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Abstract

This paper provides a simple dynamic model that explores the interdependence and dynamic properties of hate, violence and economic well-being. It shows that a time-dependent economic growth process that affects the evolution of hate can yield a long-run steady state, but this steady state will not be free of hate and violence. Moreover, we show that better (long-run) economic conditions do not necessarily result in lower equilibrium levels of hate and violence. We also show that, under reasonable conditions, cycles of hate and violence cannot occur. Consequently, the dynamic properties of hate and violence themselves cannot result in cyclical patterns of (net) economic well-being. While stable and unstable equilibria are possible, the most likely equilibrium is a saddle point.

We provide several numerical examples demonstrating the implications of psychological attributes such as congruence (reciprocity), long memory and jealousy on the nature of the steady state and stability of the equilibria. These examples also consider the role of responsiveness to economic conditions, externalities and susceptibility to violence.

Given its nature, the paper is an example of a formal model for the ideas of the "dynamical system" literature in psychology.

JEL Classification: D74; H56; C;61.
Keywords: Hate, Violence, Dynamics, Steady State, Stability, Genuine Peace.

1 Introduction

Hate and violence have been common features of human history. Secular and religious scholars, as well as leaders and warriors, have discussed their nature throughout history. As far back as the 5th century BC, Thucydides wrote about the nature of violence, observing that wars lead to even worse wars. The Bible, likewise, recognized that[1] "violence begets violence." The relationship between hate and violence is complex. What is clear, however, is that the two are very closely intertwined: they affect and are affected by each other. Making matters more complicated is that they are both affected by other factors, which, in turn, also impact their evolution. In the aftermath of 9/11, hate and violence have become the subject of renewed endless debates and extensive academic research. Indeed, a vast literature on the subject encompasses historical, philosophical, social, religious, psychological, political, economic and cultural aspects of hate and violence.[2]

Naturally, each discipline has its focus, tools and perspectives in the academic literature, possibly its own pre-conceptions. Thus, for example, many sociologists, psychologists and political scientists explain hate by underlying "root causes."[3] Economists, on the other hand, tend to explain all phenomena, including hate and violence, as the outcome of underlying optimal decision-making processes (in addition to root causes).[4] Therefore, individuals' or governments' behaviour is explained as the outcome of underlying strategic decisions.[5] Interestingly, even non-economists often argue that economic considerations such as competition over scarce resources explain hate, extremism and violence.[6] Here, too, government policies, including incitement in the face of such rivalry, are considered strategic.

Recently, psychologists began looking at the dynamic properties of conflicts as an important element in explaining what is referred to as "intractable conflicts."[7] These types of conflicts, typically vicious, persistent, costly and difficult to resolve, seem to defy rationality and are difficult to explain using standard models (in psychology, games theory, bargaining, economics, conflict resolution, etc.). In fact, they argue that the apparent lack of rationality suggests that such conflicts are driven by "a psychology-dynamics" that seems to have a "life of its own." As Vallacher et al. (2010) put it: "It is as though the conflict acts like a gravity well into which the surrounding mental, behavioral, and social-structural landscape begins to slide." Thus, in their view, the nature of such conflicts can only be understood within a dynamic framework, or what they

[1] Matthew 26:52, King James Bible version: "for all they that take the sword shall perish with the sword."

[2] For example, Sternberg and Sternberg (2008) provide a comprehensive discussion of the psychological aspects of hate, and Nozick (1997) and Breton (2002) provide a general discussion of political, economic and philosophical aspects of extremism.

[3] For examples of studies of root causes, see Blomberg and Hess (2002), Blomberg et al. (2004), Bandarage (2004) and Sandler and Enders (2004).

[4] See discussions in Glaeser (2005) and Cameron (2009).

[5] See, for example, Schelling (1960), Atkinson, Sandler and Tschirhart (1987), Enders and Sandler (2006), Sørli, Gleditsch and Strand (2005), Collier and Hoeffler (1998), Esteban and Ray (1999), Wintrobe (2006), Appelbaum and Katz (2007), Appelbaum (2008), Medoff (1999).

[6] See, for example, Sternberg and Sternberg (2008), Piazza (2006), Eizenstat Porter and Weinstein (2005) and Pape (2003). See also Sherif (1966), who proposes a "realistic conflict theory."

[7] See Vallacher et al. (2010), (2013).

call a “dynamical system.”[8] Specifically, they suggest we can view intractable conflicts as particular equilibria, or "strong attractor states," of a dynamic interaction process between the groups involved, their emotions, actions and histories.

The nonlinear dynamical systems theory was adopted to provide a framework for analyzing chaotic processes in conflicts, psychology, political science and other social sciences (see Guastello (2005)). For example, Guastello (2008) provides conflicts with alternative pathways to chaotic behaviour and empirical techniques for learning the nature of the underlying conflicts.

This paper aims to develop a simple model that can be viewed as an example of such an intractable conflict and study the dynamic properties of hate and violence. Instead of focusing on possible root causes or strategic policy determinants of hate, we focus on its evolution and properties. We do not argue that root causes or strategic considerations do not play a role; we acknowledge that they do. But, although they may affect hate, the level of hate and its evolution are not a matter of choice by individuals; motion (evolution) equations govern them. Even strategic policymakers must take these motion equations into account. We can think of strategic policymakers as using these motion equations to "their advantage." That is, a strategic policy is a way of "manufacturing" root causes: it can induce hate directly (e.g., by incitement) or indirectly (by affecting the root causes). Paraphrasing Herman and Chomsky (1988), these strategic policies serve to "manufacture dissent." Nevertheless, since the evolution of hate is still governed by the motion rules, strategic policies must also consider these rules.

The paper examines the dynamics of hate and violence in a conflict between two "entities" (which can be viewed as countries, rival groups, communities, etc.) with a history of conflict, hate and violence.[9] Ethnic, racial or religious conflicts may explain the history of the relationship. It may also be due to geographical, ideological, or economic conflicts. In general, the root causes of the conflict reflect a combination of these underlying factors. But, regardless of the underlying root causes, hate and violence are clearly interdependent within such a relationship: they affect and are affected by each other. Both the literature on the psychology of hate and the political/sociological theory of conflicts recognize that violence and hate may result in a "vicious circle": violence breeds hate, but hate breeds violence.[10] Moreover, economic considerations also become part of this vicious circle. Economic well-being directly affects the evolution of hate (presumably, an improvement in economic well-being mitigates the evolution of hate). But, economic well-being itself is affected by costly violence,[11] which in turn is affected by hate.

This paper captures the interdependent relationships among these factors and their effect on hate evolution within a system of differential equations. We use the model to study the dynamic properties of hate and

[8] See Vallacher, et al. (2010), (2013), Nowak and Lewenstein, M. (1994).

[9] In principle, a similar model applies to personal conflicts.

[10] See Minow (2002), Levin and Rabrenovic (2001).

[11] It is also possible that “net economic well-being” is affected by captured resources, hence giving rise to a strategic motive for the conflict.

violence. We begin by defining an ideal state of "genuine peace,"[12] in which there is neither hate nor violence. Given a time-dependent economic growth process (that affects the evolution of hate), we then show that a long-run steady-state is possible, but, generally, such a steady-state will not be characterized by genuine peace. Moreover, we show that a better long-run economic environment does not necessarily result in lower equilibrium levels of hate and violence. Next, we examine the stability properties of the (hate and violence) equilibrium. First, we show that, under reasonable conditions (when both rivals are "congruent" or reciprocating countries who are attuned to each other's nature), cycles of hate and violence cannot occur. Consequently, the dynamic properties of hate and violence cannot give rise to cyclical patterns of (net) economic well-being; such cyclical patterns can only arise due to cyclical patterns of gross growth rates. We demonstrate that for cycles to occur, we would need to have a case where, for one and only one country, reciprocity (congruency) does not hold. Second, we show that while it is possible to have either stable or unstable equilibria under reasonable conditions, it is more likely to have unstable ones. Specifically, the most likely outcome is a saddle point.

In its focus on the dynamics of hate and violence, this paper provides an example of a formal model that incorporates some of the ideas discussed in psychology's "dynamical system" literature (see references above). In particular, this paper (i) demonstrates the dynamic interaction of hate and violence and its evolution, (ii) provides conditions required for equilibria of different types, (iii) examines the stability properties of the equilibria and (iv) shows how these are all affected by exogenous variable. Consequently, issues that are otherwise difficult to resolve can be naturally resolved within this model. It should be noted, however, that the main focus of this paper is not on the possibility of or the routes to chaotic patterns.

2 The Model

Consider the dynamic relationship between two countries (rival groups, communities, etc.) with a history of conflict, hate and violence. To model the interdependence of the factors that affect the evolution of hate, we begin by looking at the determinants of violence. Let Country $i's$ hate toward Country j, $(i \neq j)$, at each point in time, t, be given by $h^i(t)$. We allow for the possibility that hate may be negative. In such a case, we can think of it as "love." We view hate as a state (stock) variable that summarizes the state of a country's antagonism in the conflict. Let Country $i's$ violence toward Country j, $(i \neq j)$, at each point in time, t, be given by $v_i(t)$.[13] We allow for the possibility that violence may be negative, in which case we can think of it as "benevolence." We view violence as a "flow variable." To capture the "vicious circle" aspect of violence,[14] we take a country's violence toward its rival to depend on three variables: its hate toward the rival, the rival's

[12] Discussed, for example, in Levi (2008).

[13] For example, the WHO defines violence as "the intentional use of physical force or power, threatened or actual, against oneself, another person, or against a group or community, that either results in or has a high likelihood of resulting in injury, death, psychological harm, maldevelopment, or deprivation."

[14] See, for example, Vallacher et al. (2010), (2013) and Sprott (2004).

level of hate and the rival's level of violence.

The following two continuously differentiable functions describe these relationships:

$$v_1(t) = V^1[h_1(t), h_2(t), v_2(t)] \quad (1)$$
$$v_2(t) = V^2[h_1(t), h_2(t), v_1(t)]$$

We assume that when hate in both countries is zero, the solution to the two equations is $v_1 = v_2 = 0$, namely: $V^i[0, 0, 0] = 0,\ i = 1, 2$.

Country $i's$ violence toward Country j ($\neq i$) is assumed to be an increasing function of its hate, the other country's hate and the other country's violence:

$$\frac{\partial V^i[h_1(t), h_2(t), v_2(t)]}{\partial h_j} > 0,\ i, j = 1, 2 \quad (2)$$
$$\frac{\partial V^i[h_1(t), h_2(t), v_2(t)]}{\partial v_j} > 0,\ i, j = 1, 2,\ i \neq j,$$

Let us now examine the role of economic considerations. Consider some measure of economic performance at time t, in Country i. For example, this may be Country $i's$ GDP, denoted as $x_i(t)$. We assume that any type of violence in the conflict (regardless of its source) has an economic cost. Suppose that a fraction of gross GDP is lost due to violence in the conflict. Let this fraction, denoted as c^i, be captured by the continuously differentiable cost function,

$$c^i = C^i[v_1(t), v_2(t)]),\ i = 1, 2$$

We assume that the cost function is increasing in violence (in either country), but when there is no violence, the costs are zero. Namely,

$$\frac{\partial C^i[v_1(t), v_2(t)]}{\partial v_j} > 0,\ i, j = 1, 2 \quad (3)$$
$$C^i[0, 0] = 0,\ i = 1, 2$$

Thus, *net* (of the cost of violence) GDP, denoted as $y_i(t)$, is given by:

$$y_i[v_1(t), v_2(t), t] \equiv x_i(t)(1 - C^i[v_1(t), v_2(t)]),\ i = 1, 2 \quad (4)$$

Now, define Country $i's$ *net* economic growth rate as:

$$w_i[v_1(t), v_2(t), t] \equiv d \ln[y_i[v_1(t), v_2(t), t]]/dt = \quad (5)$$
$$d \ln[x_i(t)]/dt + d \ln[1 - C^i[v_1(t), v_2(t)]]/dt \equiv g_i(t) + r^i[v_1(t), v_2(t), t]$$

where $g_i(t)$ is the gross rate of economic growth, and $r^i[v_1(t), v_2(t), t]$ is the rate of growth in the fraction of GDP that is *not* lost due to violence $(1 - C^i)$. Note that since $\partial C^i[v_1(t), v_2(t)]/\partial v_j > 0$, we have:

$$\frac{\partial r^i[v_1(t), v_2(t), t]}{\partial v_j} < 0, \;\; i, j = 1, 2 \tag{6}$$

Thus, since in the absence of violence, the net and gross rates of economic growth are the same, we have:

$$w_i[0, 0, t] = g_i(t) \tag{7}$$

Let us assume that the gross rate of economic growth converges to some "long-run" value of,

$$g_i^* = \lim_{t\to\infty} g_i(t) = \lim_{t\to\infty} w_i[0, 0, t].$$

We can now turn to the evolution of hate. We assume that the evolution of hate depends on the countries' levels of hate, violence, and net economic growth.[15]. The evolution of hate can, therefore, be described by the following two differential equations[16]:

$$\begin{aligned} \frac{dh_1}{dt} &= H^1(h_1, h_2, v_1, v_2, w_1) \\ \frac{dh_2}{dt} &= H^2(h_1, h_2, v_1, v_2, w_2) \end{aligned} \tag{8}$$

where H^1 and H^2 are continuously differentiable functions and where, for notational simplicity, the time variable, t, is dropped for the rest of the paper whenever it is not required. The dynamics of hate are complex because they depend on hate, violence and economic conditions; however, violence levels are determined simultaneously and depend on hate levels and affect net economic conditions.

