S-shaped utility, subprime crash and the black swan

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Abstract

I propose an S-shaped utility function of consumption which, combined with an heterogeneous agents and external habit setting, fits well the first order moments of the American financial and macroeconomic time series relevant for the equity premium puzzle in the second half of XX century. The average relative risk aversion of the agents remains in the 0-3 range. A "black swan"-kind phenomenon makes two of the 50 years considered (the two oil shocks) responsible for half the average of the stochastic discount factor, thus bringing the annual subjective discount factor to a very low level, around 0.5, which solves the risk-free puzzle. The shape of the relative risk aversion function of consumption suggests an explanation for the 2008 suprime crash akin to the breaking of waves on a beach in a lifecycle overlapping generations model.

KEYWORDS: financial puzzles, subprime crash, black swan, S-shaped utility

JEL classification: C0,C5,D1,D53,D91,E44,G12

1 Introduction

Friedman and Savage (1948) already called attention to the puzzle that consisted on people buying lottery tickets (whose expected return is less than

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what is paid) and, at the same time, contracting insurance (paying more for it than the expected value of the damage); in the former case, there is a risk seeking behavior, whilst, in the later, a risk aversion; in accordance, they proposed the use of utility functions that contained local convexities. Markowitz (1952) goes a bit further and proposes that the point that separates the convex from the concave parts of the utility be dependent upon the level of wealth.

Kahneman and Tversky (1979) proposed the so-called prospect theory. It presupposes that people make decisions based upon goals, defined by a mean and a variance of a target value; in this case, whenever the subjective probability density function of that target value is unimodal, the corresponding utility function will be S-shaped, which is explained as a predisposition of people to run risks, in order to reach a goal, and a risk aversion, when the task is to go beyond it. Another way to see it is the tendency of people to smooth their consumption time series, avoiding great oscillations, which presupposes the existence of a level with which they feel comfortable.

Friedman (1989) proposes that, due to bounded rationality, individuals don't know their true utility function (which would be concave) and distort it, so that they end up by using an S-shaped subjective utility. The distortion would stem from the concentration of the lotteries available for him in a small range, that is, in his words, *he will choose as if maximizing expected value for a value function V that in some sense is between his "true" fully considered function U and the cumulative distribution function F of prospective opportunities to increment wealth.*

Tummers (1992) uses an S-shaped "welfare function of income" to analyse subjective poverty line models.

The greater or smaller disposition to run risks depends upon not only the proposed lottery, but also on how many times one can play (or how many tickets you can buy); if infinite, risk neutrality is the logical consequence. That depends on the money one has: the value at risk is an important factor to be considered in investments. Worldwide, most people seem to be risk seeking, when the sum at risk is small compared to their personal wealth. Benartzi and Thaler (1995) call attention to this kind of phenomenon, which they call *myopic loss aversion* and propose an S-shaped utility centered on the consumption value of the immediately preceding period (c_{t-1}).

Hamo and Heifetz (2001) propose an explanation to the rising of spontaneous S-shaped utilities among members of a population, using evolutionary game theory. They show mathematically that society stirs its members to invest part of their resources in actuarially losing activities, because this decreases the systemic risk of the collective bet in a common direction, although at the expenses of the increase of the idiosyncratic risk to which the individual is exposed. The appearance of such S-shaped subjective utility functions would stem, at the individual level, from family tradition and would be collectively manifested by the dynamics of the evolutionary game between individuals with the most diverse types of utilities resulting on the gradual elimination of other kinds of utilities, so that, in the long run, 100% of the population would have that particular utility function.

Ternström (2001) posits a logistic utility of consumption, to analyze the tragedy of the commons. Neilson (2002) says that the hypothesis that individuals base their decisions on final wealth is rejected by the data and agrees with prospect theory in that what matters are gains and losses from a reference point; he characterizes mathematically the notion of "more S-shaped than". Levy, Giorgi, and Hens (2003) prove that the security market line theorem of the CAPM remains valid in the context of cumulative prospect theory. Levy and Levy (2004) show that the portfolio selected by the mean-variance approach belongs to the efficient set defined by prospect theory, whenever diversification between assets is allowed.

A doubt that could arise is if this class of utilities would destroy the general equilibrium, since most of the theorems in the theory use concave utilities. Xi (2007) proves that the operation point of each agent being beyond the point of tangency between the utility and the straight line that passes through the origin (0,0) is a sufficient condition for the existence of the Arrow-Debreu equilibrium, when the utility is S-shaped. Hagströmer, Anderson, Binner, Elger, and Nilsson (2007) use a combination of power utilities, to build an S-shaped utility of portfolio return and show that, in this case, the full-scale optimization approach is better then the mean-variance one.

Gerasymchuk (2007) analyses cumulative normal, logistics and arctangent as possible formulas for S-shaped utilities and relates them with attitudes towards diversification of portfolios. Gerasymchuk (2008) chooses arctangent and shows that the resulting dynamic equations for the evolution of wealth and the risky asset return exhibit chaotic regimen in a subset of its parametric space. Netzer (2008) justifies the use of S-shaped value functions (as in prospect theory) as an evolutionary adaptation. Bostian (2008) explores models of learning and utility using two experimental designs and concludes that "there is some evidence of an S-shaped utility function, suggesting that risk attitudes may change for gains and losses".

I introduce here a new utility and argue that, as it, in a way, solves the equity premium puzzle (as is empirically shown in the following sections), the case for S-shaped utility functions is reinforced. The full details can be found in de Farias Neto (2007).

The framework of the present paper is not the one of prospect theory, but what is called "reference-dependent expected utility". Plus, the reference point is supposed to be the same for all agents - the *per capita* consumption level of the country - only changing with time.

In section 2, I introduce my utility function and show some of its properties. Section 3 establishes the general framework in which the empirical work is done. Section 4 shows that the model can be narrowed. Section 5 considers the narrowed model and gets the main results. Section 6 shows results for the Brazilian market, that confirm the main conclusion obtained for the American market (namely, that my utility solves the equity premium puzzles, if, instead of considering the RRA of the average consumer, I consider the average RRA of the consumers). Section 7 draws some conclusions and comments on the results and some of their possible implications.

2 Utility function

I propose that the i-th household have the following utility function (cumulated modified Cauchy distribution):[1]

$$U(C_t^i) = \int_{-\infty}^{C_t^i} \frac{\sqrt{2}}{\pi b \left(1 + \left(\frac{\xi - X_t^i}{b}\right)^4\right)} d\xi \tag{1}$$

where

$$X_t^i = a \left(\varsigma_t^i\right)^{\kappa} \left(\varsigma_t\right)^{(1-\kappa)} \tag{2}$$

a, b, κ being the same for all households, ς_t being the time-smoothed mean consumption level of all the households in the country and ς_t^i being the time-smoothed mean consumption level of household i. Equation 2 is inspired by Abel (2007), who, using a different utility, gives it the flexibility of representing external and internal habit formation and combinations of both. Formula 1 has closed definite integral; using it, my utility can be expressed as

$$\begin{aligned} u(C_t^i) = {} & \tfrac{1}{4\pi} \ln \frac{\left(\frac{C_t^i - X_t^i}{b} + \frac{1}{2}\sqrt{2}\right)^2 + \frac{1}{2}}{\left(\frac{C_t^i - X_t^i}{b} - \frac{1}{2}\sqrt{2}\right)^2 + \frac{1}{2}} + \\ & + \tfrac{1}{2\pi}\left(\arctan\left(\sqrt{2}\frac{C_t^i - X_t^i}{b} - 1\right) + \arctan\left(\sqrt{2}\frac{C_t^i - X_t^i}{b} + 1\right) - 2\pi\right) + \tfrac{3}{2} \end{aligned} \tag{3}$$

[1] Allegations that this is an *ad hoc* formula are void, since all utility functions in economic theory are *ad hoc*. The final decision about which one to use will be taken according to their explanation powers.

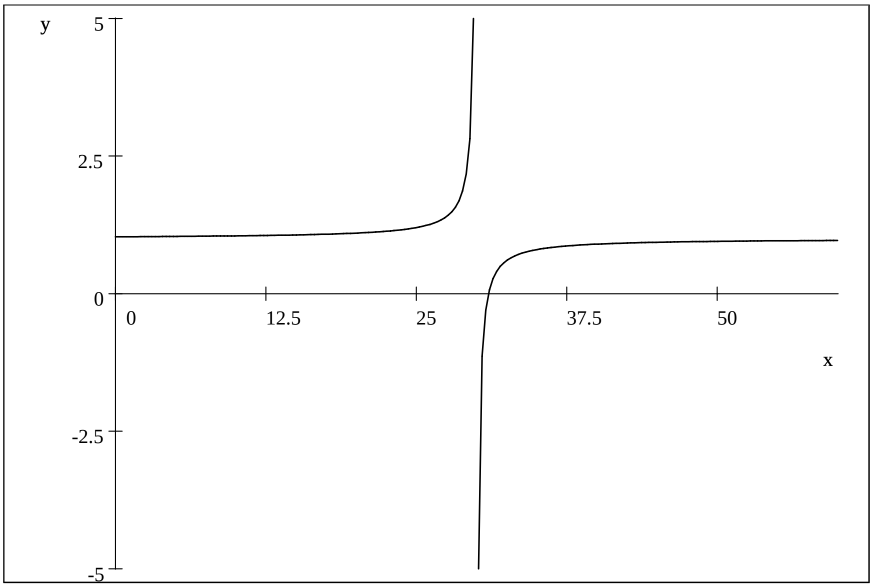

Figure 1: Constantinides-Campbell-Cochrane's utility for $\gamma = 2$ and $X = 30$.

In section 4, I group sets of households by income level, using the quintiles published by the American Bureau of Labor Statistics in the period 1984-2005; thus $i = 1, 2, 3, 4, 5$ will denote those groups of consumer units. I use annual data.

I say that this utility generalizes and regularizes the one of Constantinides-Campbell-Cochrane for the following reasons:

Constantinides-Campbell-Cochrane's utility is

$$U(C_t, X_t) = \frac{(C_t - X_t)^{1-\gamma} - 1}{1-\gamma} \tag{4}$$

If $\gamma = 2$ and $X_t = 30$, it has the graph of figure 1.

Utility 1 with $X_t = 30$ and $b = 1$ is plotted in figure 2.

