lim_{n\to\infty} \gamma^n(n, \delta_n^*(n))$. In this case $\exists\ \bar{n}(a, c, \varepsilon) > \tilde{n}(a, c, \varepsilon)$ such that for all $n > \bar{n}(a, c, \varepsilon)$, $\bar{v}(\delta_n^*(n)) > \gamma^n(n, \delta_n^*(n))$, which by (36) implies $\delta_n^*(n) < \hat{\delta}^n(n)$ and the conclusion is the same as in part (III)(iii)(b) of Proposition 4.

5.7.4 Optimal policies for the remaining regions for $n = 6$

The innovation is nondrastic, so $\varepsilon \in (0, \theta)$ where $\theta = a - c$. We look at generic magnitudes of the innovation, so only open intervals are considered.

For $n = 4, 5$, from parts (I),(II),(IV) of Proposition 4, the regions considered are: $\varepsilon < \theta/(2n-4)$, $\varepsilon > \theta/2$ and $\varepsilon \in (\theta/(2n-4), \theta/2)$, thus we have $\varepsilon \in (0, \theta/(2n-4)) \cup (\theta/(2n-4), \theta/2) \cup (\theta/2, \theta)$.

For $n \geq 7$, from parts (I)-(IV) of Proposition 4, the regions considered are: $\varepsilon < \theta/(2n-4)$, $\varepsilon > \theta/u(n)$, $\varepsilon \in (\theta/z(n), \theta/u(n))$ and $\varepsilon \in (\theta/(2n-4), \theta/z(n))$, thus we have $\varepsilon \in (0, \theta/(2n-4)) \cup (\theta/(2n-4), \theta/z(n)) \cup (\theta/z(n), \theta/u(n)) \cup (\theta/u(n), \theta)$.

For $n = 6$, from parts (I),(II),(IV) of Proposition 4, the regions considered are: $\varepsilon < \theta/(2n-4)$, $\varepsilon > \theta/2$ and $\varepsilon \in (\theta/(2n-4), \theta/z(n))$, thus we have $\varepsilon \in (0, \theta/(2n-4)) \cup (\theta/(2n-4), \theta/z(n)) \cup (\theta/2, \theta)$. Thus for $n \geq 4$, the only case that is left is $n = 6$ and $\varepsilon \in (\theta/z(n), \theta/2)$.

Let us consider the case $n = 6$ and $\varepsilon \in (\theta/z(n), \theta/2)$, that is, $\varepsilon \in (\theta/z(6), \theta/2)$, where $z(n)$ is given by (32). As mentioned before, in Table A.5 (p.183) of Sen and Tauman (2007): (i) the function $z(n)$ is denoted by $v(n)$, (ii) $z(6) = (210 + \sqrt{5642})/67$ is denoted by d_1 and (iii) $d_2 \equiv (210 - \sqrt{5642})/67$. We denote d_2 by d, that is, $d \equiv (210 - \sqrt{5642})/67$. Note that $2 < d_2 = d < d_1 = z(6)$, so that $\theta/z(6) < \theta/d < \theta/2$. It is shown (ibid., p.183) that for $n = 6$ and $\varepsilon \in (\theta/z(6), \theta/2)$:

(i) If $n = 6$ and $\varepsilon \in (\theta/z(6), \theta/d)$, the unique optimal two part tariff with per unit royalty and upfront fee has $k = 6$ and $r = \bar{\rho}_0(6)$, where $\bar{\rho}_0(n)$ is given in (31). Therefore in this case, any optimal (k, δ) for a three part tariff has $k = 6$ and $\delta = \delta_6^*(6) = \varepsilon - \bar{\rho}_0(6)$, where $\delta_n^*(n)$ is given in (34). The set of all optimal licensing policies in this case is the set of all (r, v) that supports $\delta_6^*(6)$ for which $(6, r, v)$ is both acceptable and feasible. This set is $\mathbb{S}^6(6, \delta_6^*(6))$, where $\mathbb{S}^n(k, \delta)$ is given in (20).

(ii) If $n = 6$ and $\varepsilon \in (\theta/d, \theta/2)$, the unique optimal two part tariff with per unit royalty and upfront fee has $k = 5$ and $r = \tilde{\rho}_0(6)$, where $\tilde{\rho}_0(6)$ is given in (43). In this case, any optimal (k, δ) for a three part tariff has $k = 5$ and $\delta = \delta_6^*(5) = \varepsilon - \tilde{\rho}_0(6)$, where $\delta_n^*(n-1)$ is given in (44). The set of all optimal licensing policies in this case is the set of all (r, v) that supports $\delta_6^*(5)$ for which $(5, r, v)$ is both acceptable and feasible. This set is $\mathbb{S}^6(5, \delta_6^*(5))$, where $\mathbb{S}^n(k, \delta)$ is given in (20).

Case 1: $n = 6$ and $\varepsilon \in (\theta/z(6), \theta/d)$: In this case we need to find out when $\delta_6^*(6) < \hat{\delta}^6(6)$ and when $\delta_6^*(6) > \hat{\delta}^6(6)$. Taking $n = 6$ in (42), it follows that

$$\delta_6^*(6) \lesseqqgtr \hat{\delta}^6(6) \Leftrightarrow \tau_6(\varepsilon) \gtreqqless 0 \tag{55}$$

Taking $n = 6$ in (41), we have

$$\tau_6(\varepsilon) = -5537\varepsilon^3 + 7(10023a + 393c)\varepsilon^2 + \theta(281685a - 500421c)\varepsilon - 5\theta^2(1369a + 37319c)$$

Note that $\tau_6''(\varepsilon) = 140322a + 5502c - 33222\varepsilon$. Since $a > c > \varepsilon$, we have $\tau_6''(\varepsilon) > 0$ for all $\varepsilon < c$. Thus $\tau_6'(\varepsilon)$ is increasing in ε for all $\varepsilon \in (\theta/z(6), \theta/d)$.

Since $c > \varepsilon$ and $\varepsilon > \theta/z(6) = (a-c)/z(6)$, we have $c > a/[z(6)+1]$. Noting that $z(6) = (210 + \sqrt{5642})/67$ and $d = (210 - \sqrt{5642})/67$, we observe that there are $1/[z(6)+1] < \underline{t}(6) < \bar{t}(6) < 1$ (specifically, $\underline{t}(6) \equiv k_0/(k_1 + k_2\sqrt{k_3})$ and $\bar{t}(6) \equiv k_0/(k_1 - k_2\sqrt{k_3})$, where $k_0 \equiv 84707737321$, $k_1 \equiv 127452786185$, $k_2 \equiv 93987936$ and $k_3 \equiv 5642$) such that:

(I) If $a/[z(6)+1] < c < \underline{t}(6)a$, then $\tau_6'(\theta/z(6)) > 0$ and hence $\tau_6'(\varepsilon) > 0$ for all $\varepsilon \in (\theta/z(6), \theta/d)$. In this case $\tau_6(\varepsilon)$ is increasing in ε for all $\varepsilon \in (\theta/z(6), \theta/d)$.
(II) If $\underline{t}(6)a < c < \bar{t}(6)a$, then $\tau_6'(\theta/z(6)) < 0$ and $\tau_6'(\theta/d) > 0$. In this case $\exists\ \varepsilon_0(6) \in (\theta/z(6), \theta/d)$ such that $\tau_6(\varepsilon)$ is decreasing in ε for $\varepsilon \in (\theta/z(6), \varepsilon_0(6))$ and it is increasing in ε for $\varepsilon \in (\varepsilon_0(6), \theta/d)$.
(III) If $\bar{t}(6)a < c < a$, then $\tau_6'(\theta/d) < 0$ and hence $\tau_6(\varepsilon)$ is decreasing in ε for all $\varepsilon \in (\theta/z(6), \theta/d)$.