Before considering the likely properties of the evolution equations, we should note that these properties may not be global for general and possibly nonlinear systems. Namely, properties may be different for different values (and signs) of the right-hand side variable in (8). Thus, in the following, when we refer to properties of the functional forms, they should be understood as local rather than global properties.

In principle, the response of a country's evolution of hate to its rival's hate falls into one of two categories: it may or may not be reciprocating ($\partial H^i/\partial h_{j\neq i} < 0$, or $\partial H^i/\partial h_{j\neq i} > 0$). Essentially, this captures the country's degree of congruence (attuned to its rival's nature). Similarly, the response of a country's evolution of hate to its hate falls into one of two categories: it may have a short or a long "memory," that is, high or low "depreciation" ($\partial H^i/\partial h_i < 0$, or $\partial H^i/\partial h_i > 0$). There are, therefore, four possible combinations to consider. These describe what we refer to as a country's (local) *intrinsic type.* They are shown in Table 1 below.

[15] Alternatively, rather than net economic growth, we can simply take net GDP levels or other measures of economic well-being as affecting the evolution of hate.

[16] In addition to the economic cost of violence (which enters through its effect on economic growth), it is also possible to add psychological costs of hate and violence affecting the evolution of hate. For example, defining the psychological costs in Country i as p_i, we would then have the two additional equations: $p_i(t) = P^i[h_1(t), h_2(t), v_i(t), v_2(t)]$, $i = 1, 2$. We will not pursue this further in this paper, but note that once these two equations, together with the violence equations, are solved simultaneously, the system will eventually similarly yield motion equations as in equations (17) below.

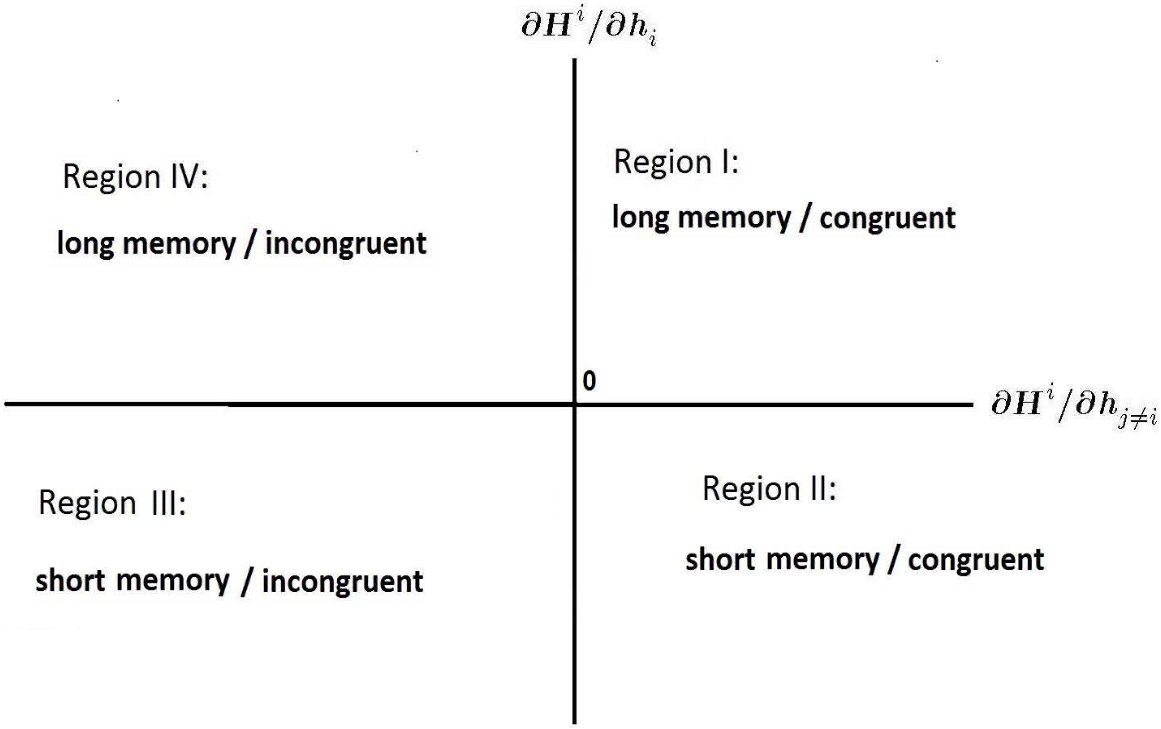


Table 1: The Nature of Country i - Effects of Hate

As we will show later, however, a country may have a (local) "perceived" type which is different from its (local) intrinsic one. This difference may occur for the following reasons. Violence and economic conditions depend on hate; consequently, a change in hate has both direct and indirect effects. The direct effect is what we referred to above as the intrinsic type (captured by $\partial H^i/\partial h_j$). The indirect effect captures the effect of a change in hate on the evolution of hate through its effects on violence and economic conditions (or other so-called "root causes"). The overall, or "perceived" effect, is the sum of the two, so there is no reason the overall effect should have the properties (say, sign) as the intrinsic type. We show this below.

As for the local effects of violence on a country's evolution of hate, here, too, we may have four cases. A country may be locally "masochistic" or vengeful ($\partial H^i/\partial v_{j\neq i} < 0$, or $\partial H^i/\partial v_{j\neq i} > 0$) with respect to violence by its rival. Moreover, a country's violence may give rise to a "need to justify" its actions by minimizing cognitive dissonance, hence (locally) affecting the evolution of hate positively ($\partial H^i/\partial v_i > 0$). On the other hand, it may exhibit (local) dissonance "affinity" or incongruence so that its own violence and the evolution of its hate are negatively related ($\partial H^i/\partial v_i < 0$). These cases are shown in Table 2:

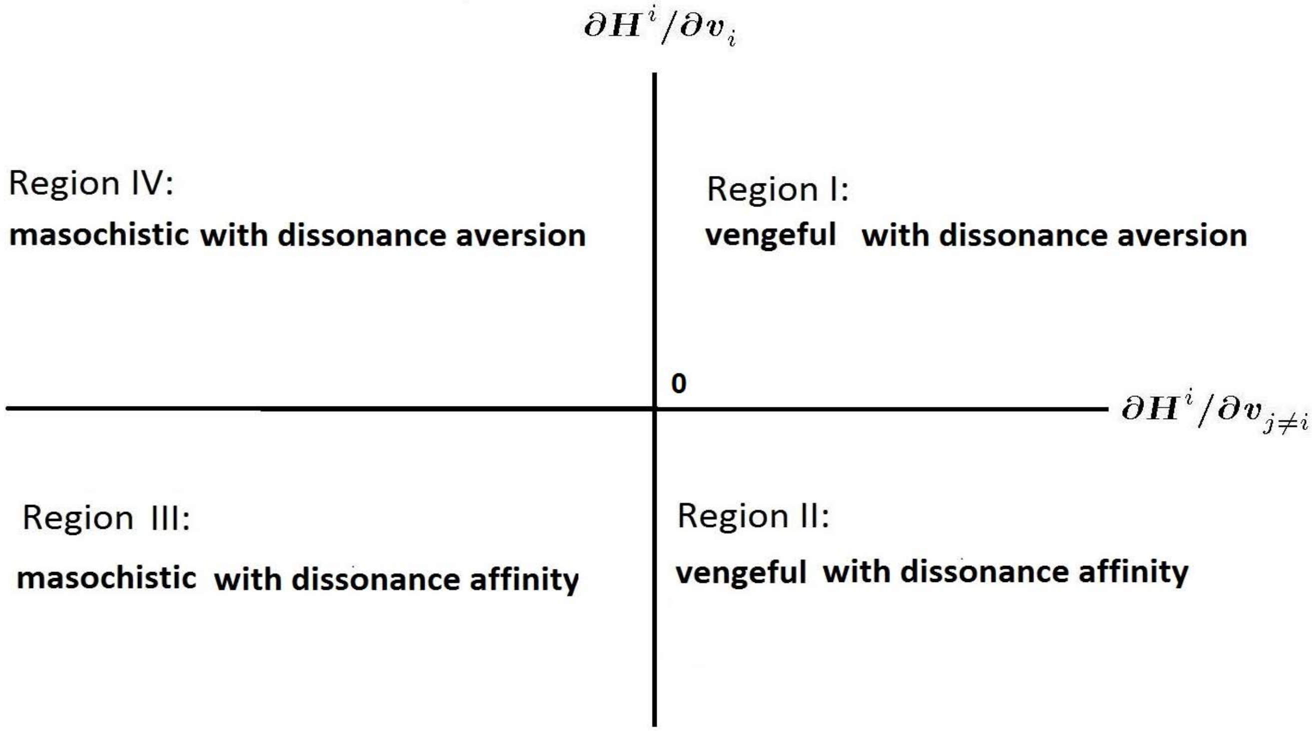


Table 2: The Nature of Country i - Effects of Violence

Although there are several possible configurations, not all are equally likely. Specifically, it does not seem likely that, in any region, an increase in the rival's levels of hate or violence would negatively affect a country's evolution of hate. Thus, we can expect that (globally),[17]

$$\frac{\partial H^i[h_1, h_2, v_1, v_2, w_1]}{\partial h_j} > 0, \quad i, j = 1, 2, \ i \neq j \tag{9}$$
$$\frac{\partial H^i[h_1, h_2, v_1, v_2, w_2]}{\partial v_j} > 0, \quad i, j = 1, 2, \ i \neq j$$

Furthermore, the literature on the psychology of hate has found that (due to the need to minimize cognitive dissonance) violence against a rival can lead to increased hate toward the rival.[18] Thus, locally, we may have[19]:

$$\frac{\partial H^i[h_1, h_2, v_1, v_2, w_i]}{\partial v_i} > 0, \quad i = 1, 2 \tag{10}$$

The local effect of a country's hate on the evolution of its hate, however, is not that clear. On the one hand, as is common in macroeconomic models, there may be persistence or inertia. On the other hand, there may be "depreciation" or willingness to forget.[20] Hence, it is not clear how the current level of a country's own hate affects its evolution of hate; $\partial H^i/\partial h_i$ may be locally positive, negative, or zero.[21]

[17] This is related to positive feedback natural systems. See DeAngelis, et. al. (1985).
[18] See Festinger and Carlsmith, (1959), Sternberg and Sternberg (2008).
[19] Note that, in principle, we could end up with a case of a vengeful but "remorseful" country.
[20] This is common in macroeconomic models and models of love. See, for example, Strogatz (1994), Rinaldi (1998), and Sprott (2004). Levy (2008), however, allows the effects of a country's own hate on its evolution of hate to be positive, negative, or zero.
[21] The Israeli-Palestinian conflict, for example, seems to be characterized by global "long memory"; $\partial H^i/\partial h_i > 0$.