The difference between figures 1 and 2 is eliminated if the right branch of the former is lifted and the vertical asymptotes are united (the resulting S-shaped graph can be shifted vertically and re-scalled to match 2). This way, the incoherence of having $U(C_1) > U(C_2)$ for $C_1 < X$ and $C_2 > X$, which violates one of the axioms of utility functions (the monotonicity condition), is eliminated; so is the need to use tricks to guarantee that $C_t > X_t$, as well as the interpretation of X_t as a subsistence level instead of habit. Abel

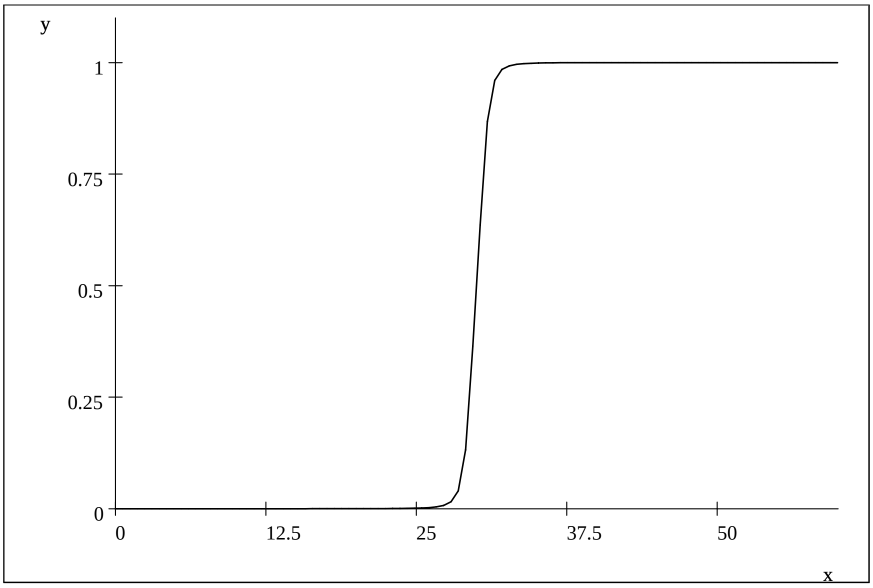

Figure 2: Our utility function.

(2007) calls attention to this problem and proposes an utility (different from ours) that remedies it. [2]

Worse problems occur in the left branch of utility 4 when γ is not an even integer. In the odd case, that branch is decreasing. For rational noninteger values of γ (for instance, 2.372 - the value estimated in the 1995 version of Campbell and Cochrane's paper), the left branch of $U(C)$ is not stable under infinitesimal changes in γ, shifting from increasing to decreasing at an infinite frequency, when γ is continuously changed. For irrational values of γ, $U(C) \notin \mathbb{R}$ in that branch.

The functional form of the RRA, denoted here by $s(C)$, shows the similarities between utilities 1 and 4.

For utility 4, one has:

$$s(C) = \frac{\gamma C}{(C - X_t)} \tag{5}$$

[2] Campbell and Cochrane arrive at a steady state X=0.95c, where c is the per capita consumption. If X is to be interpreted as a subsistence level, this means the Census Bureau is counting 150 million nonexistent Americans, since the median of the consumption distribution is around 0.7c. Alegations that X is the subsistence level of the representative agent, instead of that of a randomly chosen American, are discarded, since those authors state explicitly that theirs is an external habit model.

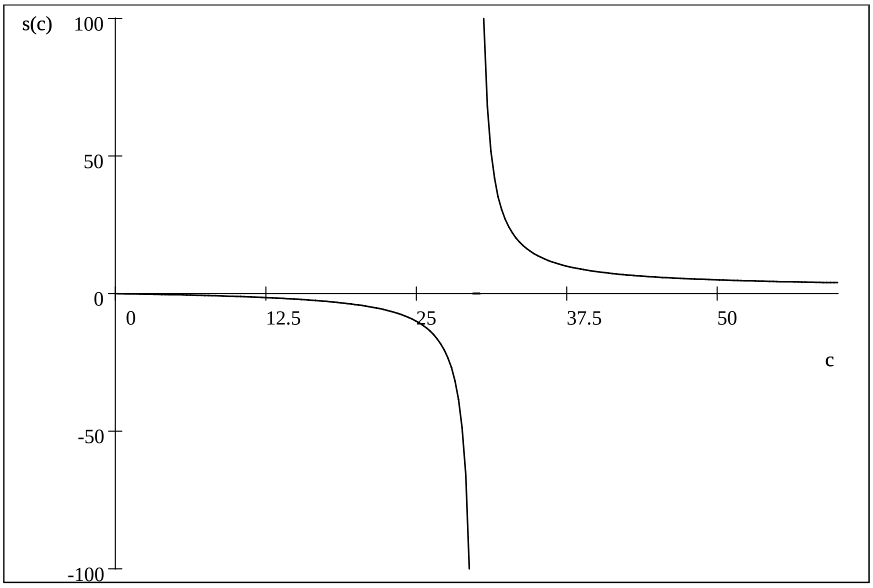

Figure 3: RRA function of Constantinides-Campbell-Cochrane utility.

For utility 1, this function is:

$$s(C) = 4C\frac{(C - X_t)^3}{(C - X_t)^4 + b^4} \tag{6}$$

So, for $b = 0$, the RRA function of my utility is equivalent to that of utility 4 with $\gamma = 4$. For $b \neq 0$, the discontinuity at $C = X$ disappears and $s(C)$ becomes a smooth function in $\mathbb{R}^+$. Now, whilst $\int_0^X \frac{\gamma C}{(C-X)} dC = \infty$ and $\int_X^\infty \frac{\gamma C}{(C-X)} dC = \infty$, my $s(C)$ function is integrable in all of $\mathbb{R}^+$. Figures 3 and 4 exhibit their graphs.

The absolute prudence, as defined in Kimball (1990), is

$$P(C) = -\frac{u'''(C)}{u''(C)} = 8\frac{(C - X_t)^3}{b^4 + (C - X_t)^4} - \frac{3}{(C - X_t)} \tag{7}$$

It's behavior for my utility - with $a = 0.92$, $b = 0.6$, $\kappa = 0$ and $\varsigma_t = 25$ - is in figure 5. It has a discontinuity on the inflexion point of the utility.

From 6, it is easy to see that $\lim_{C\to\infty} s(C) = 4$. The same happens in 5, for $\gamma = 4$.

The stochastic discount factor

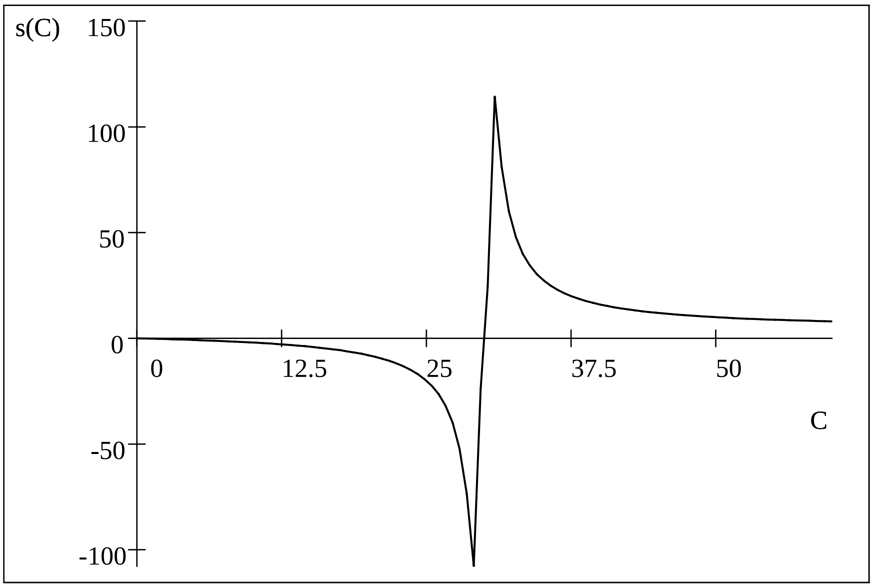

Figure 4: RRA of our utility

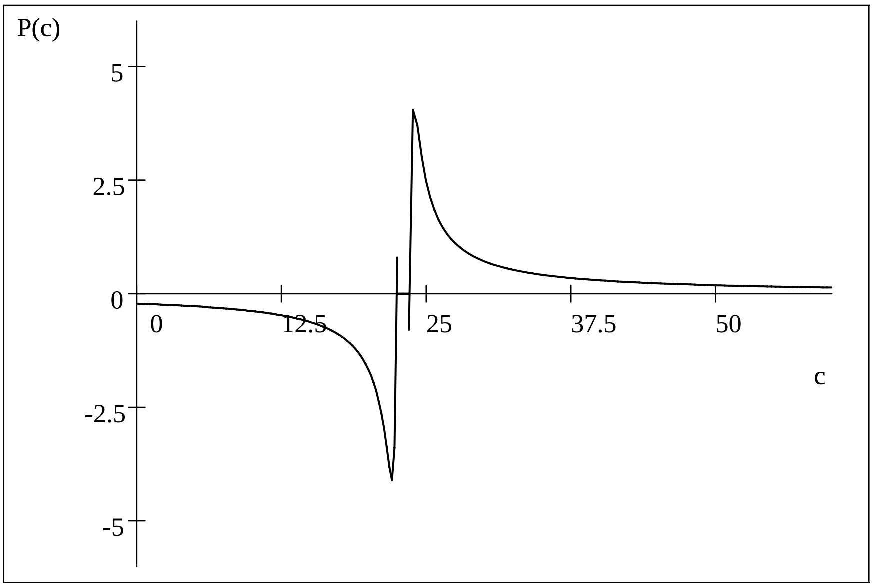

Figure 5: Absolute prudence of our utility.

$$M_{t+1} = \beta Q_{t+1} \tag{8}$$

where $\beta \in (0,1)$ is a subjective discount factor and

$$Q_{t+1} = \frac{U'(C_{t+1})}{U'(C_t)} \tag{9}$$

is also generalized from 4 to 1, since Q_{t+1} corresponding to 4 is

$$Q_{t+1} = \frac{(C_t - X_t)^\gamma}{(C_{t+1} - X_{t+1})^\gamma} \tag{10}$$

while, for my utility,

$$Q_{t+1} = \frac{(c_t - X_t)^4 + b^4}{(c_{t+1} - X_{t+1})^4 + b^4} \tag{11}$$

Equation 9 is valid here for any combination of internal and external habit, because the model is of two times (as opposed to an infinite series, which requires dynamic programming) and I presuppose the use eqs. 16-18, that, as is mentioned there, estimates ς_t as a function of $\{c_{t-2}, c_{t-3}, ...\}$. In section 5, this is not needed and 9 is always valid, as Campbell and Cochrane (1999) argue in their paper.

