The Concept of Separate needs in Cardinal Utility Theory: A Functional Form for Added Leaning-S-shaped Utlities

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14 July 2024

THE CONCEPT OF SEPARATE NEEDS IN CARDINAL UTILITY THEORY: A FUNCTIONAL FORM FOR ADDED LEANING-S-SHAPED UTILITIES

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ABSTRACT

The introduction of the concept of separate needs into cardinal utility theory requires two propositions. The first specifies that the shape of a utility function for a commodity (good, service or event) fulfilling a need should reflect the experiences of an individual as the commodity fulfils that need: deprivation, subsistence, sufficiency, finite satiation with the possibility of a surfeit, or satiation at infinity, referred to as a 'leaning-S-shaped' utility. The second is a separability rule, specifying weak separability for choices within the same need, and strong (additive) separability for those between different needs.

This paper creates a utility function for two goods fulfilling two different needs, from which the functional form for a demand equation is derived. The indifference curve map and demand and Engels curve diagrams are interpreted, and their outcomes inferred.

The main outcomes are:

- A straight-line indifference curve, BA, defined by relative-intensities-of-need, separates the concave- from the convex-to-the-origin indifference curves, and can be identified as an absolute poverty line. It leads to disequilibrium in the derived functional form diagrams.
- Concave-to-the-origin indifference curves represent dysfunctional poverty.
- The convex-to-the-origin indifference curves can be divided into four areas. Where the individual experiences a greater sufficiency in one need combined with a modest deprivation in another, s/he will respond to changes as an inferior, or even Giffen, good. Their boundaries are reflected in envelope curves in the derived functional form diagrams.
- Three types of experience can be identified: dysfunctional poverty, functional poverty and sufficiency.

JEL classification: C21, D11, J22

Keywords: Increasing marginal utility, additive utilities, absolute poverty line, disequilibrium, dysfunctional poverty, deprivation, subsistence, Giffen good.

THE CONCEPT OF SEPARATE NEEDS IN CARDINAL UTILITY THEORY: A FUNCTIONAL FORM FOR LEANING-S-SHAPED ADDITIVE UTILITIES

by

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> ‘How can we convince a sceptic that this “law of demand’ is really true of all consumers, all times, all commodities? Not by a few (4 or 4,000) selected examples, surely. Not by a rigorous theoretical proof, for none exists – it is an empirical rule. Not by stating, what is true, that economists believe it, for we could be wrong. Perhaps as persuasive proof as is readily summarised is this: if an economist were to demonstrate its failure in a particular market at a particular time, he would be assured of immortality, professionally speaking, and a rapid promotion. Since most economists would not dislike either reward, we may assume that the total absence of exceptions is not from lack of trying to find them.’
>
> (Stigler, 1966: p.24)

I. INTRODUCTION

The aim of this paper is to introduce the concept of separate fundamental human needs into cardinal utility theory. There is an extensive literature on the ontology and epistemology of needs in philosophy (Lawson (1997), Yamamori (2016)), and there are many papers about systems of needs in psychology (Maslow (1943), Doyal and Gough (1991)). Ward and Lasen (2009) provide ‘An Overview of Needs Theories behind Consumerism’, examining ‘the development of hierarchical needs theory from Maslow to Gough’. But, apart from Miller (1988) there does not seem to have been an attempt to introduce the concept of separate needs into utility theory.

Thus, the aim of this paper is to create a utility function expressing separate needs in cardinal utility theory, based on two propositions. Firstly, the shape of a utility for a single need must express the experiences of the individual undergoing the fulfilment of that need – deprivation, subsistence, sufficiency, satiation and possible surfeit. Secondly, weak separability (multiplicativity) is used for choices concerning commodities fulfilling the same need, and strong separability (additivity) for choices about commodities fulfilling two different needs.

The method used is firstly to create such a utility function for the consumption of two goods fulfilling different needs, from which, secondly, to derive the functional form for the demand equation. These equations are used to create diagrams of the indifference curve map and the demand and Engels curves for the two dependent variables. The diagrams are then interpreted, and outcomes inferred.

In section II, a brief picture of the less controversial aspects of current utility theory is presented, anticipating the role of separate needs within it.

In section III, the two propositions are presented. The first is based on the seminal work of Van Praag (1968). Figure 1 illustrates the different stages of fulfilment of a need. Eight types of leaning-S-shaped functional form for specifying the shape of the utility are identified. Of these, a two-variable, additive, normal, distribution function utility function (with satiation at infinity) is selected here for the separate needs.

In section IV, Figure 2 presents the indifference curve map for the new utility function, and its properties are noted. The utility function is maximised subject to a budget constraint using the Lagrangian multiplier method yielding the optimality condition. This is used to explore the new utility function with respect to its superior normal, inferior normal and inferior-Giffen properties.

In section V, the functional form for the demand equation is derived from first principles, together with the equation for the envelope curves on the demand curves. Figures 3 and 4 illustrate the very non-linear new demand and Engels curves, and their properties are inferred from the diagrams.

The conclusion in section VI summarises the properties of the new utility function and its derived functional form. It ends by noting how this extension of utility theory might develop alongside traditional neoclassical demand theory.

II. THE CURRENT STATE OF STATIC CARDINAL UTILITY THEORY

Individuals are endowed with some resources, of which the most important is that of time. They receive other physiological and psychological endowments (unearned consumption) from their families, communities, and education as they grow up, and they learn a variety of self-provisioning or marketable skills (Sen's capabilities, 1991) to convert time into consumption. Utility is the satisfaction experienced by an individual from the consumption of commodities (goods, services or events). It is not directly observable and thus is not measurable.

Neoclassical microeconomists recognise that most individuals would prefer to get as much satisfaction as possible from the consumption of their resources and that they have intuitive ways of attempting to achieve it. The notion of a rational *homo economicus* with perfect information, maximising utility subject to resources and prices, was used to represent this process theoretically. If/when utility maximisation is achieved, then the first order optimality condition, (the ratio of marginal utility (MU) to price is a constant across all commodities), will have occurred.

Demand theory indicates how much consumption will result from utility maximisation given prices and resources (endowments of unearned consumption, Sen's capabilities (1991), and/or a budget).

Slutsky devised a method of identifying superior-normal, inferior-normal and inferior-Giffen goods, according to how they respond to changes in prices and incomes (Deaton, *et al*. 1980; 89-93).

The assumption of maximisation of utility has been widely criticised as unrealistic in many situations. Recognising that consumers do not have perfect knowledge, the Partial Adjustment and Adaptive Expectations Models were early attempts to introduce the consumer's learning process into econometrics. Herbert Simon's bounded rationality and satisficing theories (1947) pointed out that information-gathering is time-consuming and expensive. So, individuals are more likely to choose the best of a few available options rather than spending time and resources to identify the optimal. Alternatively, in times of uncertainty, perhaps the consumer might prefer to minimise his/her maximum regret?

In the forefront of the development of behavioural economics, psychologists Kahneman and Tversky revealed that some of our decision-making processes have important weaknesses. Their prospect theory (1979) revealed that individuals tend to be loss averse. The fact that the pain of a loss is greater than the pleasure of an equivalent gain reflects the Law of Diminishing Marginal Utility. Behavioural economics identifies some of the emotional aspects of decision-making and explores their effects. It concentrates on decision-making in the absence of accurate information. Thaler and Sunstein (2008) showed that the way that information is presented to people can affect the decisions. It is impossible for a decision to be presented in a neutral way.

Utility should be the keystone of the foundation layer of economic theory. But many economists dismiss it. Vilfredo Pareto (1848-1923) was the first to dispense with unobservable, and therefore unmeasurable, cardinal utility on the grounds that it is merely a preference ordering. The axiomatic theory of demand, based on ordinal utility and observable behaviour (revealed preference) was developed by Samuelson (1947) and others. This involves the prediction of the outcomes of price and income changes driven by an unobservable utility in a black box, without a helpful theory of utility to guide them. The empirical results of axiomatic ordinal utility theory have tended to be poor predicters of consumer behaviour.

Criticisms of other aspects of utility theory have included Thurow's (1983) criticism of the equilibrium of demand and supply in microeconomics and its failure to explain the disequilibrium concept of unemployment in the labour market. Keen (2011) has pointed out that aggregation across individual utility or demands to create market functions is not valid. Roberto Fumagalli (2013) is correct in his claim that neuroscience does not provide the answer to the 'futile search for true utility'.

Neoclassical utility theory (Deaton and Muellbauer, 1980) has been barren of psychology. For many decades, it was limited entirely to the 'law of diminishing marginal utility', This was challenged by Van Praag's (1968) seminal, classic work that extended the utility function to encompass both increasing (deprivation) and diminishing (sufficiency) marginal utility within a bounded cardinal utility function for the consumption of a commodity. When Pareto and neoclassical economists dispensed with the concept of cardinal utility because it cannot be measured, in favour of formal axiomatic demand theory based on ordinal utility, they also rejected a host of potential information contained in the *shape* of an extended utility function. Van Praag's bounded cardinal utility theory also partially solves the non-measurable cardinal utility problem, enabling interpersonal welfare comparisons to take place. His resultant *multiplicative* utilities led to convex-to-the-origin indifference curves.