Finally, it is well recognized in the literature that economic well-being affects conflicts' nature, prevalence and severity.[22] Thus, we should generally expect a country's net economic growth to affect its hate evolution negatively. In other words, locally, we would have:[23]

$$\frac{\partial H^i[h_1,h_2,v_1,v_2,w_i]}{\partial w_i} < 0, \quad i=1,2$$

Since $w_i = g_i + r^i$, this implies that we would have:

$$\frac{\partial H^i[h_1,h_2,v_1,v_2,w_i]}{\partial g_i} < 0, \quad \frac{\partial H^i[h_1,h_2,v_1,v_2,w_i]}{\partial r_i} < 0, \; i=1,2$$

Our model is, therefore, described by equations (1), (5) and (8). To study its dynamic properties, we first solve for the flow (non-state) variables v_1, v_2, w_1, w_2 in terms of the state variables h_1 and h_2. Let us begin with the violence equations in (1). Write these equations as:

$$\begin{aligned} v_1(t) - V^1[h_1(t),h_2(t),v_2(t)] &\equiv F^1[v_1(t),v_2(t);h_1(t),h_2(t)] = 0 \\ v_2(t) - V^2[h_1(t),h_2(t),v_1(t)] &\equiv F^2[v_1(t),v_2(t);h_1(t),h_2(t)] = 0 \end{aligned} \tag{11}$$

The corresponding Jacobian, denoted by F, is given by:

$$F \equiv \begin{bmatrix} 1 & -\frac{\partial V^1}{\partial v_2} \\ -\frac{\partial V^2}{\partial v_1} & 1 \end{bmatrix} \tag{12}$$

For a solution to exist, we must have the following:[24]

$$|F| = 1 - \frac{\partial V^1}{\partial v_2}\frac{\partial V^2}{\partial v_1} \neq 0$$

Assuming that this condition holds, let the solution to equations (1) be given by:

$$\begin{aligned} v_1^* &= V^{*1}(h_1,h_2) \\ v_2^* &= V^{*2}(h_1,h_2) \end{aligned} \tag{13}$$

Now, consider the effects of an increase in hate on violence. Using equations (1) we have:

$$\begin{bmatrix} 1 & -\frac{\partial V^1}{\partial v_2} \\ -\frac{\partial V^2}{\partial v_1} & 1 \end{bmatrix} \begin{bmatrix} \frac{dV^1}{dh_i} \\ \frac{dV^2}{dh_i} \end{bmatrix} = \begin{bmatrix} \frac{\partial V^1}{\partial h_i} \\ \frac{\partial V^2}{\partial h_i} \end{bmatrix}, \; i=1,2$$

Thus,

$$\begin{bmatrix} \frac{dV^1}{dh_i} \\ \frac{dV^2}{dh_i} \end{bmatrix} = \begin{bmatrix} \frac{\partial V^1}{\partial h_i} + \frac{\partial V^1}{\partial v_2}\frac{\partial V^2}{\partial h_i} \\ \frac{\partial V^2}{\partial h_i} + \frac{\partial V^2}{\partial v_1}\frac{\partial V^1}{\partial h_i} \end{bmatrix} \frac{1}{|F|} \tag{14}$$

[22] For example, Muller and Weede (1990) and Blomberg, Hess and Weerapana (2004) find that high levels of economic well-being measures (e.g., high rates of economic growth) reduce the incidence of terrorism and political violence. Honaker (2004) reports that increases in Catholic unemployment lead to increases in Republican violence, and increases in Protestant unemployment lead to increases in Loyalist violence. Similarly, Santos Bravo and Mendes Dias (2006) use 1997-2004 data for two large regions of Eurasia and find that the number of terrorist incidents is negatively associated with the level of development. See also Sørli et al. (2005), Collier and Hoeffler (1998), Medoff (1999), Blomberg and Hess (2002), and Nafziger and Auvinen (2002).

[23] It is also possible that there is a "jealousy effect," where H^i is an increasing function of $w_j(t)$, $i \neq j$. We examine possible jealousy effects in the examples in Section 6 below.

[24] As required by the implicit function theorem.

Since $\frac{\partial V^1}{\partial h_i} > 0, \frac{\partial V^1}{\partial v_2} > 0, \frac{\partial V^2}{\partial h_i} > 0$, the effects of an increase in hate on violence depend on the sign of $|F|$. But what is the sign of $|F|$? To answer this question, remember that we have $V^i[0,0,0] = 0,\ i = 1,2$. Now, starting at the point where hate and violence in both countries are zero, suppose that levels of hate move away from zero in either direction. Then, if $|F| < 0$: (i) as hate increases, violence becomes negative (it becomes benevolence), and with higher levels of hate, we get higher levels of benevolence, (ii) as hate decreases (it becomes love), violence becomes positive and with higher levels of love, we get higher levels of violence. Such a scenario is clearly unreasonable. Thus, in the following, we assume that $|F| > 0$, so that.

$$\frac{dV^{*i}}{dh_j} > 0,\ all\ i,j = 1,2 \tag{15}$$

Furthermore, we also have,

$$0 = V^{*1}(0,0),\ 0 = V^{*2}(0,0)$$

In other words, if there is no hate, there is also no violence.

Now, plugging the solution for $v_i^*,\ i = 1,2$ into equations (4), (5) we get the solution for net growth as:

$$\begin{aligned} w_1^* &= g_1 + r^1[V^{*1}(h_1,h_2), V^{*2}(h_1,h_2)] \equiv g_1 + r^{*1}(h_1,h_2) \equiv w_1^*(h_1,h_2;g_1) \\ w_2^* &= g_2 + r^2[V^{*1}(h_1,h_2), V^{*2}(h_1,h_2)] = g_2 + r^{*2}(h_1,h_2) \equiv w_2^*(h_1,h_2;g_2) \end{aligned} \tag{16}$$

Then, from equations (5), (6) and (15) we have:

$$\frac{dw_i^*}{dh_j} < 0,\ all\ i,j = 1,2$$

Furthermore, when $h_1 = 0,\ h_2 = 0$, we, $w_i^*(0,0,g_i) = g_i$.

We can now plug the solutions for the variables $v_1,\ v_2,\ w_1$ and w_2 in terms of the state variables h_1 and h_2 and growth variables g_1 and g_2 into the two differential equations to obtain:

$$\begin{aligned} \frac{dh_1}{dt} &= H^1[h_1,h_2,V^{*1}(h_1,h_2),V^{*2}(h_1,h_2),g_1 + r^{*1}(h_1,h_2)] \equiv G^1(h_1,h_2;g_1) \\ \frac{dh_2}{dt} &= H^2[h_1,h_2,V^{*1}(h_1,h_2),V^{*2}(h_1,h_2),g_2 + r^{*2}(h_1,h_2)] \equiv G^2(h_1,h_2;g_2) \end{aligned} \tag{17}$$

Equations (17) capture the dynamic properties of the system. To understand the nature of the system, we need to examine the partial derivatives of the G^i functions. Define $h \equiv (h_1,h_2)$ and $G_j^i \equiv \partial G^i(h;g_i)/\partial h_j,\ i,j = 1,2$. Assuming that $G^i(h;g_i),\ i = 1,2$, is continuously differentiable, define the Jacobian matrix corresponding to these two equations, at $(h;g_i)$, as:

$$J \equiv \begin{bmatrix} G_1^1 & G_2^1 \\ G_1^2 & G_2^2 \end{bmatrix} =$$

$$\begin{bmatrix} \frac{\partial H^1}{\partial h_1} + \rho_1^1 \frac{\partial V^{*1}}{\partial h_1} + \rho_2^1 \frac{\partial V^{*2}}{\partial h_1} & \frac{\partial H^1}{\partial h_2} + \rho_1^1 \frac{\partial V^{*1}}{\partial h_2} + \rho_2^1 \frac{\partial V^{*2}}{\partial h_2} \\ \frac{\partial H^2}{\partial h_1} + \rho_1^2 \frac{\partial V^{*1}}{\partial h_1} + \rho_2^2 \frac{\partial V^{*2}}{\partial h_1} & \frac{\partial H^2}{\partial h_2} + \rho_1^2 \frac{\partial V^{*1}}{\partial h_2} + \rho_2^2 \frac{\partial V^{*2}}{\partial h_2} \end{bmatrix}$$

where,

$$\rho^i_j \equiv \frac{dH^i}{dV^{*j}} = \frac{\partial H^i}{\partial V^{*j}} + \frac{\partial H^i}{\partial w_i}\frac{\partial r^i}{\partial V^{*j}} \tag{18}$$

The determinant of J is given by:

$$|J| = G^1_1 G^2_2 - G^1_2 G^2_1$$

What can we say about the signs of the elements of the matrix J? Since $\frac{\partial H^i}{\partial V^{*j}} > 0$, $\frac{\partial r^i}{\partial V^{*j}} < 0$ for all i, j and $\frac{\partial H^i}{\partial w_i} < 0$, for all i, we have,[25]

$$\rho^i_j > 0, \quad i, j$$

But, since $\frac{\partial H^i}{\partial h_j} > 0$, $\frac{\partial V^{*i}}{dh_j} > 0$ for all $i \neq j$, we have,

$$G^i_j > 0, \text{ for all } i \neq j$$

Unfortunately, since the effect of a country's own hate on the evolution of its hate ($\partial H^i/\partial h_i$) is not that clear, we cannot determine the sign of G^i_i. Since the signs of the diagonal terms are unknown, this also means that so is the sign of the determinant of J. How likely is it for $\frac{\partial H^i}{\partial h_i}$ to be positive? We know that a sufficient condition for $G^i_i > 0$ is that $\frac{\partial H^i}{\partial h_i} > 0$ and a necessary and sufficient condition is that $\frac{\partial H^i}{\partial h_i} > -(\frac{\partial H^i}{\partial V^{*1}}\frac{\partial V^{*1}}{\partial h_i} + \frac{\partial H^i}{\partial V^{*2}}\frac{\partial V^{*2}}{\partial h_i} + \frac{\partial H^i}{\partial w^*_i}\frac{\partial r^{*i}}{\partial h_i})$. But the effects of depreciation or forgetfulness, even when they exist, may not be sufficient to outweigh the effects of violence (which consist of its direct and indirect effects through its impact on the economic cost of the conflict). The case of when $G^i_i > 0$, therefore, seems quite likely.

Finally, note that while the properties of the Jacobian matrix J are related to the intrinsic styles (properties) shown in Figures 1 and 2 above, there is no a priori reason why G^i_j and H^i_j should always have the same signs (although here we have that $H^i_{j\neq i}$ and $G^i_{j\neq i}$ are both positive). A country's intrinsic type is captured by the properties of H^i, whereas its perceived type is captured by (the properties of) G^i. The properties of G^i reflect not only the intrinsic nature of a country but also all other factors that depend on hate and play a role in its evolution. In our example, these other factors include violence and economic conditions. But in a more general model, they may also include other root causes. For example, factors that affect and are affected by hate. Some of these other factors may reflect underlying strategic (game) considerations (e.g., by political actors). Confusion over what is intrinsic and what root causes are may explain why there is so much debate and disagreement over the causes of hate, violence, extremism and terrorism.