Next we observe that there exist $1/[z(6)+1] < \underline{\lambda}(6) < \overline{\lambda}(6) < 1$ (specifically, $\underline{\lambda}(6) \equiv k_4/(k_5 + k_6\sqrt{k_3})$ and $\overline{\lambda}(6) \equiv k_4/(k_5 - k_6\sqrt{k_3})$, where $k_4 \equiv 1224822676307$, $k_5 \equiv 4722314968307$, $k_5 \equiv 15617527536$) such that

(A) If $a/[z(6)+1] < c < \underline{\lambda}a$, then both $\tau_6(\theta/z(6))$, $\tau_6(\theta/d)$ are positive.
(B) If $\underline{\lambda}a < c < \overline{\lambda}a$, then $\tau_6(\theta/z(6)) < 0$ and $\tau_6(\theta/d) > 0$.
(C) If $\overline{\lambda}a < c < a$, then both $\tau_6(\theta/z(6))$, $\tau_6(\theta/d)$ are negative.

Noting that $\overline{\lambda}(6) < \underline{t}(6)$, from (I)-(III) and (A)-(C), we conclude that:

(1) If $a/[z(6)+1] < c < \underline{t}(6)a$, then $\tau_6(\varepsilon)$ is increasing in ε for all $\varepsilon \in (\theta/z(6), \theta/d)$. Noting that $(1/[z(6)+1], \underline{t}(6)) = (1/[z(n)+1], \underline{\lambda}(6)] \cup (\underline{\lambda}(6), \overline{\lambda}(6)] \cup (\overline{\lambda}(6), \underline{t}(6))$, we have:

(a) If $a/[z(6)+1] < c < \underline{\lambda}(6)a$, then $\tau_6(\theta/z(6)) > 0$, so $\tau_6(\varepsilon) > 0$ for all $\varepsilon \in (\theta/z(6), \theta/d)$. By (55), $\delta_6^*(6) < \hat{\delta}^6(6)$ for all $\varepsilon \in (\theta/z(6), \theta/d)$.

(b) If $\underline{\lambda}(6)a < c < \overline{\lambda}(6)a$, then $\tau_6(\theta/z(6)) < 0 < \tau_6(\theta/d)$, so $\exists\ \bar{\varepsilon}(6) \in (\theta/z(6), \theta/d)$ such that $\tau_6(\varepsilon) < 0$ for $\varepsilon \in (\theta/z(6), \bar{\varepsilon}(6))$ and $\tau_6(\varepsilon) > 0$ for $\varepsilon \in (\bar{\varepsilon}(6), \theta/d)$. By (55), $\delta_6^*(6) > \hat{\delta}^6(6)$ for $\varepsilon \in (\theta/z(6), \bar{\varepsilon}(6))$ and $\delta_6^*(6) < \hat{\delta}^6(6)$ for $\varepsilon \in (\bar{\varepsilon}(6), \theta/d)$.

(c) If $\overline{\lambda}(6)a < c < \underline{t}(6)a$, then $\tau_6(\theta/d) < 0$, so $\tau_6(\varepsilon) < 0$ for all $\varepsilon \in (\theta/z(6), \theta/d)$. By (42), $\delta_6^*(6) > \hat{\delta}^6(6)$ for all $\varepsilon \in (\theta/z(6), \theta/d)$.

Again noting that $\overline{\lambda}(6) < \underline{t}(6)$, when $c > \underline{t}(6)a$, we have $c > \overline{\lambda}(6)(a)$, so both $\tau_6(\theta/z(6)), \tau_6(\theta/d)$ are negative and the following holds.

(2) If $\underline{t}(6)a < c < \bar{t}(6)a$, then $\exists\ \varepsilon_0(6) \in (\theta/z(6), \theta/d)$ such that $\tau_6(\varepsilon)$ is decreasing in ε for $\varepsilon \in (\theta/z(6), \varepsilon_0(6))$ and it is increasing in ε for $\varepsilon \in (\varepsilon_0(6), \theta/d)$. Since both $\tau_6(\theta/z(6)), \tau_6(\theta/d)$ are negative, in this case $\tau_6(\varepsilon) < 0$ for all $\varepsilon \in (\theta/z(6), \theta/d)$. By (55), $\delta_6^*(6) > \hat{\delta}^6(6)$ for all $\varepsilon \in (\theta/z(6), \theta/d)$.
(3) If $\bar{t}(6)a < c < a$, then $\tau_6(\varepsilon)$ is decreasing in ε for all $\varepsilon \in (\theta/z(6), \theta/d)$. Since $\tau_6(\theta/z(6)) < 0$, in this case $\tau_6(\varepsilon) < 0$ for all $\varepsilon \in (\theta/z(6), \theta/d)$. By (55), $\delta_6^*(6) > \hat{\delta}^6(6)$ for all $\varepsilon \in (\theta/z(6), \theta/d)$.

Recall that since $\varepsilon > \theta/z(6) = (a-c)/z(6)$ and $c > \varepsilon$, we have $c > a/[z(6)+1]$. Denote $1/\underline{\lambda}(6) \equiv \overline{\kappa}(6)$ and $1/\overline{\lambda}(6) \equiv \underline{\kappa}(6)$. Observe that $\underline{\kappa}(6) < \overline{\kappa}(6)$.

From (1)(c), (2) and (3) above, for $a < c/\overline{\lambda}(6) = \underline{\kappa}(6)c$, we have $\delta_6^*(6) > \hat{\delta}^6(6)$. From (1)(b) above, for $c/\overline{\lambda}(6) = \underline{\kappa}(6)c < a < c/\underline{\lambda}(6) = \overline{\kappa}(6)c$, $\exists\ \bar{\varepsilon}(6) \in (\theta/z(6), \theta/d)$ such that $\delta_6^*(6) > \hat{\delta}^6(6)$ for $\varepsilon \in (\theta/z(6), \bar{\varepsilon}(6))$. By Proposition 2(V), it follows that if $[a < \underline{\kappa}(6)c]$ or $[\underline{\kappa}(6)c < a < \overline{\kappa}(6)c$ and $\varepsilon < \bar{\varepsilon}(6)]$, then $\mathbb{S}^6(6, \delta_6^*(6))$ contains a policy that is a two part tariff with positive ad valorem royalty and upfront fee but no per unit royalty.

From (1)(a) above, for $a > c/\underline{\lambda}(6) = \overline{\kappa}(6)c$, we have $\delta_6^*(6) < \hat{\delta}^6(6)$. From (1)(b) above, for $c/\overline{\lambda}(6) = \underline{\kappa}(6)c < a < c/\underline{\lambda}(6) = \overline{\kappa}(6)c$, $\exists\ \overline{\varepsilon}(6) \in (\theta/z(6), \theta/d)$ such that $\delta_6^*(6) < \hat{\delta}^6(6)$ for $\varepsilon \in (\overline{\varepsilon}(6), \theta/d)$. By Proposition 2(IV), it follows that if $[a > \overline{\kappa}(6)c]$ or $[\underline{\kappa}(6)c < a < \overline{\kappa}(6)c$ and $\varepsilon > \overline{\varepsilon}(6)]$, then $\mathbb{S}^6(6, \delta_6^*(6))$ contains a policy that is a two part royalty with positive per unit and ad valorem royalties but no upfront fees.

Case 2: $n = 6$ and $\varepsilon \in (\theta/d, \theta/2)$: In this case we need to find out when $\delta_6^*(5) < \hat{\delta}^6(5)$ and when $\delta_6^*(5) > \hat{\delta}^6(5)$. Taking $n = 6$ in (52), it follows that

$$\delta_6^*(5) \lesseqqgtr \hat{\delta}^6(5) \Leftrightarrow h_6(\varepsilon) \gtreqqless 0 \tag{56}$$

Taking $n = 6$ in (51), we have

$$h_6(\varepsilon) = -8\varepsilon^3 + 4(53a - 11c)\varepsilon^2 + 2\theta(149a - 345c)\varepsilon - \theta^2(5a + 233c)$$

Note that $h_6''(\varepsilon) = 8(53a - 11c - 6\varepsilon) > 0$ (since $a > c > \varepsilon$). Thus, $h_6'(\varepsilon)$ is increasing in ε.