Associated with the paucity of psychology, there are few meaningful, psychological parameters in utility theory. Despite the assertion that

> the easy comprehension of the elasticity concept and the fact that elasticities are pure numbers has led to many economists to see the estimation of elasticities as the primary aim of empirical demand analysis (Deaton and Muellbauer, 1980: 17),

elasticities are neither easily comprehended, nor are they constants, but variables.

Maslow's (1943) theory of a hierarchy of five different needs, (physiological, safety, love and belonging, esteem and self-actualising) has been further developed by economists such as Doyal and Gough (1991) who differentiate between 'having or deficit' needs and 'being or growth' needs.

So, what is the relationship between needs and utility? As with utility, needs are not directly observable, but only by the effects of their satisfiers, or the lack thereof. A need provides the motivation for a behavioural response in terms of consumption of satisfiers, and the individual consumer experiences changes in utility reflecting the different levels of fulfilment of the need. But human needs are classifiable, implying strong separability. This paper aims to introduce more psychology into utility theory by introducing the concept of separate human needs into hitherto undifferentiated utility.

III. TWO PROPOSITIONS

The introduction of the concept of separate needs into cardinal utility theory requires two propositions.

- The first is that the shape of a utility for a commodity (good, service or event), q_i, should encompass the experience of an individual through the various stages of fulfilment of a need – deprivation (increasing marginal utility (MU)), subsistence (a point of inflection), sufficiency (diminishing MU), finite satiation with the possibility of a surfeit, or satiation at infinity. This will be referred to as 'leaning-S-shaped utility'. It will also be bounded below and above.

- The second provides a separability rule, with weak (multiplicative) separability for choices between two commodities fulfilling the same need, and strong (additive) separability for choices between two commodities fulfilling different needs.

This first proposition is based on the ground-breaking, seminal work of Bernard M S Van Praag (1968), which has been developed and applied successfully by The Leyden School (for example, Van Herwaarden and Kapteyn, 1981; Hagenaars, 1986; Van Praag and Kapteyn, 1994).

Van Praag further recognised an intermediate state between cardinal and ordinal utility in the form of bounded cardinal utility. A bounded cardinal utility function, leading to both a minimum level of utility and a maximum (satiation – at either finite or infinite consumption), enables interpersonal welfare comparisons to be made, thus partially solving the non-measurability problem of utility.

The most obvious choice for a functional form, based on a 'leaning-S-shaped' bounded cardinal utility function with satiation at infinity, is a distribution function (DF) (but it would not have any probabilistic connotations in this context). Alternatively, a scaled-down frequency function would allow for finite satiation and a surfeit.

There is also the choice between the symmetric normal distribution (for which $-\infty \leq q_i \leq +\infty$, implying that consumption could take negative values) or the (more realistic?) log normal distribution (for which $0 \leq q_i \leq +\infty$). Further, two individual utilities can be added, or more than two can be multiplied. This gives eight basic types of functional form.

Both the normal and log-normal distribution functions have parameters, μ_i, at the point of inflection (subsistence) between increasing and diminishing marginal utility, and σ_i, which becomes a measure of the intensity-of-need for the i'th commodity. That $q_i = \min_i$ could occur for $\min_i < 0$ could be explained by 'free satisfiers'. That is, the initial fulfilment of a need may be provided by natural circumstances, such as, the warmth of the sun could heat a home, before personal resources are used up.

The leaning-S-shaped functional form developed here, (and on which Figures 2 – 4 are based), is the result of adding two bounded cardinal utilities based on DFs of the normal distribution, (2-Add.N-DF). The N-DF was chosen for pragmatic reasons because it is quite tractable and is useful for illustrating many aspects of the theory, providing a reasonable approximation for that part of the leaning-S-shape around the subsistence threshold. Further, it has the added advantage that its two parameters, μ_i and σ_i, have important economic and psychological interpretations, and are potentially estimable. The parameter μ_i is the subsistence threshold or the 'survival level' estimated in many econometric models and σ_I is the intensity-of-need parameter for the i'th commodity.

Van Praag concentrated on the outcomes of an n-variable, *multiplicative*, lognormal distribution function, bounded cardinal utility function, (n-Mult.LN-DF), representing his 'leaning-S-shaped' utility, satiated at infinity.

Thus, there are two ways in which the functional form derived here differs from that of Van Praag.

- Proposition 2 provides the rule for distinguishing between utilities being additive and being multiplicative, although only the case of *additive* utilities is explored here. Van Praag assumes that the relationship between the utilities from commodities is *multiplicative*, both without and with dependence (substitutes and complements).
- Van Praag makes a case for using the *log-normal* distribution function as his functional form (1968, pp.81, 86, 119), whereas for additive utilities, the *normal* distribution function is much more tractable.

Although each functional form includes satiation at infinity, the analysis of the new functional form will concentrate on the responses around subsistence, and satiation will not be considered further here.

Proposition 1. The Leaning-S-shaped Bounded Cardinal Utility Function.

The first proposition states:

An individual's experience of consumption, q_i, of the i'th commodity, (good, service or event), $-\infty \leq q_i \leq \infty$, for i = 1, 2, …, m, can be represented by a continuous, smooth, single-valued, utility function, $u_i = u(q_i)$, $0 \leq u_i \leq 1$, that has the shape of a 'leaning-S-shaped' curve, bounded below and above, but marginal utility, u_i', is always less than infinity.

The different experiences are illustrated in Figure 1[1] and summarised in Table 1 below.

[Figure 1 near here.]

TABLE 1. Signs associated with the 'leaning S-shaped' bounded cardinal utility function, u(*qi*)

u_i''	> 0	> 0	= 0	< 0	< 0	< 0
u_i'	= 0	$0 < u_i' < \infty$	$0 < u_i' < \infty$	$0 < u_i' < \infty$	= 0	< 0
u_i	$q_i \leq 0$	$0 < u_i < 1$	$0 < u_i < 1$	$0 < u_i < 1$	= 1	$1 > u_i > 0$
q_i	$q_i = \text{min}_i$	$\text{min}_i < q_i < \mu_i$	$q_i = \mu_i$	$\mu_i < q_i < \text{sat}_i$	$q_i = \text{sat}_i$	finite $\text{sat}_i < q_i$
Individual experience:	minimum:	deprivation	inflection: subsistence	sufficiency	maximum: satiated	surfeit

The assumptions that the utility function $u_i = u(q_i)$ reaches a minimum, $u_i = 0$, at $q_i = \text{min}_i$ and a maximum, $u_i = 1$, at $q_i = \text{sat}_i$ where sat_i could be either finite or infinite, are necessary conditions for utility to be bounded below and above. It is difficult to observe either zero consumption or satiation directly. $q_i \geq 0$ represents consumption of personal resources, but a minimum could occur at $q_i \leq 0$, because individuals can receive free satisfiers from their environment – for instance, spring water from a well would reduce the amount of extra water that might need to be bought.

Similarly, each individual might be able to experience satiation, but it is not assumed that the level of utility experienced as satiation is the same for each person. However, it is assumed that it is possible to compare an individual's utility with his/her maximum attainable. These assumptions allow for the possibility of standardising utility over a range of $0 \leq u_i \leq 1$, say, for comparing the utility attained in fulfilling one fundamental need, permitting interpersonal comparisons of welfare (utility).

A point of inflection occurs at $q_i = \mu_i$, representing a 'subsistence' threshold comparable to the committed consumption parameter, or survival level, in the Stone-Geary utility function from which the Linear Expenditure System (LES) is derived. Consuming less than this, where MU is increasing, implies 'deprivation'. Consumption greater than the subsistence threshold, where MU is positive but diminishing, may be labelled 'sufficiency'. The point at which maximum utility occurs yields 'satiation' in that particular need, while consumption greater than a finite satiation point can be called a 'surfeit'. Obviously, for satiation at infinite consumption there would be no experience of a surfeit.

Parameter σ_i in Figure 1 is a measure of the intensity-of-need for the i'th commodity. The range of q_i, ($\mu_i \pm 1.96.\sigma_i$), indicates where MU is experienced most intensely. The smaller is σ_i, the steeper is the slope of the $u_i(q_i)$ function around the parameter μ_i, and the more intense is the need.

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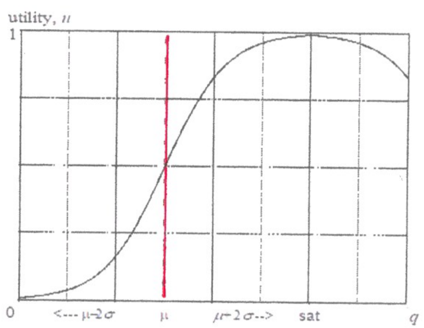
[1] Figures 1-4 were created using Seppo Mustonen's program SURVO (1992).



KEY

u	utility
q	consumption
μ	subsistence consumption
sat	satiation consumption

FIGURE 1 LEANING-S-SHAPED U-FN

> Commodities with a large variance are commodities for which satisfaction comes rather slowly … Commodities with a small variance are commodities … of which one is quickly satisfied. For instance, life necessities have presumably a small variance (Van Praag, 1968, p.34).

This raises an interesting question, which could be explored using an appropriate functional form, 'Would a commodity, or group of commodities, to which an individual is addicted, have an even smaller variance than life necessities?'

The parameters can vary over time for an individual, and between different groups of people, according to demographic variables and other experiences.

Proposition 2. The separability rule.

