Convergence Debate, Middle-Income Trap, and East Asian Miracle

zhao, guo

Southwest Minzu University

1 July 2024

Convergence Debate, Middle-Income Trap, and East Asian Miracle

Guo Zhao[1]*

[1]College of Mathematics, Southwest Minzu University; Chengdu,610041, China.

* Corresponding author. Email: zhaoguo@swun.edu.cn.

Abstract: Three key economic issues are the convergence debate, the middle-income trap, and the East Asian Miracle. Here we show that the standard Solow-Swan growth model in discrete time can simultaneously account for these key economic issues if in addition the no arbitrage constraint is imposed on the growth of total wealth. Under perfect arbitrage, club convergence is a pervasive feature of dynamic economies with realistic structural characteristics, but conditional converge may emerge as a limiting case. Following the standard Solow-Swan growth, a middle-income economy may make a miracle growth at the cost of structural unemployment if the rate of saving and investment exceed certain threshold level. These findings deepen our understanding of the impacts of capital accumulation on the dynamic evolution of economic system.

Main Text:

Three of the most interesting economic issues are the convergence debate (*1*, *2*), the middle-income trap (*3*, *4*), and the East Asian Miracle (*5*–*7*).

The convergence debate centers around the question of whether poor or middle-income economies tend to catch up with rich ones (*8*), and what determines persistent differences in economic performance between economies with basically *similar* structural characteristics and *similar* initial conditions, like the Philippines and South Korea in 1960 (*9*, *10*).

The middle-income trap is a phenomenon wherein many economies have been locked in middle-income level for several decades but very few middle-income economies managed to achieve high-income status (*11*, *12*).

The East Asian Miracle consists of the fact that a few export-oriented small economies in East Asia have graduated into high-income status due primarily to rapid capital accumulation without technological progress (*13*–*15*).

Unfortunately, neoclassical models of economic growth have difficulty in explaining these key economic issues, and models developed to reconcile these inconsistencies tend to be restrictive and often unrealistic. For example, most neoclassical growth models imply convergence to a unique balanced growth path under realistic assumptions (conditional convergence), while most empirical evidences support club convergence characterized by multiple, locally stable steady-state equilibria (*16*–*19*).

Another challenge to neoclassical growth is to formulate a coherent explanation of the distinguishing characteristic of East Asia's economic miracle, i.e., the rapid process of capital accumulation in high-savings economies to beat the law of diminishing returns (*5*, *6*, *14*). Why does the law of diminishing returns not apply to them? What policies and which factors contributed to the miracle growth, and how? And can other middle-income countries replicate those policies to stimulate equally rapid growth? In fact, the lack of a "satisfactory growth theory" to inform development in middle-income economies was the original reason for referring to a middle-income trap (*12*).

Here we show that the standard Solow-Swan growth model in discrete time can *simultaneously* account for these key economic issues if in addition the no arbitrage constraint is imposed on the growth of total wealth in each period. Historically, Solow (*20*) asked what kind of market behavior will cause full employment economy to follow the path of equilibrium growth. Solow himself analyzed the price-wage-interest behavior appropriates to the equilibrium growth path in a perfect competitive economy. In our model, the capital accumulation path of the economy still follows the standard Solow-Swan process, but the kind of market behavior appropriates to the growth path is generalized to perfect arbitrage.

As in section V of Solow (*20*), we assume that saving and investment decisions are made independently to accumulate capital. Then we explore the impact of no arbitrage on the process of capital accumulation and industrial upgrading, with the aim to replicate the East Asian Miracle. As a relaxation of the hypothesis of perfect competition, perfect arbitrage leads to a system of slightly different growth accounting equations that exhibit complex behavior. Under perfect arbitrage, the coexistence of two equilibria (club convergence) is a pervasive feature of dynamic economies with realistic structural characteristics, but conditional converge (unique and globally stable equilibrium) may also emerge as a limiting case.

We then discuss the Lyapunov asymptotic stability of these two equilibria: one capital rich and the other capital poor. Given Cobb–Douglas technology with constant return to scale, a condition for the stability of equilibrium is obtained in terms of capital/labor ratio. As a result, the capital-rich equilibrium is asymptotically stable, but the capital-poor equilibrium is unstable.

Technically speaking (*21*), we have shown that no arbitrage constraint leads to a *tangential bifurcation* from the classical production maximization point. Because bifurcation phenomenon are ubiquitous in complex systems (*22*), this finding may have considerable economic interest.

To explore the dynamics along the growth path, we further estimate the basin of attraction for these two equilibria. Numerical simulation shows that their basin of attraction exhibit phase boundaries. As the capital/labor ratio reaches certain threshold level, the economic system may undergo a phase transition from capital-poor state into capital-rich state. This phase transition is a process of qualitative change that revolutionizes the economic structure *from within*, and is consistent with the process of industrial mutation (*23*).

Since the no arbitrage constraint must be satisfied at every instance of time, we can characterize the behavior of the economic system both in the steady state and during the transition toward it. As an application, we investigate how the economic system evolves under neoclassical growth. Following the standard Solow-Swan growth process, middle-income economies may converge to capital-rich club if their rates of saving and investment exceed the corresponding phase-boundary threshold. This threshold transition enables a high-saving economy to make a miracle growth based primarily on capital accumulation without technological progress.

Neoclassical Growth under No Arbitrage Constraint

Our growth model accepts all the Solow-Swan assumptions except the kind of the market behavior. Instead we suppose that the no arbitrage constraint is imposed on the growth of total wealth in each period on the basis of the existence of risk-free assets.

Firstly, we consider an economy in which economic activity is performed over infinite discrete time. In each period t, a perfectly divisible good Q_t is produced according to an exogenously given technologies

$$Q_t = AF(K_t, L_t), \tag{1}$$

where K_t and L_t stand for the capital and labor employed in production at the *beginning* of period t. For simplicity, we shall consider a neoclassical technology with diminishing marginal rate of technical substitution in general, and a Cobb–Douglas production function $Q_t = K_t^{\alpha} L_t^{\beta}$ in particular.