Since $c > \varepsilon$ and $\varepsilon > \theta/d = (a - c)/d$, we have $c > a/(d + 1)$. Note from (54) that $h_6(\theta/2) \gtreqqless 0 \Leftrightarrow c \lesseqqgtr a/3$. We observe that $h_6(\theta/d) \gtreqqless 0 \Leftrightarrow c \lesseqqgtr \hat{\lambda}a$ where $\hat{\lambda} \equiv 352069391969/(1443487450539 - 5093934202\sqrt{5642})$. Noting that $1/(d + 1) < \hat{\lambda} < 1/3$, we conclude that:

(A) If $a/(d + 1) < c < \hat{\lambda}a$, then $h_6(\theta/d)$, $h_6(\theta/2)$ are both positive.

(B) If $\hat{\lambda}a < c < a/3$, then $h_6(\theta/d) < 0 < h_6(\theta/2)$.

(C) If $a/3 < c < a$, then $h_6(\theta/d)$, $h_6(\theta/2)$. are both negative.

We next observe that $h_6'(\theta/2) = 56\theta(9a - 13c)$, so that that $h_6'(\theta/2) \gtreqqless 0 \Leftrightarrow c \lesseqqgtr 9a/13$. We observe that $h_6'(\theta/d) \gtreqqless 0 \Leftrightarrow c \lesseqqgtr \hat{w}a$ where $\hat{w} \equiv 4102392233/(6641139661 - 9341612\sqrt{5642})$. Noting that $1/(d + 1) < \hat{w} < 9/13$, we conclude that:

(I) If $a/(d + 1) < c < \hat{w}a$, then $h_6'(\theta/d) > 0$, so $h_6'(\varepsilon) > 0$ for all $\varepsilon \in (\theta/d, \theta/2)$.

(II) If $\hat{w}a < c < 9a/13$, then $h_6'(\theta/d) < 0 < h_6'(\theta/2)$, so $\exists\ \varepsilon^* \in (\theta/d, \theta/2)$ such that $h_6'(\varepsilon) < 0$ for $\varepsilon \in (\theta/d, \varepsilon^*)$ and $h_6'(\varepsilon) > 0$ for $\varepsilon \in (\varepsilon^*, \theta/2)$.

(III) If $9a/13\tilde{\lambda} < c < a$, then $h_6'(\theta/2) < 0$, so $h_6'(\varepsilon) < 0$ for all $\varepsilon \in (\theta/d, \theta/2)$.

Noting that $1/3 < \hat{w}$, from (A)-(C) and (I)-(III), we conclude:

(1) If $a/(d+1) < c < \tilde{\lambda}a$, then $h_6(\theta/d) > 0$ and $h_6(\varepsilon)$ is increasing for all $\varepsilon \in (\theta/d, \theta/2)$, so $h_6(\varepsilon) > 0$ for all $\varepsilon \in (\theta/d, \theta/2)$.

(2) If $\tilde{\lambda}a < c < a/3$, then $h_6(\theta/d) < 0 < h_6(\theta/2)$ and $h_6(\varepsilon)$ is increasing for all $\varepsilon \in (\theta/d, \theta/2)$. So $\exists$ $\tilde{\varepsilon} \in (\theta/d, \theta/2)$ such that $h_6(\varepsilon) < 0$ for $\varepsilon \in (\theta/d, \tilde{\varepsilon})$ and $h_6(\varepsilon) > 0$ for $\varepsilon \in (\tilde{\varepsilon}, \theta/2)$.

(3) If $a/3 < c < \hat{w}a$, then $h_6(\theta/2) < 0$ and $h_6(\varepsilon)$ is increasing for all $\varepsilon \in (\theta/d, \theta/2)$, so $h_6(\varepsilon) < 0$ for all $\varepsilon \in (\theta/d, \theta/2)$.

(4) If $\hat{w}a < c < 9a/13$, then $h_6(\theta/d)$, $h_6(\theta/2)$ are both negative and $\exists\ \tilde{\varepsilon} \in (\theta/d, \theta/2)$ such that $h_6(\varepsilon)$ is decreasing in ε for $\varepsilon \in (\theta/d, \tilde{\varepsilon})$ and increasing in ε for $\varepsilon \in (\tilde{\varepsilon}, \theta/2)$. In this case $h_6(\varepsilon) < 0$ for all $\varepsilon \in (\theta/d, \theta/2)$.

(5) If $9a/13\tilde{\lambda} < c < a$, then $h_6(\theta/d) < 0$ and $h_6(\varepsilon)$ is decreasing in ε for all $\varepsilon \in (\theta/d, \theta/2)$. In this case $h_6(\varepsilon) < 0$ for all $\varepsilon \in (\theta/d, \theta/2)$.

Denoting $\tilde{\kappa} = 1/\tilde{\lambda}$, from (1), we conclude that if $a > \tilde{\kappa}c$, then $h_6(\varepsilon) > 0$ and hence $\delta_6^*(5) < \hat{\delta}^6(5)$ for all $\varepsilon \in (\theta/d, \theta/2)$. By Proposition 2(IV), in this case there is an optimal policy that is a two part royalty.

From (3)-(5), if $a < 3c$, then $h_6(\varepsilon) < 0$ and hence $\delta_6^*(5) > \hat{\delta}^6(5)$ for all $\varepsilon \in (\theta/d, \theta/2)$. By Proposition 2(V), in this case there is an optimal policy that is a two part tariff consiting of an ad valorem royalty and upfront fee.

From (2), if $\tilde{\lambda}a < c < a/3$, that is, if $3c < a < \tilde{\kappa}c$, then $\exists\, \tilde{\varepsilon} \in (\theta/d, \theta/2)$ such that $h_6(\varepsilon) < 0$ and $\delta_6^*(5) > \hat{\delta}^6(5)$ for $\varepsilon < \tilde{\varepsilon}$ and $h_6(\varepsilon) > 0$ and $\delta_6^*(5) < \hat{\delta}^6(5)$for $\varepsilon > \tilde{\varepsilon}$.

5.8 Proof of Proposition 5

For $n = 2, 3$, the function $t(n)$ is defined as

$$t(2) := 3(\sqrt{2} - 1) \text{ and } t(3) := (8 + 2\sqrt{7})/3 \tag{57}$$

We note thar $n - 1 < t(n) < 2n - 1 = 3$ for $n = 2, 3$.

(I) Part (I) follows from Proposition 3(a) (p.172) and Proposition 4(b) (p.173) of Sen and Tauman (2007, p.172).

(II) Note that $2n - 1 = 3$ for $n = 2$ and $2n - 1 = 5$ for $n = 3$. Noting from (57) that $t(2) = t \equiv 3(\sqrt{2} - 1)$ and $t(3) = d_0 \equiv (8 + 2\sqrt{7})/3$ and using these in Table A.5 (p.183) of Sen and Tauman (2007), part (II) follows.

(III) If $n = 3$ and $\varepsilon > \theta/2$, from Table A.5 (p.183) of Sen and Tauman (2007) it follows that any optimal three part tariff has $k = 2$ and $\delta = \theta/2$. Noting that $\theta/2 > \hat{\delta}^3(2)$ (see Figure 3(a)), part (III) follows by parts (II), (III) and (V) of Proposition 2.

(IV) Using $t(n)$ from (57) for $n = 2, 3$ in Table A.5 (p.183) of Sen and Tauman (2007), it follows that for $n = 2, 3$, if $\theta/(2n - 1) < \varepsilon < \theta/t(n)$, then any optimal three part tariff has $k = n$ and $\delta = \delta_n^*(n) \in (0, \varepsilon)$ (given in (34)). Then we know from Proposition 2(II) that the set of all optimal policies $\mathbb{S}^n(n, \delta_n^*(n))$ is a continuum, where $\mathbb{S}^n(k, \delta)$ is given in (20).

