Analitical Derivation of the Cobb-Douglas Function based on the Golden Rule of Capital Accumulation

Pete Yashin

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ABSTRACT

In this paper, the neoclassical model is extended for the general case of economic growth, which can be represented as the sum of cyclical and growth components. If the general formulation of the golden rule of capital accumulation is satisfied (the savings rate is equal to the capital income share), the production function takes the form of the Cobb-Douglas function. This function governs the economic growth both when the economy is growing along an equilibrium path and when the economy is departing from it (the correlation coefficient between U.S. GDP changes and calculated ones is equal to 0.91). When economy fluctuations are averaged along an equilibrium path, the Cobb-Douglas function reduces to condition, which is similar to Harrod-Domar one. The level of technology may be reasonably considered to express in terms of the wage level.

Petr Yashin
UPEC
4 Marshal Batitsky St.,
Kharkiv, 61038 Ukraine
yashin.p.v@mail.ru

I. INTRODUCTION

The development of the neoclassical theory can be traced from the Harrod-Domar model [1,2], via the classical paper of Solow [3], to the golden rule of Phelps [4]. Early conclusions of the neoclassical growth model are received for the steady state growing economy, when the level of technology and labor productivity are stable, and the economy growth rate is equal to the population growth rate and is exogenous therefore. Including non-zero technological progress is very similar to the assumption of non-zero workforce growth, in terms of "effective" or augmented labor: a new steady state is reached with constant output per worker-hour required for a unit of output. Within the Solow growth model, the Solow residual or total factor productivity is an often used measure of technological progress.

An "effective" or augmented labor is not applied to this paper, and the term L is not population or labor force, but hours worked. Hence the intensive production function $y=y(k,T)$ is not static and continuously changes its magnitude because of variations in the level of technology T (where $y=Y/L$ is the output-to-labor ratio, and $k=K/(P\times L)$ is the capital-to-labor ratio, P is the price index). Therefore the point representing the steady state growing economy in the $\{y; k\}$ plane moves with the intensive production function magnitude change. If the savings rate s and the growth rate g are considered to be constants during stable long-run economy growth, the points representing such economy in the $\{y; k\}$ plane are placed along the straight line defined by an analytical expression derived in section II. Although this expression is very similar to the Harrod-Domar condition [1,2], it has an entirely different meaning. The Harrod-Domar condition is a relation between the growth rate of total output, g, and the savings rate, s, with a priori constant capital-output ratio, whereas the derived expression determines the limit for the capital-output ratio when the mean values of growth rate g and savings rate s are constant with time. In this paper, g is the summarized long-term economy growth rate due to both labor input growth and technological change.

The general formulation of the golden rule of capital accumulation is formulated in Section II: households consume wages (include any return to labor), and the saving is equal to the capital income (return to the proprietor(s) of capital stock). In another words, the savings rate is equal to the capital income share. The growth rate of such an economy is equal to the marginal product of capital (MPK), and which is the same as that of Phelps [4].

The long-term MPK constancy during the economy steady state growth is interpreted as a result of the long-term interest rate r constancy. The long-term interest rate value is the minimum interest rate that is sufficient for households to prefer saving to current consumption and is considered to be a constant inherent to the internal nature of human beings (nearly 3%). On the other hand, according to the golden rule of capital accumulation, the MPK is equal to the economy growth rate. Consequently, the fact that the average growth rate of the real U.S. GDP constitutes approximately 3%, takes on the nature of objective laws if human beings inhere some (about 3%) long-term interest rate.

In Section III, simultaneous analysis of trend and fluctuations really does involve an integration of long run equilibrium and short run disequilibrium. The neoclassical growth model is extended to the general case of economic growth, which can be represented as the sum of cyclical and growth components. For such a model the labor L is not already smoothly varying, but shows the considerable fluctuations during business cycles.

If the general formulation of the golden rule of capital accumulation is always satisfied, the intensive production function is analytically derived in Section III and takes the form of the Cobb-Douglas function. In this function, the savings rate, s, is the exponent of capital, and $(1 – s)$ is the exponent of labor (hours worked), the level of technology corresponds to the level of wage raised to the power of $(1-s)$, and consequently the augmenting labor corresponds to the wages.

Despite all these simplifications, the relation derived shows perfect agreement with data. The correlation coefficient between GDP changes in U.S. economy and calculated one is equal to 0.91.

The equilibrium growth path of an economy is described by a straight line in the $\{y, k\}$ plane, and business cycles are represented by departure from the equilibrium point along the intensive production function $y= y(k,T)$.