3 General framework

Starting with a representative agent framework, I tested my utility function in a two-times (one period) model (Cochrane (2001), chapter 1):

$$\max_{\xi} u = U(c_t) + \beta E_t[U(c_{t+1})] \tag{12}$$

$$\text{subject to} \begin{cases} c_t = e_t - p_t\xi \\ c_{t+1} = x_{t+1}\xi \end{cases} \tag{13}$$

where β is the subjective discount factor, e_t is the initial endowment that the agent has, p_t is the price per share of the asset, $x_{t+1}$is the payoff per share (new price plus dividends in the period) and ξ is the number of shares that the agent decides to buy (thus reducing his present consumption). In equilibrium, this is valid for any asset.

The well known solution is the standard asset pricing equation:

$$E[M_{t+1}R_{t+1}|I_t] = 1 \tag{14}$$

where R_{t+1} is the return of the investment $(\frac{x_{t+1}}{p_t})$, M is defined in equations 8 and 9 and I_t is the information set available at the moment of the decision.

Subtracting equations 14 for two different assets, an equation for excess return is obtained:

$$E[M_{t+1}R^e_{t+1}|I_t] = 0 \tag{15}$$

where $R^e_{t+1} = R^i_{t+1} - R^j_{t+1}$, i and j being the assets.

I used Fama and French's 25 book to market portfolio and their three factors (MKT, SMB and HML) published in French's page

<http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/> to estimate the parameters of my utility in equations 15. In the representative agent framework, eq. 2 is reduced to $X_t = a\varsigma_t$. Care must be exerted, when using a habit formation utility: if the habit level ς_t estimation includes variables affected by the decision variable (in this case, ξ), the mathematics becomes very complicated. In order to avoid this burden, I estimate ς_t using only the series $\{c_{t-2}, c_{t-3}, ...\}$. The smoothing equations are those of the double exponential method (see the time series literature):

$$\varsigma_t = \varsigma^{(1)}_{t-2} + 2\varsigma^{(2)}_{t-2} \tag{16}$$

$$\varsigma^{(1)}_t = c_t - (1-\alpha)^2(c_t - \varsigma^{(1)}_{t-1} - \varsigma^{(2)}_{t-1}) \tag{17}$$

$$\varsigma^{(2)}_t = \varsigma^{(2)}_{t-1} + \alpha^2(c_t - \varsigma^{(1)}_{t-1} - \varsigma^{(2)}_{t-1}) \tag{18}$$

Nevertheless, as the pricing equation cannot distinguish between different methods of smoothing, in many instances I used methods that use c_t, c_{t+1} and even $c_{t+2}, c_{t+3}, ...$, like Hodrick-Prescott filter and an exponential curve adjusted to the whole consumption series (for instance: $\varsigma_t = 9e^{0.02(t-1950)}$), for reasons of convenience and/or to test the robustness of my results.

As remarked in Parker and Julliard (2005) and Belo (2007), care must be taken with respect to the time matching between returns and consumption flow. I used NIPA 1.1.5 table (personal consumption expenditures - total) from the US Bureau of Economic Analysis for aggregate consumption data and French's site for returns; the correct matching demands a lagging in nominal time of consumption (as this is a flow, convention about when computing it - at the start or at the end of the period - varies); in de Farias Neto (2007), figs. 20-22, the CRRA utility is used to establish the correct lagging. All historical series were deflated by the CPI-U (consumer price index - urban).

Now, as conditional expectation is not available, I use the time average of equation 15, that is, the unconditional expectation:

$$E[M_{t+1}R^e_{t+1}] = 0 \tag{19}$$

The excess return of asset j is $R^{e,j} = R^j - R^f$, where R^f is the return of the risk-free asset (one month t-bills). Now, $M = \beta Q_{\bar{c}}(a,b)$, where $\bar{c}$ represents the smoothing method used for the consumption series and a and b are the parameters of my utility function (see eqs. 1 and 2). Instead of eq. 19, I used eq. 20 below

$$\hat{E}(R^{e,j}) = -\frac{cov(Q, R^{e,j})}{E(Q)}, j = 1, 2, ...25 \tag{20}$$

to fit the model to Fama and French's portfolio, that is, parameters (a, b) are found minimizing

$$F\% = \frac{\sqrt{\frac{1}{25}\sum_{i=1}^{25}\Delta_i^2}}{\frac{1}{25}\sum_{i=1}^{25}E(R_e^i)} \tag{21}$$

where $\Delta_i = E(R^{e,j}) - \hat{E}(R^{e,j})$.

I also computed, as another fitting measure,

$$R^2 = 1 - \frac{var(\Delta)}{var\left[E(R_e^i)\right]} \tag{22}$$

Now notice that β is not present in eqs. 20. It is calibrated to turn true the equation

$$\beta = \frac{1}{E(QR^f)} \tag{23}$$

In the period considered (1951-2001), the mean annual risk premium was $E(R^{mkt} - R^f) = 7.87\%$ and $E(R^f) = 1.22\%$. The mean interest rate in the period is estimated by

$$E(R^f) = \frac{\frac{1}{\beta} - cov(Q, R^f)}{E(Q)} \tag{24}$$

using β given by eq. 23, with $Q = Q_{\bar{c}}(a^*, b^*)$.

4 Full model

In the full model (eqs. 1 and 2), the RRA varies according to the consumption level of the agent (person or household).

In order to estimate parameter κ, I downloaded the quintiles of annual household expenditures series from http://www.bls.gov/ in the available period (1984-2005) and aggregated the corresponding stochastic discount factors M^i, which is allowed as a consequence of adding eqs. 19 for different consumer units:

$$\bar{M}_{t+1} = \frac{1}{5}\sum_{i=1}^{5} M_{t+1}^{i} \tag{25}$$

To estimate (a, b, κ), I sought values that minimized J% as defined in 26.

$$J_{\%} = 100\sqrt{\frac{\frac{1}{15}\sum_{i=1}^{5}\sum_{j=1}^{3}\left[E(R_j^e) - \hat{E}_i(R_j^e)\right]^2}{\frac{1}{3}\sum_{j=1}^{3}E(R_j^e)^2}} \tag{26}$$

The results are $a = 0.9$, $b = 0.2$, $\kappa = 0$, $\min J\% = 6.9\%$. Smoothed series $\{\varsigma_t^i\}$ were obtained by applying the Hodrick-Prescott filter with default values of the E-views software package. This establishes the model as one of pure external habit, which is also the interpretation given by Campbell and Cochrane (1999) to theirs. Figures 6, 7 and 8 show the behavior of J%.

Figures 9 and 10 show the behavior of the local RRA for each quintile. In either cases, it is calculated at the average consumption level of each year of the corresponding quintile.

In accordance to figure 4, figure 10 shows that the RRA at the operation point is positive for the upper two quintiles and negative for the lower two; it is also very stable, for these quintiles. The interesting discovery here is that the middle quintile appears to be psychologically bipolar, oscillating between extreme risk aversion and extreme risk seeking; this is a result of its situation in the middle of the abyss of figures 2 and 4, making it extremely sensible to any small variation of its annual consumption.

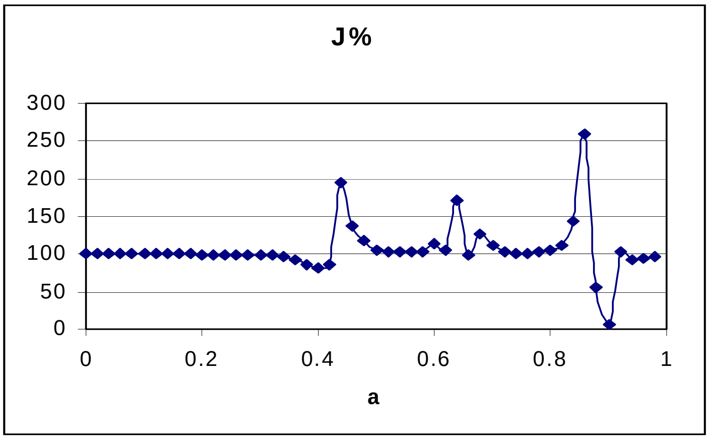

Figure 6: J% versus a, with $b = 0.2$ and $\kappa = 0$.

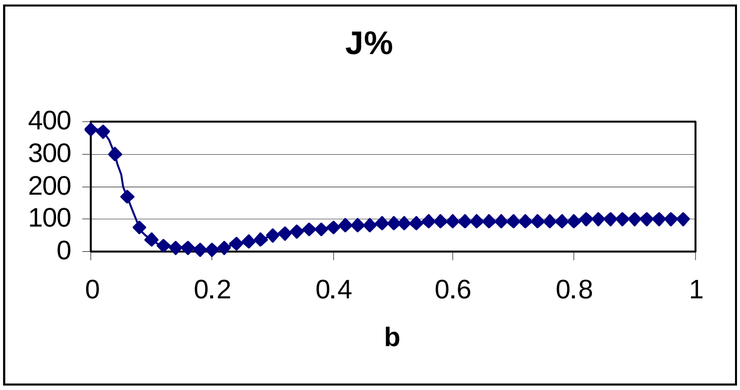

Figure 7: $J\%$ versus b, with $a = 0.9$ and $\kappa = 0$.

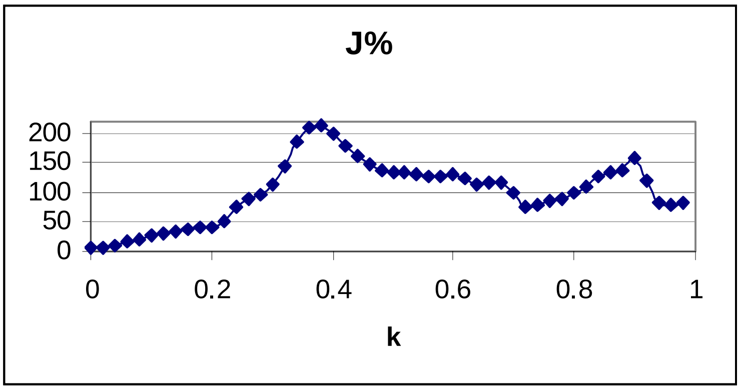

Figure 8: J% versus κ, with $a = 0.9$ and $b = 0.2$.