As in Solow (*20*), we assume that that saving and investment decisions are made independently. Consequently, we have to distinct between actual factor endowments (with a bar, like $\overline{K_t}$ and $\overline{L_t}$) and factor *in use* (without a bar, like K_t and L_t). By identifying the two we are assuming that full employment of the available labor and capital is perpetually maintained. Note that the Solow model is full employment economics—in the dual aspect of equilibrium condition and frictionless, competitive, casual system.

The good can either be consumed or saved for future investment. The economy allocates a fixed fraction $s \in (0,1)$ of output in every period to saving and investment. So the endowment of capital from a positive initial condition $\overline{K}_0 > 0$ evolves according to

$$\overline{K}_{t+1} = (1-\delta)\overline{K}_t + s\overline{Q}_t , \quad (2)$$

where $\delta \in (0,1]$ is the rate of capital depreciation.

If the neoclassical production function exhibits constant returns to scale, then we have $\overline{Q}_t = AF(\overline{K}_t, \overline{L}_t) = A\overline{L}_t F(\overline{K}_t / \overline{L}_t, 1) \equiv A\overline{L}_t f(\overline{k}_t)$, where $\overline{k}_t \equiv \overline{K}_t / \overline{L}_t$ is the actual capital/labor ratio. Assume that the available labor supply grows exponentially at a constant rate g. Then the evolution of the actual capital/labor ratio is governed by the following dynamical system

$$\overline{k}_{t+1} = \frac{(1-\delta)\overline{k}_t + sAf(\overline{k}_t)}{1+g} , \quad (3)$$

Given a positive initial endowment structure $\overline{k}_0 > 0$, it is well known that this dynamic system converges to a steady-state equilibrium that determines the balanced growth path.

Next, we shall analyze the process of adjustment to a balanced growth path. As in Solow (*20*), we assume that that saving and investment decisions are made independently and explore what kind of market behavior will cause the economy to follow the path of equilibrium growth. Most key features of neoclassical production theory are shared by our model. Typically, the market structure is characterized by perfect arbitrage with free entry and exit. Following Ventura (*14*), the economy is assumed to be small and open so the price variables are determined by the world markets. As a result, the representative firm is assumed to be a price taker, so that the price level P, the wage of labor W, and the rental price of capital i will be taken as given in each period. Moreover, there exists a risk-free asset which yields a risk-free rate of return, one of the most fundamental components of modern finance (*24*, *25*).

Since resources are scarce, firms like consumers are subject to budget constraints at any period of time (*26*–*30*). Formally, assume that the market value of the total investment at the *beginning* of period t is B_t. Then the efficient allocation of labor L_t and capital K_t must satisfy the constraint imposed by its total wealth, that is,

$$iK_t + WL_t = B_t . \quad (4)$$

According to the Efficient-Market Hypothesis (*31*), the economy is free of arbitrage opportunities, given the information available at the time the investment is made. Thus, under conditions of perfect arbitrage and with free entry and exit, the rate of return on investment is necessarily equal to the risk-free rate of return, and is the same no matter in terms of what it is measured. (Were this not so an arbitrage process would be set in motion.) To be precise, let the risk-free rate of return to be r, then the total wealth at the *end* of each period always equals $B_t(1+r)$. On the other hand, the firm's total wealth consists of two components: the physical output Q_t and the capital stock $K_t(1-\delta)$. Under conditions of perfect arbitrage, the market value of the physical output and the capital stock must add up to the total wealth at the end of period t, that is,

$$PQ_t + iK_t(1-\delta) = B_t(1+r) \equiv B_{t+1}. \quad (5)$$

Note that the equations (4) and (5) are accounting identities that must be followed at every instance of time. So we can characterize the behavior of the economic system both in the steady-state equilibrium determined by equation (3) and during the transition toward it.

Results

The key feature of the model is that it exhibits club convergence in general, and generates conditional convergence in the limit.

Tangential Bifurcation

When a particular production function $Q_t = AF(K_t, L_t)$ is specified, the basic accounting identities (4) and (5) give the two decision variables (L_t, K_t) in each period as multivariable functions of state variables (Fig. 1). No maximum problem need be studied, and no derivatives need be taken. (See Materials and Methods for the parameter settings in numerical analysis.)

In general, if the neoclassical production function is non-linear, then multiple equilibria may coexist even among economies with *identical* structural characteristics. To see why, substitute the production function $Q_t = AF(K_t, L_t)$ in the no-arbitrage constraint identity (5) to obtain the following *no arbitrage curve*

$$PAF(K_t, L_t) + iK_t(1-\delta) = B_t(1+r). \quad (6)$$

Due to the principle of diminishing marginal rate of technical substitution, the shape of the isoquant will be convex to the point of origin, so the no arbitrage curve is also convex to the origin. As a result, under a neoclassical technology with diminishing marginal productivity of capital, the no arbitrage curve intersects with the budget line twice in general (Fig. 1**B**). Specially, there is obviously an asymptotic case where the budget line and the no arbitrage curve are tangent to each other (Fig. 1A).

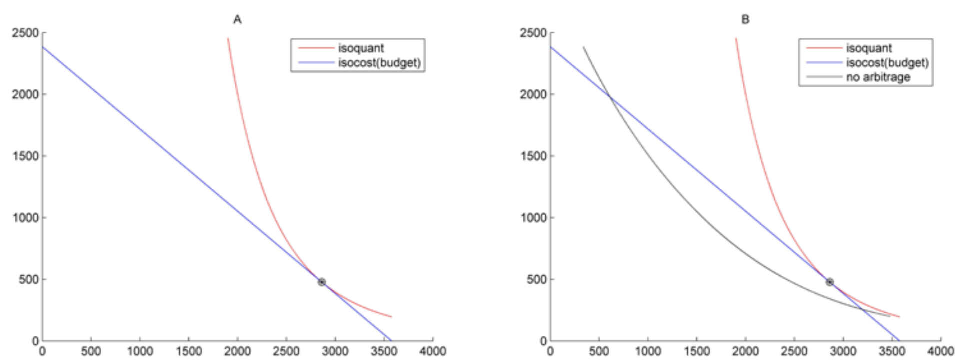

Fig. 1 Dynamic production under perfect arbitrage as a natural generalization of production maximization under budget constraint. (**A**) Shown is to maximize the production $Q_t = AK_t^{\alpha} L_t^{\beta}$ subject to the budget constraint $iK_t + WL_t = B_t$. The output is maximized at the unique tangent point, generating a linear expansion path. (**B**) The black line is a plot of the no arbitrage curve. It

is easy to see that there are two intersections between the no-arbitrage curve and the budget line. Note that a transition from capital-poor equilibrium to capital-rich equilibrium means structural unemployment.

