Two Dynamic Models of Distributive and Financial Endogenous Cycles

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Abstract

This paper proposes two theoretical dynamic models (Models A and B) to analyze the interaction between distributive and financial cycles in capitalist economies. Model A assumes investment equals savings at the aggregate level but assumes a delay between capitalists saving their income and distributing it to firms for reinvestment, leading to credit demand from a rentier class. Model B extends and modifies Model A by representing capitalist incentives to invest through an investment function and accounting for the dynamic effect of non-zero excess demand. Analytical proofs for the existence of limit cycles in both models are provided using Hopf bifurcation theorems for four-dimensional and five-dimensional dynamical systems, and numerical simulations identify stable and limit cycles, unstable cycles, damped oscillations, and multiple relevant patterns. The results suggest that the stability of cycles is significantly influenced by the distribution of bargaining power between workers and capitalists, as well as the behavior of the central bank and the rentier class. Furthermore, the paper suggests two methods to identify financing regimes within capitalist cycles and concludes by providing insights for future analytical and empirical research.

Keywords: Distributive cycles, Financial cycles, Hopf bifurcation, bargaining power, stability, interest rate

JEL codes: C61, E11, E12, E32, G01

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1. Introduction

The study of economic fluctuations in capitalism is complex due to the simultaneous interaction of multiple factors. One such example is the interaction between distributive cycles resulting from the struggle between a profit-seeking capitalist class and a working class focused on reducing labor exploitation, and financial cycles that may emerge from the dynamics associated with debt used to finance investment, along with potential movements of the interest rate depending on the dynamics of debt and inflation. These cycles are not isolated but instead interact with each other during the process of capitalist reproduction. One approach to studying this interaction is through the combination of insights from the Marxian model of distributive cycles proposed by Goodwin (1967) and financial components inspired by the works of Minsky (1976, 1986). This combination can be constructed by representing distributive and financial patterns using high-dimensional dynamical systems. Several attempts to study the interaction between distributive and financial cycles from this perspective have been made, including the works of Keen (1995, 2009), Graselli and Costa Lima (2012), Sordi and Vercelli (2014), Stockhammer and Michell (2017), Rammelt (2019), among others.[2]

This paper aims to contribute to this discussion about the interaction between distributive and financial cycles by formulating two theoretical dynamical models. The first model (Model A) represents an economy without excess demand, where investment equals savings at the macroeconomic level. However, we assume that there is a time delay between the moment capitalists receive and save their income and the moment they distribute it to firms for reinvesting. As a result, capitalist firms require debt to sustain their cash flows during economic activities, leading them to demand credit from a rentier class that borrows money and receives an interest rate. Model A also incorporates a central bank that aims to control inflation by applying a monetary policy that can adjust the general level of interest rates. We obtain a four-dimensional dynamical system from this model, analytically prove the existence of stable limit cycles by applying the Hopf bifurcation theorem (Appendix 1), and use numerical simulations to identify some relevant patterns generated by the parameters of the model that influence the stability and volatility of cycles.

The second model (Model B) extends and modifies the formulation of Model A by incorporating excess demand and the inequality between investment and savings. Model B assumes that capitalist incentives to invest can be represented through an investment function, while a capacity utilization rate adjusts due to the existence of excess demand. In this framework, firms require debt not only to sustain their cash flows during their economic activity but also to finance their investment beyond the limits of capitalist savings. Model B includes a central bank that adjusts the interest rate, as well as the reaction of the rentier class to an increasing leverage ratio. From this model, we obtain a five-dimensional dynamical system and illustrate the existence of stable cycles using numerical simulations. Additionally, we analytically prove the existence of limit cycles for a simplified version of the model by

[2] For a literature review of Goodwin-Minsky models, as well as other Minsky models, see Nikolaidi and Stockhammer (2018).

applying the Hopf bifurcation theorem (Appendix 3). From numerical simulations of Model B, we identify relevant patterns associated with the stability of cycles and propose a preliminary approach to identifying hedge, speculative, and Ponzi financing regimes within the cycles.

The paper is structured as follows. Section 2 presents the formulation of Model A, explains the construction of the four-dimensional dynamical system, demonstrates the existence of limit cycles (based on the mathematical demonstration presented in Appendix 1), and discusses relevant patterns associated with its stability. Section 3 extends Model A to Model B, allowing the existence of excess demand and an investment function. We explain the construction of the five-dimensional dynamical system and illustrate the existence of potentially stable cycles and other relevant patterns within the model using numerical simulations (and referencing the mathematical demonstration presented in Appendix 3). Section 3 also introduces a preliminary approach for identifying financing regimes within the cycles generated by Model B. Finally, Section 4 concludes the paper.

2. Model A: Cycles without Excess Demand

2.1.Formulation of Model A

In this section, we present the formulation of Model A, which is based on the assumption of a closed economy without government that includes three social classes (capitalists, workers, and rentiers) and a central bank. The economy produces a single good that can be used for either consumption or investment. Additionally, we assume that the total capital stock consists of only fixed capital that does not depreciate. In line with Kalecki (1971) and Dutt (1987), we propose that firms determine the price level p by adding a markup factor z to their primer costs, which only include the wage bill, as indicated by the behavioral price equation below:

$$p = (1 + z)\left(\frac{w}{q}\right) \quad (1)$$

Where w is the average money wage rate paid to workers and q is the labor productivity rate, which is defined as the ratio between real output Q and the level of employment L:

$$q = \frac{Q}{L} \quad (2)$$

In this model, we assume that the economy has already reached its macroeconomic equilibrium, with the magnitude of aggregate real savings S being equal to the magnitude of aggregate real investment I. We also assume that capitalists are the only social group that saves their income, which is equivalent to the aggregate net profit Π earned from their firms. These assumptions can be represented by the following equation:

$$S = \frac{\Pi}{p} = I = gK \quad (3)$$

Where Π/p is the real net profit, K is the aggregate stock of fixed capital and g is its growth rate. While the aggregate level of investment is equal to that of savings, our model assumes that there is a time lag between the moment when capitalists receive their income and save it, and when they distribute it to firms for reinvestment. Consequently, capitalist firms need to obtain new credit to maintain their cash flows during economic activities. This could mean that individual firms may require new credit to hire labor, rent capital, and start production before capitalists decide to return profits to firms. Thus, this time lag leads firms to seek credit from a rentier class that borrows money and charges an interest rate.[3] We assume that firms pay an interest rate to a rentier class that provides credit and uses all of its income for consumption. These assumptions imply that the aggregate net profit Π that capitalists receive from their firms equals the difference between production income and cost associated with wages and interest payments on debt, as indicated by:

$$\Pi = pQ - wL - \tau pD \quad (4)$$

Where τ is the nominal interest rate and D is the total stock of real debt measured in terms of the price level p. Given this definition of Π, we can represent the net profit rate r as:

$$r = \frac{\Pi}{pK} \quad (5)$$

Similarly, we can define the capital-output ratio as:

$$\sigma = \frac{K}{Q} \quad (6)$$

Here we assume σ is constant and depends on technological factors. Also, we define the leverage ratio f as the ratio between the stock of debt and the stock of fixed capital:

$$f = \frac{D}{K} \quad (7)$$

If we assume that workers and rentiers use all their income for consumption while firms invest, then we can define the real aggregate excess demand ED as:

$$ED = \left(\frac{w}{p}\right)L + \tau D + I - Q \quad (8)$$

Where w/p is the real wage rate. Substituting (3) and (4) into (8) verifies that $ED = 0$, that is, there is no excess demand. This assumption, as well as the equality between investment and savings, will change in Model B.

[3] We follow Meirelles and Lima (2006) in assuming that the rentier class is a price maker in the loan market, meaning that the supply of credit is demand driven at a given interest rate.

At this point, it is possible to obtain some relevant deductions.[4] For instance, by rearranging the terms of equation (1), we can express the real wage rate as:

$$\frac{w}{p} = \frac{q}{1+z} \quad (9)$$

Similarly, combining (2), (4) to (7), and (9) gives an expression for the net profit rate:

$$r = \left(\frac{1}{\sigma}\right)\left(\frac{z}{1+z}\right) - \tau f \quad (10)$$

As firms invest, they adjust the labor employed in production. In this sense, we define the employment rate l as:

$$l = \frac{L}{N} = \left(\frac{1}{q\sigma}\right)\left(\frac{K}{N}\right) \quad (11)$$

Where N is the labor supply. Taking logarithms and time derivatives of (11) gives:

$$\frac{\dot{l}}{l} = g - \frac{\dot{q}}{q} - \frac{\dot{N}}{N} \quad (12)$$

Where $g = \dot{K}/K$.[5] We assume that both labor productivity and labor supply have constant rates of growth respectively defined by:

$$\frac{\dot{q}}{q} = \theta, \qquad 0 \leq \theta < 1 \quad (13)$$

$$\frac{\dot{N}}{N} = n, \qquad 0 \leq n < 1 \quad (14)$$

Substituting (3), (10), (13), and (14) into (12) gives a dynamic equation for the employment rate l:

$$\frac{\dot{l}}{l} = -(\theta + n) + \left(\frac{1}{\sigma}\right)\left(\frac{z}{1+z}\right) - \tau f \quad (15)$$

As suggested by Goodwin (1967) in the context of distributive cycles, and by Stockhammer and Michell (2017) in the context of cyclical models with financial instability, changes in the bargaining power of the working class occur as the employment rate adjusts, exerting a significant influence on economic and financial cycles (reserve army effect from a Marxian perspective). To incorporate these insights into the model, we use a formulation close to Desai (1973) by assuming that the growth rate of the money wage depends on the employment rate and the inflation rate:

[4] A notebook in Wolfram Mathematica containing all mathematical deductions and numerical simulations presented in this paper is available as supplemental online material. Interested readers can access the notebook to verify the calculations and simulations. For further details or questions, please contact the author.

[5] For any variable x, $\dot{x} = dx/dt$ represents its time derivative, and $\dot{x}/x$ represents its growth rate.

$$\frac{\dot{w}}{w} = -\gamma + \rho l + \eta\left(\frac{\dot{p}}{p}\right), \qquad 0 < \rho, \gamma, \eta < 1 \quad (16)$$

Where γ represents an autonomous tendency of the money wage to fall while ρ and η respectively collect the effect on the money wage associated with the employment rate and inflation. These parameters can be interpreted as proxies for capturing the distribution of the bargaining power between workers and capitalists in determining the growth rate of the money wage. Thus, a decrease (increase) in ρ or η, or an increase (decrease) in γ, would indicate an enhanced (weakened) bargaining power for capitalists relative to the working class. As a result, the wage rate deaccelerates (accelerates) and capitalists achieve a higher (lower) labor exploitation.[6]

Regarding inflation, we adopt the approach proposed by Dutt (1987), whereby firms adjust the price level when the actual markup z they achieve in the economy differs from their desired markup ζ. The desired markup reflects the capitalist perceived degree of monopoly power and bargaining power relative to the working class. Thus, when the actual markup deviates from the desired markup, firms adjust the price level to maintain their profit margins. This idea is formalized through the following behavioral equation:

$$\frac{\dot{p}}{p} = h(\zeta - z), \qquad h > 0 \quad (17)$$

Where h accounts for the inflationary tendency generated by the reaction of capitalist firms to their perceived 'markup gap' defined by the term $(\zeta - z)$.

Now, if we take logarithms and time derivatives of (1), we get:

$$\frac{\dot{p}}{p} = \frac{\dot{z}}{1+z} + \frac{\dot{w}}{w} - \frac{\dot{q}}{q} \quad (18)$$

Substituting (13), (16), and (17) into (18) gives a dynamic equation for the actual markup z:

$$\frac{\dot{z}}{1+z} = \gamma + \theta + h(1-\eta)(\zeta - z) - \rho l \quad (19)$$

Concerning the dynamics of debt, we draw on the work of Keen (1995) and Meirelles and Lima (2006) to analyze the cash flows of capitalist firms. This analysis indicates the need to balance the sources of funds with their uses. In other words, the sum of net profits and new borrowing should be equal to the funds employed for investment and debt service (interest payments). This cash flow identity can be expressed in real terms as follows:

$$\frac{\Pi}{p} + \dot{D} = gK + \tau D \quad (20)$$

[6] For similar interpretations within the context of Marxian models of endogenous cycles, see Cajas Guijarro and Vera (2022) and Cajas Guijarro (2023).

Substituting (3) into (20) gives $\dot{D} = \tau D$, that is, the growth rate of the stock of debt is equal to τ. Therefore, in the context where investment equals savings, the aggregate level of the new borrowing undertaken by firms is ultimately equivalent to the interest payments associated with credit extended by the rentier class, which serves to support the cash flows of firms during the time lag between the receipt of profits by capitalists and their subsequent distribution to firms for reinvestment.

Now, if we note that taking logarithms and time derivatives of equation (7) gives:

$$\frac{\dot{f}}{f} = \frac{\dot{D}}{D} - g \quad (21)$$

Then, by substituting (2), (3), (5), (7), (10), and (20) into (21), we get a dynamic equation for the leverage ratio f:

$$\frac{\dot{f}}{f} = \tau(f + 1) - \left(\frac{1}{\sigma}\right)\left(\frac{z}{1 + z}\right) \quad (22)$$

Finally, we include the existence of a central bank that operates intending to achieve a specific inflation rate π^o. To attain this inflation-targeting objective, the central bank has the authority to adjust the general level of interest rates through monetary policy that affects the decisions of the rentier class. For the sake of simplicity, we express this concept using a simplified behavioral equation:

$$\dot{\tau} = m\left(\frac{\dot{p}}{p} - \pi^o\right), \qquad m > 0 \quad (23)$$

Where m represents the reaction of the central bank to the gap between the actual inflation rate and its inflation target ('inflation gap'). Thus, substituting (20) into (26) results in a dynamic equation for the interest rate τ:

$$\dot{\tau} = m[h(\zeta - z) - \pi^o] \quad (24)$$

Equations (15), (19), (22), and (24) represent a four-dimensional dynamical system for the state variables l, z, f, and τ. This system is hereby referred to as Model A.