Applying the same reasoning as in the proof of part (III)(iii) of Proposition 4, taking $n = 2$ and $n = 3$ in (42), we conclude that

$$\text{for } n = 2, 3\text{:}\ \delta_n^*(n) \lesseqqgtr \hat{\delta}^n(n) \Leftrightarrow \tau_n(\varepsilon) \gtreqqless 0 \tag{58}$$

where $\tau_n(\varepsilon)$ is given in (41). Taking $n = 2, 3$ in (41), we have

$$\tau_2(\varepsilon) = 27\varepsilon^3 + 333(a - c)\varepsilon^2 + 3(a - c)(83a - 227c)\varepsilon - (25a + 119c)(a - c)^2$$

$$\tau_3(\varepsilon) = 8[-4\varepsilon^3 + 4(72a - 51c)\varepsilon^2 + 6(a - c)(73a - 157c)\varepsilon - (25a + 248c)(a - c)^2]$$

Since $\varepsilon > \theta/(2n-1)$ and $c > \varepsilon$, we have $c > a/2n$. Noting that for $n = 2, 3$, $\tau_n''(\varepsilon) > 0$ for all $0 < \varepsilon < c < a$, it follows that $\tau_n'(\varepsilon)$ is increasing in ε. Denoting $\rho(2) \equiv 10/19$, $\rho(3) \equiv 47/87$, note that for $n = 2, 3$: if $a/2n < c < \rho(n)a$, then $\tau_n'(\theta/(2n-1)) > 0$ and hence $\tau_n'(\varepsilon) > 0$ for all $\varepsilon > \theta/(2n-1)$. Thus for all $\varepsilon > \theta/(2n-1)$, $\tau_n(\varepsilon)$ is increasing in ε when $a/2n < c < \rho(n)a$.

Note that we have $\theta/(2n-1) < \varepsilon < \theta/t(n)$ (where $t(n)$ is given in (57)). We observe that $\tau_n(\theta/(2n-1)) < 0$. Regarding $\tau_n(\theta/t(n))$, we observe that $\exists\ \lambda(n) \in (1/2n, \rho(n))$ (specifically $\lambda(2) \equiv 2689/(2185 + 2664\sqrt{2})$ and $\lambda(3) \equiv 22513/(81313 + 14556\sqrt{7})$) such that $\tau_n(\theta/t(n)) \gtreqqless 0$ iff $c \lesseqqgtr \lambda(n)a$. Thus, if $a/2n < c < \lambda(n)a$, then $\tau_n(\theta/t(n)) > 0$. Since $\tau_n(\varepsilon)$ is increasing in ε when $c \in (a/2n, \lambda(n)a) \subset (a/2n, \rho(n)a)$, we conclude that when $a/2n < c < \lambda(n)a$, $\exists\ \underline{\varepsilon}(n) \in (\theta/2n, \theta/t(n))$ such that:

(1) If $\theta/(2n-1) < \varepsilon < \underline{\varepsilon}(n)$, then $\tau_n(\varepsilon) < 0$ and by (58), we have $\delta_n^*(n) > \hat{\delta}^n(n)$.
(2) If $\underline{\varepsilon}(n) < \varepsilon < \theta/t(n)$, then $\tau_n(\varepsilon) > 0$ and by (58), we have $\delta_n^*(n) < \hat{\delta}^n(n)$.

Take $\hat{\kappa}(n) := 1/\lambda(n)$, part (IV)(ii) (the case $a > \hat{\kappa}(n)c$ or equivalently $c < \lambda(n)a$) of Proposition **??** follows by (1)-(2) above by using the conclusions of parts (IV) and (V) of Proposition 2.

To prove (IV)(ii), consider $a < \hat{\kappa}(n)c$ (that is, $c > \lambda(n)a$). In this case $\tau_n(\theta/(2n-1))$, $\tau_n(\theta/t(n))$ are both negative. There are following possibilities.

If $\lambda(n)a < c < \rho(n)a$, then $\tau_n(\varepsilon)$ is increasing and $\tau_n(\theta/t(n)) < 0$. In this case for all $\varepsilon \in (\theta/(2n-1), \theta/t(n))$, we have $\tau_n(\varepsilon) < 0$ and by (58): $\delta_n^*(n) > \hat{\delta}^n(n)$.

If $c > \rho(n)a$, then $\exists\ \tilde{\lambda}(n) > \rho(n)$ (where $\tilde{\lambda}(2) \equiv 3689/(9665 - 2880\sqrt{2})$, $\tilde{\lambda}(3) \equiv 12289/(19177 + 1212\sqrt{7})$) such that:

(A) If $\rho(n)a < c < \tilde{\lambda}(n)a$, then $\tau_n'(\theta/(2n-1)) < 0 < \tau_n'(\theta/t(n))$. Since $\tau_n'(\varepsilon)$ is increasing in ε, in this case $\exists\ \varepsilon_0(n) \in (\theta/(2n-1), \theta/t(n))$ such that $\tau_n(\varepsilon)$ is decreasing for $\varepsilon \in (\theta/(2n-1), \varepsilon_0(n))$ and increasing for $\varepsilon \in (\varepsilon_0(n), \theta/t(n))$. Since $\tau_n(\theta/(2n-1))$ and $\tau_n(\theta/t(n))$ are both negative it follows that $\tau_n(\varepsilon) < 0$ for all $\varepsilon \in (\theta/(2n-1), \theta/t(n))$. By (58), $\delta_n^*(n) > \hat{\delta}^n(n)$ for all $\varepsilon \in (\theta/(2n-1), \theta/t(n))$.
(B) If $\tilde{\lambda}(n)a < c < a$, then $\tau_n'(\theta/t(n)) < 0$. Since $\tau_n'(\varepsilon)$ is increasing in ε, in this case $\tau_n'(\varepsilon) < 0$ for all $\varepsilon \in ((\theta/(2n-1), \theta/t(n))$. Hence $\tau_n(\varepsilon)$ is decreasing for all $\varepsilon \in ((\theta/(2n-1), \theta/t(n))$. Since $\tau_n(\theta/(2n-1)) < 0$, it follows that $\tau_n(\varepsilon) < 0$ for all $\varepsilon \in (\theta/(2n-1), \theta/t(n))$. By (58), $\delta_n^*(n) > \hat{\delta}^n(n)$ for all $\varepsilon \in (\theta/(2n-1), \theta/t(n))$.

From (A)-(B) and the paragraph before (A)-(B), it follows that when $c > \lambda(n)a$ (equivalently, $a < \hat{\kappa}(n)c$), we have $\delta_n^*(n) > \hat{\delta}^n(n)$ for all $\varepsilon \in (\theta/(2n-1), \theta/t(n))$. Using this, part (IV)(i) of Proposition **??** follows by part (V) of Proposition 2. ■

5.9 An incumbent innovator in a Cournot duopoly

Consider a Cournot duopoly with two firms $1, 2$ under demand $p(Q) = \max\{a - Q, 0\}$, where $Q = q_1 + q_2$ (the sum of the quantities produced by firms $1, 2$). Initially both firms have constant marginal cost c, where $0 < c < a$. Firm 1 has a cost-reducing innovation of magnitude ε $(0 < \varepsilon < c)$ that reduces the marginal cost from c to $c - \varepsilon$. The innovation is nondrastic: $\varepsilon < a - c$.

Firm 1 licenses the innovation to firm 2 using a three part tariff with per unit royalty $r \in [0, \varepsilon]$, ad valorem royalty $v \in [0, 1)$ and upfront fee $f \geq 0$. The payoff of firm 2 at the Cournot stage is

$$\pi_2 = (1-v)p(Q)q_2 - (c-\varepsilon)q_2 - rq_2 - f = (1-v)[p(Q) - (c-\delta)]q_2 - f \tag{59}$$

where $\delta = \delta(r, v)$ given in (7). Thus, at the Cournot stage firm 2 solves the same problem as a firm that has marginal cost $c-\delta$. The payoff of firm 1 at the Cournot stage is the sum of (i) the operating profit of firm 1, (ii) the revenue from ad valorem royalty, (iii) the revenue from unit royalty and (iv) fee f, which is

$$\pi_1 = [p(Q) - (c-\varepsilon)]q_1 + vp(Q)q_2 + rq_2 + f \tag{60}$$

Observe that the revenue from unit royalty rq_2 is not affected by q_1, but q_1 affects the price $p(Q)$, which in turn affects the ad valorem royalty revenue $vp(Q)q_2$. For this reason, the (unique) Cournot price and quantities are functions of both δ and v.