Proposition 2 states that

> a group of commodities that satisfy the same need are weakly separable, that is, based on multiplicative utilities (with or without dependence), and groups of satisfiers, each group satisfying a different need, are strongly separable, that is, based on additive utilities.

The discussion of 'separability' and 'the grouping of commodities' in the economics literature (Green, 1976; Deaton and Muellbauer, 1980) often comes across as though they are secondary afterthoughts, and it tends to centre on whether the utilities gained from the consumption of different commodities are additive or multiplicative. However, it is not a question of either/or, but rather 'when should utilities be added, and when multiplied?'

The separability proposition gives rise to two very different types of indifference curve maps. The multiplicative one is similar to the familiar representative convex-to-the-origin indifference curves found in textbooks, (some sample diagrams of which can be seen in Van Praag, 1968, p.88), and will not be discussed further here.

It is assumed here, following Mallman and Nudlar (1986), and to a lesser extent Maslow (1943), that there is a finite number (though few) of separable, fundamental human needs and that these are universal and a-historic. Needs are satisfied by an infinite diversity of culturally determined satisfiers. Needs cannot be observed directly, but only through the effects of their satisfiers, or lack thereof.

Max-Neef proposed a system,

> composed of nine fundamental human needs: permanence (or subsistence), protection, affection, understanding, participation, leisure, creation, identity (or meaning) and freedom … fundamental needs are finite, few and classifiable … fundamental needs are the same in all cultures and all historical periods. What changes, both over time and through cultures, is the form or the means by which these needs are satisfied (Max-Neef, 1986: 49-50).

IV. THE INDIFFERENCE CURVE MAP

The 2.Add.N-DF utility function is defined as the sum of two distribution functions for the normal distribution (which have no statistical connotations in the present context), representing consumption, q_i, $-\infty < q_i < +\infty$, i = 1, 2, where the i'th commodity fulfils the i'th need. The sum is scaled equally such that utility, u, lies between 0 and 1.

The '2.Add.N-DF' **utility function** is given as:

$$u(q_1, q_2) = \tfrac{1}{2}\, F_1(q_1) + \tfrac{1}{2}\, F_2(q_2)$$

$$u(q_1, q_2) = \frac{1}{2}\int_{-\infty}^{q_1} \frac{\exp\left[-(R_1 - \mu_1)^2/2\sigma_1^2\right]}{\sigma_1.\sqrt{2\pi}}\, dR_1 + \frac{1}{2}\int_{-\infty}^{q_2} \frac{\exp\left[-(R_2 - \mu_2)^2/2\sigma_2^2\right]}{\sigma_2.\sqrt{2\pi}}\, dR_2 \qquad (1)$$

where u, $0 \le u \le 1$, is utility,

$\mu_1, \mu_2 \ge 0$ are subsistence parameters representing 'survival level' thresholds, and $\sigma_1, \sigma_2 > 0$ are parameters representing intensity-of-need for commodities 1 and 2.

It would be almost impossible to use equation (1) to create an indifference curve map. Fortunately, as Johnson and Kotz (1970; 244) state 'The shape of this [logistic] distribution is quite similar to that of the normal density function'.

$$P(t) = \frac{e^t}{[1 + e^t]^2} = \frac{e^{-t}}{[1 + e^{-t}]^2}.$$

This was used to create the indifference curve map in Figure 2, adjusted for location and scale.

$$q_2 = \mu_2 - \{\log [\ (0.5 * \text{bracket}) / (u * \text{bracket} - 0.5) - 1] \} / (1.82/\sigma_2), \qquad (2)$$

where u is utility and bracket = $(1 + \exp(-(1.82/\sigma_1) * (q_1 - \mu_1)))$.

[Figure 2 near here]

The indifference curve map has q_1 on the horizontal and q_2 on the vertical axis. The map is divided into four quadrants by the subsistence parameters, μ_1 and μ_2, with point (μ_1, μ_2) labelled as E. This leaves a border of deprivation along the insides of the two axes.

Secondly, there is a straight-line indifference curve, BA, which passes through point E, separating the concave-to-the-origin indifference curves in the triangular area B0A surrounding the origin, from the convex indifference curves further from the origin. The equation for BA is $q_2 = \mu_2 - (\sigma_2/\sigma_1).(q_1 - \mu_1)$.

Individuals receive endowments during their lifetimes, from family, local communities, and education. Let C_1 and C_2 be endowments of q_1 and q_2.

The concave-to-the-origin indifference curves represent dysfunctional poverty, because the individual is unable to make any optimisation decisions that would

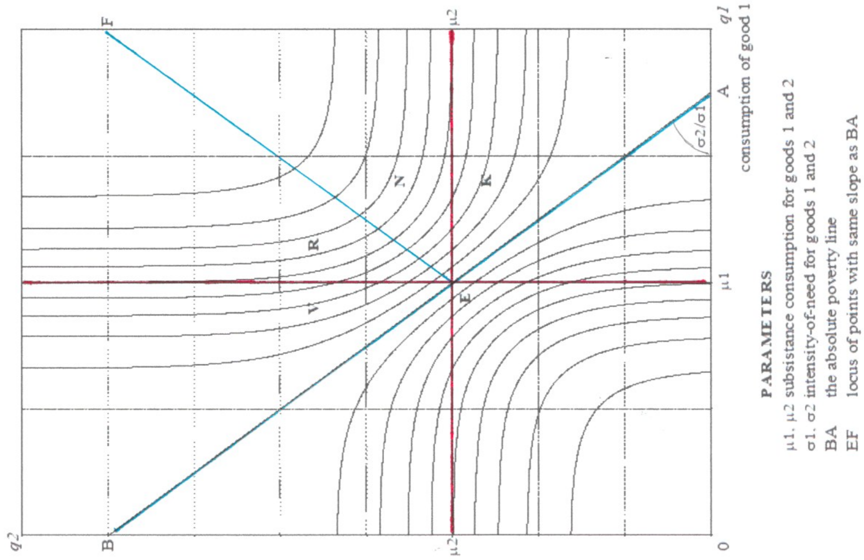


FIGURE 2 INDIFFERENCE CURVES FOR SATIATION AT INFINITY

increase his/her utility, unless s/he faces very favourable relative prices that would enable him/her to trade up to a convex-to-the-origin situation.

It follows that if area B0A represents dysfunctional poverty, and the convex-to-the-origin indifference curves present optimisation choices for the individual, then the straight-line indifference curve, BA, can be identified as an absolute poverty line for those two needs. The endowments, $C_2 = q_2$ at B and $C_1 = q_1$ at A, represent survival endowments.

EF is the locus of points where the slope of the convex-to-the-origin indifference curves is the same as that of BA,

The area of the indifference curve map containing convex-to-the-origin curves can be divided into four:

- a rhomboid bounded by the q_2-axis, BE and the μ_1 parameter, labelled V, in which the individual has sufficient of q_2, but is deprived of q_1;
- a triangular area bounded by the parameter μ_1 and EF, labelled R;
- a triangular area bounded by EF and the parameter μ_2, labelled N, and
- a rhomboid bounded by the parameter μ_2, EA and the q_1-axis, labelled K, in which the individual has sufficient in q_1, but is deprived of q_2.

This reveals that there are three levels of fulfilment where two needs are concerned:

- The lowest level leads to dysfunctional poverty, from which it is very difficult to extract oneself without favourable relative prices or extra endowments.
- An intermediate level occurs where the individual is deprived in one need, while having sufficient in the other to enable him/her to face options which could improve his/her situation, as in areas K and V.
- In the optimum level, the individual experiences sufficiency in both needs, as in areas N and R.

The budget constraint and properties of the areas K, N, R and V in the convex-to-the-origin indifference curves

Let full income, $M = C_1.p_1 + C_2.p_2$, where C_1 and C_2 are endowments of q_1 and q_2, valued at prices p_1 and p_2 respectively, where M, p_1 and $p_2 \geq 0$.
Survival income = $\mu_1.p_1 + \mu_2.p_2$.
Supernumerary income, $Z = M$ – survival income = $(C_1–\mu_1).p_1 + (C_2–\mu_2).p_2.$

A linear budget constraint is expressed in the form of the allocation of income, M, on expenditure for the consumption of the two commodities, q_1 and q_2, at prices p_1 and p_2 respectively.

$$M = q_1.p_1 + q_2.p_2.$$

$$q_2 = (M - q_1.p_1)/p_2, \tag{3}$$

The utility function, together with the budget constraint, represents the structural form of the model.

Maximising $u(q_1, q_2)$ subject to the budget constraint M, and using the Lagrangian multiplier method leads to:

$$\frac{du(q1,q2)}{dq_1} = \frac{\frac{1}{2}.exp[-(q_1 - \mu_1)^2 / 2\sigma_1^2]}{\sigma_1.\sqrt{2\pi}} - \lambda . p_1 = 0. \tag{1a}$$

$$\frac{du(q1,q2)}{dq_2} = \frac{\frac{1}{2}.exp[-(q_2 - \mu_2)^2 / 2\sigma_2^2]}{\sigma_2.\sqrt{2\pi}} - \lambda . p_2 = 0. \tag{1b}$$

$$\frac{exp[-(q_1 - \mu_1)^2 / 2\sigma_1^2]}{exp[-(q_2 - \mu_2)^2 / 2\sigma_2^2 \ ]} = \frac{\sigma_1.p_1}{\sigma_2.p_2}. \tag{1c}$$

Thus, this yields the **optimality condition**:

$$\left(\frac{q_2 - \mu_2}{\sigma_2}\right)^2 - \left(\frac{q_1 - \mu_1}{\sigma_1}\right)^2 = ln\left(\frac{\sigma_1.p_1}{\sigma_2.p_2}\right)^2 \tag{4}$$

The optimality condition describes a family of hyperbolae with respect to own price, whose asymptotes are the straight-line indifference curve and its mirror image, EF. It also describes the *income-consumption locus* for a given price ratio, p_1/p_2, on the indifference curve map.