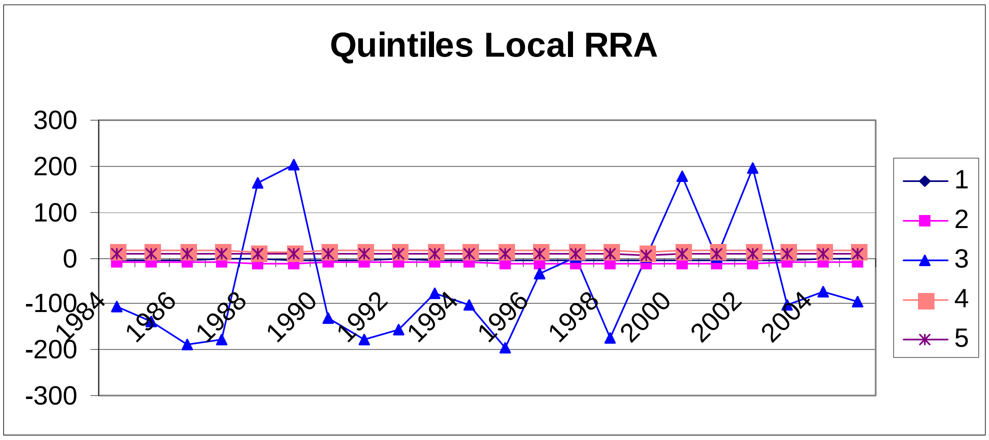

Figure 9: The third quintile's local RRA varies wildly, compared to the other four.

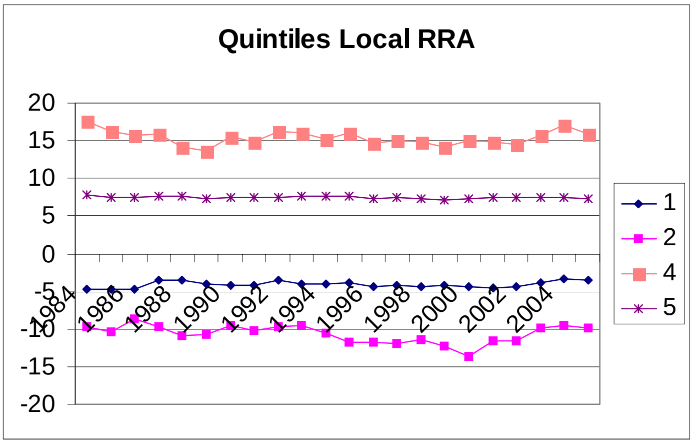

Figure 10: Behavior of the local RRA for the quintiles, excluded the third one.

5 External habit

Having established, in the precedent section, that $\kappa = 0$, I can restrict the research to the simpler model, called in the literature *external habit formation.*

Table of figure 11 shows the results, using four different ways of consumption smoothing. The relatively low values of β are due to the oil shocks in the period considered (1974, 1979 and 1991), which are revealed as peaks in the time series of the pricing kernel (M_t); this is how the low probability of disastrous events is internalized in my model.

Estimating the parameters by applying GMM (generalized method of moments) to Euler equations 19, with $R^{e,j}$, j=MKTRF,SMB and HML (the three factors of Fama and French), I got $a = 0.924 \pm 0.02$, $b = 0.59 \pm 0.16$ and J=0.0090. Now, recalling that the period considered was 1951-2001, I have $50J \simeq \chi^2_{0.5}(1)$, so the model passes the overidentification test, that is, the three equations are not mutually incompatible. The p-values are 0.0000, for a, and 0.0005, for b.

The parameters of the last line of the table in figure 11 (which is the best fitting) were used to obtain the scatter plot of figure 12. For comparison, points corresponding to Fama and French's three factor models are in the graph. M FF uses the standard pricing equations $E[M_{t+1}R^e_{t+1}] = 0$, that is, with M and R contemporaneous, and pricing kernel defined by

$$M_t = b_1 R_t^{e,mkt} + b_2 R_t^{smb} + b_3 R_t^{hml} \tag{27}$$

Figure 11: Results with our utility function.

	a	b	F%	R^2	Beta	$\hat{E}(R^e)$=7.87%	$\hat{E}(R^f)$=1.22%
C=9exp[(t-1950)/50]	0.924	0.1882	23%	0.46	0.416	9.82%	1.32%
C=9exp[(t-1950)/52]	0.946	0.2060	20%	0.54	0.418	9.81%	1.34%
Exponential Smoothing Alfa=0.1	0.946	0.6436	17%	0.71	0.671	8.21%	1.25%
Exponential Smoothing Alfa=0.2	0.924	0.4852	14%	0.76	0.462	9.16%	1.14%

where $R_t^{e,mkt} = R_t^{mkt} - R_t^f$ (all R's taken from French's page).
FF regres uses the arbitrage regression

$$R_e^i(t) = a_1^i R_{mkt}^e(t) + a_2^i R_{smb}(t) + a_3^i R_{hml}(t) \tag{28}$$

i=1,2,...,25, which has 75 free parameters estimated by ordinary least squares.

Figures 13 and 14 show the best fitting of power and recursive utility to Fama and French's portfolio. Their pricing capabilities are visibly poorer than the one of my utility function.

The RRA calculated at the per capita consumption level (that is, at the representative agent's level) is exhibited in figure 22. Notice how its mean level is about the same as the one of Campbell and Cochrane (1999). Thus, in the representative agent framework, the equity premium puzzle persists; the risk free puzzle disappeared, since both those authors and us succeeded in explaining the historical mean interest rate with $\beta \in (0,1)$ (in my case, $\beta \approx 0.5$, as shown in the table of figure 11).

Figures 15 and 16 show the situation for 2004: the representative agent operates at the edge.

5.1 Main result

Dragulesco and Yakovenko (2001) show that the cross-sectional distribution of income in US can be well modelled by the usual exponential; Husby (1971) exhibits a linear equation relating consumption with income:

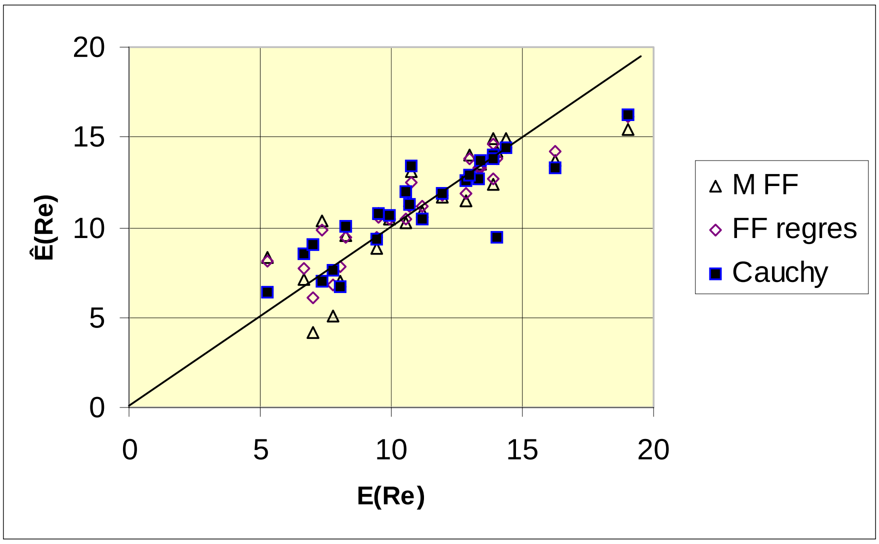

Figure 12: Fitting to Fama and French's portfolio. Results as good as the two models based upon the three factors of those authors.

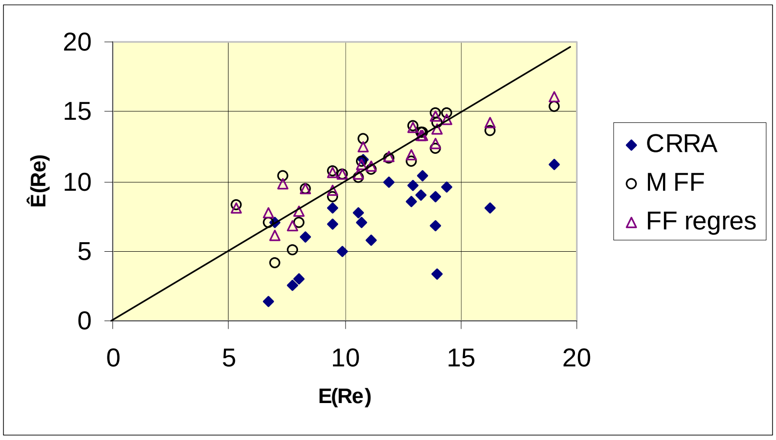

Figure 13: Best feasible ($\beta < 1$) fitting of power utility. RRA=80, F%=50%, $R^2 = 0.17$.

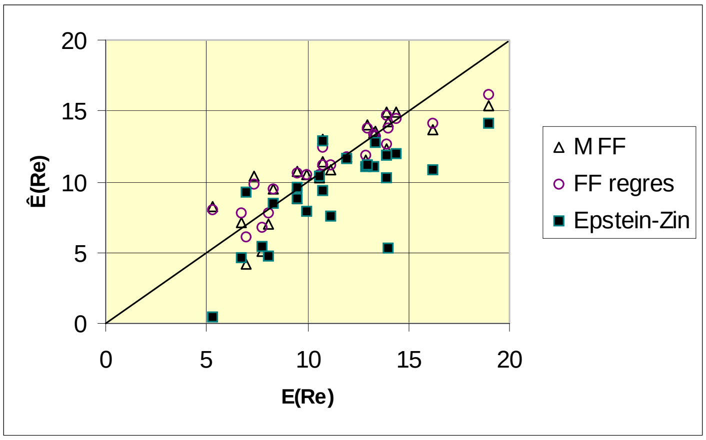

Figure 14: The best fitting of Epstein-Zin-Weil utility that allows $\beta < 1$: RRA=87. $\beta = 0.84 ==> \hat{E}(R^f) = 1.22\%$. F%=31%. $\mathrm{R}^2 = 0.35$.

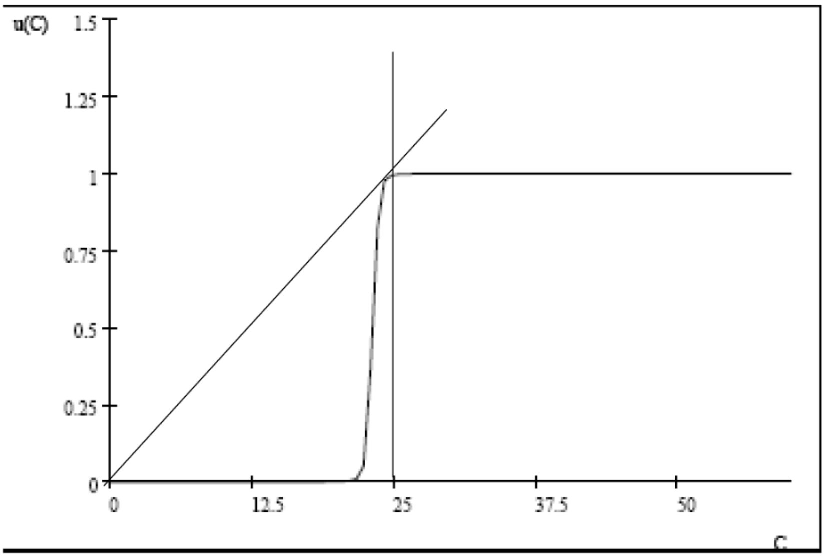

Figure 15: The operation point is near the edge of the cliff. This graph shows that, in 2004, Xi's condition is satisfied, since the operation point is beyond the tangency point.