2.2.Equilibrium, stability, cycles, and relevant patterns

In the steady state $\dot{l} = \dot{z} = \dot{f} = \dot{\tau} = 0$, Model A has one non-trivial equilibrium point (l^*, z^*, f^*, τ^*) defined by:

$$l^* = \frac{\gamma + \theta + \pi_o(1 - \eta)}{\rho} \quad (25)$$

$$z^* = \frac{h\zeta - \pi^o}{h} \quad (26)$$

$$f^* = \frac{h\zeta - \pi^o}{\sigma(\theta + n)(h + h\zeta - \pi^o)} - 1 \quad (27)$$

$$\tau^* = \theta + n \quad (28)$$

This equilibrium point (l^*, z^*, f^*, τ^*) is positive when:

$$h > \frac{\pi^o}{\zeta}, \qquad \theta + n < \frac{h\zeta - \pi^o}{\sigma(h + h\zeta - \pi^o)} \quad (29)$$

Appendix 1 proves that, if γ, σ, and h are sufficiently high, and θ, n are sufficiently low, then the equilibrium point (l^*, z^*, f^*, τ^*) of Model A is locally asymptotically stable when:

$$m < m^{HB} = \frac{(\theta + n)H_4[H_5 - H_4(\theta + n)]}{\sigma H_1^2 H_2 H_3} \quad (30)$$

Where:

$$H_1 = h + h\zeta - \pi^o, \;\; H_2 = h\zeta - \pi^o - \sigma(\theta + n)(h + h\zeta - \pi^o), \;\; H_3 = \gamma + \theta + \pi^o(1 - \eta)$$

$$H_4 = \sigma H_1^2(1 - \eta) - H_2, \qquad H_5 = hH_3 - H_1H_2(1 - \eta)$$

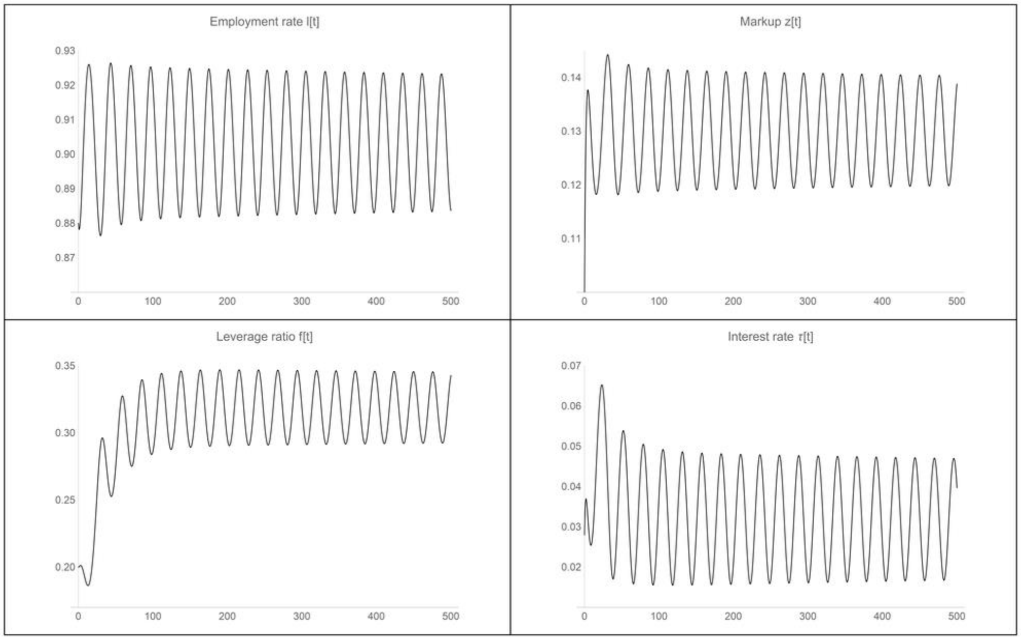
Additionally, Appendix 1 applies the Hopf bifurcation theorem for four-dimensional dynamical systems to prove that Model A exhibits limit cycles near its positive equilibrium point when m is close to the critical value m^{HB}, as defined in expression (30). Figure 1 illustrates the presence of these limit cycles when $m \approx m^{HB}$. These limit cycles can be interpreted as arising from a combination of distributive cycles, which involve clockwise oscillations in the phase plane of the employment rate l and the actual markup z,[7] and financial cycles, which involve clockwise oscillations in the phase plane of the leverage ratio f and the interest rate τ. Furthermore, Figure 2 indicates that these cycles increase their amplitude and display heightened volatility as the central bank reacts more strongly to its perceived 'inflation gap' ($m \approx m^{HB} + \varepsilon$), while Figure 3 portrays the existence of damped oscillations with decreasing volatility when the central bank exhibits a weaker reaction ($m \approx m^{HB} - \varepsilon$).[8] In other words, these findings suggest that when the central bank endeavors to manage inflation by adjusting the interest rate, it has the potential to exacerbate (reduce) the volatility of cycles when the magnitude of its reaction surpasses (goes below) the critical value m^{HB}.

[7] This observed pattern closely aligns with the distributive struggle elucidated by Goodwin (1967), albeit with the distinction that in this paper, we use the markup as a state variable instead of the wage share.

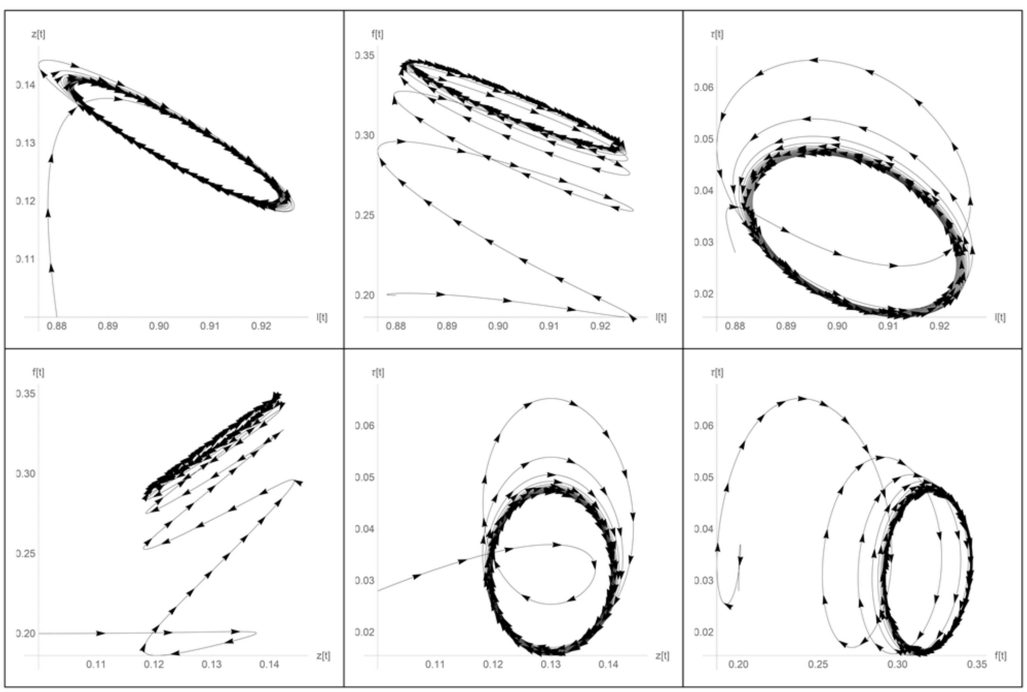
[8] Appendix 2 presents two-dimensional parametric plots that describe the effect of multiple values of m on simulated trajectories of Model A.

Figure 1. Simulation of Model A with limit cycles at critical value $(m \approx m^{HB})$

1A. Time series



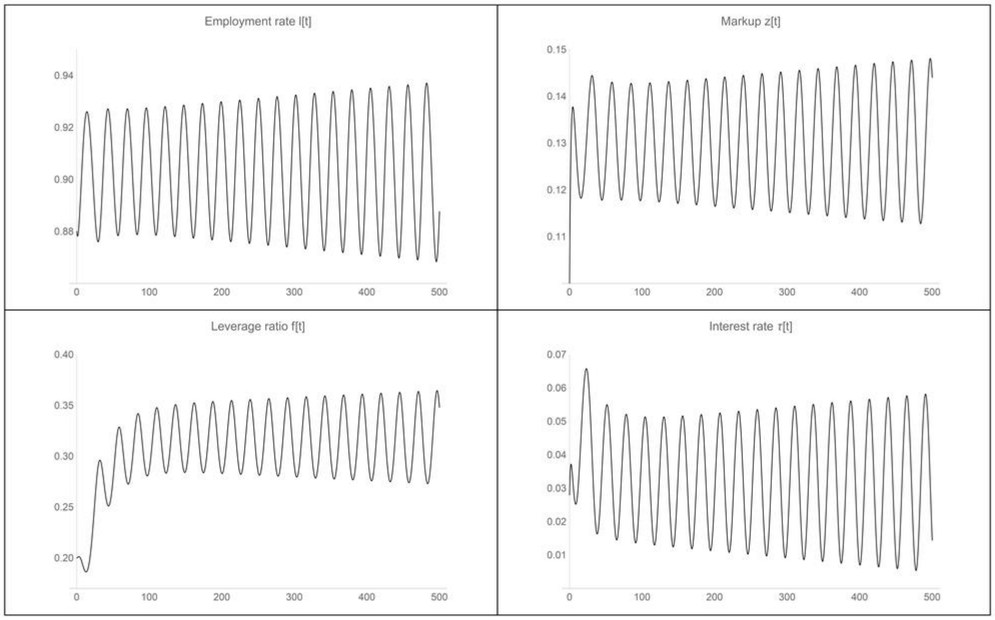
1B. Two-dimensional parametric plots



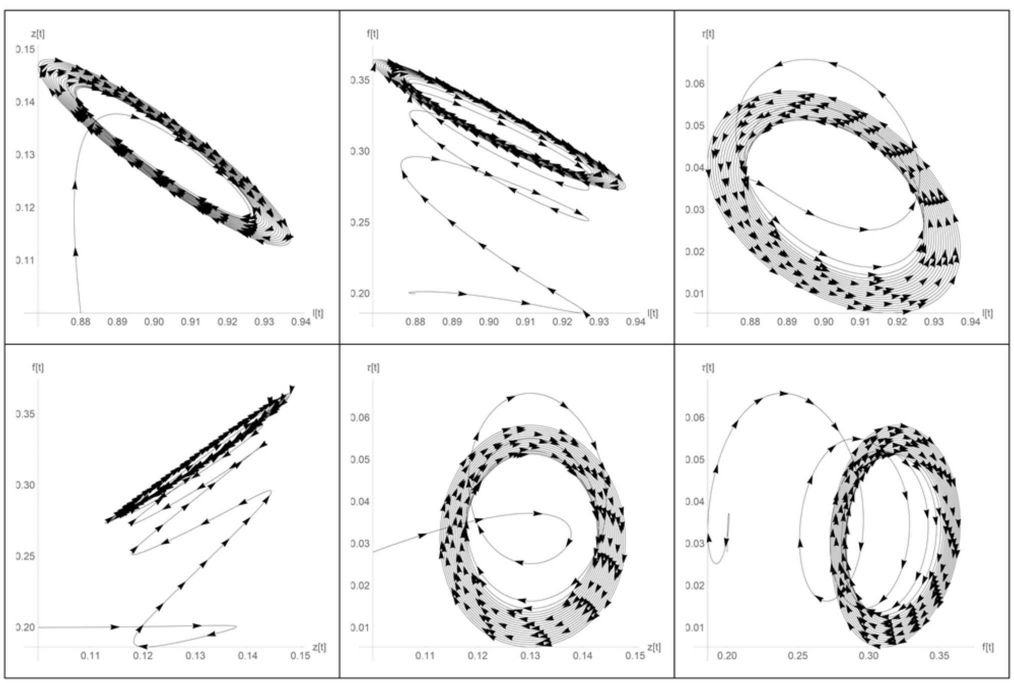
Note: Simulation of Model A with parameter values $h = 1, \eta = 0.5, \sigma = 2.725, \theta = 0.016, n = 0.016, \pi^o = 0.02, \zeta = 0.15, \gamma = 0.227, \rho = 0.28, m = m^{HB} \approx 0.35634$ and initial conditions $l_0 = 0.88, z_0 = 0.1, f_0 = 0.2, \tau_0 = 0.028$. Equilibrium point: $l^* = 0.9035, z^* = 0.13, f^* = 0.31931, \tau^* = 0.032$.

Figure 2. Simulation of Model A with unstable cycles above the critical value ($m \approx m^{HB} + \varepsilon$)

2A. Time series



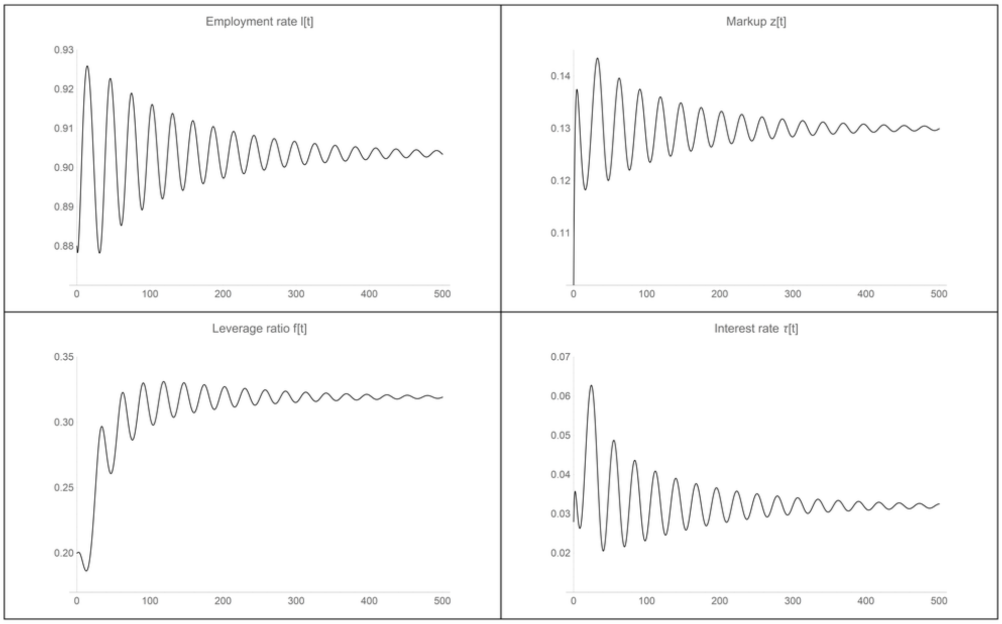
2B. Two-dimensional parametric plots



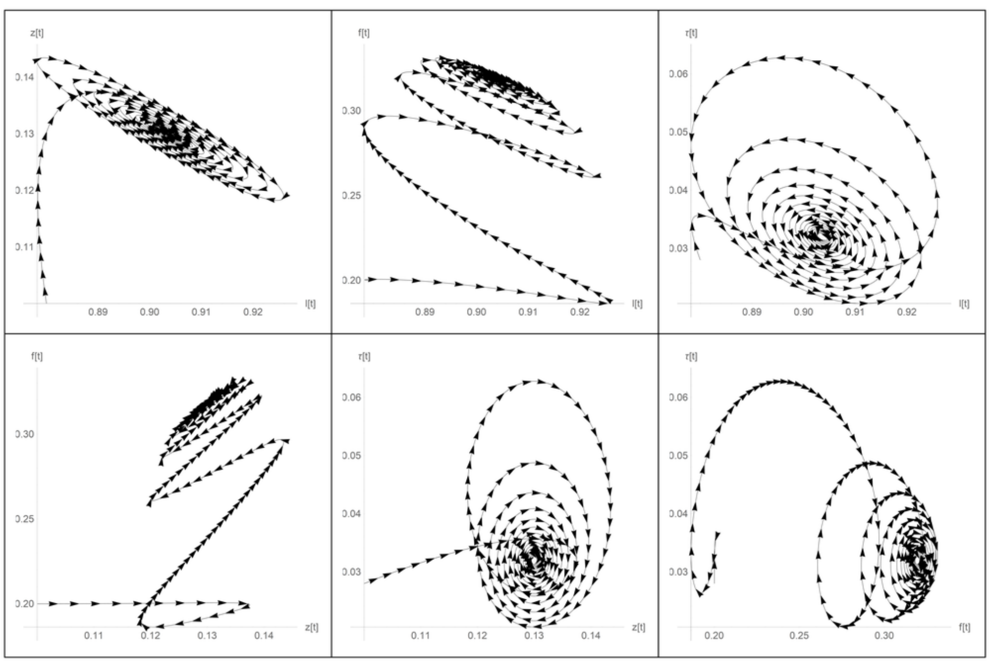
Note: Simulation of Model A with parameter values $h = 1, \eta = 0.5, \sigma = 2.725, \theta = 0.016, n = 0.016, \pi^o = 0.02, \zeta = 0.15, \gamma = 0.227, \rho = 0.28, m = m^{HB} + \varepsilon \approx 0.35634 + 0.01$ and initial conditions $l_0 = 0.88, z_0 = 0.1, f_0 = 0.2, \tau_0 = 0.028$. Equilibrium point: $l^* = 0.9035, z^* = 0.13, f^* = 0.31931, \tau^* = 0.032$.