[25] Note that if we include a jealousy effect, then when violence increases in any single country, as a result, in both countries: economic costs increase, thus reducing net growth rates and hence affecting the evolution of hate. It is reasonable, however, that the impact of the reduction in a country's growth is greater than that of the reduction in the rival's growth (the jealousy effect). Thus, we may have: $\rho^i_j > 0, \quad i \neq j$, if $\frac{\partial H^1}{\partial w_1}\frac{\partial r^1}{\partial V^{*1}} > -\frac{\partial H^1}{\partial w_2}\frac{\partial r^2}{\partial V^{*1}}$.

3 Steady State

Let us first look at the special case when the two growth rates are constant. We assume that for any given (finite column) vector $h = (h_1, h_2)$, when g_i is "sufficiently" low, we have $\frac{dh_1}{dt} > 0$ and when g_i is "sufficiently" high we have $\frac{dh_1}{dt} < 0$. Given continuous $G^i(h_1, h_2; g_i)$ functions and since $\partial H^i/\partial g_i < 0$, for all h, g_i, this implies that for any given (finite) h, there exists, at least, one fixed rate of economic growth in Country i, given by g_i^h, such that,[26]

$$\frac{dh_i}{dt} = G^i(h; g_i) \begin{cases} > 0, & \text{for all } g_i < g_i^h \\ = 0, & \text{for all } g_i = g_i^h \\ < 0, & \text{for all } g_i > g_i^h \end{cases}$$

Specifically, this implies that, in the absence of hate or violence in either country, there exists some fixed rate of economic growth in Country i, given by g_i^0, such that,

$$\frac{dh_i}{dt} = H^i[0,0,0,0,g_i] \equiv G^i(0,0;g_i) \begin{cases} > 0, & \text{for all } g_i < g_i^0 \\ = 0, & \text{for all } g_i = g_i^0 \\ < 0, & \text{for all } g_i > g_i^0 \end{cases}$$

Namely, the fixed growth rates g_1^0, g_2^0 are consistent with zero hate in both countries. We refer to them as "normal" growth rates, but only in the sense that they are required to keep hate levels in both countries at zero. Clearly, in general, there is no reason why g_i^0 should be the same as the (fixed) long-run growth rate (LRGR) g_i^*.[27]

Let us now examine the existence and nature of a steady-state (SS) solution. First, time-dependent growth rates cannot yield a SS. Second, for fixed growth rates g_1, g_2, a SS (if it exists) is defined by the two conditions:

$$\begin{aligned} G^1(h_1, h_2; g_1) &= 0 \\ G^2(h_1, h_2; g_2) &= 0 \end{aligned} \tag{19}$$

Hence, if $g_i = g_i^0 = constant$, $i = 1, 2$, then $h_1 = h_2 = 0$ satisfies equations (19). In other words, if growth rates in both countries are constant and equal to the normal rates, then we have a SS with zero hate. But can we have a SS with zero hate when we do not have constant normal growth rates in both countries? Since $G^i(0,0;g_i^0) = 0$ and $\partial G^i(0,0;g_i)/\partial g_i < 0$ for all g_i (and regardless of what $g_{j\neq i}$ is), we know that there is no other fixed value of g_i that satisfies $G^i(0,0;g_i) = 0$. Hence, we can have a SS with zero hate if and only if both countries have (fixed) normal growth. Since there is no reason why growth in both countries should be fixed and equal to the normal growth rates, it is clear that, in general, we will not have a SS with zero hate. In other words, genuine peace is not likely as a SS.

But does a SS with *any* level of hate exist? Since the only constant growth rates are the two long-run

[26] In the following discussion we assume that g_i^h is unique (which will be the case with monotonicity of the G^i functions). We examine the uniqueness question in the examples provided below.

[27] There is also no reason why, for general non-linear functions, g_i^0 should be zero for both countries.

growth rates g_1^* and g_2^*, the question is whether there exists a vector $h^* \equiv (h_1^*, h_2^*)$ such that:

$$\begin{aligned} G^1(h^*; g_1^*) &= 0 \\ G^2(h^*; g_2^*) &= 0 \end{aligned} \tag{20}$$

Define the corresponding Jacobian matrix evaluated at $h^*; g^*$ as

$$A \equiv \begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix}$$

where $a_j^i \equiv G_j^i(h^*; g_i^*)$. Assuming that the determinant of the Jacobian A does not vanish at $h^*; g^*$, there exists a vector $h^* \equiv [h_1^*(g^*),\ h_2^*(g^*)]$, where $h^*(g^*)$ is continuously differentiable and satisfies the two conditions in equations (20). Although it is obvious that

$$h_i^*(g_1^0, g_2^0) = 0,\ i = 1, 2,$$

in general, we expect to have $g_i^* \neq g_i^0$, so we conclude that it is not likely that we will have: $h_i^* = 0$, for $i = 1, 2$.

To be able to understand the nature of the SS, we examine the properties of the two conditions (isoclines, or demarcation curves) $G^1(h; g_1^*) = 0$ (denotes as IC_1) and $G^2(h; g_2^*) = 0$ (denoted as IC_2) around the SS h^* (and given the long-run growth rates g_1^*, g_2^*). Total differentiation yields the slopes of IC_1 and IC_2, at $h^*,\ g^*$, denoted as S_1 and S_2, respectively, as:

$$S_1 \equiv \frac{dh_2}{dh_1}\Big|^1_{(h^*, g_1^*)} = -\frac{a_1^1}{a_2^1}$$
$$S_2 \equiv \frac{dh_2}{dh_1}\Big|^2_{(h^*, g_2^*)} = -\frac{a_1^2}{a_2^2}$$

While we know that $a_j^i > 0,\ for\ i \neq j$, we do not know the sign of a_i^i. Thus, we do not know the signs of S_1 and S_2 at h^*. Moreover, comparing the two slopes, we get that:

$$S \equiv S_1 - S_2 = -\frac{|A|}{a_2^1 a_2^2}$$

Again, since we do not know the signs of a_2^2 and $|A|$, we do not know which isocline is steeper (we do not know the sign of S). It may be possible, however, to infer the likelihood of the two possible cases ($S > 0$ and $S < 0$) by examining the effects of a change in long-run growth rates on the SS. A Change in g_i^* shifts IC_i, thus affecting the SS values of *both* h_1^* and h_2^*. From equations (20) we get:

$$A\frac{dh^*}{dg^*} = -A_g$$

where the 2X2 matrices $\frac{dh^*}{dg^*}$ and A_g are given by:

$$\frac{dh^*}{dg^*} = \begin{bmatrix} \frac{dh_1^*}{dg_1^*} & \frac{dh_1^*}{dg_2^*} \\ \frac{dh_2^*}{dg_1^*} & \frac{dh_2^*}{dg_2^*} \end{bmatrix}$$

$$A_g = \begin{bmatrix} a^1_{g_1} & 0 \\ 0 & a^2_{g_2} \end{bmatrix}$$

where $a^i_{g_i} \equiv \partial G^i(h^*; g^*_i)/\partial g_i$. Thus, the effect of a change in long-term growth rates is given by:

$$\frac{dh^*}{dg^*} = -A^{-1}A_g \tag{21}$$

where A^{-1} is the inverse of A. Since $\partial G^i(h; g_i)/\partial g_i = \partial H^i/\partial w_i\, \partial w_i/\partial g_i < 0$, we also have: $a^i_{g_i} \equiv \partial G^i(h^*; g^*_i)/\partial g_i < 0$.

We know that an increase in g^*_i shifts IC_i. In fact, since the off-diagonal elements in A are positive and $a^i_{g_i} < 0$, we know that when g^*_1 increases, IC_1 shifts up, and when g^*_2 increases, IC_2 shifts to the right. But, unfortunately, since (as was shown above) we do not know the sign of the determinant of A or the signs of its diagonal elements, we do not know if the two isoclines are upward or downward sloping, and we do not know which is steeper. Consequently, even though we know in which direction isoclines shifts, we cannot tell what happens to the SS (the intersection of IC_1 and IC_2). Thus, the effects of a change in long-run growth rates are generally ambiguous. Nevertheless, let us examine these effects further.

From equation (21) we get:

$$\frac{dh^*_i}{dg^*_i} = -\frac{a^i_{g_i}\ a^j_j}{|A|},\ i = 1, 2 \tag{22}$$

which is ambiguous. From equation (21) we also get:

$$\frac{dh^*_i}{dg^*_j} = \frac{a^j_{g_j} a^i_j}{|A|},\ i \neq j,\ i, j = 1, 2 \tag{23}$$

Since $a^j_{g_j} a^i_j < 0$, we know that $sign(dh^*_i/dg^*_i) = -sign(|A|\,,\ i \neq j$. Thus, for both countries, the effects in equation (23) must have the same sign: $sign(dh^*_1/dg^*_2) = sign(dh^*_2/dg^*_1)$. There are, therefore, two possible cases: (1) both dh^*_1/dg^*_2 and dh^*_2/dg^*_1 are negative, (2) both dh^*_1/dg^*_2 and dh^*_2/dg^*_1 are positive.

Although it may be obvious, it is still useful to note that (assuming that $a^i_{g_i}$ and a^i_j, $i, j = 1, 2$, are all non-zero) a change in the LRGR in *one* country affects the SS hate levels in *both* countries. This "cross effect" is simply a reflection of the interaction of the effects on the two isoclines. However, it introduces two important elements to the analysis. First, even though g^*_j does not appear in country $i's$ motion equation, a higher LRGR in country j will decrease or increase the SS level of hate in country $i \neq j$. In other words, "indirect" elements of envy/compassion are introduced into the model by the interaction of the two isoclines. Second, even though, by assumption, g^*_i negatively affects a country's hate *evolution* ($a^i_{g_i} < 0$), an increase in g^*_i may either decrease or *increase* its *own* SS level of hate. In other words, the SS equilibrium response to an increase in its growth rate may be inconsistent with its "inherent" response (as captured by $a^i_{g_i}$; the response of its evolution of hate).

Altogether, there are six possible cases, depending on the properties of the A matrix (which determine the slopes, S_i, and the difference in slopes, S, of the isoclines). These cases are summarized in Table 3 below.[28]

$\|A\| \backslash S_i$	$a_i^i < 0 \rightarrow S_i > 0$	$a_i^i > 0 \rightarrow S_i < 0$	$a_1^1 > 0,\, a_2^2 < 0$	$a_1^1 < 0,\, a_2^2 > 0$
			$\rightarrow S_1 < 0, S_2 > 0$	$\rightarrow S_1 > 0, S_2 < 0$
$\|A\| > 0$	Case (i) $\frac{dh_1^*}{dg_1^*} < 0$ $\frac{dh_i^*}{dg_j^*} < 0,\, S > 0$	case (iv) $\frac{dh_1^*}{dg_1^*} > 0$ $\frac{dh_i^*}{dg_j^*} < 0,\, S < 0$	− − − − − − − − − −	− − − − − − − − − − −
$\|A\| < 0$	Case (iii) $\frac{dh_1^*}{dg_1^*} > 0$ $\frac{dh_i^*}{dg_j^*} > 0,\; S < 0$	Case (iia) $\frac{dh_1^*}{dg_1^*} < 0$ $\frac{dh_i^*}{dg_j^*} > 0,\; S > 0$	Case (iib) $\frac{dh_1^*}{dg_1^*} > 0, \frac{dh_2^*}{dg_2^*} < 0$ $\frac{dh_i^*}{dg_j^*} > 0,\; S < 0$	Case (iic) $\frac{dh_1^*}{dg_1^*} < 0, \frac{dh_2^*}{dg_2^*} > 0$ $\frac{dh_i^*}{dg_j^*} > 0,\; S > 0$

Table 3: Properties of the A Matrix

Although there are six possible cases, not all seem equally "reasonable." Specifically, it seems reasonable that a country's SS level of hate should decrease when its own LRGR increases. This property should be true at least for one of the two countries (namely, we should have $dh_i^*/dg_i^* < 0$ for at least one country). It is, of course, possible that this may not be true, but it would indeed seem unreasonable if, for both countries, h_i^* should increase with g_i^*. If this requirement is true for both countries, all cases except for cases (i) and (iia) would be eliminated. If it is true for at least one country, cases (iii) and (iv) would be eliminated. What about the "cross effect" of a change in g_i^* on h_j^*? Do we expect $dh_i^*/dg_j^*,\ i \neq j$ to be positive or negative? If $dh_i^*/dg_j^* > 0$, we have what we referred to above as indirect envy/compassion (when g_j^* increases/decreases h_i^* increases/decreases). Although the envy part does not sound appealing, it does not seem unreasonable. What about the other case where $dh_i^*/dg_j^* < 0$? In this case, a country's SS level of hate will be low when its rival's growth rate is high and high when its rival's growth rate is low. Regardless of ethical considerations, this seems less "reasonable." If, in addition to eliminating cases without $dh_i^*/dg_i^* < 0,$ for at least one country, we were also to eliminate the cases where $dh_i^*/dg_j^* < 0$, we would end up with three cases only: cases (iia), (iib) and (iic).