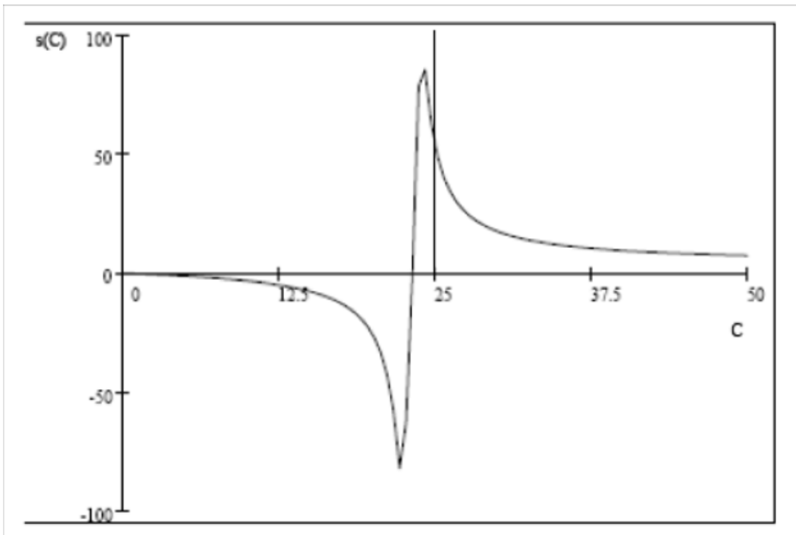

Figure 16: The operation point in 2004 corresponds to RRA$\sim$ 50.

$$C = 3.64 + 0.9016Y \tag{29}$$

Eq. 29 is expressed in aggregate values of 1961; in per capita values, using dollars of the year 2000, the intercept 3.64 represents about US$400 per year, against a per capita consumption of about US$10,000 in 1961. Thus, the intercept can be despised and I can adopt the exponential distribution for consumption too, that is:

$$f(C) = \frac{1}{\mu} e^{-\frac{C}{\mu}} \tag{30}$$

where $\mu = c =$ per capita annual consumption.

Averaging the RRA function $s(C)$ over this distribution, that is, taking

$$E[s(c)] = \int_0^\infty \frac{1}{c} e^{-\frac{\zeta}{c}} 4\zeta \frac{(\zeta - X)^3}{(\zeta - X)^4 + b^4} d\zeta \tag{31}$$

results in the graphs of figure 23. The average RRA remains in the (0,3) interval! Considering the intercept, the consumption distribution becomes a shifted exponential and, according to my simulations, the average RRA drops a little. Notice the robustness of the main result with respect to the smoothing method used to estimate the habit level, thus showing that the exact form of equations 16-18 is not very important; the same can be said about the general level of risk aversion at the consumption level of the representative agent: figure 22 shows its robustness relative to the smoothing equations.

Figure 25 shows that the operating point of the representative agent oscillates around Xi's threshold.

Figure 20 shows that, except around the second oil shock (~1980), the habit level is above the *per capita* consumption level.

Figure 26 shows that the pricing kernel has anomalies at the oil shocks. Fama and French's kernel doesn't, since it is built as a function of the porfolio returns.

Figures 28 to 32 show the behavior of FF% and of the time average of $E[s(c)]$ in the parametric space.

The worksheets with the calculations and graphs can be freely downloaded from http://www.geocities.com/joaojfn/economia.html

6 Brazilian market

For some time, there has been a polemic about the existence or not of an equity premium puzzle in Brazil, which was finally settled down by Cysne (2006) and confirmed by us: it does exist. With quarterly data from 1993 to 2003, the usual CRRA utility ($u(c) = \frac{c^{1-\gamma}}{1-\gamma}$) demands $\gamma = 35$. As in the American market, here again there is a lag problem between consumption nominal time and returns; the correct lagging is the same in both markets (see fig. 74 of de Farias Neto (2007)).

In order to test if the low average RRA (fig. 23) was an exclusive property of the American market, I used Brazilian data (from IBGE, FGV and IPEA). Figure 19 shows that, here too, the average RRA remains in the acceptable range.

First of all, it is useful to compare the two economies; figure 17 shows the difference.

Figure 18 shows the RRA computed at the per capita consumption level.

7 Conclusions

Although a large proportion of the households ($\sim 70\%$) operate at the risk-seeking region (below the external habit level, trying to "catch up with the joneses"), the representative agent is risk-averse and operates above the habit

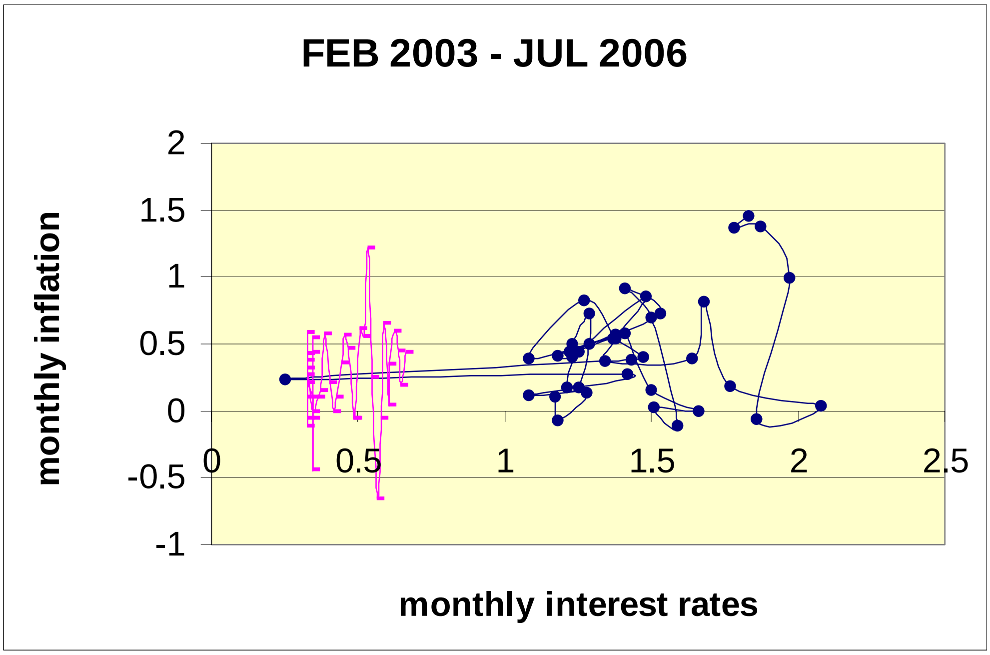

Figure 17: On the left side, USA. In the right side Brazil. Besides the higher levels of interest rates and inflation, there is a difference between almost senoidal behaviour in the former and chaos in the later. American inflation data: CPI-U. American interest rate: bank prime loan rate. Brazilian inflation: INPC IBGE. Brazilian interest rate: OVER/SELIC FGV/ANDIMA.

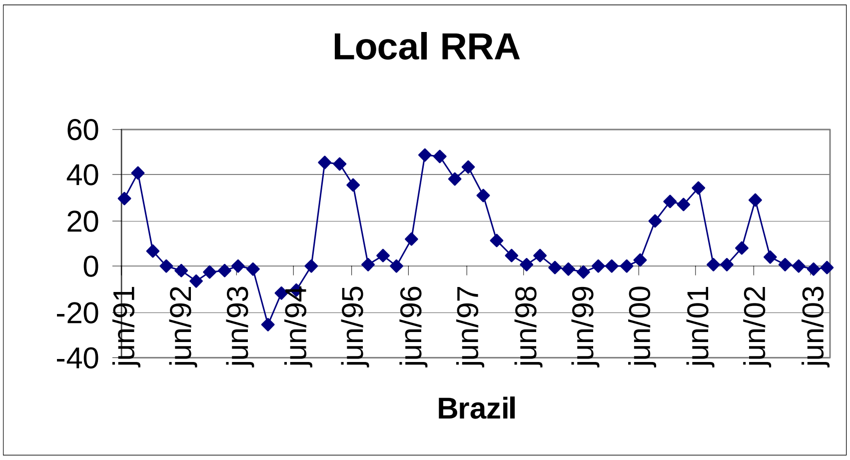

Figure 18: Brazilian RRA at the per capita consumption level. Hodrick-Prescott filter used to smooth consumption ($C \to \varsigma$).

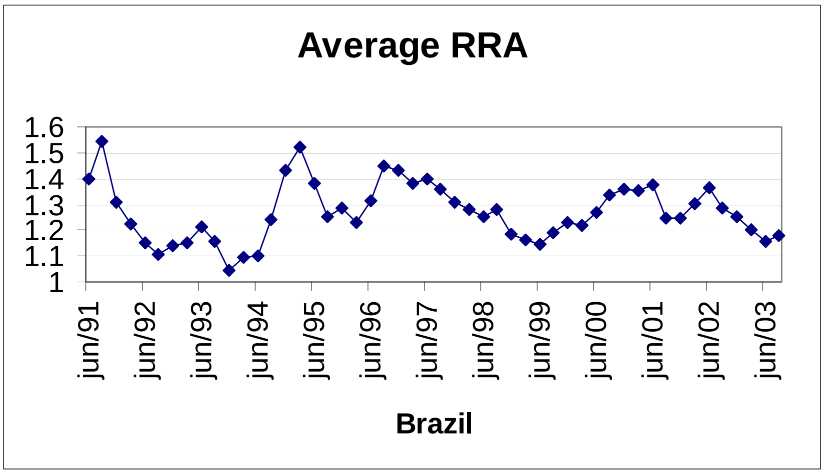

Figure 19: RRA integrated over the Brazilian consumption distribution, suposed to be exponential with mean equal to the per capita consumption level. Hodrick-Prescott filter used to smooth consumption ($C \to \varsigma$).

level $X = a\varsigma$ most of the time, thus guaranteeing the general equilibrium. The mix between risk-seeking and risk-averse agents is such that, as a whole, the economy can still find its equilibrium.