Lemma L3 *Consider the Cournot duopoly with firms* $1, 2$ *where the payoffs are given by* (59)-(60). *This duopoly has a unique (Cournot-Nash) equilibrium. Let* $q_i(v, \delta)$ *be the Cournot quantity of firm* i *for* $i = 1, 2$, $p(v, \delta)$ *be the Cournot price and* $Q(v, \delta) = q_1(v, \delta) + q_2(v, \delta)$. *Then:*

(i) $q_1(v, \delta) = [(1-v)(a-c) + 2\varepsilon - (1+v)\delta]/(3-v)$, $q_2(v, \delta) = (a-c-\varepsilon+2\delta)/(3-v)$ *and* $p(v, \delta) = [a + c - \varepsilon + (1-v)(c-\delta)]/(3-v)$.

(ii) *For* $i = 1, 2$, *let* $\phi_i(v, \delta)$ *be the operating Cournot profit of firm* i, *that is,*

$$\phi_1(v, \delta) = [p(v, \delta) - (c-\varepsilon)]q_1(v, \delta) \text{ and } \phi_2(v, \delta) = [p(v, \delta) - (c-\delta)]q_2(v, \delta)$$

Then

$$\phi_1(v, \delta) = [(1-v)(a-c) + 2\varepsilon - (1+v)\delta][a - c - (1-v)\delta + (2-v)\varepsilon]/(3-v)^2$$

$$\phi_2(v, \delta) = (a-c-\varepsilon+2\delta)^2/(3-v)^2$$

(iii) *The Cournot price* $p(v, \delta)$ *is increasing in* v *and decreasing in* δ. *Moreover* $\lim_{v\uparrow 1} p(v, 0) = p_M(\varepsilon)$, *where* $p_M(\varepsilon) = (a + c - \varepsilon)/2$ *(the monopoly price under marginal cost* $c - \varepsilon$*).*

(iv) *Let* $\psi(v, \delta) := (1-v)\phi_2(v, \delta)$. *The function* $\psi(v, \delta)$ *is decreasing in* v *and increasing in* δ.

Proof (i) Note that $p(Q) = \max\{a - Q, 0\}$, where $Q = q_1 + q_2$. We solve the problem by taking $p(Q) = a - Q = a - (q_1 + q_2)$ and confirm that in equilibrium the total quantity Q does not exceed a. From (59)-(60), we have

$$\partial\pi_1/\partial q_1 = a - c + \varepsilon - (1+v)q_2 - 2q_1 \text{ and } \partial\pi_2/\partial q_2 = (1-v)(a - c + \delta - q_1 - 2q_2)$$

As $1-v>0$, we have $\partial\pi_1/\partial q_1 \gtreqless 0 \Leftrightarrow q_1 \lesseqgtr b_1(q_2)$ and $\partial\pi_2/\partial q_2 \gtreqless 0 \Leftrightarrow q_2 \lesseqgtr b_2(q_1)$ where $b_1(q_2) := [a-c+\varepsilon-(1+v)q_2]/2$ and $b_2(q_1) := (a-c+\delta-q_1)/2$. So, the unique best response of firm 1 to any $q_2 \geq 0$ is

$$BR_1(q_2) = \begin{cases} b_1(q_2) & \text{if } q_2 < (a-c+\varepsilon)/(1+v) \\ 0 & \text{if } q_2 \geq (a-c+\varepsilon)/(1+v) \end{cases}$$

Similarly the unique best response of firm 2 to any $q_1 \geq 0$ is

$$BR_2(q_1) = \begin{cases} b_2(q_1) & \text{if } q_1 < a-c+\delta \\ 0 & \text{if } q_1 \geq a-c+\delta \end{cases}$$

Note that (I) the equation $q_1 = b_1(q_2)$ presents a downward sloping straight line in q_2 with intercepts $(a-c+\varepsilon)/2$ at the q_1-axis and $(a-c+\varepsilon)/(1+v)$ at the q_2-axis and (II) the equation $q_2 = b_2(q_1)$ presents a downward sloping straight line in q_1 with intercepts $a-c+\delta$ at the q_1-axis and $(a-c+\delta)/2$ at the q_2-axis. Since $a-c>\varepsilon$, $0 \leq \delta \leq \varepsilon$ and $0 \leq v < 1$, we have $(a-c+\varepsilon)/2 < (a-c+\delta)$ (q_1-intercept of the first line is lower) and $(a-c+\varepsilon)/(1+v) > (a-c+d)/2$ (q_2-intercept of the first line is higher). This shows that the (unique) point of intersection of the lines $q_1 = b_1(q_2)$ and $q_2 = b_2(q_1)$ has $q_1 > 0, q_2 > 0$. Consequently, the Cournot duopoly has a unique equilibrium. At the equilibrium, both q_1, q_2 are positive and they are found by solving the system of equations $q_1 = b_1(q_2)$, $q_2 = b_2(q_1)$. The unique solution to this system gives $q_1 = q_1(v,\delta)$, $q_2 = q_2(v,\delta)$ of part (i). Noting that $Q(v,\delta) = q_1(v,\delta) + q_2(v,\delta) = [(2-v)(a-c)+\varepsilon+(1-v)\delta]/(3-v)$, we have $a - Q(v,\delta) = [a+c-\varepsilon+(1-v)(c-\delta)]/(3-v) > 0$ (since $\delta \leq \varepsilon < c$ and $v < 1$). This confirms that the industry quantity $Q(v,\delta)$ does not exceed a. The Cournot price is obtained by noting that $p(v,\delta) = a - Q(v,\delta)$. This completes the proof of part (i).

(ii) Part (ii) follows from part (i) by standard computations.

(iii) The first statement of part (iii) follows by noting that $\partial p(v,\delta)/\partial v = (a-c-\varepsilon+2\delta)/(3-v)^2 > 0$ and $\partial p(v,\delta)/\partial\delta = -(1-v)/(3-v) < 0$. The second statement is immediate by using the expression of $p(v,\delta)$ from part (i).

(iv) Part (iv) follows by noting that $\partial\psi(v,\delta)/\partial v = -(1+v)(a-c-\varepsilon+2\delta)^2/(3-v)^3 < 0$ and $\partial\psi(v,\delta)/\partial\delta = 4(1+v)(a-c-\varepsilon+2\delta)/(3-v)^2 > 0$. ■

5.9.1 Detailed proof of Proposition 6

Proof of part (I) Denoting $\psi(v,\delta) := (1-v)\phi_2(v,\delta)$, using (59)-(60) and using the Cournot price and Cournot profits from Lemma L3, the payoffs of firms at (v,δ) are

$$\pi_2 = (1-v)\phi_2(v,\delta) - f = \psi(v,\delta) - f \text{ and } \pi_1 = \phi_1(v,\delta) + vp(v,\delta)q_2(v,\delta) + rq_2(v,\delta) + f \quad (61)$$

Since the innovation is nondrastic, without a license firm 2 obtains a positive profit $\underline{\phi}$, so a policy is acceptable if and only if $\pi_2 = \psi(v,\delta) - f \geq \underline{\phi}$. Thus, for a policy (v,δ), the maximum upfront fee that can be obtained from firm 2 is $\bar{f}(v,\delta) = \psi(v,\delta) - \underline{\phi}$. Noting that $(1-v)(c-\delta) = c - \varepsilon + r$ (by (7)), we have $\bar{f}(v,\delta) = (1-v)p(v,\delta)q_2(v,\delta) - (c-\varepsilon+r)q_2(v,\delta) - \underline{\phi}$. Taking $f = \bar{f}(v,\delta)$ in (61), the payoff of firm 1 is $\pi_1 = G(p(v,\delta)) - \underline{\phi}$ where $G(p)$ is given in (3).

Note from Lemma L3(iii) that for $\delta \in [0,\varepsilon]$ and $v \in [0,1)$, the Cournot price $p(v,\delta)$ is increasing in v and decreasing in δ. Since $\lim_{v\uparrow 1} p(v,0) = p_M(\varepsilon)$ (the monopoly price under marginal cost $c-\varepsilon$), we have $p(v,\delta) < p_M(\varepsilon)$ for all $v \in [0,1)$. As $G(p)$ is increasing for $p < p_M(\varepsilon)$, the payoff of firm 1 is increasing in $p(v,\delta)$. Since $p(v,\delta)$ is increasing in v, we conclude that for any δ, the payoff of firm 1 is maximum when v is maximum. Thus any optimal pair (δ, v) must be on the curve OA for $\delta \le \delta_A$ and on the curve AB for $\delta > \delta_A$ (see Figure 6).