Another way of reaching this result is to eliminate B_t from equations (4) and (6) to get the following identity

$$PAF(K_t, L_t) = iK_t(r+\delta) + WL_t(1+r). \tag{7}$$

This identity amounts to saying that the total revenue of output (PQ_t) equals the user cost of capital ($iK_t(r+\delta)$) pluses the cost of labor measured at the *end* of each period ($WL_t(1+r)$). From this *revenue-expenditure identity* it follows that the perfect arbitrage indeed imposed an essential constraint on the pattern of behavior of the economic system. This pattern is much more complicated than the relation predicted by classical production maximization (Fig. 1**a**).

If the neoclassical production function exhibits constant returns to scale, we can divide L_t and rewriting it in terms of per capita variable $k_t \equiv K_t / L_t$

$$PAf(k_t) = ik_t(r+\delta) + W(1+r). \tag{8}$$

Due to diminishing marginal productivity of capital, the revenue curve (left-hand side) passes through the origin and is convex upward, and hence intersects twice in general with the strait line corresponding to the expenditure (Fig. 2A). Note that tangency equilibrium (conditional convergence) may also emerge as a limiting case (Fig. 2B).

Unfortunately, even under Cobb–Douglas production function $Q_t = K_t^{\alpha} L_t^{1-\alpha}$, equation (8) cannot be explicitly solved for decision variables, i.e. do not have analytic solutions. For ease of reference, we denote the solution with low and high capital/labor ratio as k_t^1 and k_t^2, respectively. (Note that under constant returns to scale the economy will grow in proportion, so both low and high capital/labor ratios are stationary, corresponding to the capital-poor club k^1 and capital-rich club k^2 respectively.)

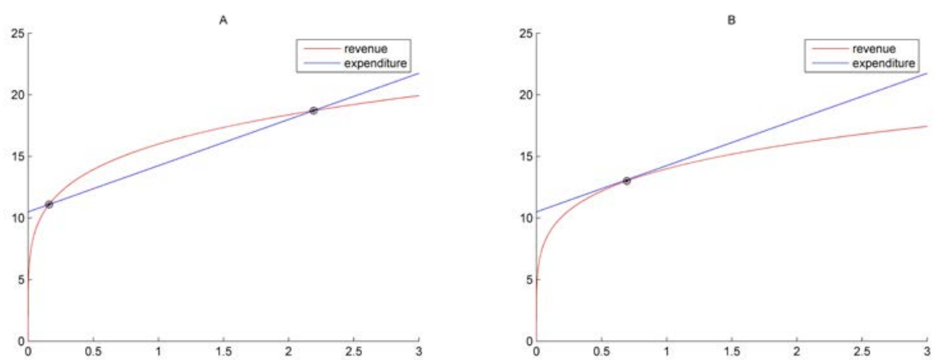

Fig. 2 Perfect arbitrage leads to multiple equilibrium in general, and tangency equilibrium in the limit. (**A**) If $P = 8$, then multiple equilibria occur (club convergence). (**B**) If $P = 6.999$, then tangency equilibrium emerges (conditional convergence).

Asymptotic Stability

Since the economic system under perfect arbitrage may have two equilibria, none of them can have global asymptotic stability in the sense of Lyapunov (*21*). Fortunately, the dynamic nature of our model enables us to discuss the asymptotic stability of these solutions.

As it turns out, both theoretical and numerical analysis indicates that the capital/labor ratio plays a major role in the long-term evolution of the economic system. Formally, given the economy's initial endowment (L_0, K_0), its capital/labor ratio $k_0 = K_0 / L_0$ is critical for determining the final state of its growth path starting from (L_0, K_0) (Fig. 3).

Specially, given Cobb–Douglas technology $Q_t = K_t^{\alpha} L_t^{1-\alpha}$ with constant return to scale, a condition for the stability of solutions is obtained in terms of capital/labor ratio (see Materials and Methods for details). Theoretical analysis shows that an economy with initial endowment (L_0, K_0) is asymptotically stable only if its capital/labor ratio k_0 satisfies

$$k_0 > \frac{\alpha W(1+r)}{(1-\alpha)i(r+\delta)} \equiv \hat{k} . \qquad (9)$$

Further, it's routine to check that the threshold $\hat{k}$ always lies *between* the two equilibrium capital/labor ratios k^1 and k^2, that is, $k^1 < \hat{k} < k^2$ (see Materials and Methods for details). As a result, the capital-rich equilibrium k^2 is asymptotically stable, but the capital-poor equilibrium k^1 is unstable (Fig. 3).

Taken together, the economic system under perfect arbitrage indeed exhibits *tangential bifurcation* at the production maximization point. Because bifurcation phenomena are ubiquitous in complex systems (*21*, *22*), this finding may have considerable economic interest.

A limiting case of importance is when $\delta \to 1$ and $r \to 0$. In such a case the threshold approaches to $\alpha W/(1-\alpha)i$, the slope of the expansion path derived from cost minimization under Cobb–Douglas production function (*32*). Further, if $r + \delta < 1$, then it is easy to see that the threshold value satisfies $\hat{k} > \alpha W/(1-\alpha)i$ (Fig. 3). It follows that the threshold may be far above the expansion path derived from cost minimization.