Figure 3. Simulation of Model A with damped oscillations below the critical value ($m \approx m^{HB} - \varepsilon$)

3A. Time series



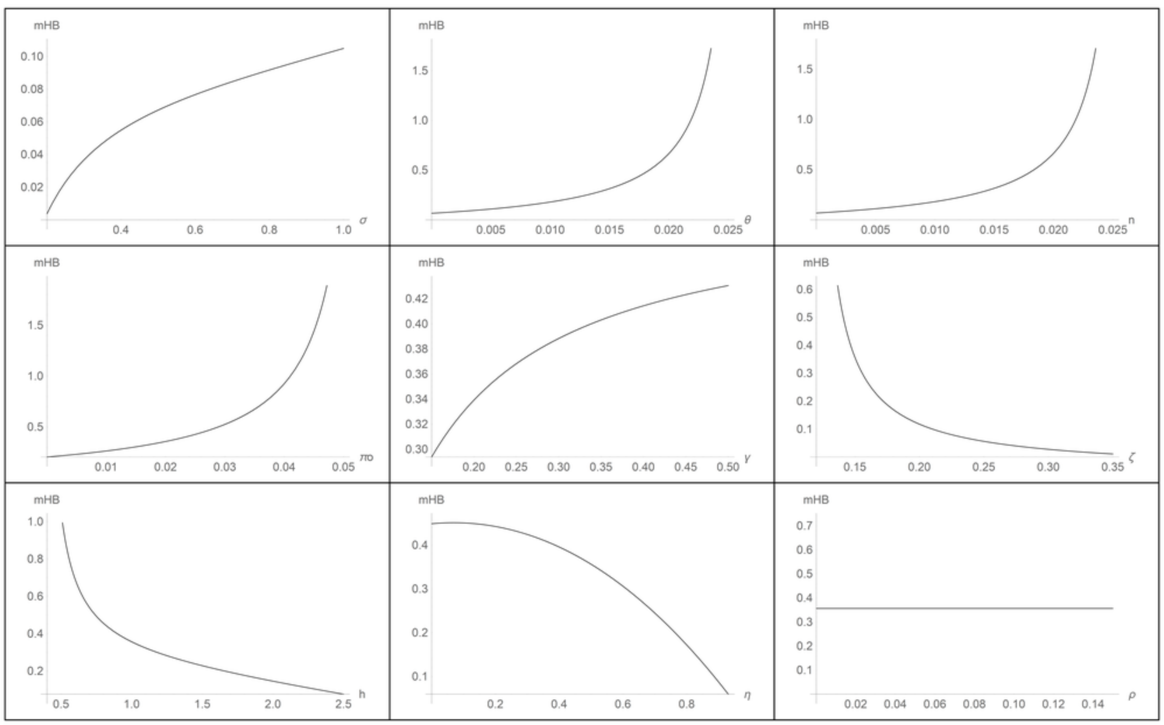
3B. Two-dimensional parametric plots



Note: Simulation of Model A with parameter values $h = 1, \eta = 0.5, \sigma = 2.725, \theta = 0.016, n = 0.016, \pi^o = 0.02, \zeta = 0.15, \gamma = 0.227, \rho = 0.28, m = m^{HB} - \varepsilon \approx 0.35634 - 0.05$ and initial conditions $l_0 = 0.88, z_0 = 0.1, f_0 = 0.2, \tau_0 = 0.028$. Equilibrium point: $l^* = 0.9035, z^* = 0.13, f^* = 0.31931, \tau^* = 0.032$.

Based on the definition of m^{HB} in equation (30), we can estimate how different parameters impact the range of values within which the central bank can effectively address its perceived 'inflation gap' without exacerbating the volatility of cycles. In this sense, Figure 4 presents numerical estimations of the critical value m^{HB} as a function of each parameter of Model A, while holding other factors constant. The figure illustrates that the ability of the central bank to control inflation is less restricted (m^{HB} increases) when there is a higher capital-output ratio (σ), higher growth rates of labor productivity (θ) and supply (n), a higher target inflation rate for the central bank (π^o), and a higher autonomous tendency of the money wage to fall (γ). Conversely, the ability of the central bank to control inflation is more limited (m^{HB} falls) when firms increase their desired markup (ζ) or their reaction to the perceived 'markup gap' (h), or when inflation has a greater effect on the money wage (η). Additionally, the effect of the employment rate on the money wage (ρ) does not affect m^{HB}.

Figure 4. Relationship between critical value m^{HB} and Model A parameters



Note: Each plot shows the estimated relationship between the critical value m^{HB} and a specific parameter of Model A, while keeping all other parameter values fixed at those listed in the caption of Figure 1. The selected parameter is varied to ensure that m^{HB} remains positive.

These results provide important insights into the relationship between the distribution of bargaining power between capitalists and workers and the ability of the central bank to control inflation by adjusting the interest rate. Here, we can remember that an increase in γ or a decrease in η implies a situation where capitalists have a relatively stronger bargaining power compared to workers, enabling them to slow down the wage rate and intensify labor exploitation. Under such circumstances, the central bank can more easily combat inflation without exacerbating distributive and financial cycles, as reflected in a higher value of m^{HB} as shown in Figure 4. This finding suggests a possible convergence between the objectives of the capitalist class and the central bank, both aimed at reducing the bargaining power of the working class to maintain stability in capitalist cycles. However, the results presented in Figure 4 also highlight potential contradictions between the capitalist objective of achieving a target markup and the goal of the central bank of adjusting the interest rate to control inflation, as the critical value m^{HB} falls when ζ or h increases.

In summary, the analysis of Model A indicates that the stability of distributive and financial cycles is closely linked to the balance of power between capitalists and workers. The interest of the central bank in achieving economic stability appears to align with the broader objective of the capitalist class to sustain and increase labor exploitation by reducing the bargaining power of workers. Nevertheless, this convergence is complex and not always consistent due to the inflationary pressures generated by the capitalist pursuit of profitability, which can conflict with the stabilization objective of the central bank.

After analyzing how interactions between workers, capitalists, and the central bank may influence distributive and financial cycles, we can now consider the potential impact of the rentier class. Similar to Keen (1995) and Charles (2008), we assume that an increasing leverage ratio could incentivize the rentier class to raise the interest rate. Building upon this observation, we propose a modification to the dynamical equation of the interest rate that takes into account both the influence of the central bank and the influence of the rentier class. To keep the equation simple, we use the following form:

$$\dot{\tau} = m\left(\frac{\dot{p}}{p} - \pi^o\right) + \lambda\left(\frac{\dot{f}}{f}\right), \qquad m > 0, \lambda > 0 \quad (31)$$

Where λ represents the reaction of the rentier class to an increasing leverage ratio.[9] Substituting (22) into (31) gives a new simplified dynamic equation for the interest rate:

$$\dot{\tau} = m[h(\zeta - z) - \pi^o] + \lambda\left[\tau(f+1) - \left(\frac{1}{\sigma}\right)\left(\frac{z}{1+z}\right)\right] \quad (32)$$

Thus, we propose a modified Model A, defined by equations (15), (19), (22), and (32). Although this modified model has the same equilibrium point as its original version, it presents an additional layer of complexity in the dynamical system. For instance, in equation (32) the term λ introduces the possibility of unstable dynamics in the interest rate τ when the leverage rate f is high enough to satisfy the following condition:

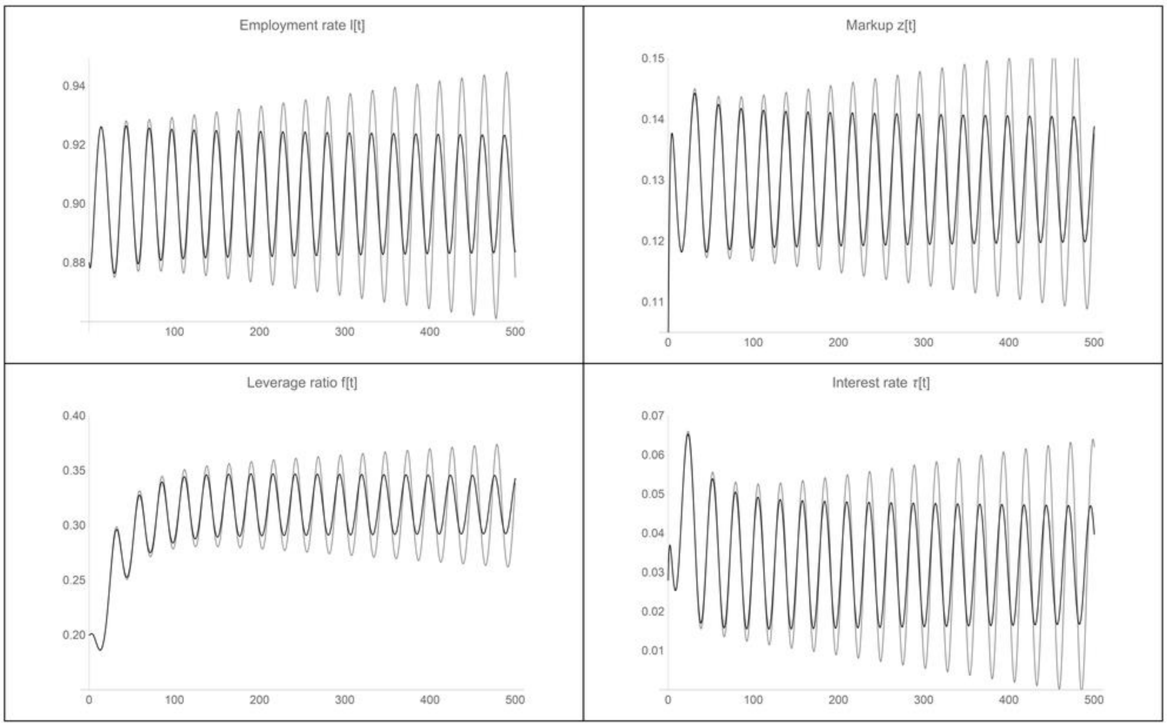
$$f > \left(\frac{1}{\sigma\tau}\right)\left(\frac{z}{1+z}\right) - 1 \quad (33)$$

Indeed, numerical simulations of the modified Model A reveal that, as the reaction of the rentier class to an increasing leverage ratio emerges and becomes more prominent (higher λ),

[9] In their original formulations, Keen (1995) and Charles (2008) assume that the interest rate is a linear function of either the debt-output ratio or the debt-capital ratio, respectively.

the limit cycles previously identified in the original version of the model when $m \approx m^{HB}$ exhibit increased amplitude and volatility, as illustrated in Figure 5.[10] The results suggest that the objective of the rentier class to gain a higher interest rate when the leverage ratio increases tends to exacerbate the volatility of cycles, thereby creating a potential contradiction with the stabilizing objective of the central bank.

Figure 5. Simulations of modified Model A for $m \approx m^{HB}$ and increasing λ



Black: $\lambda = 0$. Gray: $\lambda = 0.003$. Note: Simulation of modified Model A with parameter values $h = 1, \eta = 0.5, \sigma = 2.725, \theta = 0.016, n = 0.016, \pi^o = 0.02, \zeta = 0.15, \gamma = 0.227, \rho = 0.28, m = m^{HB} \approx 0.35634$ and initial conditions $l_0 = 0.88, z_0 = 0.1, f_0 = 0.2, \tau_0 = 0.028$. Equilibrium point: $l^* = 0.9035, z^* = 0.13, f^* = 0.31931, \tau^* = 0.032$. For a detailed illustration of the effect of an increasing λ on distributive and financial cycles, see Appendix 3.

[10] Appendix 3 presents two-dimensional parametric plots that illustrate the impact of varying values of λ on simulated trajectories of the modified model A.

3. Model B: Cycles with Excess Demand

3.1. From Model A to Model B: Investment and capacity utilization

In Model A, we have assumed that investment equals savings and, therefore, firms require new borrowing only to sustain their cash flows during their economic activity due to a time delay between the moment capitalists obtain their profits and save them, and the moment they distribute them to firms for reinvestment. However, it is also possible that firms use new borrowing to finance and expand their investment activities beyond the limits of capitalist savings. As indicated by Sordi and Vercelli (2014), in such a situation, the goods market will present a disequilibrium state, resulting in a non-zero excess demand. In this section, we present a new model, named Model B, which modifies Model A to include the possibility of financing investment through debt and the existence of a non-zero excess demand. We present Model B in a similar order as we did with Model A. Thus, we retain the assumption that firms use markup pricing, where the price level p is still defined by equation (1), while labor productivity q is still given by equation (2). Following Dutt (1988, 1990), we replace equation (3) by assuming that firms define their growth rate of fixed capital g through the following investment function:

$$g = \alpha + \beta r + \chi u, \qquad 0 < \alpha < 1, \qquad 0 \leq \beta < 1 \quad (34)$$

In this investment function, α is the autonomous component of investment (animal spirit effect), β reflects the capitalist motivation to invest in response to a given net profit rate r (profitability effect), and χ represents the capitalist motivation to invest in response to a given rate of capacity utilization u (demand effect). Building upon Keen (1995) and Grasselli and Costa Lima (2012), we assume that capitalist firms finance a fraction of their investment through debt beyond the limits of savings, thereby modifying the previous assumption of savings and investment equality in Model A. Consequently, the growth rate of capital defined by equation (34) does not need to ensure the balance between S and I.