As $U(C)$ is the utility of annual consumption, its convexity below the habit level only means that agents operating in that region prefer to throw a coin and, next year, consume $c + \varepsilon$, if it turns out head, or $c - \varepsilon$, if it turns out tail (ε small compared to c), instead of a guaranteed consumption of c. It doesn't mean that they would behave like this when confronted with any lottery offering immediate payoff of money or other benefit. They are trying to catch up with the long run standard of living of the representative agent, not necessarily seeking immediate rewards, although it is a known fact that the lower classes tend to buy (actuarially disadvantageous) lottery tickets more frequently than the upper classes....

Due to the convexity of the lower part of the utility function, consumers who operate in that part and are not close enough to the *per capita* consumption level maximize their utility not by satisfying the first order condition (eq. 14), but by restricting their consumption at t=0 to the subsistence minimum and investing all the rest, that is, for them the subsistence level is binding. As a consequence, they are willing to accept any amount of credit they are offered.

Figures 26 and 27 show the spikes in 1974 and 1980 corresponding to the two oil shocks (and a smaller one corresponding to the firs Gulf War);

notice how they are present in the models that use power (CRRA) utility and Epstein-Zin too. They alone are responsible for half of $E(Q)$ in the period considered (1951-2001), which is the hallmark of the socalled "black swan" phenomenon, as popularized by Nassim Taleb (see, for instance, Taleb (2008a) and Taleb (2008b)) and put in doubt some of the basic assumptions of the class of models considered in the present article (and most of the literature on the equity premium puzzles).

Figures 4 and 5 suggest that, as people are born and progressively have their income raised, they behave like waves that hit a beach (the central bump), thus prone to breaking and generating turbulence and, as a consequence, financial bubbles; notice that this wouldn't happen with the usual power utility (CRRA), whose RRA function is a constant horizontal line. Thus, pressure from the risk-seeking new generation driving the government to support bad credit loans via Fannie Mae and Freddy Mac (the prosaic case of the Californian strawberries picker who made US$14K per year and was offered a US$700K mortgage became a saloon anedocte) may be the ultimate responsible for the crash; further research should confirm or reject the hypothesis that all major financial crashes are caused essencialy by this shock of generations phenomenon.

The RRA function of my utility corresponds to what is observed in animal psychology: coming from higher to lower consumption values, the representative agent is first taken by panic (increased RRA), then despair (negative RRA) and, thus, disposition to risk everything, and finally desolation (negative RRA, but small in module).

The implied fact, by the final model (external habit), that $\sim$70% of the population has negative RRA, may be an explanation for the high debt levels that this extract is willing to take, which, by its turn, could explain phenomena like the so-called "credit feast" in Brazil and the "real state credit crunch" in US.

The model economy of this paper works at the edge of chaos (in a loose sense), since the operating point of the representative agent oscillates around Xi's threshold for the existence of equilibrium, with the two points being statistically indistinguishable (the sample average of their difference is within one standard deviation from zero). The median income households operate in the chaotic regime, the representative agent barely escaping this by operating slightly above the third quintile, except around the second oil shock (Iranian revolution and Iran-Iraq war). These features are in accordance with a power law behaviour for the tail of the probability distribution of the pricing kernel and the existence of very influencial spikes present in its time series.

The fact that an S-shaped utility function can, in a way, solve the equity premium puzzle pushes the research direction towards models of the economy akin to neural networks, in which neurons are fired whenever a threshold of excitation is crossed. If this is the right direction, only future works can say.

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8 Figures

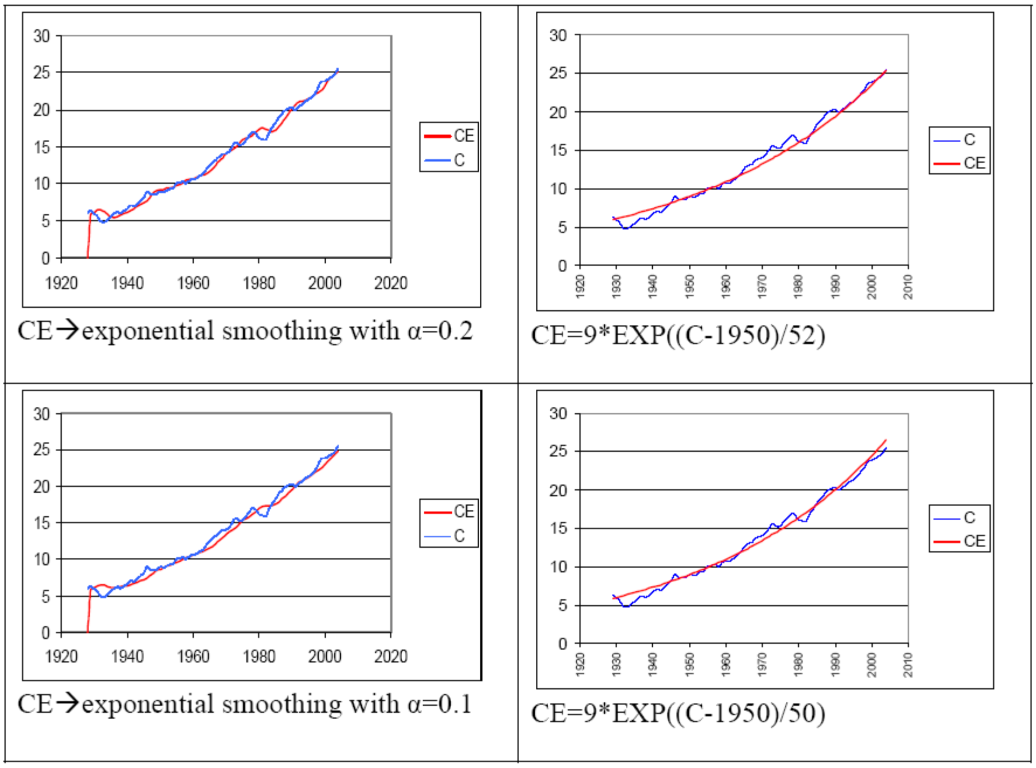


Figure 20: Consumption smoothing.

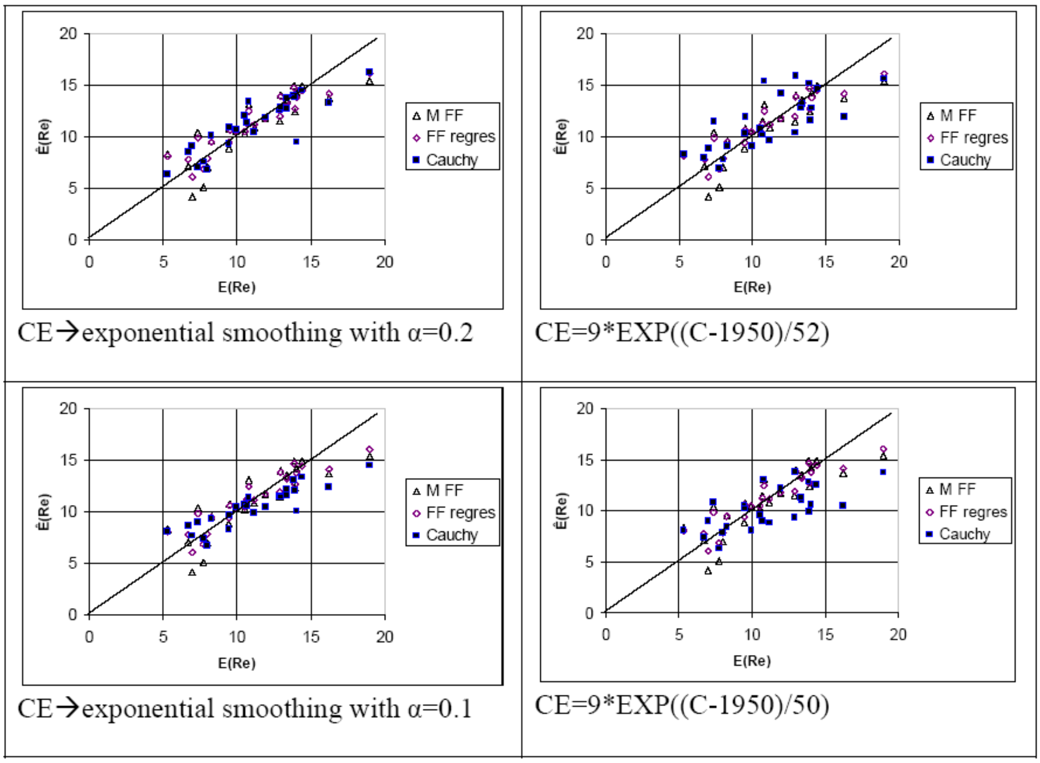
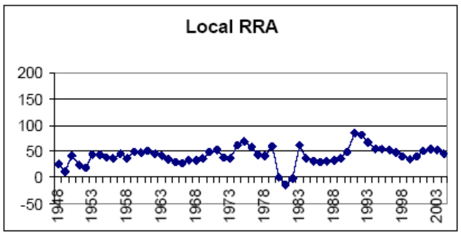
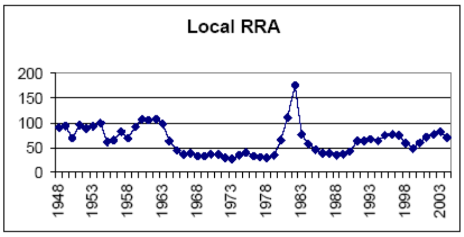


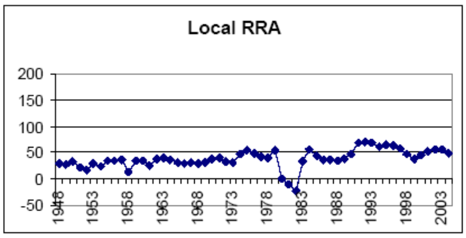
Figure 21: Fitting to Fama and French's 25 size and book-to-market portfolio.



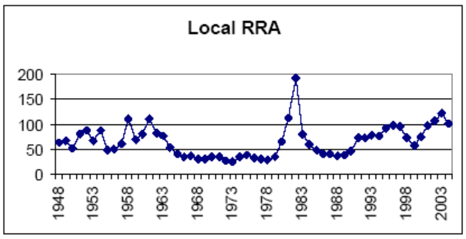
CE→exponential smoothing with α=0.2



CE=9*EXP((C-1950)/52)



CE→exponential smoothing with α=0.1



CE=9*EXP((C-1950)/50)

Figure 22: RRA of the representative agent.