Next observe that choosing $\delta > \delta_A$ cannot be optimal. This is because $p(v,\delta)$ is increasing in v, decreasing in δ and the curve AB (presenting $\bar{v}(\delta)$) is decreasing, so $p(\bar{v}(\delta_A),\delta_A) > p(\bar{v}(\delta),\delta)$ for any $\delta > \delta_A$. Therefore any optimal δ must be $\delta \le \delta_A$ and any optimal (δ, v) pair must lie on curve OA. For any such (δ, v), we have $\psi(v,\delta) = (1-v)\phi_2(v) = \underline{\phi}$, so the upfront fee $\bar{f}(v,\delta) = 0$.

Noting that $\psi(v,\delta) = (1-v)\phi_2(v,\delta) = (1-v)(a-c-\varepsilon+2\delta)^2/(3-v)^2$ and $\underline{\phi} = \psi(0,0) = (a-c-\varepsilon)^2/9$, we have

$$\psi(v,\delta) = \underline{\phi} \Leftrightarrow \delta = h(v) \text{ where } h(v) := (a-c-\varepsilon)[\{(3-v)/6\sqrt{1-v}\} - 1/2]$$

Note from Figure 6 that for any $v \in [0, \gamma(\delta_A)]$ (any such v is on the line OD on the vertical axis), $h(v)$ is given by the curve OA. When $\delta = h(v)$, the Cournot price is

$$p(v,h(v)) = (a+c-\varepsilon)/2 - \{(a-c-\varepsilon)\sqrt{1-v}\}/6$$

Observe that $p(v,h(v))$ is increasing in v. This shows that for all $(v,h(v))$ on the curve OA, the Cournot price is maximum when $v = \gamma(\delta_A)$. So the unique optimal three part tariff corresponds to the point A in Figure 6. Noting that $\gamma(\delta_A) = \bar{v}(\delta_A)$, at point A we have $v = \bar{v}(\delta_A)$ (the maximum feasible ad valorem royalty), so the unit royalty r is zero and the unique optimal three part tariff is a pure ad valorem royalty policy.

Proof of part (II) Noting that $(v=0, \delta=0)$ results in the same Cournot price as in the case of no licensing, when there is no licensing, the price is $p(0,0)$ (the price at the point O in Figure 6). The antitrust constraint requires to choose acceptable and feasible policies for which $p(v,\delta) \le p(0,0)$.

Recall that any point on OA other than O has a price higher than $p(0,0)$. So the price at A is higher than $p(0,0)$. The price at B, $p(0,\varepsilon)$, is lower than $p(0,0)$. As prices on the curve AB increase as we move from B to A, there exists a point E on AB at which the price equals $p(0,0)$, prices are higher on AE and lower on EB. The point E is identified in Figure 6 and the corresponding δ is denoted by δ_E. For any point in the region FEB other than E, the price is lower than $p(0,0)$.

For any $\delta \in (0,\delta_E)$, the maximum v that makes (v,δ) acceptable and feasible lies on OA or on AE, for which $(v,\delta) = (m(\delta),\delta)$ where $m(\delta) = \min\{\bar{v}(\delta), \gamma(\delta)\}$. As the price at any point OA (excluding O) or AE is higher than $p(0,0)$, for any $\delta \in (0,\delta_E)$: $p(0,\delta) < p(0,0) < p(m(\delta),\delta)$. So there is a unique $\hat{v}(\delta) \in (0, m(\delta))$ such that $p(v,\delta) \gtreqless p(0,0) \Leftrightarrow v \gtreqless \hat{v}(\delta)$. As $p(v,\delta)$ is increasing in v and decreasing in δ, $\hat{v}(\delta)$ is increasing in δ. The function $\hat{v}(\delta)$ is presented by the curve OE in Figure 6 ($\hat{v}(0) = 0$ and $\hat{v}(\delta_E)$ coincides with point E).

The region OEB is the set of all acceptable and feasible (v, δ) for which the post-licensing Cournot price $p(v,\delta)$ does not exceed $p(0,0)$. For any point on OE, $p(v,\delta) = p(0,0)$, whereas for any other point in the region OEB, $p(v,\delta) < p(0,0)$. Since the payoff of firm 1 is maximum when $p(v,\delta)$ is maximum, the set of all optimal policies is the continuum given by the curve OE where the price exactly equals $p(0,0)$. For any such policy, firm 2 obtains $\underline{\phi}$ and firm 1 obtains $G(p(0,0)) - \underline{\phi}$.

At O: $v = 0 = \gamma(0)$ (no ad valorem royalty and zero surplus from the license, so no upfront fee as well). As $\delta = 0$ at O, by (7), setting $\delta(r,v) = 0$ and $v = 0$ gives $r = \varepsilon$, proving (i). At E: $v > 0$ (positive ad valorem royalty), $v = \bar{v}(\delta_E)$ (maximum feasible v, so the unit royalty is zero) and $v < \gamma(\delta_E)$ (positive surplus from the license, so positive upfront fee), which is a two part tariff with ad valorem royalty and upfront fee, proving (ii).

For any point on OE excluding O and E: $v > 0$ (positive ad valorem royalty), $v < \bar{v}(\delta)$ (less than maximum feasible v, so the unit royalty is positive) and $v < \gamma(\delta)$ (positive surplus from the license, so positive upfront fee), so all three components are positive, proving (iii). ■

Table 1: Optimal three part tariffs and payoffs for an outside innovator under linear demand $p(Q) = \max\{a - Q, 0\}$ in a Cournot oligopoly with $n \geq 2$ firms, initial marginal cost c, magnitude of innovation ε, where $0 < \varepsilon < c < a$ and $\varepsilon < \theta \equiv a - c$ (nondrastic innovation)

$n = 2, 3$: less significant and intermediate innovations	$\varepsilon \in (0, \theta/(2n-1))$	$\varepsilon \in (\theta/(2n-1), \theta/t(n))$
Set of all optimal policies	unique optimal policy: pure upfront fee	$\mathbb{S}^n(n, \delta_n^*(n))$
Licenses	$k = n$	$k = n$
Payoff	$\Pi^n(n, \varepsilon)$	$\Pi^n(n, \delta_n^*(n))$
Two part tariffs FR: $r > 0, f > 0, v = 0$		such an optimal policy always exists
Three part tariffs: $r > 0, v > 0, f > 0$		a continuum of such optimal policies always exist
Two part tariffs FV: $v > 0, f > 0, r = 0$		optimal if $[a < \hat{\kappa}(n)c]$ or $[a > \hat{\kappa}(n)c$ and $\varepsilon < \underline{\varepsilon}(n)]$
Two part royalty: $r > 0, v > 0, f = 0$		optimal if $[a > \hat{\kappa}(n)c$ and $\varepsilon > \underline{\varepsilon}(n)]$

$n = 2$: significant innovations	$\varepsilon \in (\theta/t(2), \theta)$
Set of all optimal policies	unique optimal policy: pure upfront fee
Licenses	$k = 1$
Payoff	$\varepsilon(2\theta + \varepsilon)/3$

$n = 3$: significant innovations	$\varepsilon \in (\theta/t(3), \theta/2)$	$\varepsilon \in (\theta/2, \theta)$
Set of all optimal policies	unique optimal policy: pure upfront fee	$\mathbb{S}^3(2, \theta/2)$
Licenses	$k = 2$	$k = 2$
Payoff	$\varepsilon(a - c)$	$\varepsilon(a - c)$
Two part tariffs FR: $r > 0, f > 0, v = 0$		such an optimal policy always exists
Three part tariffs: $r > 0, v > 0, f > 0$		a continuum of such optimal policies always exist
Two part tariffs FV: $v > 0, f > 0, r = 0$		such an optimal policy always exists
Two part royalty: $r > 0, v > 0, f = 0$		not optimal