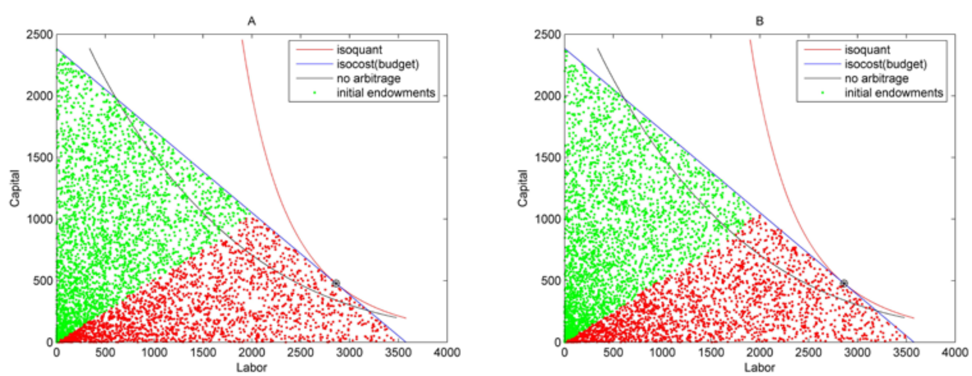

Fig. 3. Monte Carlo simulation of the basin of attraction. Shown is to invoke Matlab's fsolve function to compute the growth path under $Q_t = 2K_t^{0.2} L_t^{0.8}$. The number of samples is 5000. The growth

path starting from the red dot will converge to the capital-poor club, and the growth path starting from the green dot will converge to the capital-rich club. The resulting phase-boundary threshold is far above the slope of the expansion path derived from cost minimization (the tangency point between the isoquant and the isocost). (**A**) The basin of attraction corresponding to exponential growth. (**B**) The basin of attraction corresponding to Logistic growth.

Furthermore, numerical simulation shows that if the Cobb–Douglas production function *approximately* satisfies constant return to scale (that is, $\alpha + \beta \approx 1$), then a threshold of the capital/labor ratio also exists for the steady-state equilibrium to be asymptotically stable (Fig. S1).

Estimate Basin of Attraction

To explore the dynamics along the growth path, we further estimate the basin of attraction for the two convergence clubs.

Recall that the firm's total wealth grows at the risk-free return rate r in no arbitrage equilibrium. So the total budget follows an exponential growth $B_{t+1} = B_t(1+r)$. To estimate the basin of attraction for these two steady states, we define the system's growth path starting from (L_0, K_0) via numerical simulation: Using (L_0, K_0) as starting point, we apply numerical method to find the solution (L_1, K_1) for the system of accounting identities (4) and (5) at B_0. Recursively, by adopting (L_1, K_1) as new initial value, we continue to apply numerical method to calculate a solution (L_2, K_2) for the system (4) and (5) at $B_1 = B_0(1+r)$. Repeat this iteration process until a satisfactory growth path $\{(L_t, K_t) \mid t = 0,1,2,\cdots\}$ is obtained.

The resulting basin of attraction for the two convergence clubs is depicted in Fig. 3. As it turns out, the estimate basin of attraction exhibits phase boundaries, across which dramatic changes occur in the growth paths. Economies that are *identical* in economic structures and are on different sides of the phase boundary may diverge from one another in the long run. Due to heterogeneity in factor endowments, the world may eventually form into two convergence clubs: one capital poor and the other capital rich. This result agrees well with the observed pattern of club convergence (*16*–*19*).

It worth emphasis that under constant returns to scale the phase-boundary threshold is independent of the growth process of total budget. For example, similar phase-boundary threshold also occurs under the following Logistic process

$$B_{t+1} = B_t[1 + r(1 - B_t / TC)], \tag{10}$$

where TC is the capacity of total wealth (Fig. 3).

Making a Miracle Growth

In the absence of any external forces, which equilibrium growth path will arise depends on the economy's actual factor endowment structure, which in turn is determined by saving and investment decisions. Here we show that a miracle growth is possible when a threshold on the rate of saving and investment is overcome.

To determine the capital accumulation path, we proceed in the spirit of the Solow-Swan growth model under constant returns to scale, namely, the equation (3). A nontrivial fixed point of dynamical system (3) gives rise to a steady-state equilibrium, $\overline{k^*}$, which satisfies the following identity

$$sAf(\overline{k^*}) = (g+\delta)\overline{k^*}. \tag{11}$$

It follows that economies with different saving rates have different steady states, and higher saving rate leads to higher steady-state capital stocks (*20*).

Since the revenue-expenditure identity (8) always holds under conditions of perfect arbitrage and full employment of labor and capital (by identifying $\overline{k_t} \equiv \overline{K_t}/\overline{L_t}$ with $k_t \equiv K_t/L_t$), we can solve and then substitute for $Af(\overline{k^*})$ in equation (11) to get

$$s\frac{[i(r+\delta)\overline{k^*} + W(1+r)]}{P} = (g+\delta)\overline{k^*}. \tag{12}$$

So we can solve the actual capital/labor ratio in terms of state variables

$$\overline{k^*} = \frac{sW(1+r)}{P(g+\delta) - si(r+\delta)}. \tag{13}$$

To make a miracle growth into the capital-rich club, the steady-state equilibrium must exceed the phase-boundary threshold, that is,

$$\overline{k^*} = \frac{sW(1+r)}{P(g+\delta) - si(r+\delta)} > \frac{\alpha W(1+r)}{(1-\alpha)i(r+\delta)} \equiv \hat{k}. \tag{14}$$

Or equivalently,

$$s > \frac{\alpha P(g+\delta)}{i(r+\delta)} \equiv \hat{s}. \tag{15}$$

Monte Carlo simulation shows that such a threshold for the rate of saving and investment agrees well with the evolution of the economy with Cobb–Douglas technology under constant return to scale (Fig. 4).