Concerning the aggregate net profit Π and the net profit rate r, they are still defined by equations (4) and (5), respectively. In contrast, following Dutt (1990), to represent the economic effect of the excess demand associated with the discrepancy between savings and investment, we replace equation (6) by defining the capacity utilization rate u through the following ratio:

$$u = \frac{Q}{K} \quad (35)$$

Where u will become a new state variable that will adjust when the goods market is not in equilibrium. Regarding the leverage ratio f and the real excess demand ED, they are defined by equations (7) and (8), respectively. Additionally, the real wage rate w/p remains the same and still verifies equation (9). To obtain the equation for the profit rate, we substitute equations (2), (4), (7), (9), and (35) into (5), which yields:

$$r = \frac{uz}{1+z} - f\tau \quad (36)$$

This expression closely resembles equation (10), but replaces the inverse of the capital-output ratio σ with the capacity utilization rate u. Similarly, by substituting u for the inverse of σ in equation (11), we obtain the following expression for the employment rate l:

$$l = \frac{L}{N} = \left(\frac{u}{q}\right)\left(\frac{K}{N}\right) \quad (37)$$

Taking logarithms and time derivatives of (37) gives:

$$\frac{\dot{l}}{l} = \frac{\dot{u}}{u} + g - \frac{\dot{q}}{q} - \frac{\dot{N}}{N} \quad (38)$$

In equation (38), we observe that the dynamics of the growth rate of l must take into account the growth rate of u, as u is a new state variable that will vary over time depending on the dynamics of the goods market. Following Dávila-Fernández and Sordi (2019), we assume that capacity utilization has a positive effect on the growth rate of productivity due to learning-by-doing processes that generate economies of scale in the usage of fixed capital and multiple spillover effects. These processes are induced by the economic expansion represented by a higher capacity utilization rate, in line with the Kaldor-Verdoorn effect (Verdoorn 1949, Kaldor 1966). This pattern is captured through the following expression:

$$\frac{\dot{q}}{q} = \theta_0 + \theta_1 u, \qquad 0 < \theta_0 < 1, \qquad 0 < \theta_1 < 1 \quad (39)$$

Where θ_0 accounts for an autonomous tendency of productivity to grow and θ_1 is the influence of the rate of capacity utilization on the acceleration of productivity. Concerning the growth rate of labor supply, we keep the assumption that it is equal to an exogenous constant, as specified in equation (14).[11] Using equations (14), (34), (36), (38), and (39), we derive a new version of the dynamic equation for the employment rate l, which now includes the effect of capacity utilization:

$$\frac{\dot{l}}{l} = \alpha - (\theta_0 + n) + u\left(\frac{z\beta}{1+z} + \chi - \theta_1\right) - \beta f \tau + \frac{\dot{u}}{u} \quad (40)$$

We retain equations (16), (17), and (18) from Model A to describe the behavior of the growth rates of the money wage w, the price level p, and their impact on the markup z. By substituting equations (16), (17), and (39) into (18), we obtain a new dynamic equation for the markup z:

$$\frac{\dot{z}}{1+z} = \gamma + \theta_0 + \theta_1 u + h(1-\eta)(\zeta - z) - \rho l \quad (41)$$

As mentioned before, in contrast to Model A, in Model B we do not assume that investment equals savings. Instead, we introduce the capitalist motivations described in the investment function defined by equation (34). Nevertheless, we retain the cash flow identity defined in

[11] An alternative approach to modeling labor supply is to endogenize it based on the growth rate of capital, as suggested by Harris (1983).

equation (20) to derive debt dynamics. By combining this cash flow identity with expressions (2), (5), (7), (21), (34), and (36), we derive a new dynamic equation for the leverage ratio f:

$$\frac{\dot{f}}{f} = \tau[2 - \beta(1 - f)] - \frac{uz - (1 - f)\{u[\chi + (\beta + \chi)z] + \alpha(1 + z)\}}{f(1 + z)} \quad (42)$$

In terms of the interest rate, we adopt the dynamic equation (31), which takes into account both the response of the central bank to its perceived "inflation gap" and the response of the rentier class to changes in the leverage ratio. By substituting equations (17) and (42) into (31), we obtain a revised dynamic equation for the interest rate τ:

$$\dot{\tau} = m[h(\zeta - z) - \pi^o] + \lambda\left\{\tau[2 - \beta(1 - f)] - \frac{uz - (1 - f)\{u[\chi + (\beta + \chi)z] + \alpha(1 + z)\}}{f(1 + z)}\right\} \quad (43)$$

The inequality between investment and savings implies the existence of excess demand in the goods market. As highlighted by Sordi and Vercelli (2014), it is crucial to take into account the dynamic effect of this excess demand on economic activity. Similar to Dutt (1988), we assume that the capacity utilization rate u increases (decreases) in response to a positive (negative) excess demand. We operationalize this assumption through the following behavioral equation:

$$\frac{\dot{u}}{u} = \phi\left(\frac{ED}{K}\right), \qquad \phi > 0 \quad (44)$$

Where ϕ represents the speed at which the growth rate of capacity utilization adjusts to excess demand, measured as a proportion of the stock of capital for mathematical convenience. Substituting expressions (1), (2), (7), (8), (34), (35), and (36) into (44) results in a dynamic equation for the capacity utilization rate u:

$$\frac{\dot{u}}{u} = \phi\left\{\alpha + u\left[\chi - \frac{z(1 - \beta)}{1 + z}\right] + (1 - \beta)f\tau\right\} \quad (45)$$

Finally, by substituting (45) into (40) we can rewrite the dynamic equation for l as:

$$\frac{\dot{l}}{l} = \alpha(1 + \phi) - (\theta_0 + n) + u\left\{\chi(1 + \phi) - \theta_1 - \frac{z[\phi(1 - \beta) - \beta]}{1 + z}\right\} + [\phi(1 - \beta) - \beta]f\tau \quad (46)$$

Equations (41), (42), (43), (45), and (46) form a five-dimensional dynamical system that describes the state variables l, z, f, τ, and u. We refer to this system as Model B, which distinguishes it from Model A due to its ability to allow for investment to be partially financed through debt, by accounting for the inequality between investment and savings. Additionally, Model B takes into account the adjustment of capacity utilization as a response to excess demand in the goods market.

3.2.Equilibrium, cycles, and relevant patterns

In the steady state $\dot{l} = \dot{z} = \dot{f} = \dot{\tau} = \dot{u} = 0$, Model B has one non-trivial equilibrium point given by:

$$l^* = \frac{\theta_1\{\alpha - (1-\beta)[n - \gamma - \pi^o(1-\eta)]\} - [\gamma + \theta_0 + \pi^o(1-\eta)]\chi}{\rho[(1-\beta)\theta_1 - \chi]} \quad (47)$$

$$z^* = \frac{h\zeta - \pi^o}{h} \quad (48)$$

$$f = \frac{(h\zeta - \pi^o)[\alpha - (n+\theta_0)(1-\beta)]}{(h + h\zeta - \pi^o)[\alpha\theta_1 - (n+\theta_0)\chi]} - 1 \quad (49)$$

$$\tau^* = \frac{\alpha\theta_1 - (n+\theta_0)\chi}{(1-\beta)\theta_1 - \chi} \quad (50)$$

$$u^* = \frac{\alpha - (n+\theta_0)(1-\beta)}{(1-\beta)\theta_1 - \chi} \quad (51)$$

This equilibrium point $(l^*, z^*, f^*, \tau^*, u^*)$ has positive values when:

$$\theta_1 < \frac{(h\zeta - \pi^o)[\alpha - (n+\theta_0)(1-\beta)] + (n+\theta_0)(h + h\zeta - \pi^o)\chi}{\alpha(h + h\zeta - \pi^o)} \quad (52)$$

$$\chi < \theta_1(1-\beta) \quad (53)$$

$$\alpha > \frac{(n+\theta_0)\chi}{\theta_1} \quad (54)$$

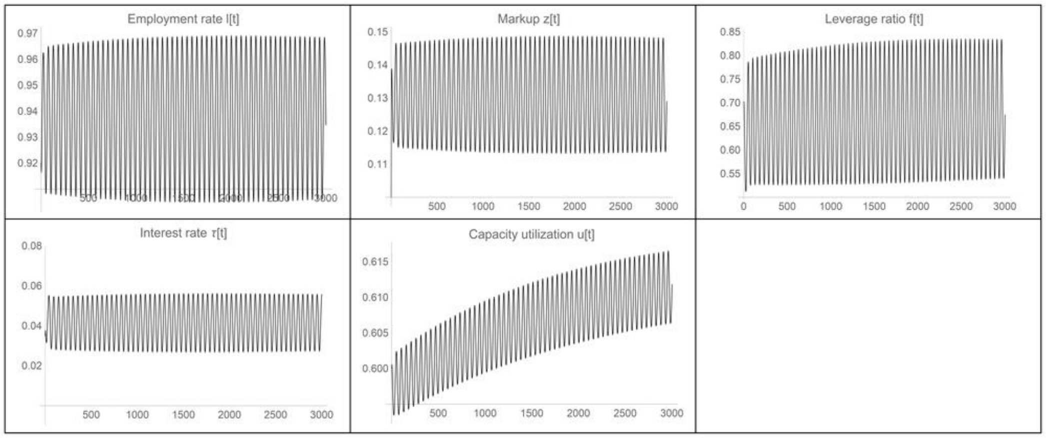
$$\alpha > (1-\beta)(n+\theta_0) \quad (55)$$

$$\alpha > (1-\beta)[n - \gamma + \pi^o(1-\eta)] + \frac{[\gamma + \theta_0 + \pi^o(1-\eta)]\chi}{\theta_1} \quad (56)$$

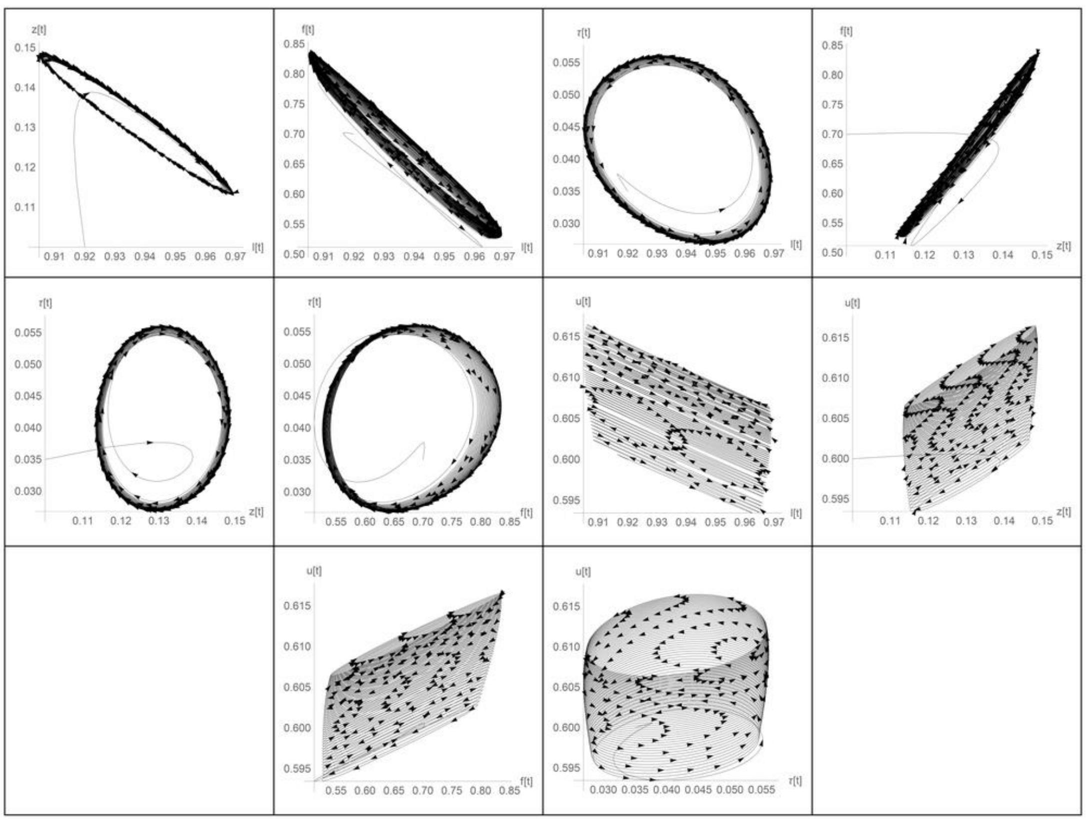
Appendix 4 proves the local stability of the non-trivial equilibrium point and applies the Hopf bifurcation theorem for five-dimensional dynamical systems to prove the existence of limit cycles for a simplified version of Model B. Based on these preliminary results, we suggest that the complete Model B tends to generate stable cycles when the value of m is close to a potential critical value m^*. Figure 6 illustrates this tendency, while Figure 7 reveals the existence of unstable cycles for larger values of m, and Figure 8 shows damped oscillations for lower values of m. These findings suggest that Model B can produce similar qualitative outcomes to Model A, indicating that a stronger adjustment in the interest rate by the central bank in response to its perceived ‘inflation gap’ could exacerbate the volatility of distributive and financial cycles.

Figure 6. Simulation of Model B with stable cycles at a potential critical value ($m = m^*$)