II. JOINT MODEL

Consider the standard neoclassical one-sector model featuring a constant-return production function and perfectly competitive (domestic) markets, so that the MPK equals the interest rate, r. The model invoked includes only a household sector and a business sector.
The model presented in this section is developed under the following simplifying assumptions. The total output Y is assumed to be steadily growing at a constant rate, g, with a constant depreciation rate, δ, and with the constant of the savings rate $s = S/(Y \times P)$. S is the gross saving, which is equal to the gross investment I along an equilibrium path.

Harrod-Domar Equilibrium Path

In an actual economy, the desire to invest is stable enough. For instance, the data from the NIPA Tables, which may be obtained on the World Wide Web on http://www.bea.gov/, show that the ratio of the sample standard deviation of the desire to invest to its mean value amounts to only 8.7% (Table I). It can be shown (Appendix A, Eq (A1)) that the ratio of capital stock, K, accumulated over a long (greater than $1/g$) period of time to total output is equal to

$$\frac{K^*}{Y^* \times P} = \frac{s}{g+\delta} \tag{1}$$

For an economy growing along an equilibrium path, levels of capital and output carries the meaning of the potential one, Y^* and K^*. Eq (1) corresponds to the conclusion following from the neoclassical growth model regarding the constancy of the capital-output ratio. This point of view finds many types of empirical evidence.

Although Eq (1) is very similar to the Harrod-Domar condition [1,2], it has an entirely different meaning. The Harrod-Domar condition is a relation between the growth rate of total output, g, and the savings rate, s, with a priori constant capital-output ratio, whereas Eq (1) determines the limit for the capital-output ratio when the mean values of g and s are constant with time because their oscillations are averaged in the course of a business cycles. So, the constancy of the capital-output ratio is the corollary of steadiness of the savings rate and the output growth rate mean values. The reasons of the long-term output growth rate constancy are considered later.

Let's rewrite Eq (1) accordingly to the conventional neoclassical notation: $k=K/(L\times P)$, $y=Y/L$.

$$y^* = \frac{g+\delta}{s} \times k^* \tag{2}$$

The long-term constancy of g, s and δ implies the economy steady state growth with capital-output ratio steadiness at the same level as according to the Harrod-Domar condition.

The point representing the equilibrium state of the economy on the graph of the production function in the neoclassical growth model is determined as that in which the saving done by households equal the investment required for supporting the existing capital-to-labor ratio. If the exogenous growth rate and the savings rate are invariable, the equilibrium point on the graph of intensive production function is fixed (Figure 1, equilibrium k at point c is determined: $s\times y=(g+\delta)\times k$). During non-zero technological progress, a new steady state is reached with constant output per worker-hour required for a unit of output in terms of "effective" or augmented labor.

In this paper, L is not effective labor force but total hours worked. Hence the intensive production function continuously changes its magnitude because of variations in the level of technology. Figure 1 illustrates the growth of the intensive production function due to the growth in the level of technology, from curve $y(k,T_a)$ to $y(k,T_b)$. The points a and b that correspond to the equilibrium state of the economy during a phase of stable long-run growth satisfy Eq (2), and hence they are on the straight line R defined by this equation. For the labor cost share, w/y, and the capital cost share, $(r+\delta)\times k/y$, to remain constant in total output, the MPK= r, must also be constant, during a period when the economy is growing along an equilibrium path. The labor cost share is stable enough in the U.S. total output (see Table I). Long-term real interest rate constancy is in practice observed in the developed economy also. This point of view is discussed in detail below. Hence the MPK= r, is considered to be a constant, during a period when the economy is growing along an equilibrium path, $r=\tau$. The constant τ is simply interpreted as the minimum real interest rate that is sufficient for households to prefer saving to current consumption.

So the slope of tangent to the intensive production functions for steady state growing economy is a constant (and is equal to $\tau+\delta$, Figure 1, points a and b).

Since $y = w + (r+\delta)\times k$, substitution of Eq (1) into this relation gives the following relation between the capital-to-labor ratio and the level of wage in the economy during a phase of steady growth at a constant rate:

$$k^*(T) = w^*(T)\times\frac{s}{g+\delta-s\times(\tau+\delta)} \tag{3}$$

And similarly the equilibrium labor productivity is:

$$y^* = w^*(T)\times\frac{g+\delta}{g+\delta-s\times(\tau+\delta)} \tag{4}$$

Eq (3) shows that during a period when the economy is growing along an equilibrium path the level of wage $w^*=W^*/P$ is proportional to the capital-to-labor ratio since the MPK, which is equal to the interest rate, is a constant. Therefore, if the economy is steadily growing at a constant rate, then its state can be represented by the level of wage instead of by capital-to-labor ratio, and the average labor productivity for such an economy is a linear function of the real wage, according to Eq (4). It is quite reasonable because in an actual economy the level of wage shows some rigidity; it fluctuates weakly during business cycle as compared to the capital-to-labor ratio (see standard deviations of real wage and real capital-to-labor ratio in Table I).