The boundary between q_1 being superior or inferior in equation (5), is derived in the Appendix, as is an equation for the boundary between its being inferior normal or inferior-Giffen in equation (7) (which must be solved numerically).

The **superior-inferior boundary** for q_1 is $q_2 = \mu_2$ for $q_1 > \mu_1$, that is, the boundary between areas labelled K and N on the indifference curve map. Thus, in area K, q_1 will react to price changes as an inferior good.

Similarly, the boundary for q_2 is $q_1 = \mu_1$ for $q_2 > \mu_2$, which is the boundary between areas labelled V and R on the indifference curve map. Thus, in area V, q_2 will react to price changes as an inferior good.

- It can be shown that in the top right-hand quadrant of Figure 2, both commodities are experienced as superior normal goods, (additivity and positive diminishing marginal utilities always yield superior normal characteristics). With additive utilities, the two goods are net substitutes for each other.
- Inferior normal and inferior-Giffen responses occur for a need that is experienced as sufficient but is combined with a moderate deprivation in another, as anticipated by Berg (1987). Good 1 responds as inferior in the rhomboid area labelled as K in Figure 2, bounded by EA, the q_1-axis and $q_2 = \mu_2$ for $q_1 > \mu_1$, (Dougan, 1982; Silberberg *et al*,1984). That the Giffen experience is associated with a straight-line indifference curve, adjacent to a triangular non-solution space, was anticipated by Davies (1994).
- In area V, in that part of the left-hand border where the indifference curves are convex-to-the-origin, the consumer is deprived of good 1, (with increasing MU), and, following Hirschleifer's terminology (1976, chap.4), good 1 is here termed an ultra-superior good. Kohli (1985) calls this experience an 'anti-Giffen good', but 'anti-inferior' would be more accurate. *Good 2* is experienced as an inferior good in area V.

The boundary for q_1 between its responding as inferior normal or inferior-Giffen is more complex.

$$\left[\left(\frac{q_2-\mu_2}{\sigma_2}\right)^2 - \left(\frac{q_1-\mu_1}{\sigma_1}\right)^2\right].\left(2.\,ln\left(\frac{q_1}{\sigma_1}\right) + 2.\,ln\left(\frac{q_1-\mu_1}{\sigma_1}\right)\right) = 0,\ \text{, for } q_2 < \mu_2. \quad (7)$$

(This equation has not been confirmed yet.)

It is merely a more extreme reaction to greater deprivation in q_2. Individuals experiencing deprivation of a need react differently compared with those who are more generously endowed.

Rather than categorising the commodity, it is the consumer's *experience of, and response to, the fulfilment of a need* by a commodity, in combination with a good that fulfils another need, that should be categorised as ultra-superior, superior-normal, inferior-normal or inferior-Giffen. This would appear to confirm Spiegel's belief 'that Giffen goods are far more pervasive than is generally believed' (1994, p.137; Weber 1997). That the challenge of formulating a utility function for the elusive 'Giffen *good*' (as opposed to the pervasive Giffen *experience*) continues to engage economists is evidenced by Sørensen (2007), Jensen *et al* (2008), Moffatt (2012), Haagsma (2012) and Biederman (2015).

V. THE DEMAND EQUATION

The (very non-linear) demand equation is derived in the Appendix, together with an equation for the envelope curve on the demand curves representing the boundary between superior and inferior responses. The main results are presented here with equations numbered according to their derivation in the Appendix.

Let $x = p_1/p_2$ (relative prices); b = σ_2/σ_1 (relative intensities-of-need);
C_1 and C_2 are endowments of unearned consumption.

Using the following short-hand notation for elements that appear frequently,

$M = C_1.p_1 + C_2.p_2.$ (full income)
$Z = (C_1 - \mu_1).p_1 + (C_2 - \mu_2).p_2$
$= M - \mu_1.p_1 - \mu_2.p_2,$ (supernumerary expenditure),
$M = q_1.p_1 + q_2.p_2$ (budget equation)

and substituting for $q_2 = (M - q_1.p_1)/p_2$, from the budget constraint,
and for $M = Z + \mu_1.p_1 + \mu_2.p_2$ from the supernumerary expenditure equation,
into optimality condition, equation (4), yields an '**implicit demand equation**' (8):

$$\left(\frac{q_2-\mu_2}{\sigma_2}\right)^2 - \left(\frac{q_1-\mu_1}{\sigma_1}\right)^2 = ln\left(\frac{\sigma_1.p_1}{\sigma_2.p_2}\right)^2 \quad (4)$$

$$\left[\frac{\frac{Z}{p_2} - (q_1 - \mu_1).x}{\sigma_2}\right]^2 = \left[\frac{(q_1-\mu_1)}{\sigma_1}\right]^2 + ln\left[\frac{x}{b}\right]^2 \quad (8)$$

which is a quadratic equation in (q_1 – μ_1), which is solved using the negative square root, yielding **demand equation** (9) for commodity, q_1:

$$q_1 = \mu_1 + \frac{\left(\frac{Z}{p_2}\right).x \; - b.\sqrt{\left[\left(\frac{Z}{p_2}\right)^2 + \; (x^2 - b^2).\left(\sigma_1^2.2.ln\left(\frac{x}{b}\right)\right)\right]}}{(x^2 - b^2)} \qquad (9)$$

$Z/p_2 = (C_1 – \mu_1).p_1/p_2 + (C_2 – \mu_2).p_2/p_2$.

Thus, an alternative version of the demand equation is given by:

$$q_1 = \mu_1 + \frac{((C_1 - \mu_1)..x + (C_2 - \mu_2)).x \; - b.\sqrt{\left[((C_1 - \mu_1).x + (C_2 - \mu_2))^2 + \; (x^2 - b^2).\left(\sigma_1^2.2.ln\left(\frac{x}{b}\right)\right)\right]}}{(x^2 - b^2)} \qquad (10)$$

This demonstrates that the dependent variable, q_1, is a non-linear function of the independent variables, 'own' relative price, ($x = p_1/p_2$), and C_1 and C_2, with parameters, μ_1, μ_2, σ_1 and σ_2/σ_1. It describes a family of hyperbolae with respect to own price, whose asymptotes are the straight-line indifference curve and its mirror image, EF. Equation (9) was used to create Figures 3 and 4.

To accommodate the effect of constraining $q_2 \geq 0$, equation (9) must be qualified such that $0 \leq q_1 \leq M/p_1$. Thus, if $q_1 < 0$, put $q_1 = 0$, and if $q_1 > M/p_1$, put $q_1 = M/p_1$. Similarly, for $0 \leq q_2 \leq M/p_2$. These give corner solutions on the axes bordering the non-solution space. However, this has not been followed through on the diagrams in Figures 3 and 4.

Equation (9) is the negative root to the solution to a quadratic equation (8) in (q_1 – μ_1) and gives two solutions. The expenditure equations for q_1 and q_2 are symmetric and homogeneous of degree zero in p_1, p_2 and Z. The two demand equations, for q_1 and q_2, represent the reduced form of the model.

When both $M \geq Z$, and the budget line is parallel to the straight-line indifference curve, and thus $x^2 = b^2$, and using the negative root, equation (9) simplifies to

$$q_1 = \mu_1 + \frac{\left(\frac{Z}{p_2}\right).x - \mathrm{b}.\sqrt{\left(\frac{Z}{p_2}\right)^2}}{(\mathrm{x} - \mathrm{b}).(x + b)}.$$

$$q_1 = \mu_1 + \frac{Z/p_2}{(x + b)} \qquad (11)$$

The strong separability assumption means that the demand equation for any two commodities can be estimated independent of any other commodity fulfilling another need.