6A. Time series



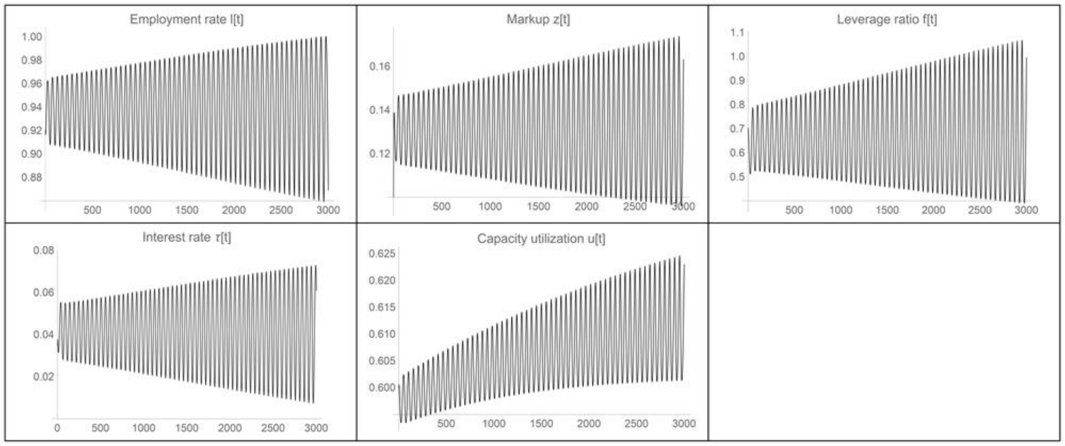
6B. Two-dimensional parametric plots



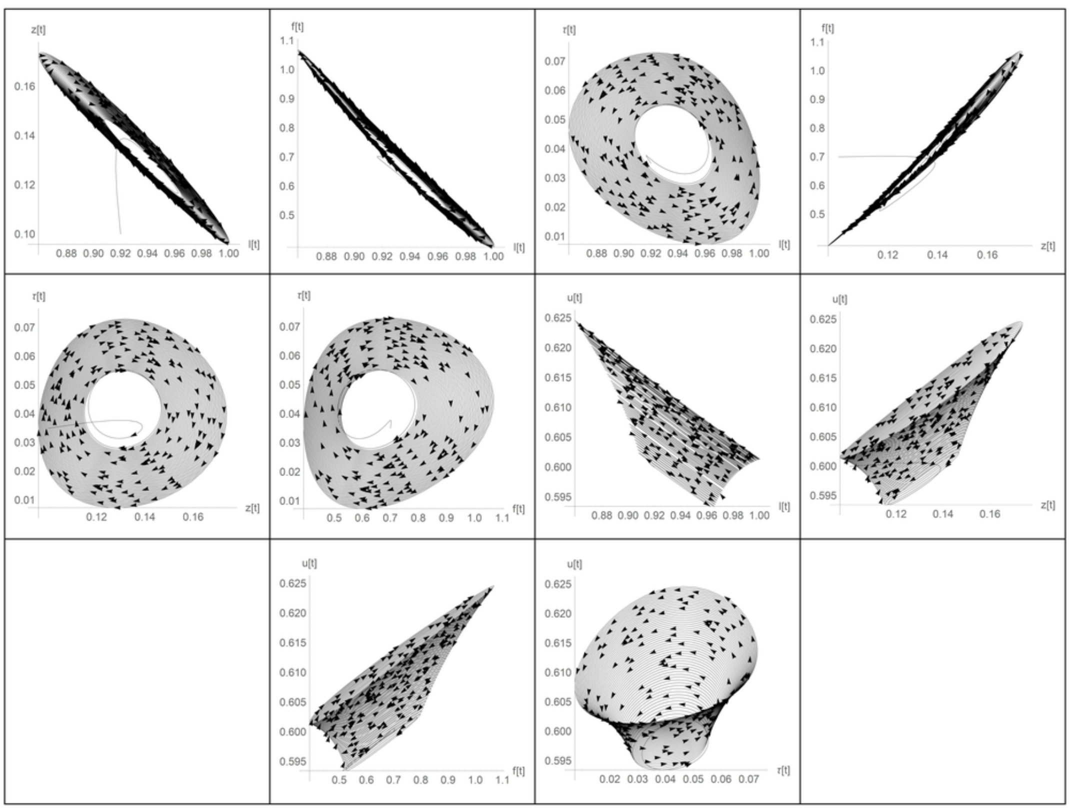
Note: Simulation of Model B with parameter values $h = 1, \eta = 0.5, \theta_0 = 0.016, \theta_1 = 0.01625, n = 0.016, \pi o = 0.02, \zeta = 0.15, \gamma = 0.227, \rho = 0.28, m = m^* = 0.1, \alpha = 0.017, \beta = 0.5, \chi = 0.0065, \lambda = 0.003, \phi = 0.2$ and initial conditions $l_0 = 0.92, z_0 = 0.1, f_0 = 0.7, \tau_0 = 0.035, u_0 = 0.6$. Equilibrium point: $l^* = 0.9392, z^* = 0.13, f^* = 0.6856, \tau^* = 0.042, u^* = 0.6153$.

Figure 7. Simulation of Model B with unstable cycles above a potential critical value $(m = m^* + \varepsilon)$

7A. Time series



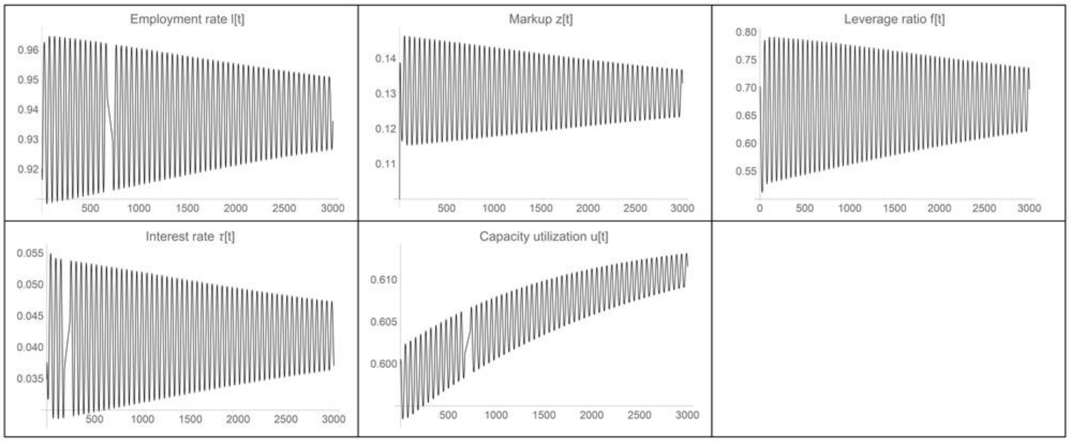
7B. Two-dimensional parametric plots



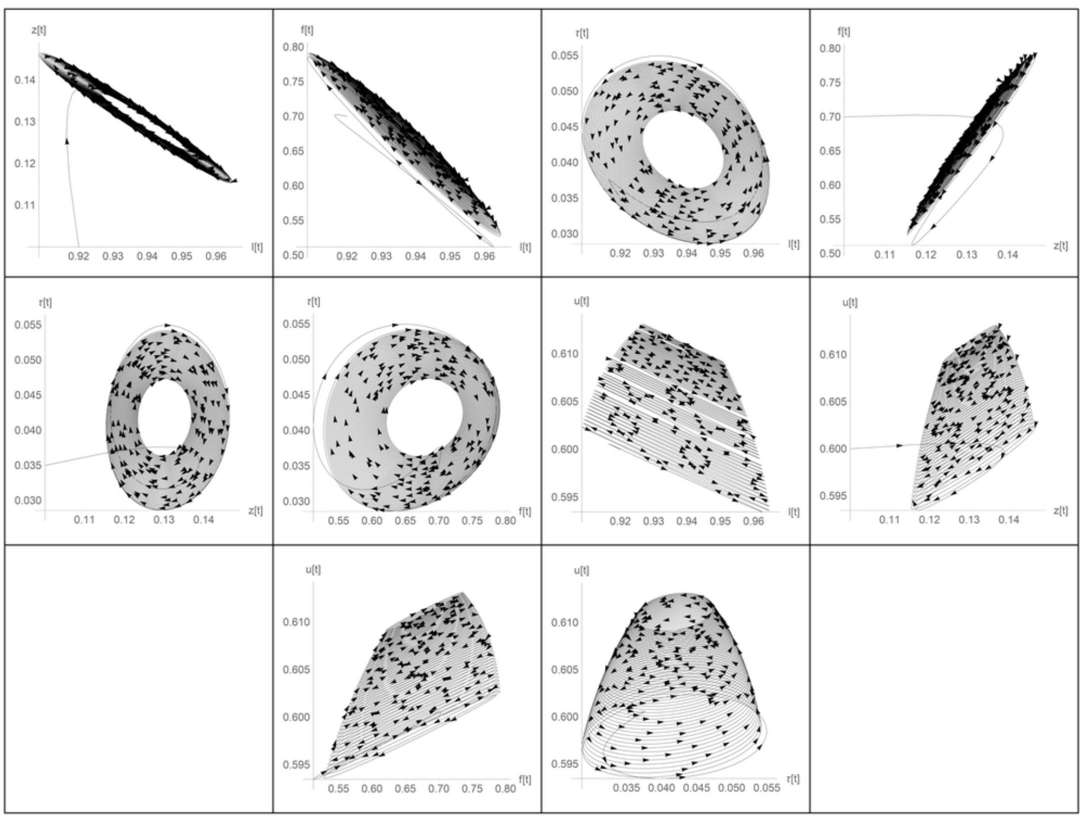
Note: Simulation of Model B with parameter values $h = 1, \eta = 0.5, \theta_0 = 0.016, \theta_1 = 0.01625, n = 0.016, \pi o = 0.02, \zeta = 0.15, \gamma = 0.227, \rho = 0.28, m = m^* + \varepsilon = 0.1 + 0.0025, \alpha = 0.017, \beta = 0.5, \chi = 0.0065, \lambda = 0.003, \phi = 0.2$ and initial conditions $l_0 = 0.92, z_0 = 0.1, f_0 = 0.7, \tau_0 = 0.035, u_0 = 0.6$. Equilibrium point: $l^* = 0.9392, z^* = 0.13, f^* = 0.6856, \tau^* = 0.042, u^* = 0.6153$.

Figure 8. Simulation of Model B with damped oscillations below a potential critical value $(m = m^{*} - \varepsilon)$

8A. Time series



8B. Two-dimensional parametric plots

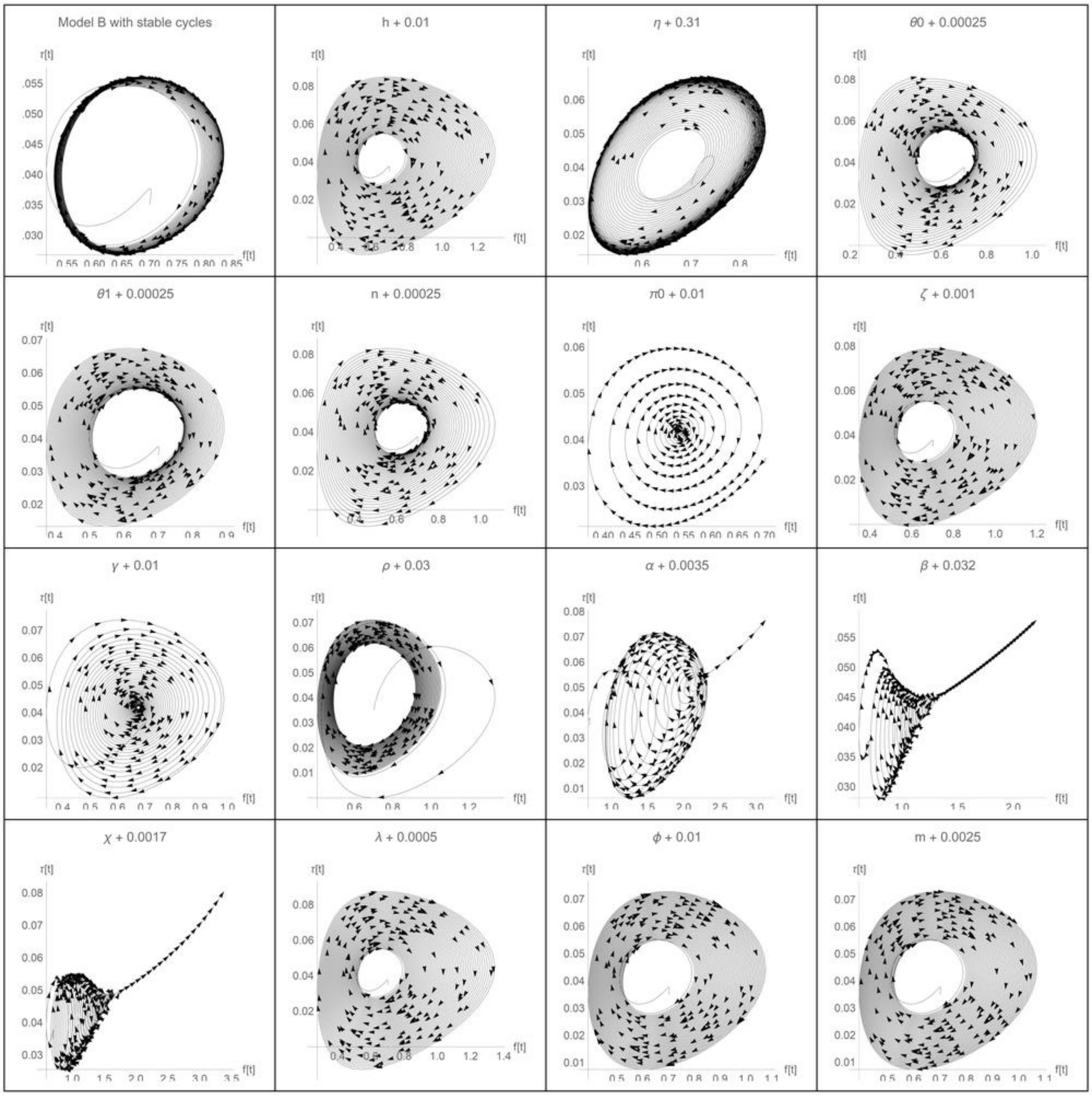


Note: Simulation of Model B with parameter values $h = 1, \eta = 0.5, \theta_0 = 0.016, \theta_1 = 0.01625, n = 0.016, \pi o = 0.02, \zeta = 0.15, \gamma = 0.227, \rho = 0.28, m = m^{*} - \varepsilon = 0.1 - 0.0025, \alpha = 0.017, \beta = 0.5, \chi = 0.0065, \lambda = 0.003, \phi = 0.2$ and initial conditions $l_0 = 0.92, z_0 = 0.1, f_0 = 0.7, \tau_0 = 0.035, u_0 = 0.6$. Equilibrium point: $l^{*} = 0.9392, z^{*} = 0.13, f^{*} = 0.6856, \tau^{*} = 0.042, u^{*} = 0.6153$.

Regarding the effects of parameter changes in the dynamics of Model B, numerical simulations presented in Figure 9 suggest that cycles become less volatile when the central bank has a higher target inflation rate (π^o), when capitalists have stronger incentives to invest (represented by higher parameters α, β, χ), and when there is a higher autonomous tendency of the money wage to fall (γ). Conversely, cycles become more volatile when firms increase their desired markup (ζ) or their reaction to the perceived 'markup gap' (h), when there is a higher growth rate of labor supply (n), when labor productivity has a higher exogenous tendency to grow (θ_0) or a higher tendency to grow for a given capacity utilization rate (θ_1), when there is a stronger effect of the employment rate on the money wage (ρ), when the rentier class has a more significant reaction to a higher leverage ratio (λ), when the capacity utilization rate is more sensitive to excess demand (ϕ), and when the central bank implements a stronger adjustment in response to its perceived 'inflation gap' (m).

In the particular case of the parameters representing the multiple incentives of capitalist firms to invest (α, β, χ), it is worth noting that the simulations presented in Figure 9 suggest that these parameters can reduce the volatility of cycles, but they also may lead to an increasing leverage ration and interest rate over the long run. This pattern may indicate a tendency toward a 'debt crisis' when investment is financed through debt and is too high, leading to a never-ending pattern of new borrowing to sustain capital accumulation. However, this interpretation is preliminary and requires a more in-depth analytical investigation of Model B and its sensitivity to different specifications of the investment function. Concerning the parameters representing the distribution of bargaining power between workers and capitalists (γ, ρ, η), our findings seem to reinforce the intuition identified in Model A: a weakened working class appears to be beneficial for the general stability of distributive and financial cycles. But, again, a more comprehensive analytical discussion of Model B is needed to generalize these results.

Figure 9. Simulation of financial cycles with changes in selected parameters (Model B)



Note: The first plot depicts the simulation of Model B with parameter values $h = 1, \eta = 0.5, \theta_0 = 0.016, \theta_1 = 0.01625, n = 0.016, \pi o = 0.02, \zeta = 0.15, \gamma = 0.227, \rho = 0.28, m = m^* = 0.1, \alpha = 0.017, \beta = 0.5, \chi = 0.0065, \lambda = 0.003, \phi = 0.2$ and initial conditions $l_0 = 0.92, z_0 = 0.1, f_0 = 0.7, \tau_0 = 0.035, u_0 = 0.6$. The subsequent plots use these values as a baseline while varying the parameter indicated in their respective titles.