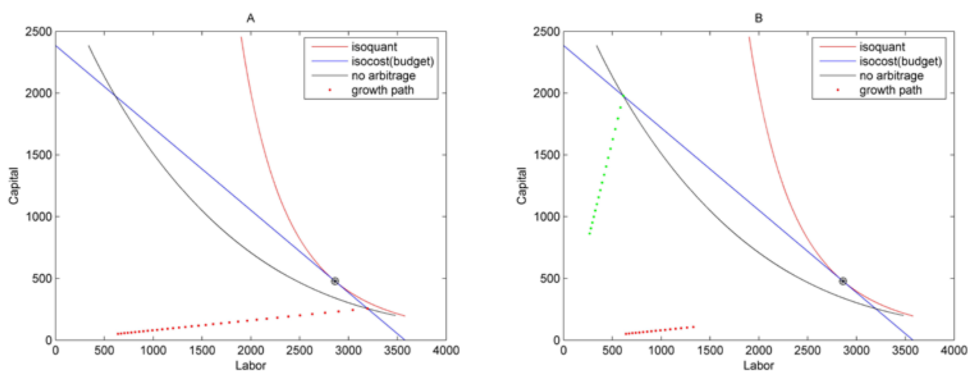

Fig. 4. A simulation of phase transition. (**A**) The saving rate $s = 0.08$. The threshold cannot be overcome, and the economy will be locked in the capital-poor state. (**B**) The saving rate $s = 0.09$. Following its comparative advantage in labor-intensive technologies to accumulate capital, the capital-poor economy successfully transited into capital-rich state at the cost of technological unemployment.

As a result, a minimum rate of investment is a necessary (though not sufficient) condition for a phase transition from capital-poor equilibrium into capital-rich equilibrium. Failing to overcome this threshold, middle-income economies will inevitably fall into the capital-poor club. Once the rate of saving and investment exceed the threshold value $\hat{s}$, the rapid process of capital accumulation makes it possible for capital-poor economy to achieve the threshold for the capital/labor ratio (i.e., $\overline{k^*} > \hat{k}$). Beyond this phase-boundary threshold, a low- or middle-income economy may shift to a capital-intensive technology and "takes off" toward the capital-rich club, though technological unemployment may be inevitable in the short run. Such virtuous cycle of higher saving and higher growth is consistent with the distinguishing characteristic of the East Asian Miracle (*5*).

However, that phase transition alone was not enough for miracle growth, and higher-for-longer rate of saving and investment is required to continue the climb to the capital-rich club. Under the assumption of full employment, the rate of saving and investment must be higher enough to ensure that the steady-state endowment reaches the capital-rich status, that is, $\overline{k^*} \geq k^2 > \hat{k}$. (We shall discuss structural unemployment below on the basis of stylized facts about industry dynamics.) Further, without technological progress, the rapid process of capital accumulation must be sustained long enough until the actual factor endowment achieved the capital-rich status.

Unfortunately, fast and sustained growth is not easy—the phase-boundary threshold is far above the neoclassical expansion path, as shown in Fig. 3. In fact, since World War II *only* 13 fastest-growth economies have achieved grow rate of 7 percent for more than 25 years (*33*, *34*).

Discussion

I have reported that it is possible to simultaneously account for the middle-income trap, the convergence debate, and the East Asian Miracle by combing a standard Solow-Swan growth model in discrete time with a no arbitrage constraint imposed on the growth of total wealth in each period. As we have seen, dynamic production under perfect arbitrage may exhibit complex behaviors, including tangential bifurcation and phase transition. Because the phenomena of bifurcation and phase transition exist widely in complex systems (*21*, *22*), these findings may have considerable economic interest. Now we shall discuss the implication of these findings for economic growth and industrial upgrading, with the aim to replicate the East Asian Miracle.

Our finding of tangential bifurcation indicates that perfect arbitrage may lead to club convergence in general, as well as conditional convergence in the limit. Thanks to the occurrence of tangential bifurcation, two equilibria may coexist even in homogeneous environments. Due to the difference in factor endowments, the world may eventually form into two convergence clubs: one capital poor and the other capital rich, with the middle class vanishing (*35*). This result agrees well with the observed pattern of club convergence (*2*, *16*–*19*).

Our results of asymptotic stability show that the capital/labor ratio plays a key role in the dynamical evolution of the economic system. Both theoretical and numerical analysis indicates that there exists a critical threshold level for the capital/labor ratio such that, when the capital/labor ratio is above the threshold, the economy is asymptotically stable, while below the threshold the economy may exhibit instability. In theory, the inherent instability of capital-poor state means that low- or middle-income economy has a strong incentive to improve its endowment structure during the dynamic process of capital accumulation. As the capital/labor ratio reaches this threshold level, the economy may undergo a phase transition from a capital-poor state into capital-rich state.

Our finding of phase transition indicates that capital accumulation may result in a process of industrial mutation (*23*). Numerical simulation shows that dynamic production under perfect arbitrage exhibits phase boundaries, across which dramatic changes occur in the growth paths. In practice, a capital-poor economy can follow its comparative advantage in labor-intensive industries to accumulate capital and then update its endowment structure. When the endowment-structure threshold has been achieved, the economy may adopt capital-intensive technologies and "takes off" toward the capital-rich club, though technological unemployment may be inevitable in the short run. In essence, capital accumulation serves as the fundamental driving force to revolutionize the industry structure *from within*. Such a process of Creative Destruction is consistent with the process of industrial mutation (*23*).

However, the possibility of phase transition does not mean that industrial mutation will occur automatically. In fact, it is easy to see from Fig. 3 that the phase-boundary threshold may be far above the expansion path derived from cost minimization (*32*). This means that phase transition from a labor-intensive state into capital-intensive state can only be found beyond the product-maximization regime, implying that the critical endowment-structure threshold is not easy to overcome. As a result, industrial upgrading is not a simple and automatic process. This result explains why low-income economies may be persistently locked in poverty trap (*3*, *36*), and why so few economies succeed in climbing out of the middle-income trap (*4*).