Let us consider the implications of cases (i)-(iic) above regarding the nature of the SS. Specifically, we are interested in finding out the likely SS solutions and, in particular, whether they are likely to involve no hate. First, as was pointed out above, in general, there is no reason why we should have $g_i^* \neq g_i^0$, so it is not likely that we will have zero hate in both countries. Second, whether the SS levels of hate are positive or negative depends on (i) whether the long-run growth rates are higher or lower than the normal rates and (ii) which of the six cases above occurs. The possible cases are summarized in Table 4 below (an entry/pair $+-$ means that $h_1^* > 0$ and $h_2^* < 0$; more than one entry means that, at least for one country, the sign is ambiguous - it depends on $g_1^* - g_1^0$, relative to $g_2^* - g_2^0$).

[28] Note that when a_1^1 and a_2^2 have opposite signs, then $|A| < 0$; thus, there are no entries in the last two columns of the first row in Table 3.

	$g_1^* > g_1^0,\ g_2^* = g_2^0$	$g_1^* > g_1^0,\ g_2^* > g_2^0$	$g_1^* < g_1^0,\ g_2^* = g_2^0$	$g_1^* < g_1^0,\ g_2^* < g_2^0$	$g_1^* > g_1^0,\ g_2^* < g_2^0$
case (i)	$--$	$--$	$++$	$++$	$-+/--/++$
case (iia)	$-+$	$-+/+-/++$	$+-$	$-+/+-/--$	$-+/+-/++$
case (iib)	$++$	$++/+-$	$--$	$--/-+$	$-+/++$
case (iic)	$-+$	$-+/++$	$+-$	$+-/--$	$-+/--$

Table 4: The Signs of the SS levels of hate

As Table 4 shows, a given configuration $(g_1^*,\ g_2^*)$ and (g_1^0, g_2^0) does not necessarily tell us if h_1^* and h_2^* are positive or negative. Specifically, just because $g_1^* > g_1^0$, it does not mean that $h_1^* < 0$ (e.g., in case (iib), we have $h_1^* > 0$). Furthermore, a SS generally involves non-zero levels of hate (love). Table 4 also shows that, in general, a SS involves non-zero levels of hate (love).

4 Stability

Rather than examining the details of the solution for the two differential equations system, we now focus on the system's stability (the solution is given in the Appendix).

We begin the stability analysis by examining the solution to the homogeneous system (HS), given by $\frac{dh}{dt} = Ah$ (see Appendix). The characteristic equation corresponding to the HS is given by,

$$\lambda^2 - (a_1^1 + a_2^2)\lambda + |A| = 0,$$

and its characteristic roots are:

$$\begin{aligned} \lambda_1, \lambda_2 &= \frac{1}{2}[(a_1^1 + a_2^2 \pm \sqrt{(a_1^1 + a_2^2)^2 - 4\ |A|}] \\ &= \frac{1}{2}[(a_1^1 + a_2^2) \pm \sqrt{\Delta}] \end{aligned}$$

where $\Delta \equiv (a_1^1 + a_2^2)^2 - 4\ |A|$ is the corresponding discriminant. But, since $a_{ij} \equiv \partial G^i(h^*, g_i^*)/\partial h_j > 0,\ i \neq j$, $i, j = 1, 2$, we have:

$$\Delta \equiv (a_1^1 + a_2^2)^2 - 4\ |A| = (a_1^1 - a_2^2)^2 + 4a_2^1 a_1^2 > 0$$

Thus, we conclude that the solution involves two distinct real roots. Consequently, a cyclical pattern of hate in the HS is not possible. Moreover, given that cyclical patterns of hate cannot occur in the HS and given the monotonic relationship between violence and hate (in equation (15), we have $dV^{*i}/dh_j > 0,\ all\ i, j = 1, 2$), it also follows that the HS (itself) cannot give rise to cyclical patterns of violence. Similarly, this also implies that, *for fixed values* of *gross* rates of economic growth, we cannot have cyclical patterns of *net* rates of economic growth $(w_1^*(h_1, h_2; g_1)$. Cyclical patterns of hate, violence and net economic growth can, therefore, occur only due to possible cyclical patterns in the non-autonomous component, $\eta_i(t) = q^i + a_{g_i}^i g_i(t)$, which in turn reflect cyclical patterns in $g_i(t)$.

Therefore, the only question is whether the SS equilibrium in the HS is stable or unstable. In principle, the following cases are possible in the HS: (I) if both roots are negative, we have a stable node; (II) if both roots are positive, we have an unstable node; and (III) if the roots have opposite signs, we have a saddle point.

Let us, therefore, look at the six possible cases above. In the three "most reasonable" cases ((iia), (iib) and (iic)), we have $|A| < 0$. Since

$$\lambda_1 \lambda_2 = |A| < 0$$

λ_1 and λ_2 must have opposite signs. Hence, we conclude that the solution to the HS is a saddle point in all three "most reasonable" cases.

In cases (i), (iii), and (iv) that were deemed to be "less reasonable," we have the following stability properties. In case (i), we have $|A| > 0$, so λ_1 and λ_2 must have the same sign. But, since $a_j^j < 0$, we have $\lambda_1 + \lambda_2 = a_1^1 + a_2^2 < 0$. Consequently, λ_1 and λ_2 must both be negative; hence, the solution to the HS is a stable node. In case (iii), we have $a_i^i < 0$, and $|A| < 0$, so we have a saddle-point. In case (iv), we have $|A| > 0$ and $a_i^i > 0$, so λ_1 and λ_2 must be both positive, so we have an unstable node.

Since all three reasonable cases have a saddle point, we conclude that the most likely outcome is, in fact, a saddle point. Namely, the HS is most likely to be unstable. But, since the HS is unstable in the most likely cases, so is the corresponding NAS. At the same time, given growth rates that converge to fixed (finite) long-run rates, it follows that, in the less likely case, when the HS is stable (case(i)), so is the NAS.

Can we refine this even further? Remember that in order to have $a_i^i < 0$, we require that (at $h^*; g_i^*$): $\frac{\partial H^i}{\partial h_i} < -(\frac{\partial H^i}{\partial V^{*1}} \frac{\partial V^{*1}}{\partial h_i} + \frac{\partial H^i}{\partial V^{*2}} \frac{\partial V^{*2}}{\partial h_i} + \frac{\partial H^i}{\partial w_i^*} \frac{\partial r^{*i}}{\partial h_i})$; namely, the effects of depreciation, or forgetfulness, if they exist, must outweigh the effects of persistence plus the direct and indirect effects of violence. Since this is likely not to occur, cases (iib) and (iic) are not likely to occur. Therefore, the most likely case is (iia), which yields a saddle point. We should remember, however, that hate neither conforms to nor is based on reason. Asking whether a particular case is reasonable or not may, in itself, not be desirable or even reasonable. Hence, it is perhaps best not to exclude cases that seem unreasonable.

We showed above that since $a_{12} > 0$ and $a_{21} > 0$, cyclical patterns of hate cannot occur in the HS. Let us briefly comment on the circumstances (assumptions) under which cycles may occur in the HS. Suppose that we introduce direct jealousy effects. Specifically, suppose the evolution of hate in each country depends on economic conditions in both countries so that the differential equations are given by:

$$\frac{dh_i}{dt} = H^i(h_1, h_2, v_1, v_2, w_1, w_2), \; i = 1, 2 \tag{24}$$

A jealousy effect in country i occurs if $\frac{\partial H^i}{\partial w_j} > 0$, $i \neq j$. Given equation (24) we now have to re-write equation (18) above as:

$$\rho_j^i \equiv \frac{dH^i}{dV^{*j}} = \frac{\partial H^i}{\partial V^{*j}} + \frac{\partial H^i}{\partial w_i} \frac{\partial r^i}{\partial V^{*j}} + \frac{\partial H^i}{\partial w_j} \frac{\partial r^j}{\partial V^{*j}}, \; i \neq j$$

Thus, for example, if country i has a jealousy effect, we may not have $\rho_j^i > 0$, $i \neq j$, and consequently, we may (but not necessarily) have $a_j^i \equiv G_j^i(h^*; g_i^*) < 0$, $i \neq j$. We say that jealousy in country i is "sufficiently strong" if it yields $a_j^i < 0$, $i \neq j$. Hence, if both countries have sufficiently strong jealousy, no cycles are possible (with

$a_2^1 < 0$ and $a_1^2 < 0$, we still have $\Delta = (a_1^1 - a_2^2)^2 + 4a_2^1 a_1^2 > 0$). But, if only one country has sufficiently strong jealousy, cycles may occur (since a_2^1 and a_1^2 have opposite signs, we may have $\Delta = (\gamma_1^1 + \gamma_2^2)^2 + 4\gamma_2^1\gamma_1^2 < 0$). Sufficiently strong jealousy in one and only one country is, therefore, a necessary condition for cycles in the HS.

Note that if the evolution of hate in country i is negatively affected by the level of hate in country $j \neq i$ (i.e., we have $\partial H^i/\partial h_j < 0$), we may also get $a_j^i < 0$, $i \neq j$, possibly leading to cycles in the HS, which is not very reasonable. Moreover, even if the evolution of hate in country i is negatively affected by the level of hate in country j, this would still need to be true for one and only one country so that a_2^1 and a_1^2 would have opposite signs.

Finally, it would be possible to obtain more complex dynamic patterns, including chaotic ones. But, for this to be the case, we would have to study the nonlinear system itself (rather than the linearized system at the SS point),[29] either for continuous functions capable of producing more complex, possibly chaotic, patterns (and for that, we would need added dimensions, as is required by the Poincaré–Bendixson theorem). Alternatively, we could introduce discreetness. Examples of specific nonlinear (higher dimensional continuous) systems are provided in the next section.

5 Examples

This section provides examples of a nonlinear higher dimensional system that can provide richer dynamical patterns of hate and violence. For each country, we introduce a third "stock variable" that measures the country's "economic well-being" (EWB) or "wealth." Let these be given by n_1 and n_2, where both are positive. We assume that the evolution of EWB is affected by the level of current EWB (e.g., the "return" on current wealth) and the cost of violence (captured by the reduction in the value of the current net domestic product (NDP) in the two countries, through the $r^{*1}(h_1, h_2)$, $r^{*2}(h_1, h_2)$ functions as described above). In addition, we allow for direct economic spillovers, or externalities, across the two countries (not directly tied to the cost of violence). Specifically, the evolution of EWB in each country is affected by EWB in the other country for purely economic reasons. We assume that, in each country, the evolution of hate is responsive to economic conditions and possibly exhibits jealousy. The two motion equationss for the evolution of hate are, therefore, similar to the ones described above, except that now we use the measure of EWB instead of economic growth in equations (17). Namely, in each $H^i(\cdot)$ equation, we use n_1, n_2 instead of w_i^*.

[29] Although (by the Hartman–Grobman Theorem), when all the characteristic roots of the Jacobian, evaluated at the equilibrium, have nonzero real parts so that the equilibrium is hyperbolic, the dynamic behaviour of the nonlinear system near the equilibrium is qualitatively the same as the dynamic behaviour of the linearized model near the equilibrium.