3.3.Identifying financing regimes within cycles: A preliminary proposal

Following Meireles and Lima (2006), in Model B we can identify three distinct financing regimes, based on the taxonomy proposed by Minsky (1982) and the formalization presented by Foley (2003). These are the hedge financing regime when $\dot{D} \leq 0$, the speculative regime when $0 < \dot{D} < gK$, and the Ponzi regime when $\dot{D} \geq gK$. To analyze these regimes, we can define two auxiliar variables:

$$\psi = \frac{\dot{D}}{D} \quad (57)$$

$$\omega = \frac{g}{f} = \frac{\dot{K}}{D} \quad (58)$$

Thus, the financing regimes mentioned by Meireles and Lima (2006) can be rewritten as:

$$Hedge\text{:} \quad \psi \leq 0$$

$$Speculative\text{:} \quad 0 < \psi < \omega$$

$$Ponzi\text{:} \quad \psi \geq \omega$$

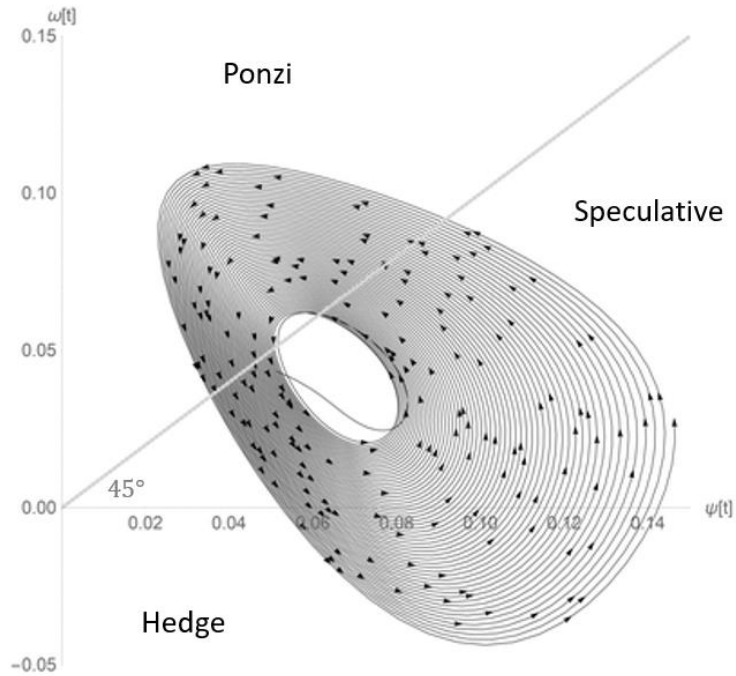
By substituting equations (2), (5), (7), (20), (34), and (36) into (57) and (58), we get:

$$\psi = \frac{(1+z)[\alpha + (2-\beta)f\tau] - u[(1-\beta-\chi)z - \chi]}{f(1+z)} \quad (59)$$

$$\omega = \frac{u[\chi + (\beta+\chi)z] + (1+z)(\alpha - \beta f\tau)}{f(1+z)} \quad (60)$$

Equations (59) and (60) can be used in combination with simulated values of the state variables z, f, τ, and u to identify financing regimes within economic cycles. To this end, we propose analyzing the counterclockwise trajectories of Model B that are generated within the plane formed by the auxiliary variables ψ (growth rate of the stock of debt) and ω (investment expressed as a proportion of the stock of debt). Figure 10 provides an example of this identification process using the simulation of unstable cycles. Specifically, we identify a hedge financing regime when the trajectories fall below the horizontal axis ($\psi < 0$), a speculative regime when the trajectory lies above the horizontal axis but below the 45-degree line ($0 < \psi < \omega$), and a Ponzi regime when the trajectory is above the 45-degree line ($\psi \geq \omega$). The analytical exploration of the relationships between each financing regime and the other cycles identified in Model B is left for future research.

Figure 10. Identification of financing regimes within simulated cycles (Model B)



Note: Simulation of Model B with parameter values $h = 1, \eta = 0.5, \theta_0 = 0.016, \theta_1 = 0.01625, n = 0.016, \pi o = 0.02, \zeta = 0.15, \gamma = 0.227, \rho = 0.28, m = m^* = 0.1, \alpha = 0.017, \beta = 0.5, \chi = 0.0065, \lambda = 0.0035, \phi = 0.2$ and initial conditions $l_0 = 0.92, z_0 = 0.1, f_0 = 0.7, \tau_0 = 0.035, u_0 = 0.6$.

Finally, an alternative approach to identifying financing regimes can be suggested by analyzing the dynamics of the interest rate. To elaborate, by substituting (59) and (60) into the respective regime definitions and solving for the interest rate, we obtain:

$$Hedge: \quad \tau \leq \tau^H$$

$$Speculative: \quad \tau^H < \tau < \tau^P$$

$$Ponzi: \quad \tau \geq \tau^P$$

Where the bounds τ^H and τ^P that distinguish each financing regime are given by:

$$\tau^H = \frac{u[(1-\beta-\chi)z-\chi]-\alpha(1+z)}{f(2-\beta)(1+z)} \quad (61)$$

$$\tau^P = \frac{uz}{2f(1+z)} \quad (62)$$

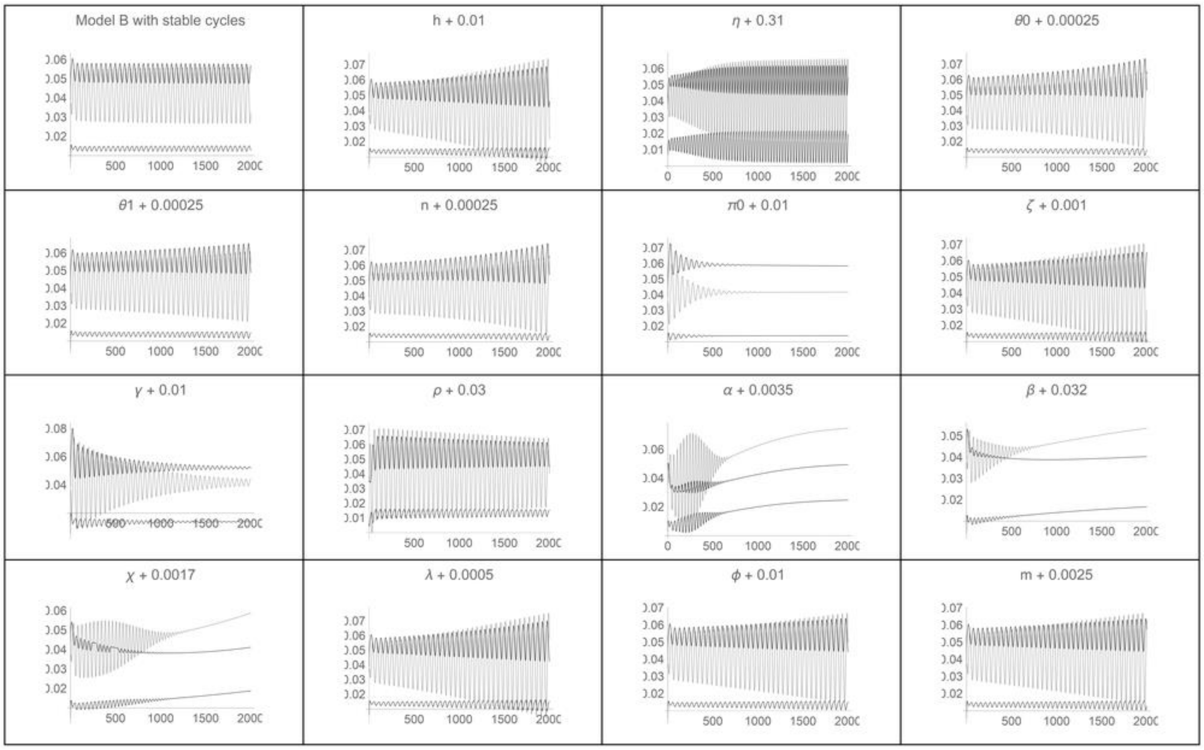
In the steady state $\dot{l} = \dot{z} = \dot{f} = \dot{\tau} = \dot{u} = 0$, these bounds are equal to:

$$\tau^{H*} = \frac{(1-\beta)[\alpha\theta_1-(n+\theta_0)\chi]}{(2-\beta)[(1-\beta)\theta_1-\chi]} \quad (63)$$

$$\tau^{P*} = \frac{(h\zeta - \pi^o)[\alpha\theta_1 - (n+\theta_0)\chi][\alpha - (1-\beta)(n+\theta_0)]}{2[(1-\beta)\theta_1 - \chi]\{(h\zeta - \pi^o)[\alpha(1-\theta_1) - (n+\theta_0)(1-\beta-\chi)] - h[\alpha\theta_1 - (n+\theta_0)\chi]\}} \quad (64)$$

Although the dynamics of τ^H and τ^P in Model B are complex, numerical simulations suggest that these bounds also exhibit cyclical trajectories around their 'equilibrium values' τ^{H*} and τ^{P*}. Figure 11 illustrates the simulation results for the dynamics of the actual interest rate τ and the bounds τ^H and τ^P, with changes in selected parameters and using the case of stable cycles as a baseline. Although it is challenging to draw definitive conclusions from these simulations, at least they suggest that the same parameters that increase the volatility of cycles (as shown in Figure 9) also tend to increase the likelihood that the economy will fall into the Ponzi financing regime, where the actual interest rate τ exceeds the upper bound τ^H. Conversely, parameters that reduce the volatility of cycles tend to make the economy less prone to the Ponzi financing regime. Notable, the parameters capturing the capitalist incentive to invest (α, β, χ) tend to lower cycle volatility but also push the interest rate above the upper bound τ^H and increase the probability of the economy falling into a Ponzi financing regime. In contrast, the parameters associated with the distribution of bargaining power between workers and capitalists (γ, ρ, η) suggest that a weaker working class tends to reduce the likelihood of the economy falling into a Ponzi regime. The simulations also show that the economy rarely reaches a hedge financing regime, and when it does, it is usually a result of unstable cycles that also are more likely to fall into Ponzi regimes. However, as mentioned before, it is crucial to note that these results are preliminary and require a more detailed analytical discussion before being generalized.

Figure 11. Dynamics of the interest rate with changes in selected parameters (Model B)



Black: lower bound τ^H and upper bound τ^P. Gray: actual interest rate τ. Note: The first plot depicts the simulation of Model B with parameter values $h = 1, \eta = 0.5, \theta_0 = 0.016, \theta_1 = 0.01625, n = 0.016, \pi o = 0.02, \zeta = 0.15, \gamma = 0.227, \rho = 0.28, m = m^* = 0.1, \alpha = 0.017, \beta = 0.5, \chi = 0.0065, \lambda = 0.003, \phi = 0.2$ and initial conditions $l_0 = 0.92, z_0 = 0.1, f_0 = 0.7, \tau_0 = 0.035, u_0 = 0.6$. The subsequent plots use these values as a baseline while varying the parameter indicated in their respective titles.

4. Conclusion

This paper has proposed two theoretical models (Models A and B) to examine the interaction between distributive and financial cycles in capitalist economies. Model A assumes an economy without excess demand but with a time delay between capitalists saving their income and distributing it to firms for reinvestment. This delay leads to firms needing debt to sustain their cash flows, resulting in credit demand from a rentier class that borrows money and receives an interest rate. Model A also includes a central bank that adjusts the interest rate to control inflation, and an extended version of the model considers the reaction of the rentier class to changes in the leverage ratio. Model B extends and modifies Model A by representing capitalist incentives to invest through an investment function, while a capacity utilization rate adjusts due to the existence of excess demand. In Model B, firms require debt not only to sustain their cash flows but also to finance their investment beyond the limits of savings.

Analytical proofs for the existence of limit cycles are provided in both models. Specifically, we show the existence of limit cycles in the complete four-dimensional dynamical system associated with Model A (Appendix 1) and the existence of limit cycles in a simplified five-dimensional dynamical system associated with Model B (Appendix 4). We also identify cyclical dynamics from numerical simulations of these models, including stable and limit cycles, unstable cycles, and damped oscillations, as well as several relevant patterns.

For instance, our analysis suggests that the distribution of bargaining power between workers and capitalists plays a crucial role in determining the stability of distributive and financial cycles. Specifically, we found that a weakened working class can lead to more stable cycles, reduce the risk of a debt crisis and the emergence of Ponzi financing regimes, and facilitate the objective of the central bank of controlling inflation without exacerbating the volatility of cycles. Thus, the interest of the central bank in achieving stability appears to align with the broader objective of the capitalist class to sustain and increase labor exploitation by reducing the bargaining power of workers. Nevertheless, this convergence is complex and not always consistent due to the inflationary pressures generated by the capitalist pursuit of profitability. Additionally, our simulations indicate that the various incentives of capitalist firms to invest (animal spirit effect, profitability effect, demand effect), can reduce the cycle volatility, but may lead to an increasing leverage ratio and interest rate over the long run, implying a potential tendency toward a debt crisis. Furthermore, we observe that when the rentier class has a stronger reaction to an increasing leverage ratio, it can also exacerbate cycle volatility.

We also propose two methods to identify financing regimes within capitalist cycles. The first method examines the counterclockwise trajectories of Model B within the plane formed by the auxiliary variables ψ (growth rate of the stock of debt) and ω (investment expressed as a proportion of the stock of debt). The second method focuses on the dynamics of the interest rate, including the estimation of lower and upper bounds that differentiate each financing regime. Through numerical simulations, we have found that both methods may be useful to identify hedge, speculative, and Ponzi financing regimes in Model B.

Finally, this paper provides insights for future investigations. One potential avenue for future research is to further examine the dynamics of Model B, particularly concerning the properties of its non-trivial equilibrium point, the analytical identification of critical values that may generate limit cycles, and the behavior of the interest rate bounds that distinguish each financing regime. Building on the mathematical proof presented in Appendix 4, future studies could delve deeper into these aspects to enhance our understanding of distributive and financial cycles in the case of non-zero excess demand. Another area for exploration is the empirical testing of the counterclockwise cycles and the empirical identification of financing regimes in the plane defined by the auxiliary variables ψ and ω for various capitalist economies. By contrasting the identified regimes with historical periods of financial crises, it may be possible to assess the correlation between the two and establish the relevance of the approach presented in this work. Thus, we hope that the paper will contribute to the ongoing debate on the sources of macroeconomic instability associated with the complexity inherent in capitalist economies and their multiple contradictions.

Disclosure statement

The author reports there are no competing interests to declare.