Furthermore, it is worth emphasis that in the absence of technological progress structural unemployment is inevitable during threshold transition from capital-poor equilibrium into capital-rich equilibrium (Fig. 1). Fortunately, stylized fact about industry dynamics indicate that (i) there exists tremendous cross-industry heterogeneity both in capital intensities and productivities and (ii) more capital-intensive industry reaches its threshold later (*37*). As a result, the threshold transition for a single industry does not imply a discontinuous transition for the economy as a whole. At the macro-level, the economy may gradually and smoothly shift from the labor-intensive state to more capital-intensive state as the economy develops. So, structural unemployment may be a reasonable cost that has to be accepted if a middle-income economy wants to replicate the East Asia Miracle without technological progress. As pointed out by Solow in (*20*): "*It may take deliberate action to maintain full employment. But the multiplicity of routes to full employment, via tax, expenditure, and monetary policies, leaves the nation some leeway to choose whether it wants high employment with relatively heavy capital formation, low consumption, rapid growth; or the reverse, or some mixture.*" (italic added)

Note that the production function remains *unchanged* during the threshold transition process from capital-poor state into capital-rich state. This shuts down the productivity-driven mechanism of structural change. Instead, the comparative-advantage-following approach enables a capital-poor economy to improve its endowment structure as capital accumulation reaches certain threshold level. Long-term sustainable growth itself is the driving force for phase transition into capital-intensive economies. In theory, our results echo previous argument that capital accumulation itself serves as a fundamental mechanism that drives the industrial dynamics, which was referred to as endowment-driven structural change in the literature (*33*, *37*).

Also note that after the phase transition the rate of return on investment still remains *the same*. As a result, small and open economies with high saving rate can beat the curse of diminishing returns at the cost of structural unemployment. The "smallness" assumption allows us to model the representative firm as a price taker who can export any amount without affecting world prices. The crucial role of international trade is that it converts the excess production of capital intensive goods into exports, and as the economy is small, their prices in the world do not fall. Otherwise, if an excess production of capital-intensive goods cannot be converted into exports, then instead it will force price level down, say, from $P = 8$ to $P = 6.999$. Then, other things equal, the economy may exhibit tangency equilibrium (conditional convergence) rather than tangential bifurcation (club convergence), as we have seen in Fig. 2. Note that in the case of conditional convergence no phase transition will take place, but miracle growth is still possible for high-saving economies if they are able to trade without falling prices (*14*, *15*).

A potential criticism of our study arises from the difficulty of deciding optimal investment budget. Note that in much of classical production theory, only consumers, not producers, face budget constraints. But the assumption that producers are unconstrained is made merely for convenience, since most of classical production theory is not concerned with the relationship between finance and production, where such constraints come into play (*30*). In practice, the existence and importance of a budget constraint becomes patently clear, and the traditional distinction between firms and households is blurred and perhaps vanished (*26*). In fact, much work in corporate finance has been devoted to the study of the firm's budget constraint (*27–29*). And the longstanding Capital Structure Puzzle (*38*, *39*) means that a significant gap still exists in understanding how financial decisions influences and is influenced by production decisions (*40–44*).

Another shortcoming of our model is how to determine a satisfactory rate of return on investment. With free entry and exit, risk-neutral firms will enter an industry only if they are satisfied with the expected rate of return on investment, given the information available at the time the investment is made. However, if representative firms are not risk neutral, then the degree of uncertainty (measured by Variance) will affect investment decisions. In this circumstance, risk premium of different industries must be considered on the basis of the Capital Asset Pricing Model (*45*).

In conclusion, these findings are consistent with the empirical pattern of modern growth, and hence deepen our understanding of the impacts of capital accumulation on the dynamic evolution of economic system. To justify our model of economic growth as a useful framework, a natural next step is to test it using real national accounts data, such as the Penn World Table. For example, even a rough estimate of the phase-transition threshold for a low- or middle-income economy may be helpful to avoid or escape the growth traps. However, this work is extremely time-consuming, and we have to leave it to the future.

References and Notes

1. W. J. Baumol, Productivity Growth, Convergence, and Welfare: What the Long-Run Data Show. *American Economic Review*. **76**(5), 1072-85(1986).
2. J. B. De Long, Productivity growth, convergence, and welfare: comment. *American Economic Review*. **78**(5), 1138-1154(1988).
3. R. Nelson, S. Winter, *An Evolutionary Theory of Economic Change* (Harvard University Press, Cambridge, MA, 1982).

4. I. S. Gill, H. Kharas, *An East Asian Renaissance: Ideas for Economic Growth* (Washington,DC: World Bank, 2007).
5. World Bank, *The East Asian Miracle: Economic Growth and Public Policy* (Oxford University Press, New York, 1993).
6. P. Krugman, The Myth of Asia's Miracle. *Foreign Affairs*. **73**(November/December), 62-78 (1994).
7. E. A. Hanushek, L. Woessmann, Knowledge capital, Growth,and the East Asian miracle. *Science*. **351**(6271), 344-345 (2016)
8. R. J. Barro, Xavier Sala-i-Martin, Convergence. *Journal of Political Economy*. **100**(2), 223-251 (1992).
9. C. Azariadis, A.Drazen, Threshold Externalities in Economic Development. *The Quarterly Journal of Economics*. **105**(2), 501-526(1990).
10. R. Lucas, Making a Miracle. *Econometrica*. **61**（2）, 251–72(1993).
11. H. Kharas, and H. Kohli, What Is the Middle-Income Trap, Why Do Countries Fall into It, and How Can It Be Avoided? *Global Journal of Emerging Market Economies.* **3**(3), 281–89(2011).
12. I. S. Gill, and H. Kharas, The Middle-Income Trap Turns Ten. Policy Research Working Paper 7403 (World Bank,Washington, DC, 2015).
13. A. Young, The Tyranny of Numbers: Confronting the Statistical Realities of the East Asian Growth Experience. *Quarterly Journal of Economics*. **110**, 641–680 (1995).
14. J. Ventura, Growth and interdependence. *Quarterly Journal of Economics.* **112**,57–84(1997).
15. J. Braude, Y. Menashe. The Asian miracle: Was it a capital-intensive structural change? *The Journal of International Trade & Economic Development*. **20**(1), 31-51(2011).
16. S. N. Durlauf, P. A. Johnson, Multiple Regimes and Cross-Country Growth Behaviour. *Journal of Applied Econometrics*. **10**, 365–384 (1995).
17. O. Galor. Convergence? Inferences form theoretical models. *Econ J*. **106**(437), 1056–1069 (1996).
18. D. Quah, Empirics for Economic Growth and Convergence. *European Economic Review*. **40**, 1353–1375(1996).
19. P. C. B. Phillips, D. Sul, Transition modeling and econometric convergence tests. *Econometrica.* **75**, 1771–1855(2007).
20. R. M. Solow, A Contribution to the Theory of Economic Growth. *The Quarterly Journal of Economics.* **70**(1), 65-94 (1956).
21. R. C. Robinson, *An Introduction to Dynamical Systems: Continuous and Discrete* (American Mathematical Society, Second Edition, 2012).
22. W. B. Arthur, Complexity and the Economy. *Science.* **284**(5411), 107-9(1999).
23. J. A. Schumpeter, *Capitalism, Socialism and Democracy. 3rd edition* (London: Allen and Unwin, 1950).