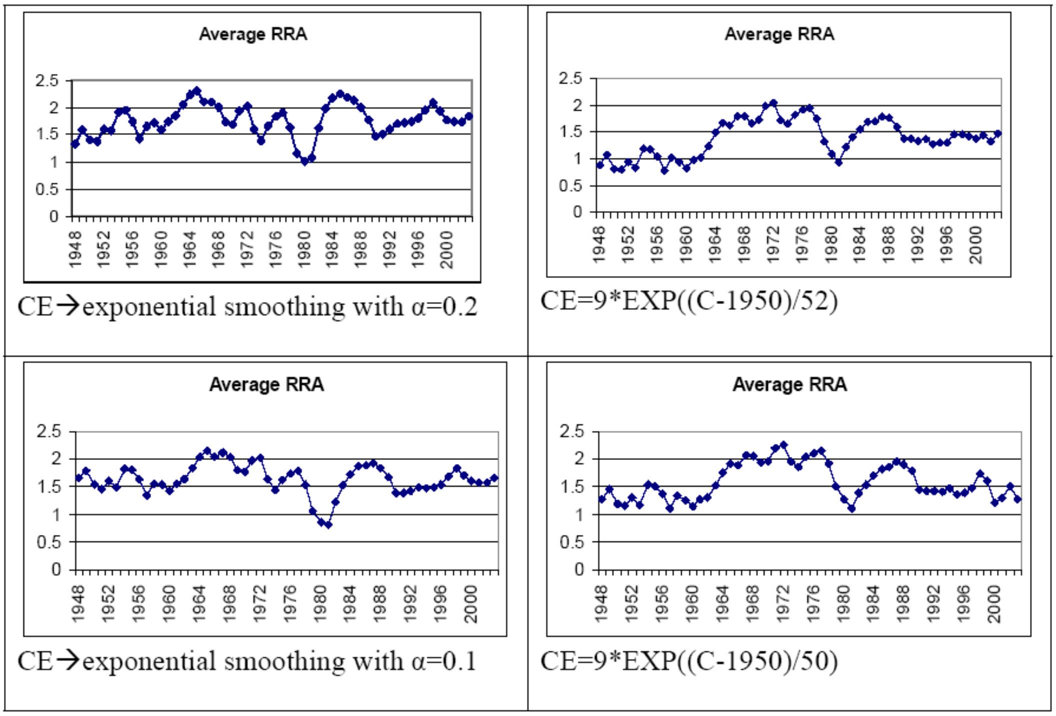


Figure 23: Expected value of the RRA over the consumption distribution.

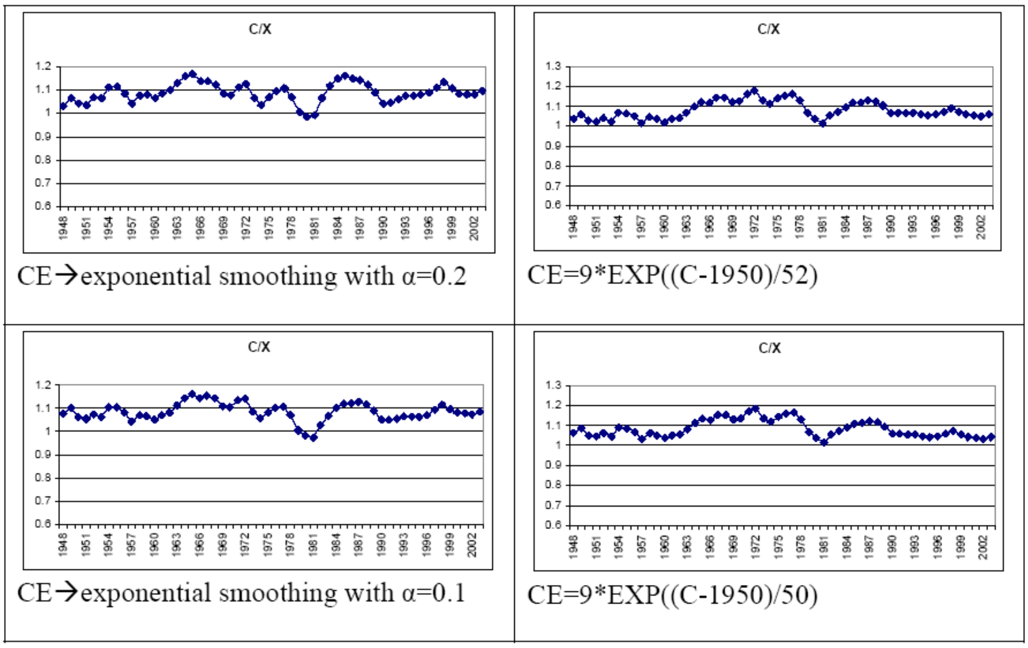


Figure 24: Consumption of the representative agent over the habit level $X = a\varsigma$.

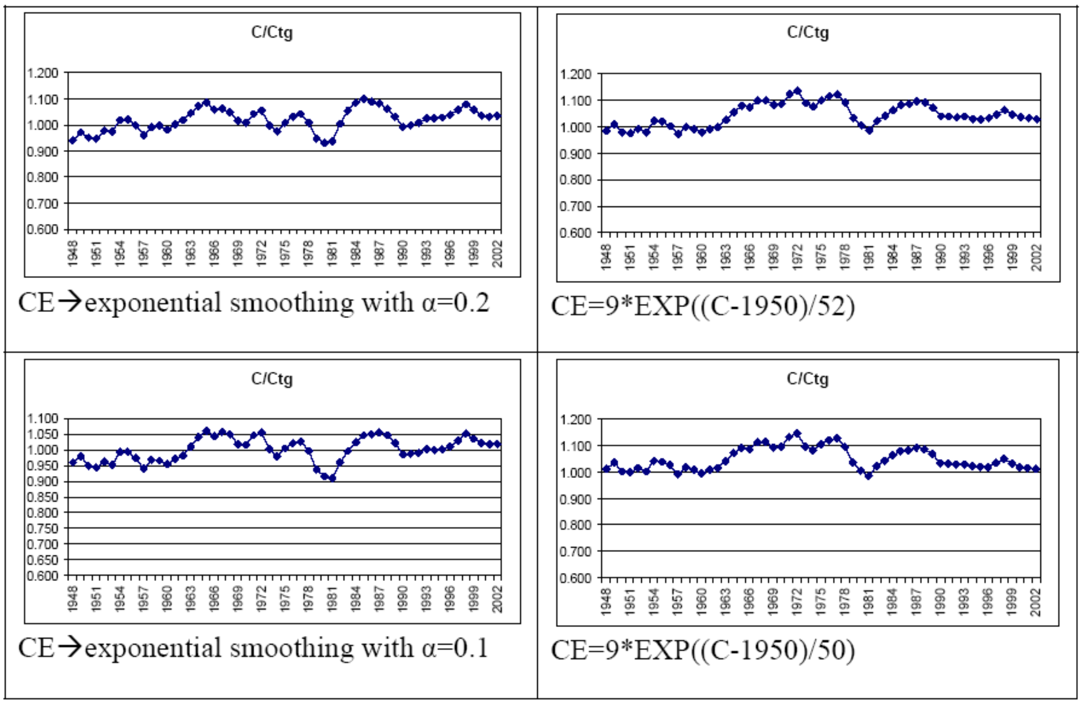


Figure 25: Consumption of the representative agent over Xi's threshold C_{tg}. When $C > C_{tg}$, the general equilibrium is guaranteed to exist.

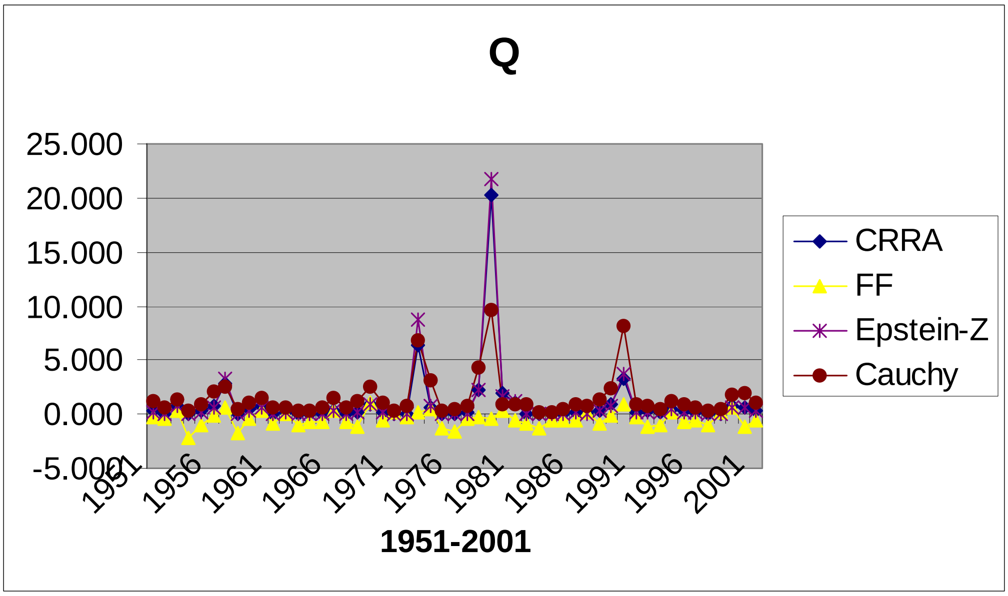


Figure 26:

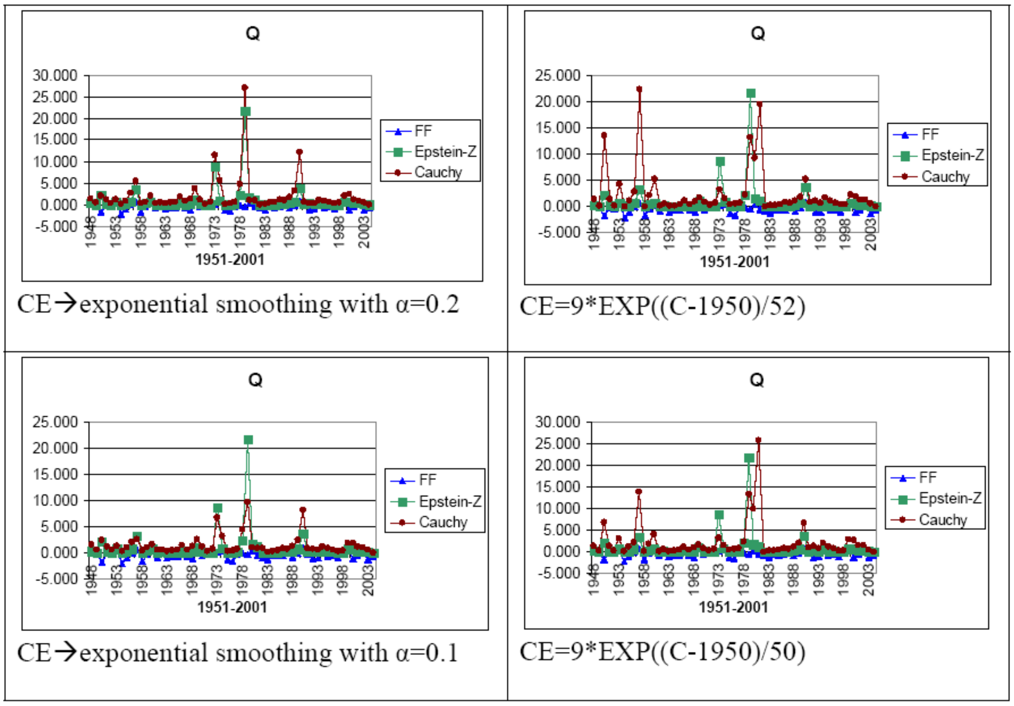


Figure 27: Q=M/β (pricing kernel over the subjective discount factor).