Thus, our system is described by the following four differential equations:

$$
\begin{aligned}
\frac{dh_1}{dt} &= H^1[h_1, h_2, V^{*1}(h_1,h_2), V^{*2}(h_1,h_2), n_1, n_2] \equiv M^1(h_1,h_2,n_1,n_2) \\
\frac{dh_2}{dt} &= H^2[h_1, h_2, V^{*1}(h_1,h_2), V^{*2}(h_1,h_2), n_1, n_2] \equiv M^1(h_1,h_2,n_1,n_2) \\
\frac{dn_1}{dt} &= N^1(h_1,h_2,\ n_1,n_2) \\
\frac{dn_2}{dt} &= N^2(h_1,h_2,\ n_1,n_2)
\end{aligned}
\tag{25}
$$

One question that needs to be addressed when we specify functional forms for these equations is whether we want a country's perceived type to be locally or globally determined. Namely, should its perceived type be the same for all values of the variables h_1, h_2, n_1, n_2? This question becomes even trickier since hate variables may be either positive (hate) or negative (love).[30] Thus, the signs of the effects of hate on the evolution of hate may be different for positive and negative values of h_1, h_2. For example, with a quadratic function, the sign of $\partial H^1/\partial h_2$ will depend on the signs of h_1 and h_2. Specifically, if h_1 and h_2 are negative, we will have $\partial H^1/\partial h_2 < 0$, but if h_1 and h_2 are positive, we will have $\partial H^1/\partial h_2 > 0$ (and similarly, for the N^i functions). In the following, we address this issue by using a cubic or exponential function, which guarantees that the sign of $\partial H^i/\partial h_j$ is not affected by the signs of h_1, h_2.

5.1 Example 1

In the first example, we use the following specification for the M^i, N^i functions:

$$
\begin{aligned}
\frac{dh_1}{dt} &= .2 + h_1 + .5h_2 + h_2^3 - n_1 - n_1^2 + .1n_2 \\
\frac{dh_2}{dt} &= 2 + h_1 + h_2 + h_1^3 - n_2 - n_2^2 + n_1 \\
\frac{dn_1}{dt} &= 2 - 1.1h_1 - h_2 + .5n_1 + .1n_2 \\
\frac{dn_2}{dt} &= 10 - h_2 - {h_1}^3 + .1n_1 + .1n_2
\end{aligned}
\tag{26}
$$

Note that,

$$
\begin{aligned}
\frac{\partial H^1}{\partial h_1} &= 1,\ \frac{\partial H^1}{\partial h_2} = .5 + 3h_2^2 \geq 0 \text{ for all } h_2,\ \frac{\partial H^2}{\partial h_1} = 1 + 3h_1^2 \geq 0 \text{ for all } h_1,\ \frac{\partial H^2}{\partial h_2} = 1, \\
\frac{\partial H^1}{\partial n_1} &= -(1+2n_1) < 0 \text{ for all } n_1 \geq 0,\ \frac{\partial H^1}{\partial n_2} = .1 > 0,\ \partial\frac{\partial H^2}{\partial n_1} = 1,\ \frac{\partial H^2}{\partial n_2} = -(1+2n_2) < 0 \text{ for all } n_1 \geq 0 \\
\frac{\partial N^1}{\partial h_1} &= -1.1,\ \frac{\partial N^1}{\partial h_2} = -1,\ \frac{\partial N^2}{\partial h_1} = -3h_1^2 \leq 0 \text{ for all } h_1,\ \frac{\partial N^2}{\partial h_2} = -1 \\
\frac{\partial N^1}{\partial n_1} &= .5,\ \frac{\partial N^1}{\partial n_2} = .1,\ \frac{\partial N^2}{\partial n_1} = .1,\ \frac{\partial N^2}{\partial n_2} = .1
\end{aligned}
$$

Therefore, the countries are perceived to be congruent, with long memories, and responsive to economic conditions and jealousy. In addition, the conflict is economically costly (because hate induces violence and

[30] In the linear case, this is not an issue since the first derivatives are constants.

violence affects economic well-being): the cost of hate is captured here by the cost functions ($h_1 + 1.1h_2^3$ and $h_2 + h_1^3$) in the motion equations for economic well-being. The specification also reflects the (positive) effects of the current EWB on its evolution, as well as positive economic externalities.

We find that the system has four real (but two of them have negative values for one of the wealth variables z_i) and multiple (thirteen) complex steady-state equilibria. Not surprisingly, none of the equilibria are at $h_1 = h_2 = 0$; namely, none represents genuine peace. It may be ironic that most of the equilibria in this dynamic hate model are indeed "imaginary." The real equilibria, for which both wealth measures are positive, are given by:

$$h_1 = 2.102857273,\ h_2 = 1.323176106,\ n_1 = 2.535724568,\ n_2 = 3.684568212 \quad (27)$$

$$h_1 = 1.978696368,\ h_2 = 3.272057023,\ n_1 = 6.073740480,\ n_1 = 4.117527876 \quad (28)$$

It can be easily verified that the system is very sensitive to the choice of initial conditions. This sensitivity can be seen in the following four phase diagrams for slightly modified initial conditions.

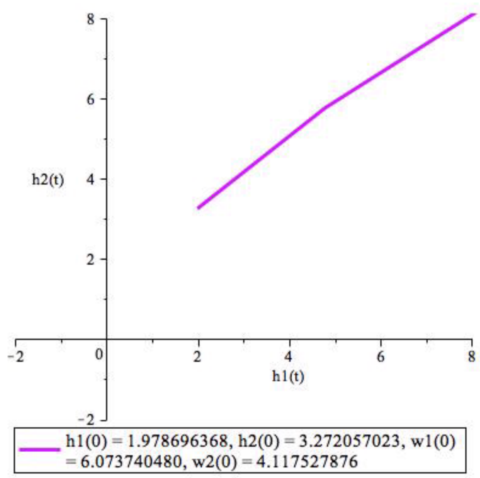


Figure 1a: Phase Diagram Real Equilibrium 1

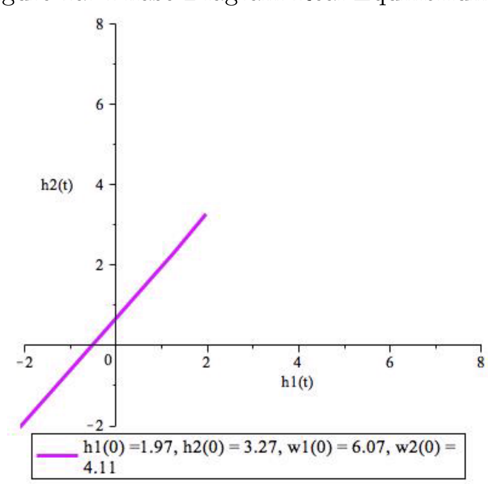


Figure 1b: Phase Diagram Real Equilibrium 1 - Slightly Modified Initial Conditions

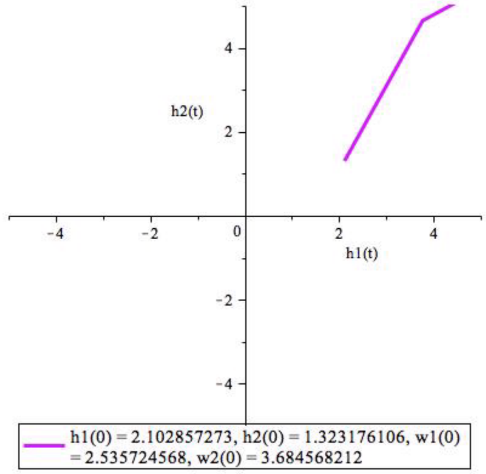


Figure 2a: Phase Diagram Real Equilibrium 2

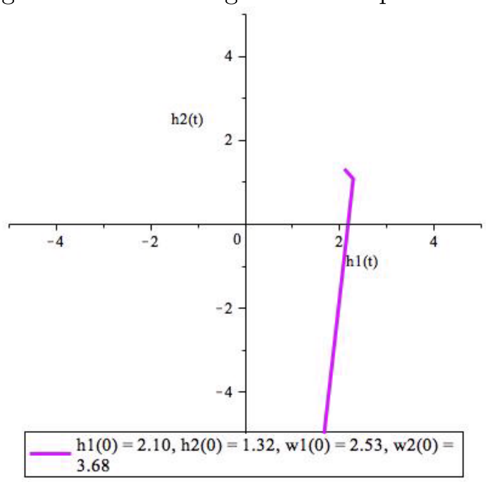


Figure 2b: Phase Diagram Real Equilibrium, with Initial Conditions, Slightly Modified

The corresponding characteristic roots, evaluated at the two equilibria above, are given by:

$$-10.2108166665249,$$

$$6.65908487806529 + 4.17324495551472i,$$

$$6.65908487806529 - 4.173244956i$$

$$-.507353089605624$$

$$25.2011274398189$$

$$-11.3687442866440 + 2.89652517087097i$$

$$-11.3687442866440 - 2.896525171i$$

$$.136361133469076$$

Since at least one of the roots is either real and positive or has a positive real part, the equilibria are unstable (the same is true for all other equilibria).

Hence, all equilibria are, indeed, unstable.

5.2 Example 2

In the second example, we use exponential terms in the specification for the M^i and N^i functions:

$$\begin{aligned} \frac{dh_1}{dt} &= -1 + h_1 + e^{h_2} - n_1 + .1n_2 \\ \frac{dh_2}{dt} &= 2 + e^{h_1} + h_2 + .1n_1 - 2n_2 \\ \frac{dn_1}{dt} &= 1 - e^{h_1+h_2} + n_1 + .1n_2 \\ \frac{dn_2}{dt} &= 2 - e^{h_1} - 2h_2 + .05n_1 + .1n_2 \end{aligned} \tag{29}$$

Note that,

$$\begin{aligned} \frac{\partial H^1}{\partial h_1} &= 1\ ,\ \frac{\partial H^1}{\partial h_2} = e^{h_2},\ \frac{\partial H^2}{\partial h_1} = e^{h1},\ \frac{\partial H^2}{\partial h_2} = 1,\ \frac{\partial H^1}{\partial n_1} = -1,\ \frac{\partial H^1}{\partial n_2} = .1,\ \partial\frac{\partial H^2}{\partial n_1} = .1,\ \frac{\partial H^2}{\partial n_2} = -2 \\ \frac{\partial N^1}{\partial h_1} &= -e^{h_1+h_2},\ \frac{\partial N^1}{\partial h_2} = -e^{h_1+h_2},\ \frac{\partial N^2}{\partial h_1} = -e^{h_1},\ \frac{\partial N^2}{\partial h_2} = -2,\ \frac{\partial N^1}{\partial n_1} = 1\ ,\ \frac{\partial N^1}{\partial n_2} = .1,\ \frac{\partial N^2}{\partial n_1} = .05,\ \frac{\partial N^2}{\partial n_2} = .1 \end{aligned}$$

Therefore, the countries are perceived to be congruent, with long memories, and responsive to economic conditions and jealousy. In addition, the conflict is economically costly: the cost of hate is captured here by the cost functions ($e^{h_1+h_2}$ and $e^{h_1} + 2h_2$) in the motion equations for economic well-being. The specification also reflects the (positive) effects of the current EWB on its evolution and positive economic externalities.

We find that the system has three real steady-state equilibria. Again, none of the equilibria are at $h_1 = h_2 = 0$. Namely, none represents genuine peace. The equilibria are given by:

$$\begin{aligned} [h_1 &= .5704652985,\ h_2 = .2447803840,\ n_1 = 1.053768419,\ n_2 = 2.059623620], \\ [h_1 &= .8623376887,\ h_2 = -0.4780944922e-1,\ n_1 = 1.036881624,\ n_2 = 2.212285101], \\ [h_1 &= -2.929754818,\ h_2 = 1.117959449,\ n_1 = 2.377464559,\ n_2 = 1.704558018] \end{aligned}$$

For example, Figure 3 shows the phase diagram with initial conditions near the third equilibrium.