24. W. F.Sharpe, Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk. *Journal of Finance.* **19**(3), 425–42(1964).

25. S. A. Ross, *Neoclassical Finance* (Princeton University Press, 2004).

26. G. S. Becker, Irrational Behavior and Economic Theory. *Journal of Political Economy.* **70**(1), 1-13(1962).

27. J. Kornai, Resource-Constrained versus Demand-Constrained Systems. *Econometrica.* **47**(4), 801-819 (1979).

28. P. Bolton, M. Dewatripont, The time and budget constraints of the firm. *European Economic Review.* **39**(3), 691-699(1995).

29. K. Kuga, Budget constraint of a firm and economic theory. *Economic Theory.* **8**(1),137-153(1996).

30. J. E. Kornai, Maskin, G. Roland, Understanding the Soft Budget Constraint. *Journal of Economic Literature.* **41**(4), 1095-1136(2003).

31. E. Fama, Efficient Capital Markets: A Review of Theory and Empirical Work. *Journal of Finance.* **25** (2), 383–417 (1970).

32. A. Goolsbee, S. Levitt, C. Syverson, *Microeconomics* (Worth Publishers, Third edition, 2019).

33. Justin Yifu Lin, *New structural economics : a framework for rethinking development* (The World Bank, Washington, D.C., 2011).

34. World Bank, *The Growth Report: Strategies for Sustained Growth and Inclusive Development* (The World Bank, Washington, D.C., 2008).

35. G. Garrett, Globalization's Missing Middle. *Foreign Affairs*. **83**(6) , 84-96 (2004)

36. S. Bowles, S. N. Durlauf, and K. Hoff, *Poverty Traps*. (Princeton University Press, Princeton, 2011).

37. J. Ju, J.Y. Lin, Y. Wang, Endowment Structures, Industrial Dynamics, and Economic Growth. *Journal of Monetary Economics*. **76**, 244-263 (2015).

38. F. Modigliani, M.H.Miller, The Cost of Capital, Corporation Finance and the Theory of Investment. *American Economic Review.* **48**(3), 261-97(1958).

39. S. Myers, The Capital Structure Puzzle. *Journal of Finance.* **39**(3), 575-592(1984).

40. P. Diamond, The Role of a Stock Market in a General Equilibrium Model with Technological Uncertainty. *The American Economic Review.* **57**(4), 759-776(1967).

41. E. Fama, Perfect Competition and Optimal Production Decisions under Uncertainty. *The Bell Journal of Econiomics and Mantagement Science.* **3**(2), 509-530(1972).

42. J.E. Stiglitz, On the Optimality of the Stock Market Allocation of Investment. *Quarterly Journal of Economics.* 86(1), 25-60(1972).

43. J.E. Stiglitz, On the Irrelevance of Corporate Financial Policy. *American Economic Review.* **64**(6), 851-866(1974).

44. S. A.Ross, The Determination of Financial Structure: The Incentive-Signalling Approach. *Bell Journal of Economics.* **8**(1), 23-40(1977).

45. W. F. Sharpe, Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk. *Journal of Finance.* **19**(3), 425–42(1964).

Acknowledgments: Part of this paper has been presented at The Chinese Economists Society 2015 Conference, held at Chongqing University on June 13-14th, 2015. The author also thanks Professor Depeng Shan for helpful discussions.

Funding: The author acknowledges financial support from Hunan University.

Author contributions: All authors contributed equally to this work.

Competing interests: The Author declare no competing interests.

Data and materials availability: All data are available in the manuscript or the supplementary materials. All code for numerical simulation and figure production is available upon request from the author.

Supplementary Materials

Materials and Methods

Figs. S1

Materials and Methods

Parameter settings.

Throughout this paper, we consider the following instance of production maximization

$$\begin{cases} \max \quad Q_t = 2K_t^{0.2}L_t^{0.8} \\ s.t. \quad 15K_t + 10L_t = 35774 \end{cases}.$$

Note that this production maximization problem is dual to the cost minimization problem subject to $Q_t = 4000$, an example taken from the textbook of (*32*).

Without further declaration, we consider the following instance of the basic accounting identities (corresponding to equations (4) and (5))

$$\begin{cases} 15K_t + 10L_t = B_t \\ 8 \times 2K_t^{0.2}L_t^{0.8} + 12K_t = (1+0.05)B_t \end{cases}.$$

In the more general case of neoclassical technology, the production function is replaced by $Q_t = 2K_t^{0.2}L_t^{0.81}$, which *approximately* satisfies constant return to scale.

Lyapunov asymptotic stability

We shall consider the Lyapunov stability of the system of accounting identities (corresponding to equations (4) and (5))

$$\begin{cases} iK_t + WL_t = B_t \\ PQ_t + i(1-\delta)K_t = B_t(1+r) \equiv B_{t+1} \end{cases}$$

under Cobb–Douglas technology with *constant return to scale*, that is, $Q_t = AK_t^{\alpha}L_t^{1-\alpha}$ for some $\alpha \in (0,1)$.