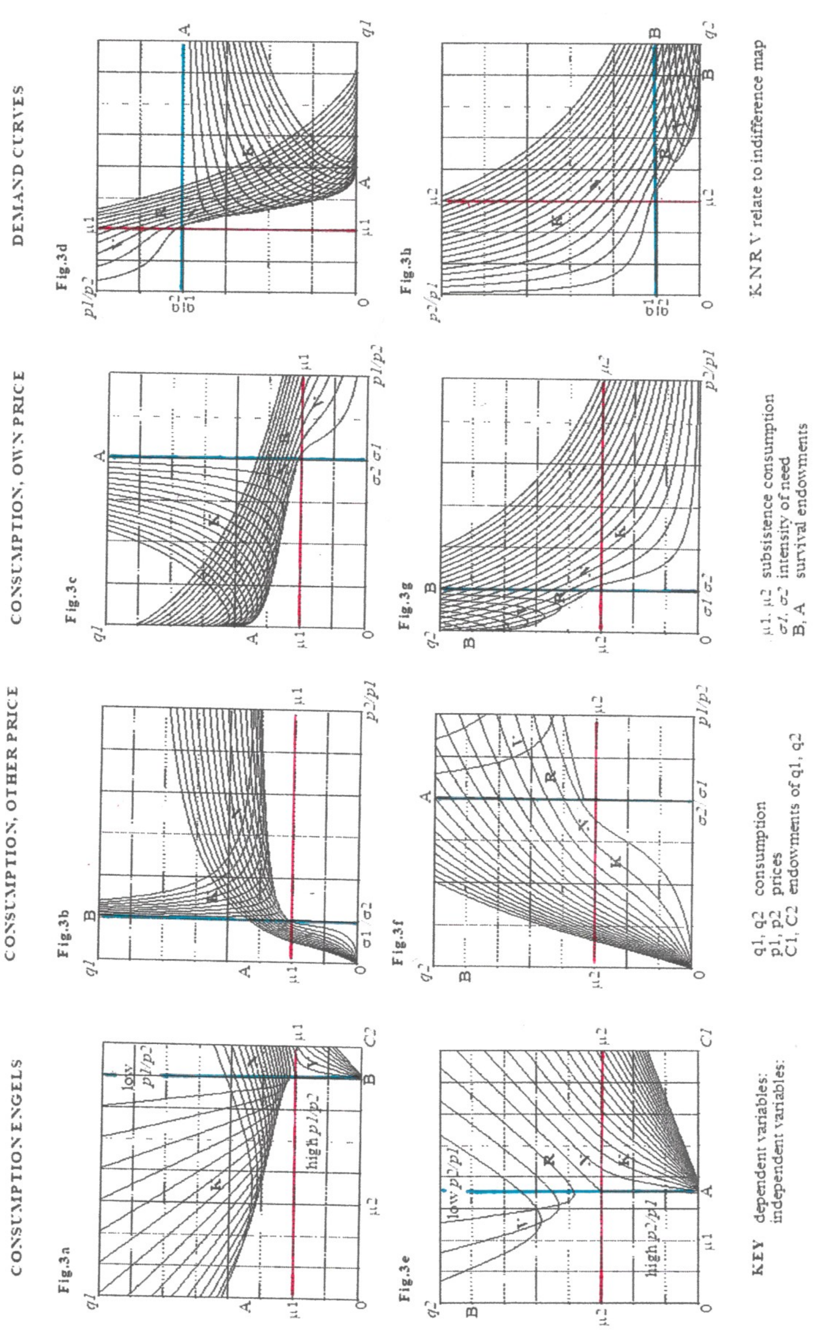


FIGURE 3 DERIVED FUNCTIONAL FORMS

In each of the eight diagrams of Figure 3, the dependent variable is either q_1 or q_2. The first diagram is of Engels curves, where consumption is plotted against endowments of the other good (or money). In the second and third diagrams, the dependent variable is plotted against other price and own price. The fourth diagram in each row rotates the axes to present the demand equations in the more familiar orientation.

These diagrams identify the areas K, N, R and V from the indifference curve map. Table 2 indicates the boundaries of these four areas *where q_1 is the dependent variable*. For instance, it indicates that K is always an area which has a sufficiency of q_1, combined with deprivation of q_2 and is facing low own relative price in the derived functional form diagrams. R is always in an area which has sufficiency of both q_1 and q_2 and faces high relative prices of q_1.

TABLE 2 Characteristics of areas K, N, R and V on the derived functional form diagrams, where q_1 is the dependent variable

	$> \mu_1$	$< \sigma_2/\sigma_1$	$> \sigma_2/\sigma_1$	$> \mu_2$
K	Yes	Yes	No	No
N	Yes	Yes	No	Yes
R	Yes	No	Yes	Yes
V	No	No	Yes	Yes

One of the aims of presenting eight diagrams together in Figure 3 is to enable patterns to be discerned. Although the demand curves have essentially the same pattern, they can appear to be very different in figs 3b, 3c, 3f and 3g.

Each diagram manifests the following:

- Each is divided into four quadrants by the dependent variable's own subsistence parameter and by a line representing survival endowments, A or B, or the relative intensities of need (σ_1/σ_2), all associated with the straight-line indifference curve, BA, which creates a disequilibrium in each diagram.
- In each, one quadrant is completely blank. For the Engels and own price diagrams this is the quadrant surrounding the origin and represents dysfunctional poverty.
- For the Engels and own price diagrams, the top left-hand quadrant displays a series of U-shaped curves creating an envelope curve that represents the boundaries between superior and inferior, or inferior and Giffen, responses respectively.
- In own price and Engels diagrams, the top two quadrants, (representing sufficiency in the dependent variable), display superior normal responses to changes in price and endowments, representing the responses of an individual when not deprived in either need, (areas N and R).
- The bottom right-hand quadrant represents sufficiency in the other dependent variable.

A few economists at various times in the past (Stonier and Hague, 1980, p.77; Hirschleifer, 1976, pp. 98 and 114) have tried to draw a series of demand curves for a commodity as it transforms from superior to inferior-normal (or from an inferior-normal to an inferior-Giffen good). With hindsight, it should have been intuitively obvious that there might be an envelope curve on demand curves, if one assumes that the demand curves slope down from left to right. If they shift to the right as unearned income increases for superior goods, and they also shift to the right as unearned income decreases for inferior goods, then the envelope must occur on the boundary between a good being inferior and its being superior.

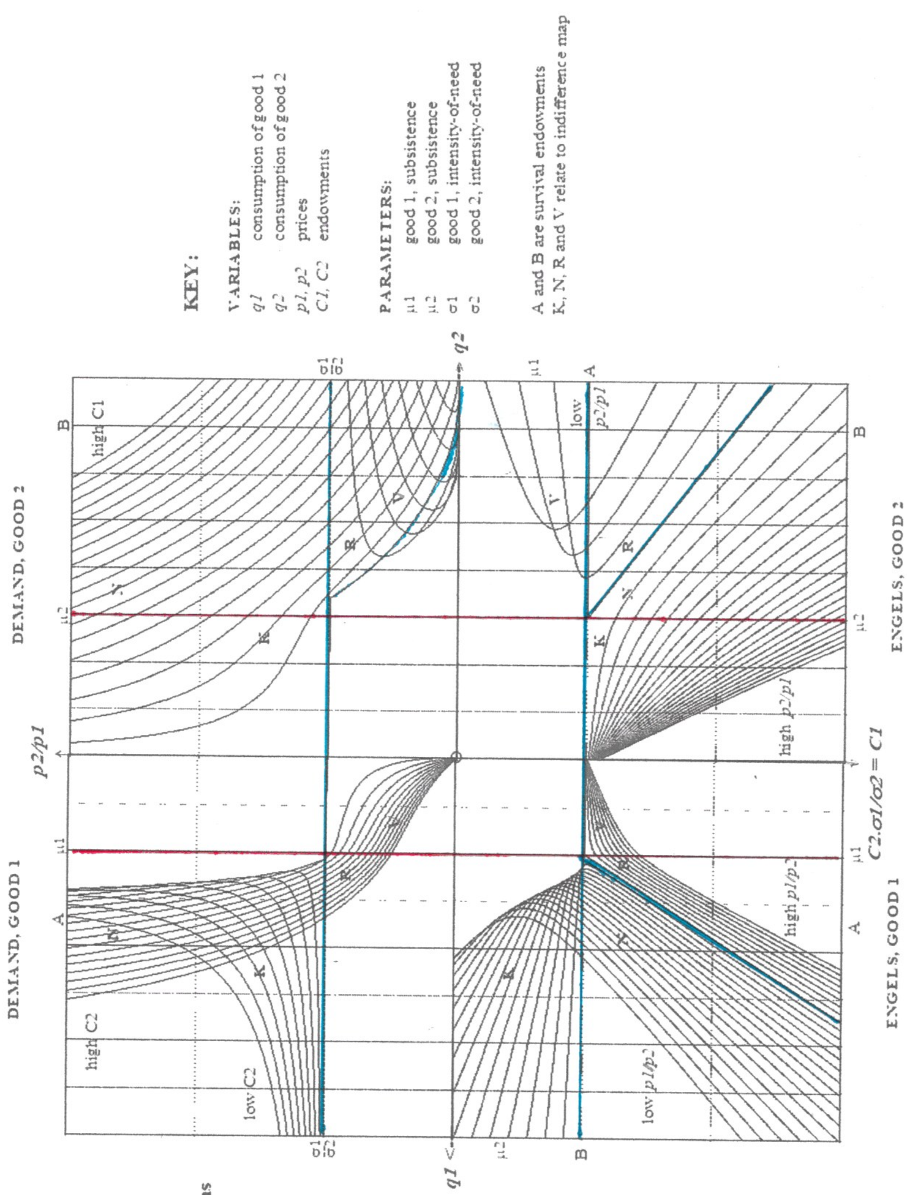


FIGURE 4.
Derived functional forms for the general case

Figure 4 has been created from Figures 3b, 3h, 3a and 3e, but with their axes rotated so that they can share the axes for q_1, p_2/p_1, q_2 and endowments. The straight-line indifference curve BA determines the survival endowments in figures 3a and 3e, and the scales have been adjusted to reflect this. The purpose of Figure 4 is to give a visual impression of how the different diagrams fit together, and to observe the patterns emerging. The four blank or nearly blank quadrants are now distinct as a 'slough of despond' in the middle, the only way out of which is to trade out with favourable relative prices, or for endowments to increase to survival level.