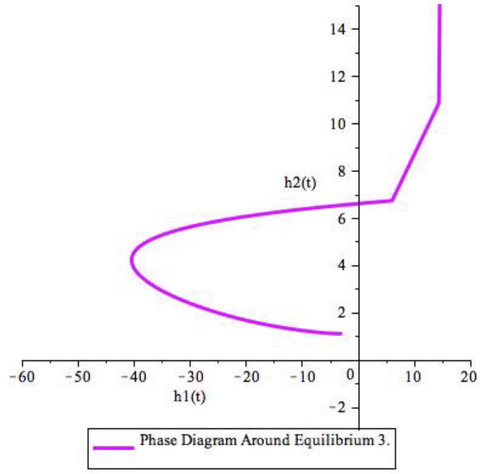
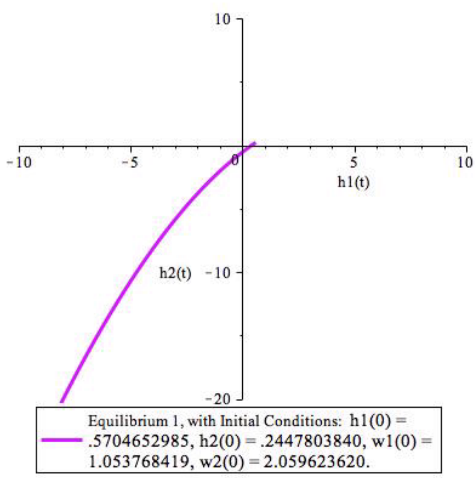


Figure 3

Again, it is easily verified that the system is very sensitive to the choice of initial conditions. This sensitivity can be seen in the phase diagrams in Figures 4a and 4b; for very slightly modified initial conditions around equilibrium1 (instead of $w_1(0) = 1.053768419$, we take $w_1(0) = 1.05376841$):



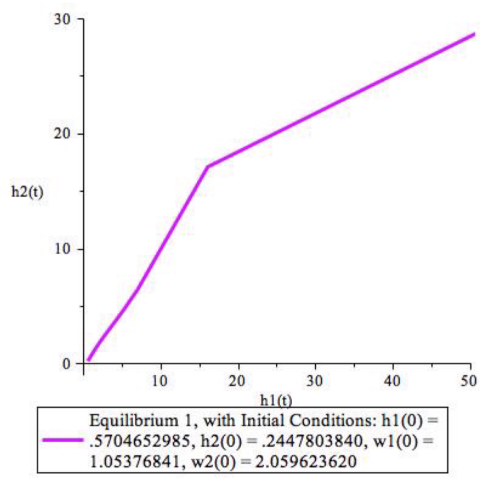


Figure 4a **Figure 4b**

As the figures show, a negligible change of 9.000000000 10^{-9} percent in $n_1(0)$ results in a radical change in the evolution of hate.

The corresponding characteristic roots, evaluated at the three equilibria, are given by:

$$4.09166204775407$$

$$-1.38955176133993$$

$$-0.301778839419891$$

$$0.699668553005753$$

4.18027127510310

$0.168648615150435 + 0.454957635681980\, i,$

$0.168648615150435 - 0.454957635681980\, i$

-1.41756850540397

2.70113826296994

1.33116754813230

0.565189947508794

-1.49749575861103

Since at least one of the roots is positive or has a positive real part, the equilibria are unstable.

5.3 Example 3

In this example, we use functional forms that are not restricted to being exponential or cubic. Specifically, we use the following example (for the M^i, U^i functions):

$$\frac{dh_1}{dt} = .1 + .01h_1 + h_2^2 - .5n_1^2 + .01n_2 \tag{30}$$

$$\frac{dh_2}{dt} = .2 + h_1^2 + .5h_2 + .2n_1 - .8n_2$$

$$\frac{dn_1}{dt} = 1.2 + .05n_1 + .01n_2 - 4h_1 - h_2 \tag{31}$$

$$\frac{dn_1}{dt} = .4 + .2n_1 + .01n_2 - h_1 - 4h_2 \tag{32}$$

Thus, the countries are responsive to economic conditions, but they also exhibit jealousy. In addition, the conflict is economically costly (because hate induces violence, and violence affects economic well-being: the cost of hate is captured here by the cost function: $(h_1 + h_2)^2$), which appears in the motion equations for economic well-being).

First, we find that the system has four real steady-state equilibria (but only one has positive values for both n_1 and n_2). These equilibria are given by:

$$h_1(t) = .2919235121,\ h_2(t) = 0.4740605113e-1,\ n_1(t) = .4429841140,\ n_2(t) = -.7049106196, \tag{33}$$

$$h_1(t) = .2947612489,\ h_2(t) = 0.5184059633e-1,\ n_1(t) = .4749202829,\ n_2(t) = .7139577619 \tag{34}$$

$$h_1(t) = .2923365680,\ h_2(t) = 0.3539371936e-2,\ n_1(t) = -.4426105874,\ n_2(t) = -.4983826811 \tag{35}$$

$$h_1(t) = .2943230973,\ h_2(t) = 0.4428036536e-2,\ n_1(t) = -.4645678812,\ n_2(t) = .4948819652 \tag{36}$$

The real parts of the corresponding characteristic roots (evaluated at these equilibria) are not all negative; thus, none of the equilibria are stable.

Figures 5a and 5b show phase diagrams for two alternative initial conditions.

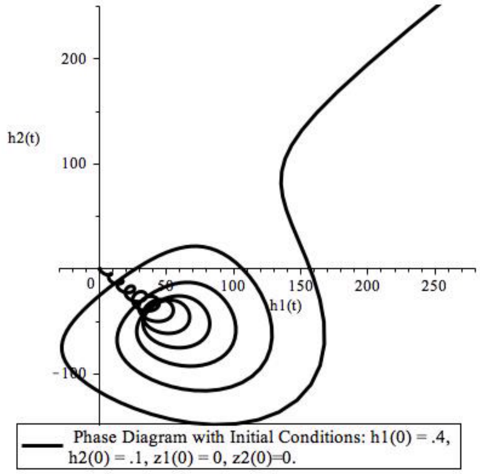


Figure 5a

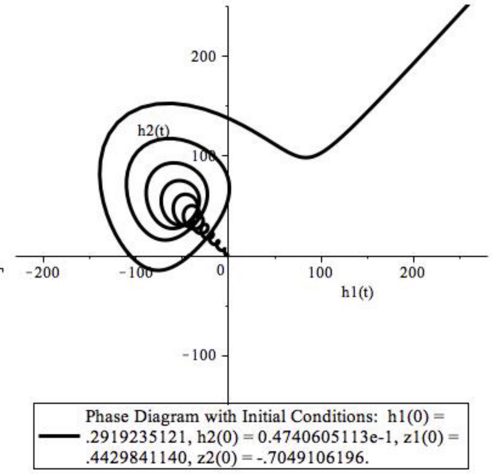


Figure 5b

5.4 Example 4

In the last example, again, we use functional forms which are not restricted to being exponential or cubic but in addition, we also assume that the two countries are jointly affected by the same economic conditions, denoted by n. The system, therefore, has only three differential equations. These are given by:

$$\frac{dh_1}{dt} = M^1(h_1, h_2, n) \quad (37)$$

$$\frac{dh_2}{dt} = M^2(h_1, h_2, n)$$

$$\frac{dn}{dt} = N(h_1, h_2, n) \quad (38)$$

We use the following example (for the M^i, N functions):

$$\frac{dh_1}{dt} = .01 + h_2^2 - n^2 \quad (39)$$

$$\frac{dh_2}{dt} = h_1^2 + h_2 - .5n$$

$$\frac{dn}{dt} = 2 - 4h_1 - h_2 \quad (40)$$

First, we find that the system has four real steady-state equilibria (the first and fourth with positive values for n). These equilibria are given by:

$$h_1(t) = .4713467788, h_2(t) = .1146128850, n(t) = .2151098017],$$

$$[h_1(t) = .3744026630, h_2(t) = .5023893478, n(t) = -.7244239875],$$

$$[h_1(t) = -1.549373958, h_2(t) = 8.197495833, n(t) = -11.59387234],$$

$$[h_1(t) = -7.296375484, h_2(t) = 31.18550193, n(t) = 44.10318653$$

Figure 6a shows the phase diagram with initial conditions at the first equilibrium, and Figure 6b shows the phase diagram with initial conditions at the origin.

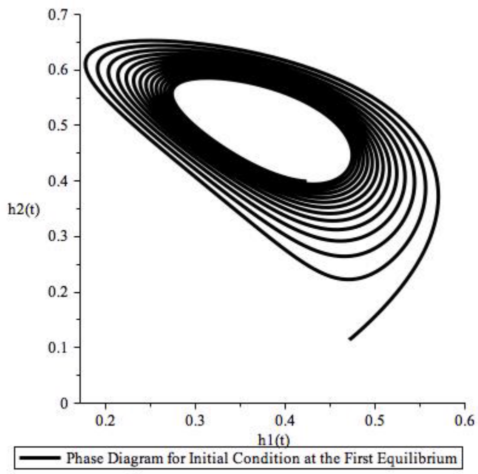


Figure 6a

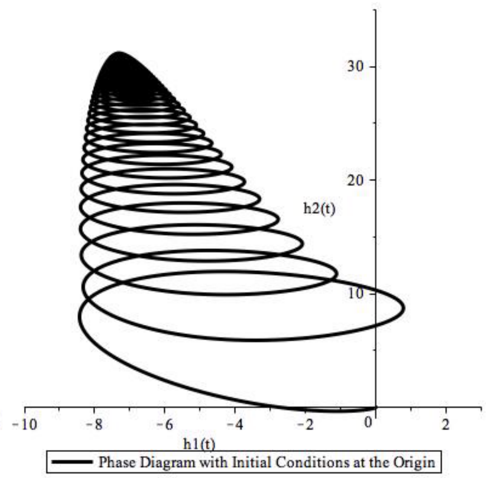


Figure 6b

for two alternative initial conditions. The corresponding characteristic roots (evaluated at these equilibria) are given by:

$$\begin{aligned} \lambda_1 &= 1.528359197 + 1.978292793i,\ \lambda_2 = 1.025184608 - 1.368424023i, \\ \lambda_3 &= 1.019062464 + .1013277939i,\ \lambda_4 = -.5785092486 - .5327886482i, \end{aligned}$$

$$\begin{aligned} \lambda_1 &= 1.140652216 + .6454729334i,\ \lambda_2 = 1.140652216 - .6454729334i, \\ \lambda_3 &= -.3302177999 + 1.240348770i,\ \lambda_4 = -.3302177999 - 1.240348770i, \end{aligned}$$

$$\begin{aligned} \lambda_1 &= 1.438995340 + .8645493539i, \lambda_2 = 1.438995340 - .8645493539i, \\ \lambda_3 &= -2.634413403 + 1.847967441i,\ \lambda_4 = -2.634413403 - 1.847967441i, \end{aligned}$$

$$\begin{aligned} \lambda_1 &= 1.607179756 - 1.768296648i,\ \lambda_2 = .7932834382 + 1.395986020i, \\ \lambda_3 &= .9869075648 - 0.9666523606e - 1i,\ \lambda_4 = -.3783784862 + .5356386167i, \end{aligned}$$

6 Conclusion

This paper provides a simple model that studies the interdependence between hate, violence and economic conditions and their dynamic properties. It provides a formal model for some of the ideas of the "dynamical system" literature in psychology. We define the ideal state of genuine peace and show that such a state is not likely to be an equilibrium state. Although we show that a time-dependent economic growth process that affects the evolution of hate (and converges to some long-run value) can yield a long-run steady state, this steady state will not be one with zero hate and zero violence. Moreover, we show that a better long-run economic environment does not necessarily result in lower equilibrium levels of hate and violence. We examine the dynamic properties of hate and violence and show that, under reasonable conditions, cycles of hate and violence cannot occur. This result implies that the dynamic properties of hate and violence (themselves) cannot result in cyclical (net) economic well-being patterns. Specifically, cyclical patterns of (net) economic well-being can occur only due to cyclical patterns of gross growth rates. Specifically, cyclical patterns of (net) economic well-being can occur only due to cyclical patterns of gross growth rates. We also show that while it is possible to have either stable or unstable equilibria, the most likely equilibrium is unstable (a saddle point).