In general, the system has two steady-state solutions (L^1, K^1) and (L^2, K^2) (Fig. 1**b**). (Note that under constant returns to scale the economy will grow in proportion, so both of these two solutions are stationary.) Consequently, none of them can have global asymptotic stability in the sense of Lyapunov (*20*).

Fortunately, the dynamic nature of our model enables us to discuss the asymptotic stability of these two solutions. To proceed, eliminate B_t from the system to get

$$PAK_t^{\alpha}L_t^{1-\alpha} = i(r+\delta)K_t + W(1+r)L_t. \quad (16)$$

Dividing L_t and rewriting it in terms of per capita variable $k_t = K_t / L_t$, we have

$$PAk_t^{\alpha} = i(r+\delta)k_t + W(1+r). \quad (17)$$

Due to the convexity of the function k_t^{α}, equation (17) has two solutions in general (see Fig. 2(A)), denoted as k^1 and k^2, corresponding to the capital-poor equilibrium (L^1, K^1) and capital-rich equilibrium (L^2, K^2) respectively.

To continue, rearrange equation (17) as follows

$$k_t = \frac{PA}{i(r+\delta)}k_t^{\alpha} - \frac{W(1+r)}{i(r+\delta)} \equiv \phi(k_t). \quad (18)$$

The evolution of the fixed point $k_t = \phi(k_t)$ is governed by the dynamic system $k_{t+1} = \phi(k_t)$. To analyze the stability of k_t under iteration, take first-order derivative with respect to k_t

$$\frac{d}{dk_t}\phi(k_t) = \frac{\alpha}{i(r+\delta)} PAk_t^{\alpha-1}. \tag{19}$$

Since identity (17) always holds under perfect arbitrage, we can solve and then substitute for $PAk_t^{\alpha-1}$ to get

$$\frac{d}{dk_t}\phi(k_t) = \alpha\left(1 + \frac{W(1+r)}{i(r+\delta)} \times \frac{1}{k_t}\right). \tag{20}$$

It is well known (*20*) that the fixed point $k_t = \phi(k_t)$ is stable under iteration if its first-order derivative satisfies

$$\frac{d}{dk_t}\phi(k_t) = \alpha\left(1 + \frac{W(1+r)}{i(r+\delta)} \times \frac{1}{k_t}\right) < 1. \tag{21}$$

Solving for k_t we obtain a condition for k_t to be asymptotically stable

$$k_t > \frac{\alpha W(1+r)}{(1-\alpha)i(r+\delta)} \equiv \hat{k}. \tag{22}$$

Further, it is routine to check that the threshold $\hat{k}$ always lies *between* the two equilibrium k^1 and k^2. To this end, we shall show that threshold value $\hat{k}$ occurs at the most significant gap between the revenue PAk_t^{α} and the expenditure $i(r+\delta)k_t + W(1+r)$ (see Fig. 2(A)). So we take the first-order derivative of the gap with respect to k_t

$$\frac{d}{dk_t}[PAk_t^{\alpha} - i(r+\delta)k_t - W(1+r)] = \alpha APk_t^{\alpha-1} - i(r+\delta). \tag{23}$$

Eliminating the term $PAk_t^{\alpha-1}$ by substituting identity (17) into equation (23), we obtain

$$\frac{d}{dk_t}[PAk_t^{\alpha} - i(r+\delta)k_t - W(1+r)] = \alpha[i(r+\delta) + \frac{W(1+r)}{k_t}] - i(r+\delta). \tag{24}$$

It's easy to see that the corresponding stationary point of equation (24) coincides with the threshold value $\hat{k}$. So the threshold value $\hat{k}$ indeed maximizes the difference between the revenue PAk_t^{α} and the expenditure $i(r+\delta)k_t + W(1+r)$, and hence always lies *between* the two intersections k^1 and k^2, as desired.

Taken together, the capital-rich club $k^2 = K^2 / L^2$ is asymptotically stable, but the capital-poor club $k^1 = K^1 / L^1$ is unstable (Fig. 3). In conclusion, we have shown that perfect arbitrage indeed leads to a *tangential bifurcation* (*20*).

Fig. S1.

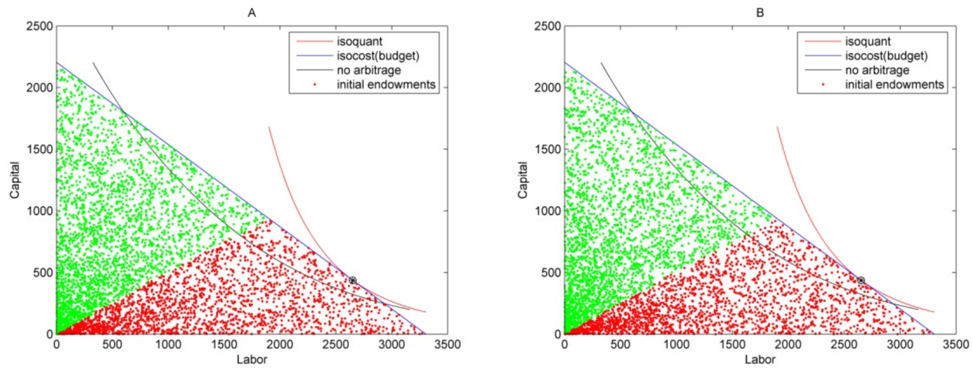

Fig. S1. Monte Carlo simulation of the basin of attraction. Shown is to invoke Matlab's fsolve function to compute the growth path under $Q_t = 2K_t^{0.2} L_t^{0.81}$. The number of samples is 5000. The growth path starting from the red dot will converge to the capital-poor club, and the growth path starting from the green dot will converge to the capital-rich club. The resulting phase-boundary threshold is far above the slope of the expansion path derived from cost minimization (the tangency point between the isoquant and the isocost). (**A**) The basin of attraction corresponding to exponential growth. (**B**) The basin of attraction corresponding to Logistic growth.