The top two diagrams (A and B) illustrate the corner solutions clearly. In top-right diagram B, if the relative price of good 2 is free, (p_2/p_1 = 0), then the individual can consume as much as s/he wants, but the relative price of good 1 is infinite and the individual cannot afford any of it. This a case of the poverty-stricken consumer choosing his/her own cheaper deprivation.

In the top-right diagram, (diagram B), the familiar downward-sloping demand curves for good 2 are obvious in all three quadrants furthest away from the origin. However, where the individual has sufficient in one good, combined with low own price, (in the lower right-hand quadrant), the familiar downward sloping demand curves are overlain by upward-sloping curves. In diagram B, as endowments (unearned income) increase, these curves shift leftwards. The individual consumes less of good 2, while still avoiding deprivation, and can afford a little more of good 1 in diagram A. As own price increases for the same level of endowments, demand for good 2 increases, displaying inferior good responses. This pattern continues until own price reaches the slope of BA, (p_2/p_1 = σ_1/σ_2). This pattern is replicated in each of the diagrams of the derived functional forms.

In the lower two diagrams (C and D), the consumption of each good is assumed to be dependent on *endowments of the other good*. If prices favour the dependent variable, then this can be translated into own consumption, but it decreases as own price increases, until it reaches the disequilibrium level.

Envelope curves are obvious in each of the four diagrams.

Demand curves associated with deprivation tend to be more elastic compared with those of the individual when experiencing sufficiency in both needs.

VI. CONCLUSION

The observations noted in previous sections can be summarised in Table 3.

TABLE 3 A comparison of the features of the indifference curve map and derived functional form diagrams

	Indifference curve map	**Diagrams of derived functional form with the dependent variable on the vertical axis**
1	The map is divided into four quadrants by the subsistence parameters, μ_1, μ_2.	Each diagram is divided into four quadrants by the dependent variable's own subsistence parameter, μ, and the relative intensity-of-need parameters, σ_1 *and* σ_2.

2.	The concave-to-the-origin indifference curves surrounding the origin represent dysfunctional poverty.	In the demand and Engels diagrams dysfunctional poverty leads to the quadrant surrounding the origin being blank. Corner solutions can be identified.
3.	The straight-line indifference curve, BA, separating concave- from convex-to-the-origin indifference curves, is an absolute poverty line with respect to these two needs. B and A represent survival endowments of the two goods.	BA leads to a disequilibrium in each of the derived functional form diagrams.
4.	The convex-to-the-origin indifference curves can be divided into four areas: K, N, R and V.	These four areas can be identified in the derived functional form diagrams.
5.	Areas K and V represent a sufficiency in one need combined with a relatively modest deprivation in the other, resulting in 'functional poverty'.	The curves in areas K and V are more elastic than in N and R, where the individual is not deprived in either need.
6.	A sufficiency in one need, combined with a relatively modest deprivation in the other, leads to 'inferior' responses, by the good that is fulfilled sufficiently, to changes in prices or endowments.	In own-price and Engels curves diagrams, a quadrant representing sufficient consumption combined with low own-price, or low endowments, displays a series of U-shaped curves creating an envelope curve that represents the boundaries between superior and inferior, or inferior and Giffen, responses respectively.
7.	In areas N and R, the individual is not deprived of either need.	In this same quadrant in own-price and Engels diagrams, the familiar superior normal curves representing *sufficiency in both needs* is overlain by the less familiar inferior response curves representing *deprivation in one of the needs*.

The most important outcome is the recognition that there are three different situations that an individual could face, resulting in different reactions to changes in prices and endowments.

- In dysfunctional poverty, s/he is trapped in a situation where (nearly) all of his/her options can only make him/her worse off.
- In functional poverty, the individual will have higher utility than in dysfunctional poverty and can optimise his/her situation, but s/he will remain deprived in one need.
- Only in sufficiency, not deprived in either need, will s/he enjoy true freedom of choice.

The introduction of separate needs explains some former anomalies in utility theory such as the incidence of inferior and Giffen responses, and why, for some commodities, responses to both low- and high-prices are very elastic.

Table 4 shows how the two propositions presented in this paper provide an integrating framework encompassing all three related parts of utility theory – the new ‘needs’ theory, together with Van Praag’s seminal work and the traditional neoclassical demand theory. The latter is based on the Law of Diminishing Marginal Utility, which cannot distinguish between multiplicative and additive versions. It remains very important for the analysis of products, firms and markets. Where the emphasis shifts to people and the fulfilment of their needs, Van Praag’s multiplicative function with bounded cardinal utilities is useful for analysing responses to commodities fulfilling a given need, and the additive bounded cardinal utilities can analyse utility and demands for commodities fulfilling different needs.

Table 4 TWO PROPOSITIONS FOR UTILITY THEORY

<table>
<tr><th>SHAPE of UTILITY →

SEPARABILITY below</th><th>Increasing MU
(deprivation of need)</th><th colspan="2">Diminishing MU
(sufficiency)</th></tr>
<tr><td>Added utilities</td><td>Extension of utility</td><td>theory</td><td rowspan="2">Traditional neoclassical theory</td></tr>
<tr><td>Multiplicative utilities</td><td colspan="2">Van Praag and the Leyden School.</td></tr>
</table>

The theory presented here was based on plausible psychological assumptions, specifying the shape of the utility function for the fulfilment of a need and enabling utility to be separated into classifiable needs. It can predict the responses of individuals when deprived in one or two dimensions of need. The two propositions together integrate many current, already well-established, *de facto* piecemeal results, explain some of the anomalies that arise with the traditional theory, and offer some further insights and novel predictions of its own. It extends the paradigm of traditional utility and demand theory, offering an additional perspective for analysis and application.

This utility theory integrates the analysis of demand where the emphasis is on people and the satisfaction of their needs, with traditional demand focusing on firms, production and goods satisfying the same need.

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Stern in Blundell and Walker – for 6c paper

APPENDIX

The Appendix includes the following:

- The derivation of an equation for the boundary between superior and inferior responses.
- However, the derivation for an equation for the boundary between inferior normal and inferior-Giffen responses, which must be solved numerically, does not yield credible observations yet.
- The derivation of the (very non-linear) functional form for the demand equation.
- The derivation of an equation for the envelope curve on the demand curves associated with the boundary between superior and inferior responses.
- However, a derivation of an equation for the envelope curve on an Engels curves diagram, associated with the boundary between inferior normal and inferior-Giffen responses, has not been forthcoming yet.

Boundary between superior and inferior responses

The optimality condition in equation (4) gives the locus of points describing the *income-consumption locus* for a given price ratio, p_1/p_2, on the indifference curve map.

By expressing equation (4) in terms of q_1, and differentiating with respect to q_2, dq_1/dq_2 can be found.

$$q_1 = \mu_1 + \sigma_1.\left[\left(\frac{q_2-\mu_2}{\sigma_2}\right)^2 - 2.\,ln\left(\frac{\sigma_1.p_1}{\sigma_2.p_2}\right)\right]^{\frac{1}{2}} \tag{4}$$

$$\frac{dq_1}{dq_2} = \frac{1}{2}.\left[\left(\frac{q_2-\mu_2}{\sigma_2}\right)^2 - 2.\,ln\left(\frac{\sigma_1.p_1}{\sigma_2.p_2}\right)\right]^{\frac{-1}{2}}.\left(\frac{2q_2-2\mu_2}{\sigma_2}\right).\left(\frac{\sigma_1}{\rho_2}\right) = 0.$$

$$\frac{dq_1}{dq_2} = \frac{\left(\frac{q_2-\mu_2}{\sigma_2}\right).\left(\frac{\sigma_1}{\sigma_2}\right)}{\sqrt{\left[\left(\frac{q_2-\mu_2}{\sigma_2}\right)^2 - 2.ln\left(\frac{\sigma_1.p_1}{\sigma_2\,.p_1}\right)\right]}} = 0. \tag{5}$$

By setting $dq_1/dq_2 = 0$, in equation (5), the **locus for the threshold between q_1 being superior and its being inferior** on the indifference curve map, is found to be coincidental with $q_2 = \mu_2$, for $q_1 > \mu_1$. This is the boundary between areas K and N. Similarly, $q_1 = \mu_1$, for $q_2 > \mu_2$ is the boundary between q_2 being inferior in area V and its being superior in area R.

Boundary between inferior normal and inferior-Giffen responses

The optimality condition in equation (4) gives the locus of points describing the *price ratio-consumption locus* for a given income, *M*, on the indifference curve map.

To obtain the **locus of points for the threshold between q_1 being inferior normal and its being inferior-Giffen**, the following procedure is adopted.