$n \geq 4$: less significant innovations	$\varepsilon \in (0, \theta/\ell(n))$	$\varepsilon \in (\theta/\ell(n), \theta/(2n-4))$
Set of all optimal policies	unique optimal policy: pure upfront fee	
Licenses	$k = n$	$k = n - 1$
Payoff	$\Pi^n(n, \varepsilon)$	$\Pi^n(n - 1, \varepsilon)$

$n = 4, 5$: intermediate and significant innovations	$\varepsilon \in (\theta/(2n-4), \theta/2)$	$\varepsilon \in (\theta/2, \theta)$
Set of all optimal policies	$\mathbb{S}^n(n-1, \delta_n^*(n-1))$	$\cup_{k=2}^{n-1} \mathbb{S}^n(k, \theta/k)$
Licenses	$k = n-1$	$k \in \{2, \dots, n-1\}$
Payoff	$\Pi^n(n-1, \delta_n^*(n-1))$	$\varepsilon(a-c)$
Two part tariffs FR: $r > 0, f > 0, v = 0$	such an optimal policy always exists	$n-2$ such optimal policies always exist
Three part tariffs: $r > 0, v > 0, f > 0$	a continuum of such optimal policies always exists	
Two part tariffs FV: $v > 0, f > 0, r = 0$	optimal if $[a < 3c]$ or $[a > 3c$ and $\varepsilon < \underline{\varepsilon}(n)]$	such an optimal policy always exists
Two part royalty: $r > 0, v > 0, f = 0$	optimal if $[a > 3c$ and $\varepsilon > \underline{\varepsilon}(n)]$	not optimal

$n = 6$: intermediate and significant innovations	$\varepsilon \in (\theta/8, \theta/z(6))$	$\varepsilon \in (\theta/z(6), \theta/d)$	$\varepsilon \in (\theta/d, \theta/2)$	$\varepsilon \in (\theta/2, \theta)$
Set of all optimal policies	$\mathbb{S}^6(5, \delta_6^*(5))$	$\mathbb{S}^6(6, \delta_6^*(6))$	$\mathbb{S}^6(5, \delta_6^*(5))$	$\cup_{k=2}^{5} \mathbb{S}^6(k, \theta/k)$
Licenses	$k = 5$	$k = 6$	$k = 5$	$k \in \{2, 3, 4, 5\}$
Payoff	$\Pi^6(5, \delta_6^*(5))$	$\Pi^6(6, \delta_6^*(6))$	$\Pi^6(5, \delta_6^*(5))$	$\varepsilon(a-c)$
Two part tariffs FR: $r > 0, f > 0, v = 0$	such an optimal policy always exists			4 such optimal policies always exist
Three part tariffs $r > 0, v > 0, f > 0$	a continuum of such optimal policies always exists			
Two part tariffs FV: $v > 0, f > 0, r = 0$	optimal if $[a < \kappa(6)c]$ or $[a > \kappa(6)c$ and $\varepsilon < \underline{\varepsilon}(6)]$	optimal if $[a < \underline{\kappa}(6)c]$ or $[a \in (\underline{\kappa}(6)c, \overline{\kappa}(6)c)$ and $\varepsilon < \overline{\varepsilon}(6)]$	optimal if $[a < 3c]$ or $[a \in (3c, \tilde{\kappa}c)$ and $\varepsilon < \tilde{\varepsilon}]$	such an optimal policy always exists
Two part royalty: $r > 0, v > 0, f = 0$	optimal if $[a > \kappa(6)c$ and $\varepsilon > \underline{\varepsilon}(6)]$	optimal if $[a > \overline{\kappa}(6)c]$ or $[a \in (\underline{\kappa}(6)c, \overline{\kappa}(6)c)$ and $\varepsilon > \overline{\varepsilon}(6)]$	optimal if $[a > \tilde{\kappa}c]$ or $[a \in (3c, \tilde{\kappa}c)$ and $\varepsilon > \tilde{\varepsilon}]$	not optimal

$n \geq 7$: intermediate and significant innovations	$\varepsilon \in (\theta/(2n-4), \theta/z(n))$	$\varepsilon \in (\theta/z(n), \theta/u(n))$	$\varepsilon \in (\theta/u(n), \theta)$
Set of all optimal policies	$\mathbb{S}^n(n, \delta_n^*(n-1))$	$\mathbb{S}^n(n, \delta_n^*(n))$	$\cup_{k=2}^{n-1} \mathbb{S}^n(k, \theta/k)$
Payoff	$\Pi^n(n, \delta_n^*(n-1))$	$\Pi^n(n, \delta_n^*(n))$	$\varepsilon(a-c)$
Licenses	$k = n-1$	$k = n$	$k \in \{2, \dots, n-1\}$
Two part tariffs FR: $r > 0, f > 0, v = 0$	such an optimal policy always exists		$n-2$ such optimal policies always exist
Three part tariffs: $r > 0, v > 0, f > 0$	a continuum of such optimal policies always exists		
Two part tariffs FV: $v > 0, f > 0, r = 0$	optimal if $[a < \kappa(n)c]$ or $[a > \kappa(n)c$ and $\varepsilon < \underline{\varepsilon}(n)]$	optimal if $[a < \underline{\kappa}(n)c]$ or $[a \in (\underline{\kappa}(n)c, \overline{\kappa}(n)c)$ and $\varepsilon < \overline{\varepsilon}(n)]$	such an optimal policy always exists
Two part royalty: $r > 0, v > 0, f = 0$	optimal if $[a > \kappa(n)c$ and $\varepsilon > \underline{\varepsilon}(n)]$	optimal if $[a > \overline{\kappa}(n)c]$ or $[a \in (\underline{\kappa}(n)c, \overline{\kappa}(n)c)$ and $\varepsilon > \overline{\varepsilon}(n)]$	not optimal

Notes: (i) Two part tariffs FR: combinations of per unit royalties and upfront fees; (ii) Two part tariffs FV: combinations of ad valorem revenue royalties and upfront fees.

Effective magnitude of the innovation under optimal policies:

$$\delta_n^*(k) := \begin{cases} (\theta + 2\varepsilon)/2(n-1) & \text{if } k = n-1 \\ [(n-1)\theta + (n+1)\varepsilon]/2(n^2 - n + 1) & \text{if } k = n \end{cases}$$

Payoffs:

$$\Pi^n(k, \varepsilon) = k[\{\theta + (n-k+1)\varepsilon\}^2 - \{\theta - (n-1)\varepsilon\}^2]/(n+1)^2 \text{ for } k = n-1, n$$

$$\Pi^n(n-1, \delta_n^*(n-1)) = [\theta^2 + 4n\theta\varepsilon + 4\varepsilon^2]/4(n+1)$$

$$\Pi^n(n, \delta_n^*(n)) = n[(n-1)^2\theta^2 + 2(2n^3 + n^2 + 1)\theta\varepsilon + (n+1)^2\varepsilon^2]/4(n+1)^2(n^2 - n + 1)$$

$$t(n) := \begin{cases} 3(\sqrt{2} - 1) & \text{if } n = 2 \\ 2(4 + \sqrt{7})/3 & \text{if } n = 3 \end{cases}$$

$$\ell(n) := (n^2 + n - 3)/2, d \equiv (210 - \sqrt{5642})/67,$$

$$z(n) := [n^3 - n + \sqrt{(n+1)(n^2 - n + 1)(n^3 - 6n^2 + 5n - 4)}]/(2n^2 - n + 1)$$

$$u(n) := (n+1)(1 + \sqrt{n^2 - n + 1})^2/n(n-1)^2$$

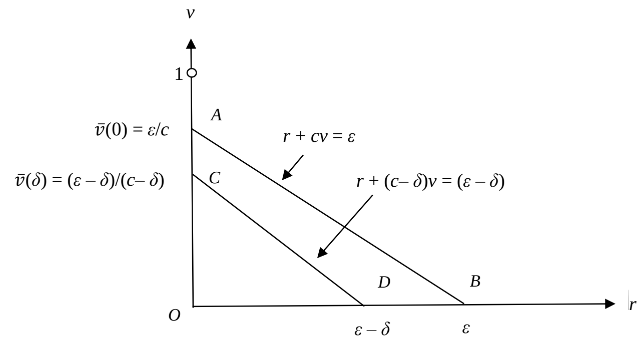
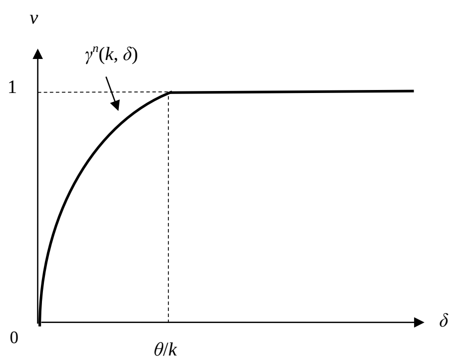
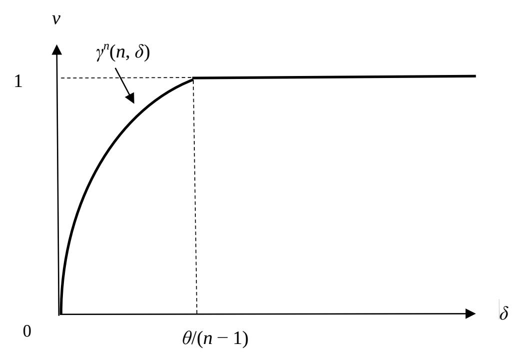