The general formulation of the golden rule of capital accumulation.

Let's recall the golden rule of capital accumulation now. Phelps [4] shows that consumption arrives at its maximum, when the MPK is equal to the economy growth rate:

$$r = g \tag{5}$$

Eq (5) was derived for the steady state growing economy. For such economy the MPK is constant during equilibrium growth, $r=\tau$. The economy is considered to be competitive so that the MPK is equal to the interest rate. Long-term real interest rate constancy is observed in practice in developed economies. It is commonly believed that high interest rates motivate households to save and do not motivate businesses to invest; however, the data presented in Table II do not support this point of view. The correlation between the long-term interest rate and the fluctuations in investment and saving is absent. It may be noted that the real long-term interest rate is approximately equal to 3% in a period of stable long-run growth. For instance (e.g. [5], Figure 21), the real interest rate was negligibly different from this value in Japan and

Germany in the 1980th. Adam Smith considered such a value to be a standard in his *Wealth of Nations*.

The long term interest rate is considered to be a constant inherent to the internal nature of human beings (nearly 3%). It is the minimum interest rate that is sufficient for households to prefer saving to current consumption. On the other hand, according to the golden rule of capital accumulation (5), the MPK is equal to the economy growth rate. Consequently, the fact that the average growth rate of the real U.S. GDP constitutes approximately 3%, takes on the nature of objective laws if human beings inhere some (about 3%) long-term interest rate.

According to the neoclassical model the equilibrium capital-to-labor ratio for the steady state growing economy is determined by following condition: the saving done by households equal to the investment required for supporting the existing capital-to-labor ratio:

$$s \times y = (g + \delta) \times k \ ,$$

combining this equation and Eq (5),

$$s \times y = (r + \delta) \times k, \text{ or} \tag{6}$$

$$s = (r + \delta) \times k / y \tag{6a}$$

The left side of Eq(6) is the saving and the right side is equal to the capital income. Eq (6) (the saving is equal to the capital income) is one of the possible formulations of the golden rule of capital accumulation. In another words, the savings rate is equal to the capital income share (Eq 6a). In this paper this statement is named “the general formulation of the golden rule of capital accumulation” since it will be extended for the general case of economic growth, which can be represented as the sum of cyclical and growth components in the next section. The saving and the capital income values equity during stable long-run economy growth is extra discussed in Appendix B.

IV. The Production Function for an Economy Departing from an Equilibrium Growth Path

Let’s examine the economy departing from an equilibrium growth path in this section. Here the case when the economy shows cyclically deviations from steady growth path during business cycles is considered. Reasons of these deviations (social upheavals, incorrect expectations that lead to short-term inequality of saving and investment) are not regarded in this paper.

The average labor productivity y and the capital-to-labor ratio k are not equilibrium ones during business cycles. A MPK= r is not constant for such an economy either.

Under the assumption of constant returns to scale, the intensive production function is a function of the capital-to-labor ratio and the level of technology $y = f(k,T)$. y can be considered to be only a function of the capital-to-labor ratio over a period short enough for the level of technology to remain unchanged.

Let the general formulation of the golden rule of capital accumulation be met, i.e., let the household’s saving be equal to the capital income. Then, at the production function point that corresponds to the state of the economy, the saving $s \times y(k,T)$ must be equal to the capital income $(r + \delta) \times k$. In the plane $\{y;\ k\}$, the value of the capital-to-labor ratio satisfying the general formulation of the golden rule of capital accumulation is found at the intersection of the saving per labor unit graph $s \times y(k,T)$ and the capital income per labor unit straight line $(r + \delta) \times k$ in accordance with Eq (6) (Figure 1, point c).

The partial derivative of the intensive production function $y(k,T)$ is equal to the slope of a tangent to the graph of this function, i.e.,

$$\frac{\partial y(k,T)}{\partial (k)} = r + \delta \ ,$$

or

$$r = \frac{\partial y(k,T)}{\partial (k)} - \delta \ ,$$

Having substituted value r received in last equation to the general formulation of golden rule (6), we have:

$$\frac{\partial y(k,T_a\ )}{\partial (k)} = s \times \frac{y(k,T)}{(k)} \qquad (7)$$

If savings rate s does not depend on the capital-to-labor ratio k (as mentioned above, the U.S. statistics show that the gross investment-to-GDP ratio is fairly constant), the following relation holds:

$$y(k,T) = \beta(T) \times k^s \qquad (8)$$

where β is an arbitrary constant which value depends on the level of technology and determines the magnitude of the intensive production function.