The budget equation is rearranged as $p_1 = (M - q_2.p_2)/q_1$, and p_1 is substituted into the optimality condition, equation (4), eliminating p_1 from equation (6),

$$\left(\frac{q_2 - \mu_2}{\sigma_2}\right)^2 - \left(\frac{q_1 - \mu_1}{\sigma_1}\right)^2 = ln\left(\frac{\sigma_1.p_1}{\sigma_2.p_2}\right)^2 \tag{4}$$

$$exp\left[\left(\frac{(q_2 - \mu_2)}{\sigma_2}\right)^2 - \left(\frac{(q_1 - \mu_1)}{\sigma_1}\right)^2\right] = \left[\left(\frac{(M - q_2.p_2)}{q_1.p_2}\right).\left(\frac{\sigma_1}{\sigma_2.}\right)\right]^2 . \tag{6}$$

To simplify the notation, let $\left(\frac{q_2 - \mu_2}{\sigma_2}\right)^2 - \left(\frac{q_1 - \mu_1}{\sigma_1}\right)^2$ = S

Re-arranging equation (6) in terms of M, gives:

$$M = q_2.p_2 + (q_1.p_2).\left(\frac{\sigma_2}{\sigma_1}\right).\sqrt{[exp(S)]}.$$

Differentiating M with respect to q_1 and q_2, yields the following:

$$\frac{dM}{dq_1} = p_2.\left(\frac{\sigma_2}{\sigma_1}\right).\sqrt{[exp(S)]} + (q_1.p_2).\left(\frac{\sigma_2}{\sigma_1}\right).\frac{1}{2}.[exp(S)]^{-\frac{1}{2}}.\exp(S).(-2)(q_1 - \mu_1)/\sigma_1^2.$$

$$\frac{dM}{dq_1} = p_2.(1 - q_1.(q_1 - \mu_1)/\sigma_1^2).\left(\frac{\sigma_2}{\sigma_1}\right).\sqrt{[exp(S)]}.$$

$$\frac{dM}{dq_2} = p_2 + (q_1.p_2).\left(\frac{\sigma_2}{\sigma_1}\right).\frac{1}{2}.[exp(S)]^{-1/2}exp(S).(+2)(q_2 - \mu_2)/\sigma_2^2.$$

$$\frac{dM}{dq_2} = p_2\left[1 + q_1(q_2 - \mu_2)/(\sigma_2^2).\left(\frac{\sigma_2}{\sigma_1}\right).\sqrt{[exp(S)]}\right].$$

Using implicit differentiation, and dividing through by p_2, dq_2/dq_1 is obtained and set equal to zero, eliminating M, and resulting in equation (7).

$$\frac{dq_2}{dq_1} = \frac{dM}{dq_2} \Big/ \frac{dM}{dq_1}.$$

$$\frac{dq_2}{dq_1} = \frac{(1 + q_1.(q_2 - \mu_2)/\sigma_2^2).\left(\frac{\sigma_2}{\sigma_1}\right).\sqrt{[exp(S)]}}{\left[1 - q_1(q_1 - \mu_1)/(\sigma_1^2).\left(\frac{\sigma_2}{\sigma_1}\right).\sqrt{[exp(S)]}\right]} = 0.$$

$$\sqrt{\left[exp\left(\left(\frac{q_{2}-\mu_2}{\sigma_2}\right)^2-\left(\frac{q_1-\mu_1}{\sigma_1}\right)^2\right)\right]}\ .(q_1).\left(\frac{\sigma_2}{\sigma_1}\right).\frac{(q_2-\mu_2)}{\sigma_2^2}=-1$$

$$\left[exp\left(\left(\frac{q_{2}-\mu_2}{\sigma_2}\right)^2-\left(\frac{q_1-\mu_1}{\sigma_1}\right)^2\right)\right]\left[\left(\frac{q_1}{\sigma_1}\right).\left(\frac{q_2-\mu_2}{\sigma_2}\right)\right]^2=+1$$

$$[\left(\frac{q_{2}-\mu_2}{\sigma_2}\right)^2-\left(\frac{q_1-\mu_1}{\sigma_1}\right)^2].2.\ln\left[\left(\frac{q_1}{\sigma_1}\right).\left(\frac{(q_2-\mu_2)}{\sigma_2}\right)\right]=0, \text{ for } q_2<\mu_2. \qquad (7)$$

Equation (7) must be solved numerically to find the solutions for q_2 for given values of q_1, or *vice versa*.

However, this does not give credible solutions!

Derivation of the demand equation

Let $x = p_1/p_2$ (relative prices); b = σ_2/σ_1 (relative intensities-of-need);
a = (C_1 - μ_1) and g = ($C_2 - \mu_2$) (supernumerary endowments).

Using the following short-hand notation for elements that appear frequently,

$M = C_1.p_1 + C_2.p_2.$ (full income)
$Z = (C_1 - \mu_1).p_1 + (C_2 - \mu_2).p_2$
$= M - \mu_1.p_1 - \mu_2.p_2,$ (supernumerary expenditure),
$M = q_1.p_1 + q_2.p_2$ (budget equation)

and substituting for $q_2 = (M - q_1.p_1)/p_2$, from the budget constraint,
and for $M = Z + \mu_1.p_1 + \mu_2.p_2$ from the supernumerary expenditure equation,
into equation (4), yields an '**implicit demand equation**' (8):

$$\left(\frac{q_{2}-\mu_2}{\sigma_2}\right)^2 - \left(\frac{q_1-\mu_1}{\sigma_1}\right)^2 = ln\left(\frac{\sigma_1.p_1}{\sigma_2.p_2}\right)^2 \qquad (4)$$

$$\left[\frac{\frac{Z}{p_2}-(q_1-\mu_1).x}{\sigma_2}\right]^2 = \left[\frac{(q_1-\mu_1)}{\sigma_1}\right]^2 + ln\left[\frac{x}{b}\right]^2 \qquad (8)$$

$$\frac{\sigma_1^2.\left(\frac{Z}{p_2}\right)^2-2.\sigma_1^2(q_1-\mu_1).x.\left(\frac{Z}{p_2}\right)+\sigma_1^2.(q_1-\mu_1)^2.x^2-\sigma_2^2.(q_1-\mu_1)^2-2.\sigma_1^2.\sigma_2^2.ln\left(\frac{x}{b}\right)}{\sigma_1^2.\sigma_2^2} = 0.$$

$$\frac{(\sigma_1^2.x^2-\sigma_2^2).(q_1-\mu_1)^2 - 2.\sigma_1^2.\left(\frac{Z}{p_2}\right).x.(q_1-\mu_1) + \sigma_1^2.\left(\frac{Z}{p_2}\right)^2-2.\sigma_1^2.\sigma_2^2.ln\left(\frac{x}{b}\right)}{\sigma_1^2.\sigma_2^2} = 0.$$

which is a quadratic equation in ($q_1 - \mu_1$), which is solved using the negative square root, yielding **demand equation** (9) for the first commodity:

Let the root be $(q_1 - \mu_1) = \frac{-b \pm \sqrt{b^2 - 4.a.c}}{2.a}$;

$$a = (\sigma_1^2.x^2 - \sigma_2^2)\,; \qquad b = -2.\sigma_1^2.x.Z/p_2\,; \qquad c = \sigma_1^2.\left(\left(\frac{Z}{p_2}\right)^2 - \sigma_2^2.2.ln(x/b)\right).$$

$$(q_1 - \mu_1) = \frac{2.\sigma_1^2.\left(\frac{Z}{p_2}\right).x \pm \sqrt{\left[4.\sigma_1^4\left(\frac{Z}{p_2}\right)^2.x^2 - 4.(\sigma_1^2.x^2 - \sigma_2^2).\sigma_1^2.\left(\left(\frac{Z}{p_2}\right)^2 - .\sigma_2^2.2.ln(x/b)\right)\right]}}{2.(\sigma_1^2.x^2 - \sigma_2^2)}$$

$$(q_1 - \mu_1) = \frac{\sigma_1^2.\left(\frac{Z}{p_2}\right).x \pm \sqrt{\left[.\sigma_1^2.\sigma_2^2.\left(\frac{Z}{p_2}\right)^2 + (\sigma_1^2.x^2 - \sigma_2^2).\left(\sigma_1^2.\sigma_2^2.2.ln(x/b)\right)\right]}}{(\sigma_1^2.x^2 - \sigma_2^2)}$$

Using the negative square root and dividing numerator and denominator by σ_1^2.

$$(q_1 - \mu_1) = \frac{\left(\frac{Z}{p_2}\right).x - \sqrt{\left[b^2.\left(\frac{Z}{p_2}\right)^2 + (x^2 - b^2).\left(\sigma_2^2.\left(\frac{\sigma_1^2}{\sigma_1^2}\right)2.ln(x/b)\right)\right]}}{(x^2 - b^2)}$$

$$q_1 = \mu_1 + \frac{\left(\frac{Z}{p_2}\right).x - b.\sqrt{\left[\left(\frac{Z}{p_2}\right)^2 + (x^2 - b^2).\left(\sigma_1^2.2.ln\left(\frac{x}{b}\right)\right)\right]}}{(x^2 - b^2)} \qquad (9)$$

q_1 is a function of relative prices, x, and supernumerary income, Z.