Figure 1: Effective magnitude of the innovation $\delta(r, v)$

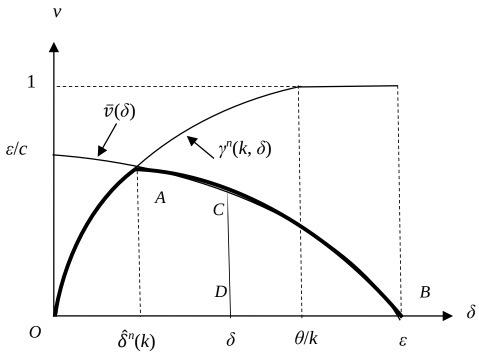


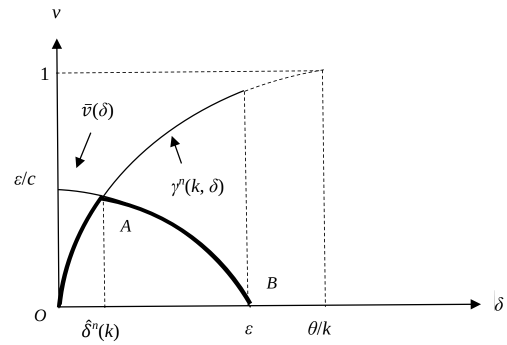


Acceptable v for (k, δ)

Figure 2(a): $\gamma^n(k, \delta)$ for $1 \leq k \leq n - 1$

Figure 2(b): $\gamma^n(n, \delta)$

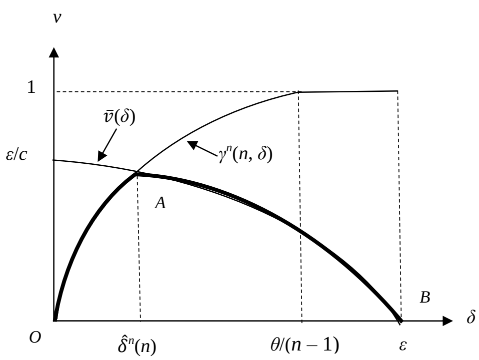


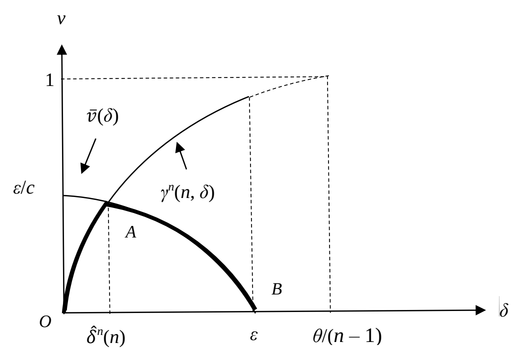


Acceptable versus feasible v for (k, δ), $1 \leq k \leq n - 1$

Figure 3(a): $\varepsilon \geq \theta/k$

Figure 3(b): $\varepsilon < \theta/k$





Acceptable versus feasible v for (n, δ)

Figure 3(c): $\varepsilon \geq \theta/(n-1)$

Figure 3(d): $\varepsilon < \theta/(n-1)$

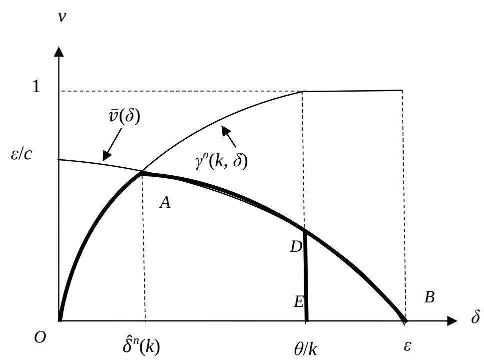


Figure 4(a): $DE = \mathbb{S}^n(k, \theta/k)$, $k \in \{2,\ldots,n-1\}$; part (II) of Proposition 4

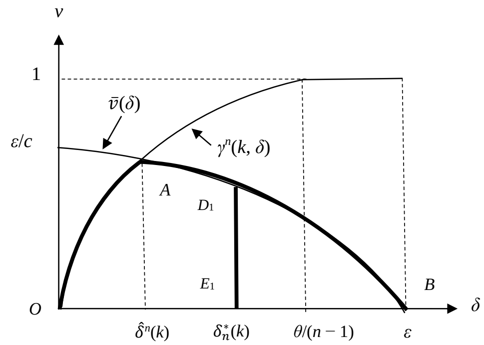


Figure 4(b): $D_1E_1 = \mathbb{S}^n(k, \delta_n^*(k))$; $k = n$ for (III)(iii)(a) and $k = n - 1$ for (IV)(iii)(a) of Proposition 4

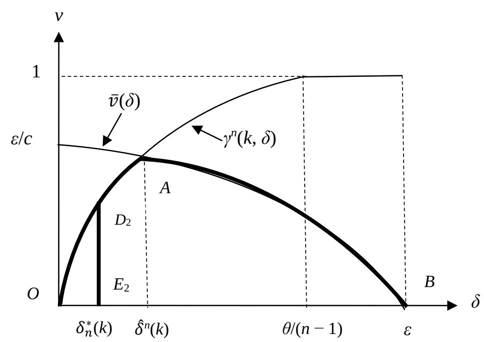


Figure 4(c): $D_2E_2 = \mathbb{S}^n(k, \delta_n^*(k))$; $k = n$ for (III)(iii)(b) and $k = n - 1$ for the last part of (IV)(iii)(b) of Proposition 4

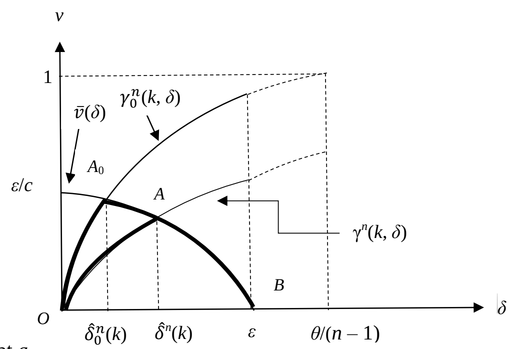


Figure 5: Effects of change in demand intercept *a*

OA: $v = \gamma(\delta)$

AB: $v = \bar{v}(\delta)$

OE: $v = \hat{v}(\delta)$

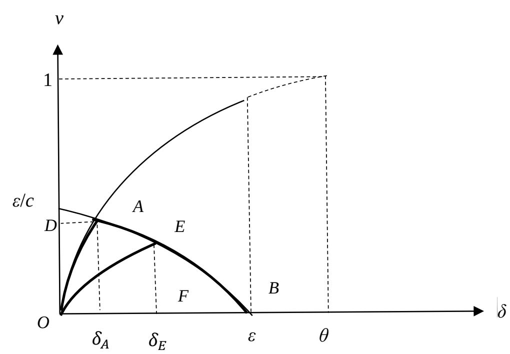


Figure 6: An incumbent innovator in a Cournot duopoly: acceptable versus feasible *v*

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