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Appendix 1. Stability and Hopf bifurcation in Model A

Linearizing the dynamical system defined by (15), (19), (22), and (24) near its non-trivial equilibrium point (l^*, z^*, f^*, τ^*) gives the following Jacobian matrix J:

$$J = \begin{bmatrix} 0 & \frac{h^2 H_3}{H1^2 \rho\sigma} & -\frac{(H_1 - h - H_2)H_3}{H_1 \rho\sigma} & -\frac{H_2 H_3}{(H_1 - h - H_2)\rho} \\ -\frac{H_1 \rho}{h} & -H_1(1-\eta) & 0 & 0 \\ 0 & -\frac{h^2 H_2}{H_1^2 (H_1 - h - H_2)\sigma} & \frac{H_2}{H_1 \sigma} & \frac{(H_1 - h)H_2}{(H_1 - h - H_2)^2} \\ 0 & -hm & 0 & 0 \end{bmatrix}$$

Where:

$$H_1 = h + h\zeta - \pi^o, \;\; H_2 = h\zeta - \pi^o - \sigma(\theta + n)(h + h\zeta - \pi^o), \;\; H_3 = \gamma + \theta + \pi^o(1-\eta)$$

The characteristic equation of J is given by $\lambda^4 + b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4 = 0$, where λ represents the eigenvalues of J. A necessary and sufficient condition for the local stability of the model is that all the eigenvalues λ have negative real components. According to the Routh-Hurwitz criteria, this stability condition is met if the coefficients b_1, b_2, b_3, b_4 are positive and if $y = b_1 b_2 b_3 - b_1^2 b_4 - b_3^2 > 0$.

Concerning the coefficients b_1, b_2, b_3, b_4, they depend on the trace T, the determinant Δ, and the minors of the matrix J, as indicated by the following equations:

$$b_1 = -\mathrm{T} = \frac{H_4}{\sigma H_1}$$

$$b_2 = \begin{vmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{vmatrix} + \begin{vmatrix} J_{11} & J_{13} \\ J_{31} & J_{33} \end{vmatrix} + \begin{vmatrix} J_{11} & J_{14} \\ J_{41} & J_{44} \end{vmatrix} + \begin{vmatrix} J_{22} & J_{23} \\ J_{32} & J_{33} \end{vmatrix} + \begin{vmatrix} J_{22} & J_{24} \\ J_{42} & J_{44} \end{vmatrix} + \begin{vmatrix} J_{33} & J_{34} \\ J_{43} & J_{44} \end{vmatrix}$$

$$b_2 = \frac{H_5}{\rho\sigma H_1}$$

$$b_3 = -\begin{vmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{vmatrix} - \begin{vmatrix} J_{11} & J_{12} & J_{14} \\ J_{21} & J_{22} & J_{24} \\ J_{41} & J_{42} & J_{44} \end{vmatrix} - \begin{vmatrix} J_{11} & J_{13} & J_{14} \\ J_{31} & J_{33} & J_{34} \\ J_{41} & J_{43} & J_{44} \end{vmatrix} - \begin{vmatrix} J_{22} & J_{23} & J_{24} \\ J_{32} & J_{33} & J_{34} \\ J_{42} & J_{43} & J_{44} \end{vmatrix}$$

$$b_3 = \frac{m H_1 H_2 H_3}{H_1 - h - H_2}$$

$$b_4 = \Delta = \frac{m H_2 H_3}{\sigma}$$

Where:

$$H_4 = \sigma H_1^2(1-\eta) - H_2, \qquad H_5 = hH_3 - H_1 H_2 (1-\eta)$$

Condition (29) that guarantees a positive equilibrium point also verifies that b_3 and b_4 are positive. In the case of b_1 and b_2, they are positive if we assume that σ and h are sufficiently high to verify:

$$\sigma > \frac{H_2}{H_1^2(1-\eta)} \quad \rightarrow \quad H_4 > 0 \quad (A1)$$

$$h > \frac{H_1 H_2(1-\eta)}{H_3} \quad \rightarrow \quad H_5 > 0 \quad (A2)$$

Regarding the term $y = b_1 b_2 b_3 - b_1^2 b_4 - b_3^2$, it is equal to:

$$y = \frac{mH_2H_3\{(\theta+n)H_4[H_5 - H_4(\theta+n)] - m\sigma H_1^2 H_2 H_3\}}{\sigma^3 H_1^2(\theta+n)^2} \quad (A3)$$

For the sake of simplicity, we assume h is sufficiently high to satisfy:

$$h > \frac{H_1 H_2(1-\eta) + H_4(\theta+n)}{H_3} \quad (A4)$$

Where we note that condition (A4) is sufficient to verify condition (A2). Thus, given conditions (A1) and (A4), $y > 0$ is verified when m is sufficiently low such that:

$$m < m^{HB} = \frac{(\theta+n)H_4[H_5 - H_4(\theta+n)]}{\sigma H_1^2 H_2 H_3} \quad (A5)$$

To summarize, Model A exhibits a positive equilibrium point that is locally asymptotically stable when γ, σ, and h are sufficiently high, and θ, n, and m are sufficiently low, satisfying conditions (29), (A1), (A4), and (A5).

Now, to identify the presence of a Hopf bifurcation in Model A, we have to verify two conditions (Asada and Yoshida 2003). Firstly, all the coefficients of the characteristic equation b_1, b_2, b_3, and b_4 must be positive, something that is confirmed by conditions (29), (A1), and (A4). Secondly, if we designate m as a bifurcation parameter, we need to prove that $y(m^{HB}) = 0$ and $\left.\frac{dy}{dm}\right|_{m=m^{HB}} \neq 0$. In this sense, we can verify that $y(m^{HB}) = 0$ by substituting (A5) into (A3). Finally, differentiating (A3) with respect to m gives:

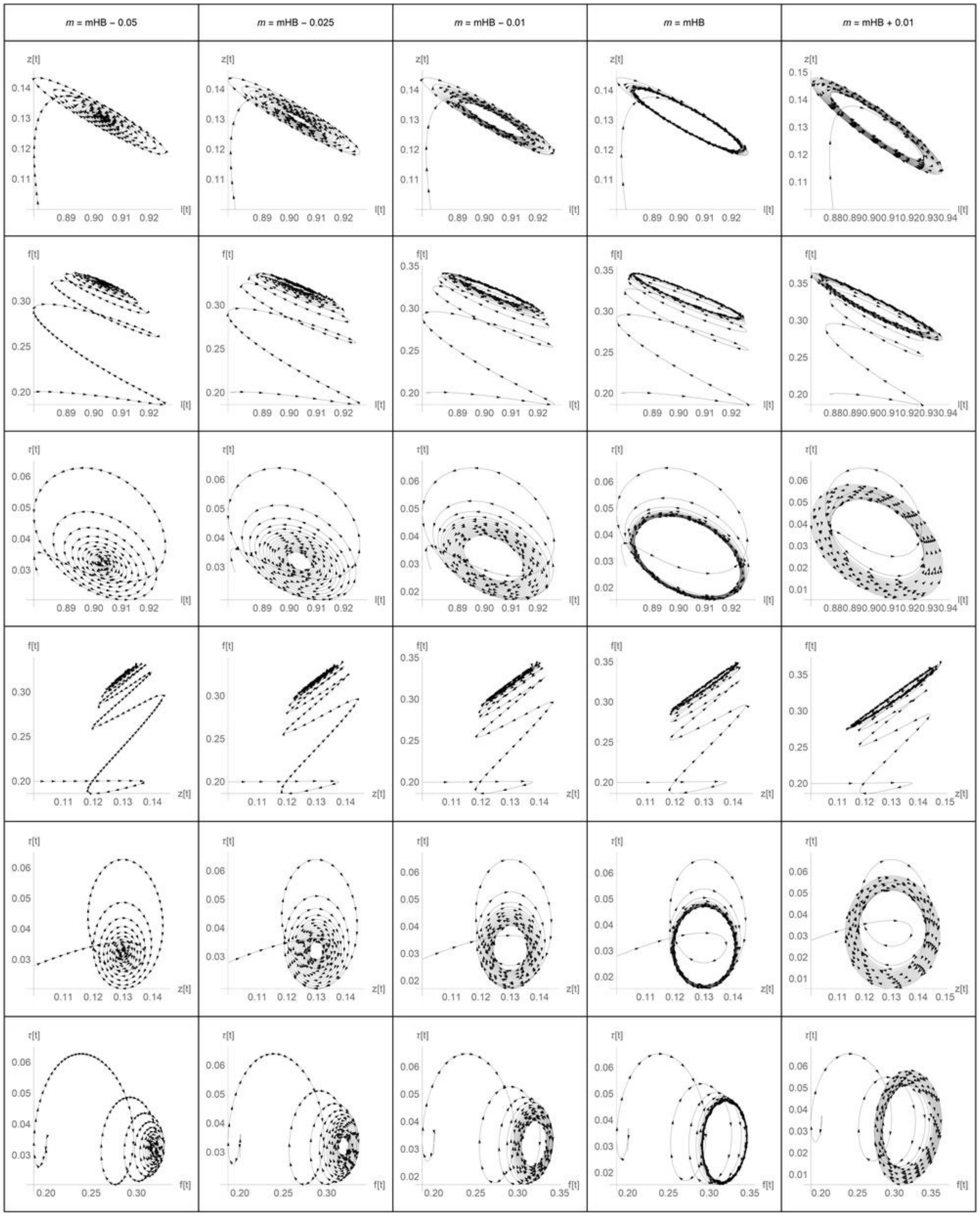
$$\frac{dy}{dm} = \frac{H_2H_3\{(\theta+n)H_4[H_5 - H_4(\theta+n)] - 2m\sigma H_1^2 H_2 H_3\}}{\sigma^3 H_1^2(\theta+n)^2} \quad (A6)$$

And, by substituting (A5) into (A6) we get:

$$\left.\frac{dy}{dm}\right|_{m=m^{HB}} = -\frac{H_2H_3H_4[H_5 - H_4(\theta+n)]}{\sigma^3 H_1^2(\theta+n)} < 0 \quad (A7)$$

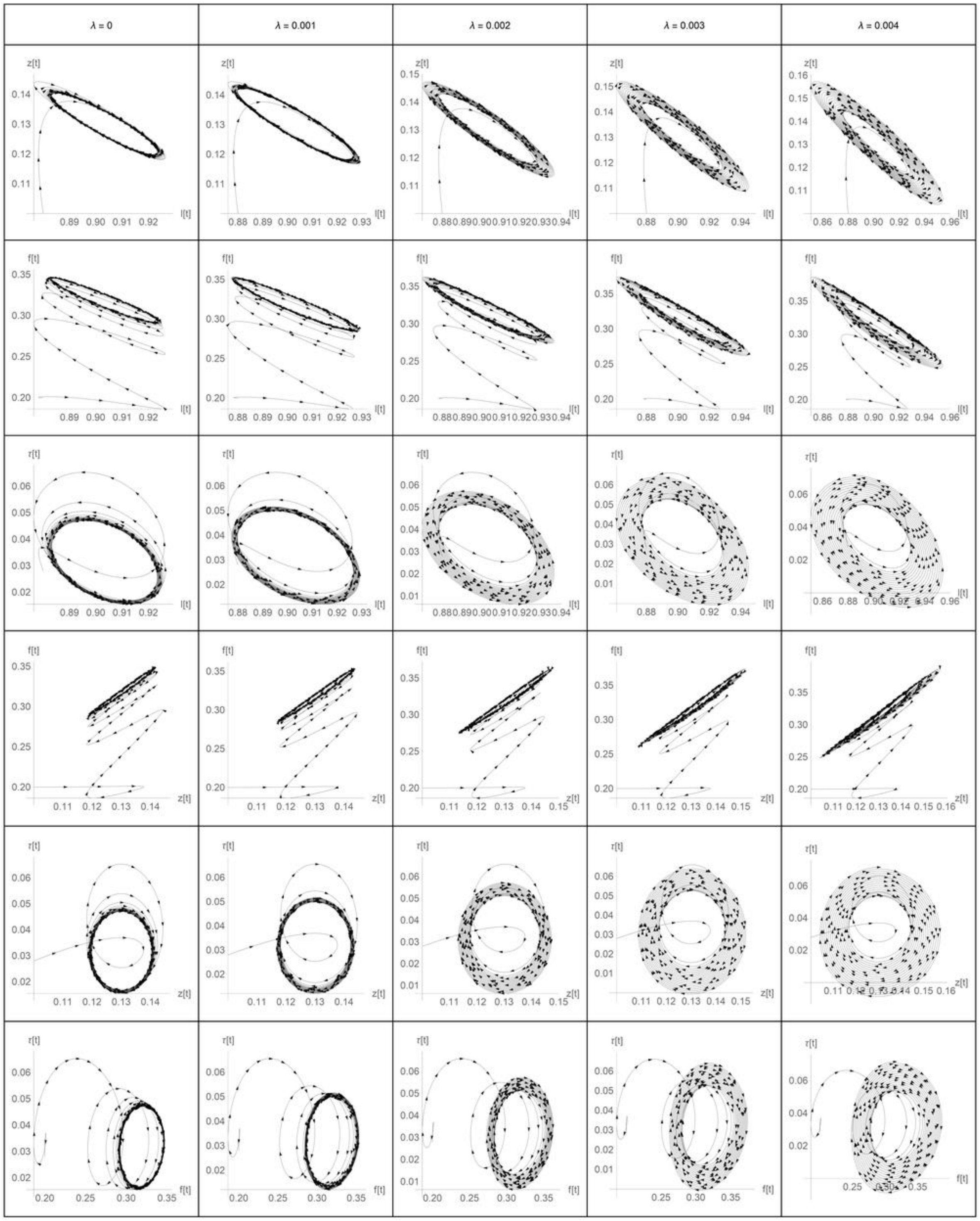
Since this derivative is different from zero, we conclude that Model A undergoes a Hopf bifurcation, that is, the model generates limit cycles near the non-trivial equilibrium point (l^*, z^*, f^*, τ^*) in the neighborhood of the critical value m^{HB}.

Appendix 2. Simulations of Model A for different values of m



Note: Simulation of Model A with parameter values $h = 1, \eta = 0.5, \sigma = 2.725, \theta = 0.016, n = 0.016, \pi^o = 0.02, \zeta = 0.15, \gamma = 0.227, \rho = 0.28, m^{HB} \approx 0.35634$ and initial conditions $l_0 = 0.88, z_0 = 0.1, f_0 = 0.2, \tau_0 = 0.028$. Equilibrium point: $l^* = 0.9035, z^* = 0.13, f^* = 0.31931, \tau^* = 0.032$.

Appendix 3. Simulations of modified Model A with increasing values of λ



Note: Simulation of modified Model A with parameter values $h = 1, \eta = 0.5, \sigma = 2.725, \theta = 0.016, n = 0.016, \pi^o = 0.02, \zeta = 0.15, \gamma = 0.227, \rho = 0.28, m = m^{HB} \approx 0.35634$ and initial conditions $l_0 = 0.88, z_0 = 0.1, f_0 = 0.2, \tau_0 = 0.028$. Equilibrium point: $l^* = 0.9035, z^* = 0.13, f^* = 0.31931, \tau^* = 0.032$.