Finally, several numerical examples show the effects of psychological attributes, and responsiveness to economic conditions, externalities and susceptibility to violence on the nature of the equilibria.

7 Appendix: The Solution of the Differential Equations

To solve the two differential equations, we linearize the equations $G^i(h_1, h_2, g_i)$ at the SS point (h^*, g^*). The linear approximation of the two hate motion equations is then:

$$\begin{aligned} G^1(h,g) &\simeq q^1 + a_1^1 h_1 + a_2^1 h_2 + a_{g_1}^1 g_1 \\ G^2(h,g) &\simeq q^2 + a_1^2 h_1 + a_2^2 h_2 + a_{g_2}^2 g_2 \end{aligned}$$

where $q^i \equiv -\left({a^i}_1 h_1^* + {a^i}_2 h_2^* + a_{g_i}^i g_i^*\right),$[31] so that the (linearized) differential equations can be written as the non-autonomous system (NAS),

$$\frac{dh}{dt} = Ah + \eta \tag{41}$$

where $\frac{dh}{dt}$ and η are the 1X2 column vectors $\left(\frac{dh_1}{dt}, \frac{dh_2}{dt}\right)^T$ and $(q^1 + a_{g_1}^1 g_1, q^2 + a_{g_2}^2 g_2)^T$, respectively.

First, we find the solution to the homogeneous system (HS) $\frac{dh}{dt} = Ah$. Let the vectors $h^1 = \begin{bmatrix} h_1^1 \\ h_2^1 \end{bmatrix}$ and $h^2 = \begin{bmatrix} h_1^2 \\ h_2^2 \end{bmatrix}$ be two linearly independent solutions to the HS (note that h^i with a superscript stands for the solution vector, whereas h_i with a subscript stands for the state variable -level of hate- in country i). Then, $h^m = \begin{bmatrix} h_1^m \\ h_2^m \end{bmatrix}$, where $h_i^m = c_1 h_i^1 + c_2 h_i^2$, is the general solution to the HS.

The two corresponding eigenvectors, defined as $z^1 = \begin{bmatrix} z_1^1 \\ z_2^1 \end{bmatrix}$ and $z^2 = \begin{bmatrix} z_1^2 \\ z_2^2 \end{bmatrix}$, are obtained from the following equations:

$$\begin{aligned} Az^1 &= \lambda_1 z^1 \\ Az^2 &= \lambda_2 z^2 \end{aligned}$$

[31] And, of course, $G^i(h^*; g_i^*) = 0$.

Hence, the general solution to the HS is:

$$h^m = \begin{bmatrix} h_1^m \\ h_2^m \end{bmatrix} = \begin{bmatrix} c_1 z_1^1 e^{\lambda_1 t} + c_2 z_1^2 e^{\lambda_2 t} \\ c_1 z_2^1 e^{\lambda_1 t} + c_2 z_2^2 e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} z_1^1 e^{\lambda_1 t} & z_1^2 e^{\lambda_2 t} \\ z_2^1 e^{\lambda_1 t} & z_2^2 e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \equiv \Psi c$$

If the determinant of Ψ is non-zero, we refer to it as the fundamental matrix corresponding to the HS above. We assume that the initial time is $t_0 = 0$ and the vector of initial values of hate, $h(0)$, is given by,

$$h(0) \equiv [h_1(0), h_2(0)] = [h_1^0, h_2^0] \equiv h^0$$

Given the fundamental matrix and using the initial conditions, we can obtain the values of the elements in the vector c by solving the equations $h^0 = \Psi(0)c$ to obtain:

$$c = \Psi^{-1}(0)\ h^0$$

The solution to the HS is, therefore, given by

$$h^m = \Psi(t)\Psi^{-1}(0)\ h^0$$

Now, define the (transition) matrix $\Phi(t, \tau)$ as:

$$\Phi(t, \tau) = \Psi(t)\Psi^{-1}(\tau)$$

then, the solution to the HS is given by:

$$h^m = \Phi(t, 0)\ h^0$$

The general solution to the NAS in (41), defined as h^s, is given by the sum of the solutions to the HS (h^m) and the particular solution to the NAS, denoted as h^p :

$$h^s = h^m + h^p$$

where h^p is given by:

$$\begin{aligned} h^p &= \Psi(t) \int_0^t \Psi^{-1}(\tau)\eta(\tau)d\tau \\ &= \int_0^t \Phi(t, \tau)\eta(\tau)d\tau \end{aligned}$$

Hence, the solution to the non-autonomous is given by (the variation of parameter formula):

$$\begin{aligned} h^s(t) &= \Psi(t)\Psi^{-1}(0)\ h^0 + \Psi(t) \int_0^t \Psi^{-1}(\tau)\eta(\tau)d\tau = \\ & \Phi(t, 0)\ h^0 + \int_0^t \Phi(t, \tau)\eta(\tau)d\tau \end{aligned}$$

8 References

Appelbaum, E. and E. Katz, (2007), “Political Extremism in the Presence of a Free Rider Problem,” Public Choice 133, 1-2, 31-40.

Appelbaum, E., (2008), “Extremism as a strategic tool in conflicts,” Journal of Economic Behavior & Organization 68, 2, 352-364.

Atkinson, S. E., Sandler, T. and Tschirhart, J., (1987), “Terrorism in a bargaining framework,” Journal of Law and Economics, 30, 1, 1-21.

Bandarage, A., (2004), “Beyond Globalization and Ethno-Religious Fundamentalism,” Development, 47, 1, 35-41.

Blomberg, S. B., Hess, G. D. and Weerapana, A., (2004), “Economic Conditions and Terrorism,” European Journal of Political Economy, 20, 2, 463-478.

Blomberg, B.S. and G.D. Hess, (2002), “The temporal links between conflict and economic activity,” Journal of Conflict Resolution, 46, 74-90.

Cameron, S., (2009), The Economics of Hate, Edward Elgar Publishing.

Breton, A., et al., eds., (2002), Political Extremism and Rationality, Cambridge University Press.

Collier, P. and A. Hoeffler, (1998), “On economic causes of civil war,” Oxford Economic Papers, 50, 4, 563-73.

DeAngelis, D.L., W. Post and C. Travis, (1985), Positive Feedback in natural Systems, Springer-Verlag.

Eizenstat, S., J. E. Porter and J. Weinstein, (2005), “Rebuilding Weak States,” Foreign Affairs, January/February.

Enders, W. and T. Sandler, (2006), The political economy of terrorism, Cambridge university press.

Esteban, J. and D. Ray, (1999), “Conflict and Distribution,” Journal of Economic Theory, 87, 2, 379-415.

Herman, E.S. and N. Chomsky, (1988), Manufacturing consent: The Political Economy of the Mass Media, Pantheon Books.

Festinger, L. and J.M. Carlsmith, (1959), “Cognitive Consequences of Forced Compliance,” Journal of Abnormal and Social Psychology, 58, 203-210.

Glaeser, E.L., (2005), “The Political Economy of Hatred,” The Quarterly Journal of Economics, 120, 1, 45-86.

Guastello, S.J. (2008), “Chaos and Conflict: Recognizing Patterns,” Emergence: Complexity & Organization,10,4, 1-9.

Guastello, S .J. (2005). “Nonlinear models for the social sciences,” in S.A. Whelan (ed.), The Handbook of Group Research and Practice, pp. 251-272.

Hirsch, M.W. and H. Smith, (2006), "Chapter 4 Monotone Dynamical Systems," Handbook of Differential Equations: Ordinary Differential Equations, 2, 239–357.

Honaker, J., (2004), "Unemployment and Violence in Northern Ireland: A Missing Data Model for Ecological Inference.," Presented at the Annual Meetings of the Midwest Political Science Association.

Levin, J. and G. Rabrenovic, (2001), "Hate crimes and ethnic conflict," American Behavioral Scientist, 45, 4, 574-587.

Levy, A. (2008), "From Hatred to Genuine Peace: A Dynamic Formulation of Conflict Resolution," International Mathematical Forum, 3, 1-4, 43-47.

Medoff, H., (1999), "Allocation of Time and Hateful Behavior: A Theoretical and Positive Analysis of Hate and Hate Crimes," American Journal of Economics and Sociology, 58, 4, 959-973.

Minow, M., (2002), Breaking the Cycles of Hatred: Memory, Law, and Repair, Princeton University Press.

Muller, E.N. and E. Weede, (1990), "Cross-National Variation in Political Violence: A Rational Action Approach," The Journal of Conflict Resolution, 34, 4, 624-651.

Nafziger, E.W. and J. Auvinen, (2002), "Economic development, inequality, war and state violence," World Development, 30, 153-163.

Nowak, A., and Lewenstein, M. (1994). Dynamical systems: A tool for social psychology? In R. R. Vallacher & A. Nowak (Eds.), Dynamical

systems in social psychology (pp. 17–53). San Diego, CA: Academic Press.

Nozick, R. ed., (1997), "The Characteristic Features of Extremism," in Socratic Puzzles, Cambridge: Harvard University Press, 296-299.

Pape, R.A., (2003), "The Strategic Logic of Suicide Terrorism," The American Political Science Review, 97, 3, 343-361.

Piazza, J.A., (2006), "Rooted in Poverty?: Terrorism, Poor Economic Development, and Social Cleavages," Terrorism and Political Violence, 18, 1, 159 - 177.

Reynal-Querol, M. (2002), "Ethnicity, political systems, and civil wars," Journal of Conflict Resolution, 46, 29-54.

Rinaldi, S., (1998), "Laura and Petrarch: An Intriguing Case of Cyclical Love Dynamics," SIAM Journal on Applied Mathematics, 58, 4, 1205-1221.

Sandler, T. and Enders, W., (2004), "An Economic Perspective on Transnational Terrorism," European Journal of Political Economy, 20, 2, 301-316.

Santos Bravo, A., B. and C. M. Mendes Dias, (2006), "Depravation, Islamism and Geopolitical Factors," Defence and Peace Economics, 17, 4, 329 - 341.

Schelling, T.C., (1960), The Strategy of Conflict, Harvard University Press.

Sherif, M. (1966), In Common Predicament: Social Psychology of Intergroup Conflict and Cooperation, Boston: Houghton Mifflin Company, 24–61.

Sørli, M. E., N.P. Gleditsch and H. Strand, (2005), "Why Is There so Much Conflict in the Middle East?," The Journal of Conflict Resolution, 49, 1, 141-165.

Sprott, J. C., (2004), "Dynamical Models of Love," Nonlinear Dynamics, Psychology, and Life Sciences, 8, 3, 303-313.

Sternberg, R.J., and K. Sternberg, (2008), The Nature of Hate, Cambridge University Press.

Strogatz, S. H. (1994). Nonlinear dynamics and chaos: With applications to physics, biology, chemistry, and engineering. Reading, MA: Addison- Wesley.

Vallacher, R. R., Coleman, P. T., Nowak, A., & Bui-Wrosinska, L. (2010). Rethinking intractable conflict: The perspective of dynamical systems. The American Psychologist, 65, 262–278.

Vallacher, R. R., & Nowak, A. (Eds.), (1994a), Dynamical systems in social psychology, San Diego, CA: Academic Press.

Vallacher, R.R. et al., (2013), Attracted to Conflict: Dynamic Foundations of Destructive Social Relations, Peace Psychology Book Series. Springer-Verlag Berlin Heidelberg.