$Z/p_2 = (C_1 - \mu_1).p_1/p_2 + (C_2 - \mu_2).p_2/p_2 = a.x + g.$ $\qquad (Z/p_2).x = a.x^2 + g.x.$

Thus, an alternative version of the demand equation is given by:

$$q_1 = \mu_1 + \frac{((C_1 - \mu_1)..x + (C_2 - \mu_2)).x - b.\sqrt{\left[((C_1 - \mu_1).x + (C_2 - \mu_2))^2 + (x^2 - b^2).\left(\sigma_1^2.2.ln\left(\frac{x}{b}\right)\right)\right]}}{(x^2 - b^2)} \qquad (10)$$

When both $M \geq Z$, and the budget line is parallel to the straight-line indifference curve, and thus $x^2 = b^2$, and using the negative root, equation (9) simplifies to

$$q_1 = \mu_1 + \frac{\left(\frac{Z}{p_2}\right).x - \mathrm{b}.\sqrt{\left(\frac{Z}{p_2}\right)^2}}{(\mathrm{x} - \mathrm{b}).(x + b)}.$$

$$q_1 = \mu_1 + \frac{Z/p_2}{(x+b)} \tag{11}$$

Envelope curve on the demand equations representing the boundary between superior and inferior responses.

By differentiating q_1 in the demand function, equation (9), with respect to Z, and setting the partial derivative equal to zero, one obtains:

$$\frac{dq_1}{dZ} = \frac{\frac{x}{p_2} - \frac{2.b.Z}{p_2^2}.\frac{1}{2}.\left[\left(\frac{Z}{p_2}\right)^2 + (x^2 - b^2).\left(\sigma_1^2 .2.ln\left(\frac{x}{b}\right)\right)\right]^{-1/2}}{(x^2 - b^2)} = 0.$$

Re-arranging this and squaring both sides, in order to express it in terms of z, gives:

$$\left(\frac{x}{p_2}\right)^2.\left[\left(\frac{Z}{p_2}\right)^2 + (x^2 - b^2).\left(\sigma_1^2 .2.\,ln\left(\frac{x}{b}\right)\right)\right] = \left(\frac{b.Z}{p_2^2}\right)^2$$

$$\left(\frac{x}{p_2}\right)^2.\left(\frac{Z}{p_2}\right)^2 - \left(\frac{b.Z}{p_2^2}\right)^2 = -\left[\left(\frac{x}{p_2}\right)^2.(x^2 - b^2).\left(\sigma_1^2 .2.\,ln\left(\frac{x}{b}\right)\right)\right]$$

$$\left(\frac{Z}{p_2^2}\right)^2 (x^2 - b^2) = -\left[\left(\frac{x}{p_2}\right)^2.(x^2 - b^2).\left(\sigma_1^2 .2.\,ln\left(\frac{x}{b}\right)\right)\right]$$

$$\left(\frac{Z}{p_2^2}\right)^2 = -\left[\left(\frac{x}{p_2}\right)^2.\left(\sigma_1^2 .2.\,ln\left(\frac{x}{b}\right)\right)\right]$$

$$\left(\frac{Z}{p_2}\right)^2 = -\left[x^2.\left(\sigma_1^2 .2.\,ln\left(\frac{x}{b}\right)\right)\right]$$

$$Z^2 = -\left[p_2^2.x^2.\left(\sigma_1^2 .2.\,ln\left(\frac{x}{b}\right)\right)\right]$$

$Z = \sigma_1.p_1\sqrt{\left[-2.\,ln\left(\frac{x}{b}\right)\right]}$, if (x/b) < 1; that is, x < b, or $p_1/p_2 < \sigma_2/\sigma_1$.

$Z = \sigma_1.p_1\sqrt{\left[+2.\,ln\left(\frac{b}{x}\right)\right]}$, if (b/x) > 1. (12)

Substituting for z from equation (12) into equation (9) gives the envelope curve on the demand equations, for $p_1/p_2 \leq \sigma_2/\sigma_1$.

$$q_1 = \mu_1 + \frac{\left(\frac{Z}{p_2}\right).x - b.\sqrt{\left[\left(\frac{Z}{p_2}\right)^2 + (x^2 - b^2).\left(\sigma_1^2.2.ln(x/b)\right)\right]}}{(x^2 - b^2)} \tag{9}$$

$$q_1 = \mu_1 + \frac{\sigma_1\left(\frac{p_1}{p_2}\right).\sqrt{\left[+2.ln\left(\frac{b}{x}\right)\right]}x \; - \; b.\sqrt{\left[-\sigma_1^2.x^2.2ln\left(\frac{x}{b}\right) + (x^2-b^2).\left(\sigma_1^2.2.ln(x/b)\right)\right]}}{(x^2 - b^2)}$$

$$q_1 = \mu_1 + \frac{\sigma_1.x^2.\sqrt{\left[+2.ln\left(\frac{b}{x}\right)\right]} \; - \; \sigma_1.b^2.\sqrt{\left[+\left(.2.ln(b/x)\right)\right]}}{(x^2 - b^2)}$$

$$q_1 = \mu_1 + \sigma_1.\sqrt{\left[+2.ln\left(\frac{b}{x}\right)\right]}\;,\; for\,\frac{p_1}{p_2} < \frac{\sigma_2}{\sigma_1}. \tag{13}$$

Envelope curve on the demand equations between Inferior and Giffen (Workings)

The derived functional form, equation (9), has been used to create the four diagrams comprising Figure 4. It illustrates the envelope curve in each diagram clearly.

The two top diagrams display the envelope curve associated with the move from inferior to superior responses. In the top left-hand diagram (diagram A), it occurs in its top left-hand quadrant. In the top right-hand diagram (diagram B), it occurs in its bottom right-hand quadrant. These envelope curves are associated with low values of own relative price.

The two lower (Engels curves) diagrams (diagrams C and D) also display envelope curves associated with the shift from inferior normal to inferior-Giffen responses, in their top left- and top right-hand quadrants respectively. These are associated with low values of endowments of unearned consumption.

However, Fig.3a is probably an easier diagram to follow.

PROCEDURE.

Starting with equation (10).

$$q_1 = \mu_1 + \frac{\left((C_1-\mu_1).x^2+(C_2-\mu_2)\right).x \; - b.\sqrt{\left[\left((C_1-\mu_1).x+(C_2-\mu_2)\right)^2 + (x^2-b^2).\left(\sigma_1^2.2.ln\left(\frac{x}{b}\right)\right)\right]}}{(x^2 - b^2)} \tag{10}$$

where:
q_1 = consumption of good 1 in Fig.3a.
p_2 = price of the other good.
$x = p_2/p_1$ (relative prices).
C_1 and C_2 are endowments of goods 1 and 2.

μ_1 and μ2 are the subsistence consumption parameters for needs 1 and 2.
σ_1 is the intensity-of-need parameter for need 1.
b is the ratio of the intensity of need parameters, σ_1/σ_2, for needs 1 and 2.

Since this is merely a scaling factor, it has been assumed to be 1.

In the general case of two needs, it is assumed that for q_1, the emphasis is on C_2, and that $C_1 = 0$. Thus, q_1 is a function of x and C_2.
Let $a = (C_1 - \mu_1)$ and $g = (C_2 - \mu_2)$

$$q_1 = \mu_1 + \frac{(a.x+g).x \; - \; \sqrt{\left[(a.x+g)^2 + (x^2-1).\left(\sigma_1^2.2.ln(x)\right)\right]}}{(x^2-1)} \qquad (14)$$

PROCEDURE: To obtain the envelope curve, differentiate q_1 in equation (14) with respect to x, and set $dq_1/dx = 0$. Re-arrange the new equation in terms of x and insert x back into equation (14), thus eliminating x, leaving q_1 as a function of C_2, (that is, $q_1 = f(C_2)$ is the envelope curve).

Differentiating equation (14) involves a chain of a square root, a product, and a natural logarithm, as follows:

Let $q_1 = \mu_1 + \frac{U - W.}{V}$

Let $U = a.x^2 + g.x$ $\quad \frac{dU}{dx} = 2.a.x + g.$

Let $V = (x^2 - 1).$ $\quad \frac{dV}{dx} = 2.x.$

Let $W = \sqrt{[\ldots]}.$ $\quad \frac{dW}{dx} = \frac{g(a+g.x) + \sigma_2^2.\left(X.2.\ln(x) + (x^2-1).2/x\right)}{W}; \quad \frac{d\ln(x)}{dx} = \frac{1}{x}, for\ X > 0.$

$\frac{dW}{dx} = \frac{H}{W}.$ H is not a function of W

$$\frac{dq_1}{dx} = \frac{V.(U'-W') - V'.(U-W)}{V^2} = 0.$$

$$\frac{dq}{dx} = \frac{V.\left(U' - \frac{H}{W}\right) - V'.(U - W)}{V^2} = 0$$

Multiplying numerator and denominator by W gives:

$$\frac{dq}{dx} = \frac{V.(U' \cdot W - H) - V'.(U - W).W}{W.V^2} = 0$$

$$V.U'.W - V.H - V'.U.W - V'.W^2 = 0.$$

$$V'.W^2 - (V.U' - V'.U).W + V.H = 0$$

This equation is a quadratic in W.

$$W = \frac{-b \pm \sqrt{(b^2 - 4ac)},}{2a}$$

Where a = V'; b = (V'.U – V.U'); c = V.H.

$$W = \frac{-(V'.U - V.U') \pm \sqrt{[(V'.U - V.U')^2 - 4.V'.V.H]}}{2.V'} \qquad (15)$$

Equation (15) is deemed too complex a function to proceed further.

However, equation (9) produces the inferior normal to inferior-Giffen envelope curves naturally in fig.3a, fig.3e and in the two lower diagrams, C and D, of Figure 4.