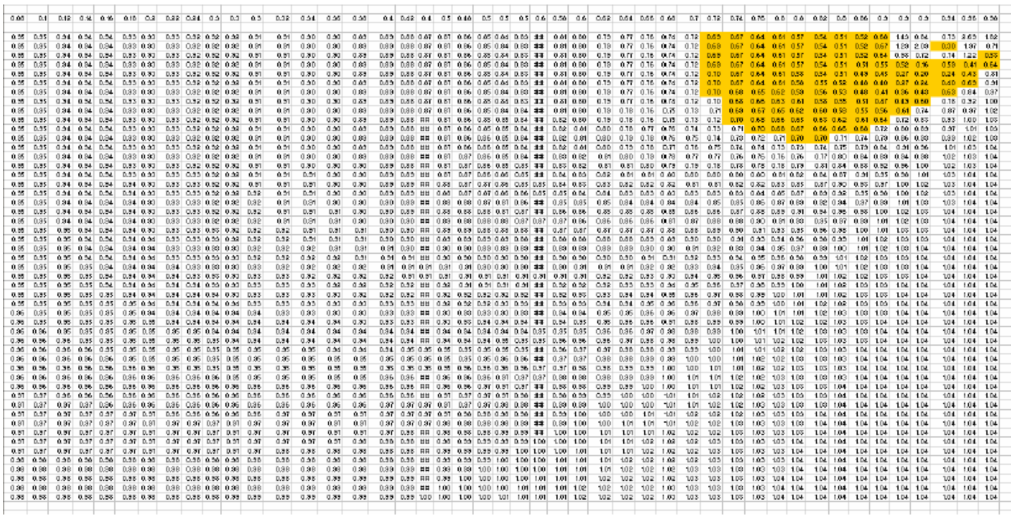


Figure 28: Parametric space. Parameter a of our utility runs horizontally from 0 to 1 towards the right. Parameter b runs vertically from 0 to 10 towards the bottom. The stain is the region where FF% < 70%.

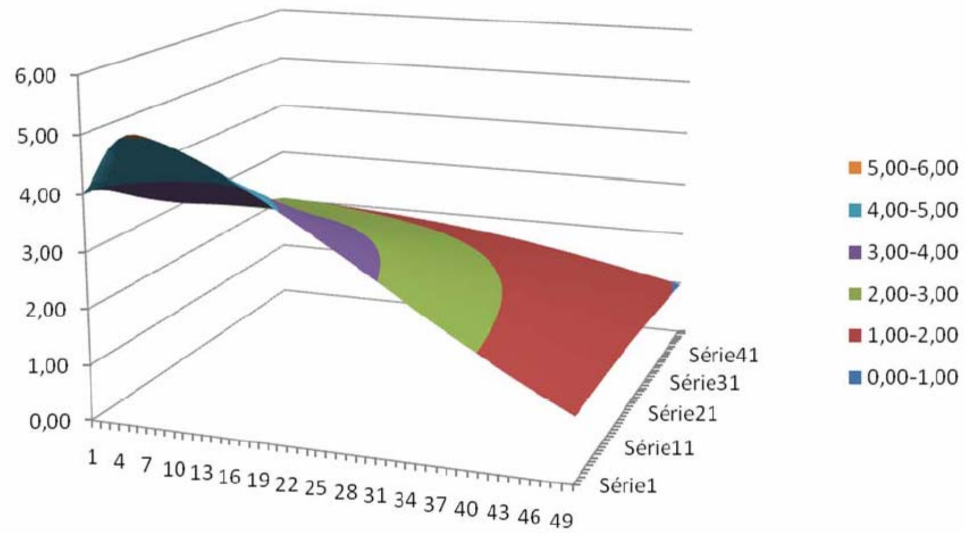


Figure 29: Parametric space. The vertical axis shows the average of E[RRA] in the period 1946-2004. The roughly horizontal one, shows parameter b running from 0 to 10 (1 to 49 in the figure). The depth axis shows parameter a running from 0 to 1 (série1 to série 41 in the figure).

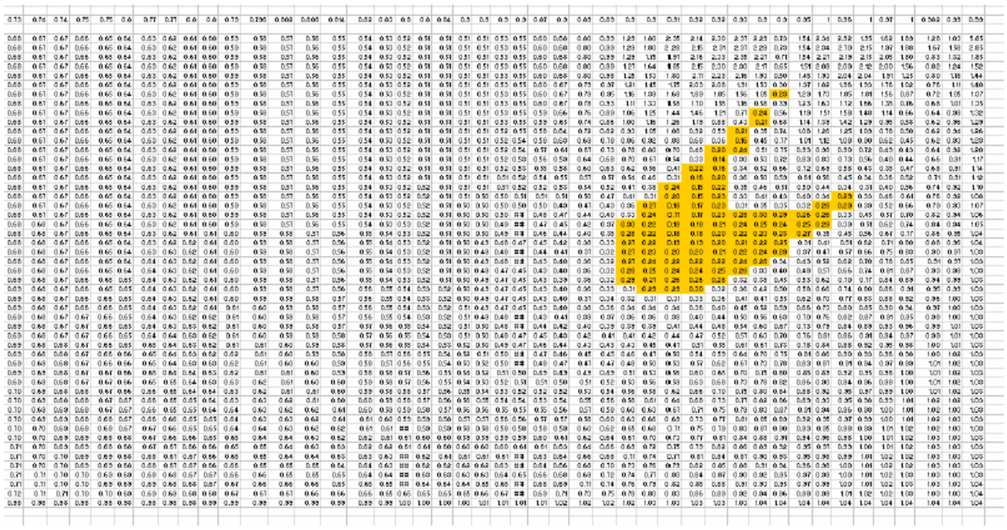


Figure 30: Parametric space. Parameter a of our utility runs horizontally from 0.7 to 1 towards the right. Parameter b runs vertically from 0 to 2 towards the bottom. The stain is the region where FF% < 30%.

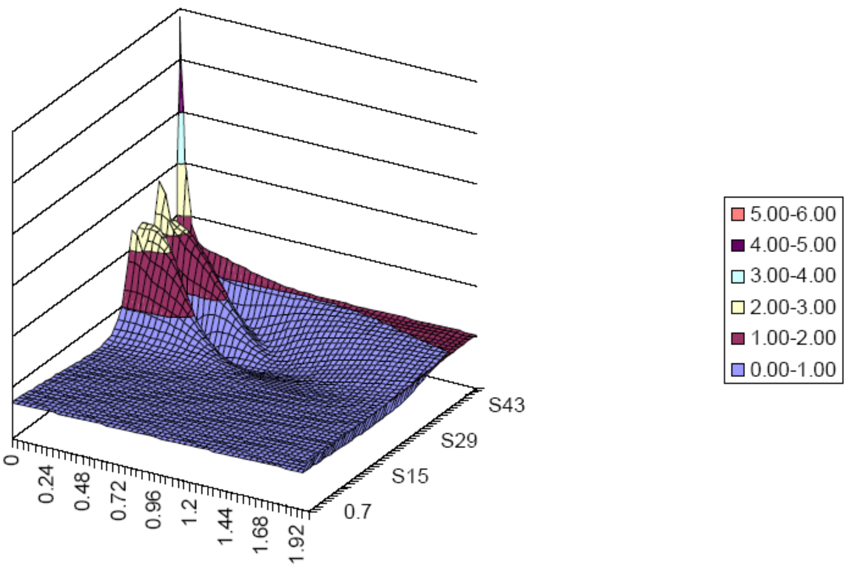


Figure 31: The precedent mapping using the vertical axis to represent FF%. The best fit to Fama-French portfolio is in the central depression (lowest FF%). Parameter b runs from 0 to 2 in the roughly horizontal axis. Parameter a runs from 0.7 to 1 in the depth axis (0.7 to S43).

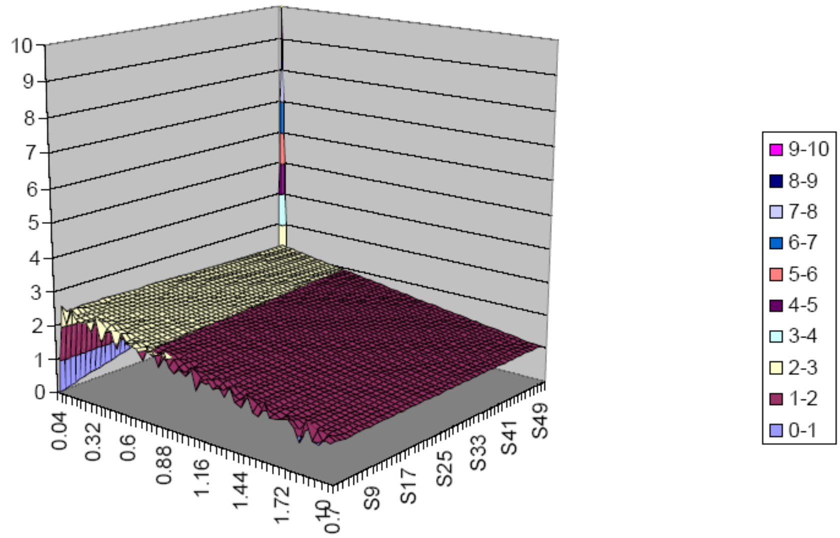


Figure 32: Parametric space. The vertical axis shows the average of E[RRA] in the period 1946-2004. The roughly horizontal one, shows parameter b running from 0 to 2 . The depth axis shows parameter a running from 0.7 to 1 (0.7 to S49 in the figure).