Appendix 4. Stability and Hopf bifurcation in a simplified version of Model B

Assume a simplified situation where $\chi = 0, \lambda = 0, \eta = 0, n = 0, \beta = 0, h = 1, \phi = 1, \gamma = 0, \pi^o = 0, \zeta = 1, \theta_0 = 0$. Under these conditions, Model B simplifies to:

$$\frac{\dot{l}}{l} = 2\alpha + u\left[\frac{1}{1+z} - (1+\theta_1)\right] + f\tau$$

$$\frac{\dot{z}}{1+z} = 1 - \rho l + \theta_1 u + z$$

$$\dot{f} = \alpha - \frac{uz}{1+z} - f(\alpha - 2\tau)$$

$$\dot{\tau} = m(1-z)$$

$$\frac{\dot{u}}{u} = \alpha - \frac{uz}{1+z} + f\tau$$

In the steady state $\dot{l} = \dot{z} = \dot{f} = \dot{\tau} = \dot{u} = 0$, this simplified system has a non-trivial equilibrium point $(l^*, z^*, f^*, \tau^*, u^*)$ defined by:

$$l^* = \frac{\alpha}{\rho}, \qquad z^* = 1, \qquad f^* = \frac{1-2\theta_1}{2\theta_1}, \qquad \tau^* = \alpha, \qquad u^* = \frac{\alpha}{\theta_1}$$

Where we assume $\theta_1 < 1/2$ to guarantee that $f^* > 0$. Linearizing the simplified dynamical system near its non-trivial equilibrium point gives the following Jacobian matrix J:

$$J = \begin{bmatrix} 0 & -\frac{\alpha^2}{4\theta_1\rho} & \frac{\alpha^2}{\rho} & \frac{\alpha}{\rho}\left(\frac{1}{2\theta_1} - 1\right) & -\frac{\alpha + 2\alpha\theta_1}{2\rho} \\ -2\rho & -2 & 0 & 0 & 2\theta_1 \\ 0 & -\frac{\alpha}{4\theta_1} & \alpha & \frac{1}{\theta_1} - 2 & -\frac{1}{2} \\ 0 & -m & 0 & 0 & 0 \\ 0 & -\frac{\alpha^2}{4\theta_1^2} & \frac{\alpha^2}{\theta_1} & \frac{\alpha - 2\alpha\theta_1}{2\theta_1^2} & -\frac{\alpha}{2\theta_1} \end{bmatrix}$$

The characteristic equation of J is $\lambda^5 + b_1\lambda^4 + b_2\lambda^3 + b_3\lambda^2 + b_4\lambda + b_5 = 0$ and its coefficients b_i are equal to:

$$b_1 = 2 + \frac{1}{2}\alpha\left(\frac{1-2\theta_1}{\theta_1}\right), \qquad b_2 = \alpha\left(\frac{1-2\theta_1}{\theta 1}\right), \qquad b_3 = \frac{\alpha^3}{2\theta_1}$$

$$b_4 = \frac{m\alpha^2(1-2\theta_1)}{\theta_1}, \qquad b_5 = \frac{m\alpha^3(1-2\theta_1)}{\theta_1}$$

From the Routh-Hurwitz criteria, it is known that all the roots of the characteristic equation have negative real parts when:

$$R_1 = b_1 > 0 \quad (A8)$$

$$R_2 = b_5 > 0 \quad (A9)$$

$$R_3 = b_1 b_2 - b_3 > 0 \quad (A10)$$

$$R_4 = b_3(b_1 b_2 - b_3) - b_1(b_1 b_4 - b_5) > 0 \quad (A11)$$

$$R_5 = (b_1 b_2 - b_3)(b_3 b_4 - b_2 b_5) + (b_1 b_4 - b_5)(b_5 - b_1 b_4) > 0 \quad (A12)$$

The assumption that $\theta_1 < 1/2$ satisfies the inequalities defined in expressions (A8) and (A9).

On the other hand, by substituting the values of b_1, b_2, and b_3 into (A10), we get:

$$R_3 = \frac{\alpha H_6}{2\theta_1^2}$$

Where $H_6 = 4\theta_1(1 - 2\theta_1) + \alpha[1 - 4\theta_1(1 - 2\theta_1)] - \alpha^2\theta_1$. Thus, inequality (A10) is satisfied if α falls within the following interval:

$$\frac{1 - 4(1 - \theta_1)\theta_1 - \sqrt{H_7}}{2\theta_1} < \alpha < \frac{1 - 4(1 - \theta_1)\theta_1 + \sqrt{H_7}}{2\theta_1} \quad (A13)$$

Where $H_7 = (1 - 2\theta_1)\{1 - 2\theta_1[3 - 2\theta_1(7 - 2\theta_1)]\}$.

In regard to expression (A11), substituting the coefficients b_i into it gives:

$$R_4 = \frac{\alpha^4 H_6 - m\alpha^2(1 - 2\theta_1)[\alpha - 2(\alpha - 2)\theta_1][\alpha - 4(\alpha - 1)\theta_1]}{4\theta_1^3}$$

If θ_1 is sufficiently low to verify that $[\alpha - 2(\alpha - 2)\theta_1]$ and $[\alpha - 4(\alpha - 1)\theta_1]$ are positive, then inequality (A11) is satisfied if m is low enough to guarantee:

$$m < \frac{\alpha^2 H_6}{(1 - 2\theta_1)[\alpha - 2(\alpha - 2)\theta_1][\alpha - 4(\alpha - 1)\theta_1]} \quad (A14)$$

Concerning expression (A12), by substituting the coefficients b_i into it, we obtain:

$$R_5 = \frac{m\alpha^4}{4\theta_1^4}(H_8 - mH_9)$$

Where:

$$H_8 = (1 - 2\theta_1)(\alpha - 2 + 4\theta_1)\{\alpha(1 - 2\theta_1)[\alpha(1 - 2\theta_1) + 4\theta_1] - \alpha^3\theta_1\}$$

$$H_9 = (1 - 2\theta_1)^2[\alpha(1 - 4\theta_1) + 4\theta_1]^2$$

If θ_1 is sufficiently low to verify that H_9 is positive, then inequality (A12) is guaranteed if m is sufficiently low to satisfy:

$$m < \frac{H_8}{H_9} \quad (A15)$$

In summary, the simplified version of Model B presented in this Appendix has a non-trivial positive equilibrium that is stable if θ_1 is sufficiently low, α falls within the interval defined in (A12), and, in particular, m is low enough to satisfy the inequalities presented in (A13) and (A14).

Regarding the identification of a Hopf bifurcation in the simplified model presented in this Appendix, we adopt the approach proposed by Douskos and Markellos (2015) for identifying Hopf bifurcations in five-dimensional dynamical systems.[12] According to this approach, if we consider m as the bifurcation parameter, we can identify a Hopf bifurcation by checking whether there exists a critical value m^{HB} that satisfies the following conditions:

$$R_5(m^{HB}) = 0, \qquad \left.\frac{dR_5}{dm}\right|_{m=m^{HB}} \neq 0 \quad (A16)$$

$$R_6 = \frac{b_5 - b_1 b_4}{b_3 - b_1 b_2} > 0, \qquad R_7 = b_3 - b_1 R_6 \neq 0 \quad (A17)$$

The non-trivial critical value that guarantees $R_5(m^{HB}) = 0$ is given by:

$$m^{HB} = \frac{H_8}{H_9} \quad (A18)$$

On the other hand, differentiating (A12) with respect to m gives:

$$\frac{dR_5}{dm} = \frac{\alpha^4}{4\theta_1^4}(H_8 - 2mH_9) \quad (A19)$$

Substituting (A18) into (A19) results in:

$$\left.\frac{dR_5}{dm}\right|_{m=m^{HB}} = -\frac{\alpha^4 H_{10}}{4\theta_1^4} < 0 \quad (A20)$$

Finally, substituting the coefficients b_i into R_6 and R_7 defined in (A17) gives:

$$R_6 = \frac{m\alpha(1-2\theta_1)[\alpha - 4(\alpha-1)\theta_1]}{\alpha - (\alpha^2 + 4\alpha - 4)\theta_1 + 4(\alpha-2)\theta_1^2} \quad (A21)$$

$$R_7 = \frac{\alpha^3 - m\alpha(1-2\theta_1)[\alpha - 4(\alpha-1)\theta_1][4\theta_1 + \alpha(1-2\theta_1)]}{2\theta_1[\alpha - (\alpha^2 + 4\alpha - 4)\theta_1 + 4(\alpha-2)\theta_1^2]} \quad (A22)$$

Substituting (A18) into (A21) and (A22) results in:

$$R_6(m^{HB}) = \frac{\alpha^2(-2 + \alpha + 4\theta_1)}{\alpha - 4(\alpha-1)\theta_1} > 0 \quad (A23)$$

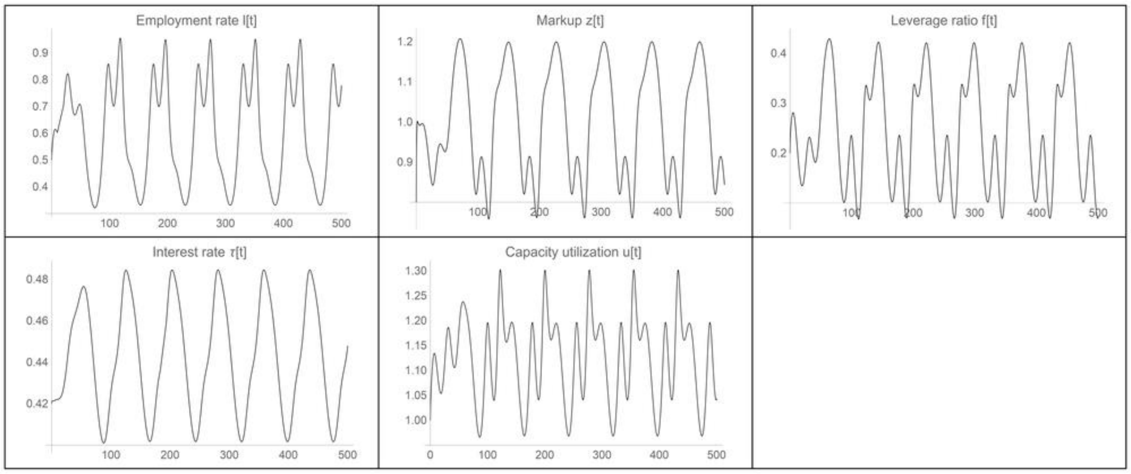
[12] For an example of the approach proposed by Douskos and Markellos (2015) being implemented in a five-dimensional dynamical system, see Al Basir et al. (2018). The procedure followed in this Appendix closely follows the approach presented in that example.

$$R_7(m^{HB}) = \frac{\alpha^2[\alpha - (\alpha^2 + 4\alpha - 4)\theta_1 + 4(\alpha - 2)\theta_1^2]}{\theta_1[\alpha - 4(\alpha - 1)\theta_1]} > 0 \quad (A24)$$

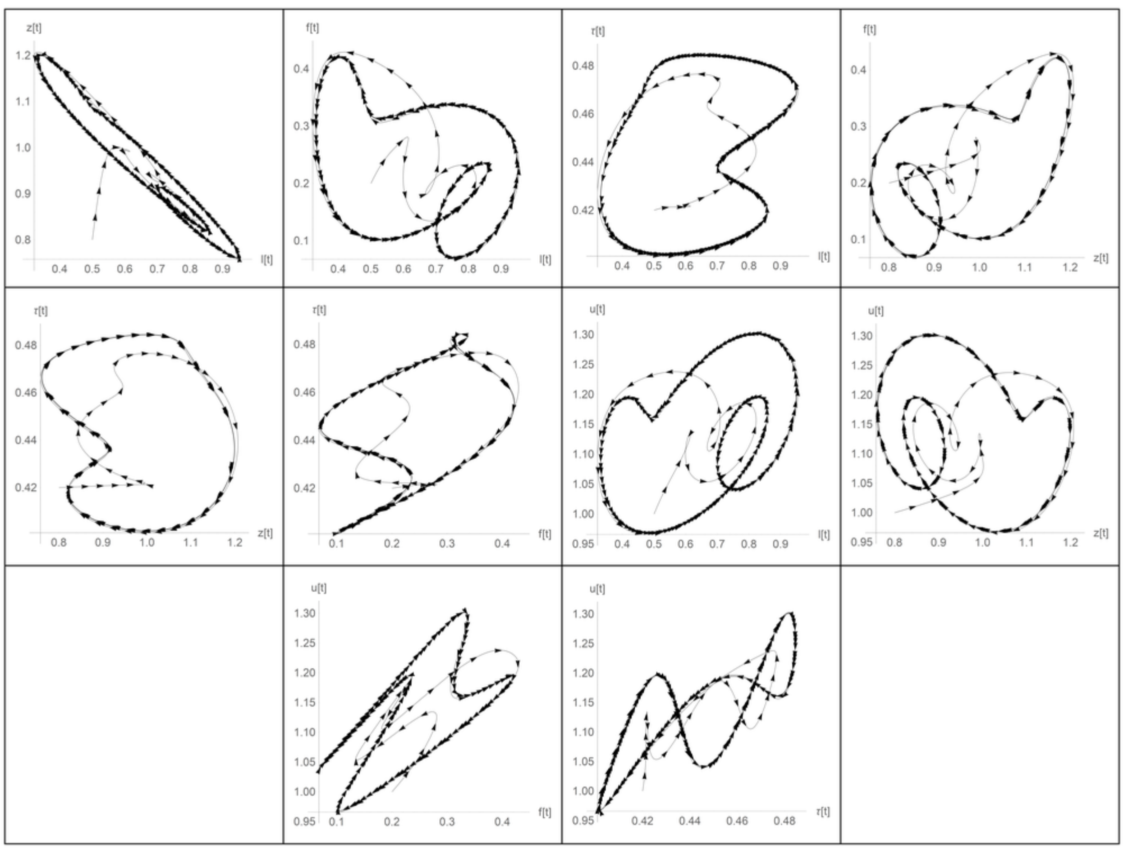
Expressions (A20), (A23), and (A24) satisfy the conditions defined in (A16) and (A17), indicating the occurrence of a Hopf bifurcation in the simplified version of Model B presented in this Appendix. Specifically, the model generates limit cycles near its positive non-trivial equilibrium point $(l^*, z^*, f^*, \tau^*, u^*)$ in the vicinity of the critical value m^{HB}. Appendix 5 provides time series and two-dimensional parametric plots that illustrate the complexity of these limit cycles.

Appendix 5. Simulation of the simplified version of Model B

Time series



Two-dimensional parametric plots



Note: Simulation of the simplified version of Model B presented in Appendix 4 with parameter values $\rho = 0.75, \theta_1 = 0.4, \alpha = 0.45, m = m^{HB} \approx 0.01634$ and initial conditions $l_0 = 0.5, z_0 = 0.8, f_0 = 0.2, \tau_0 = 0.42, u_0 = 1$. Equilibrium point: $l^* = 0.6, z^* = 1, f^* = 0.25, \tau^* = 0.45, u^* = 1.125$.