Since s is less than unity, Eq (8) exhibits the diminishing marginal returns of capital, which has been expected to be the case. Eq (8) is the Cobb-Douglas function. If we assume the production function to be given by the Cobb-Douglas function, then Eq (8) holds and consequently (7) do. Thus, the production function may be expressed in terms of the Cobb-Douglas function only when the general formulation of the golden rule of capital accumulation is satisfied. The magnitude of this intensive production function is determined by the level of technology.

Savings rate and growth rate vary during business cycles, but their mean values s and g considered being a constant during the long-term time period. This is justified because values $I/Y \times P$ and Y/L have minor standard deviations (see Table I). Therefore Eq (2) holds for the long-term period.

The value β is calculated for the equilibrium point [y^*; k^*] where the intensive production function (8) and the condition (2) for steady growth at a constant rate intersect. This intersection is represented in Figure 1 (the points a and b) by the intersection of the straight line R and two intensive production functions. The constant is obtained by equating the right-hand sides of (2) and (8), so that

$$\beta(T) = \{k^*(T)\}^{1-s} \times \frac{g+\delta}{s} \ .$$

A capital-to-labor ratio k varies during business cycle due to labor oscillations unlike the equilibrium one k^*, which is defined by the level of technology T. In an actual economy, the equilibrium capital-to-labor ratio k^* can not been easily and directly measured. That's why it is more convenient to use equilibrium wage level w^* (in (8)) that characterizes the level of technology. This is justified because the level of wage exhibits rigidity and varies weakly in the course of cyclic oscillations as compared to the capital-to-labor ratio (see Table I). Substitution of the capital-to-labor ratio k^* from Eq (3) into the relation for β yields:

$$\beta(T) = \left(w^*(T) \times \frac{s}{g + \delta - s \times (\tau + \delta)} \right)^{1-k_I} \times \frac{g + \delta}{s} .$$

Since the golden rule of capital accumulation is met, it follows that $g = r = \tau$, and Eq (8) can be re-cast in the following form:

$$y(k,T) = w^*(T)^{1-s} \times k^s \times \frac{(g + \delta)^s}{s^s \times (1 - s)^{1-s}} \tag{9}$$

Hodrick and Prescott [6] represent time series as the sum of a smoothly varying trend component and a cyclical component. They find that the nature of the co-movements of the cyclical components of macroeconomic time series is very different from the co-movements of the slowly varying components of the corresponding variables. The cyclical variations in output arise principally as the result of changes in cyclical hours worked and not as the result of changes in cyclical productivity or capital stocks. In contrast, growth is characterized by roughly proportional growth in capital stock and productivity (output per hour) and little change in the hours worked. Similarly, in this paper, gross domestic product is determined by two groups of economic factors (Table II and Figure 5). First, these are investment and hours worked, with the correlation coefficient between the investment and gross domestic product being equal to 0.70, between GDP and the hours worked being equal to 0.72, and between the investment and the hours worked being equal to 0.70. On the other hand, GDP is defined by the productivity of labor and the level of wage, with the correlation coefficient between average labor productivity and gross domestic product being equal to 0.60, between the wage level and GDP being equal to 0.68, and between the average labor productivity and the wage level being equal to 0.79. These two groups of factors are considered to be linearly independent because they show weak correlation between themselves (see Table II). This weak dependence can also be seen in Figure 5 where the variables causing changes in GDP (changes in the hours worked, in the wage level, and in the ratio of the change in investment to GDP) are shown along with GDP. Since changes in the labor productivity are determined by the growth in the level of technology, the level of wage in an actual economy is governed by the same factor and it does not depend upon labor fluctuations during business cycles. Consequently we can change $w = w^*$, then

$$y(k,T) = \lambda \times w(T)^{1-s} \times k^s \tag{10}$$

where

$$\lambda = \frac{(g + \delta)^s}{s^s \times (1 - s)^{1-s}} .$$

The capital-to-labor ratio k oscillates strongly during business cycles due to labor L fluctuations. Therefore Eq (10) may be considered reasonable to express in accordance with total variables instead convention neoclassical notation. Let’s rewrite Eq (10)

$$Y \times P = \lambda \times W^{1-s} \times K^s \times L^{1-s} \tag{11}$$

Despite all simplifications, the relation derived shows generally good agreement with data. Figure 4 depicts the U.S. GDP data and the calculations of the GDP from Eq (11) for $s = 0.3$. It can be seen that the calculations and the real data are in a remarkable agreement. The correlation coefficient between nominal US GDP and calculated one is equal to 0.91.

The production function obtained is the Cobb-Douglas function where W^{1-s} turns out to be the level of technology, and $W \times L$ the augmented labor. The value of λ changes with the coefficients weakly, and the operating value of λ is equal to approximately 0.8 in a developed economy.

Eq (10) is derived from the assumption that the general formulation of the golden rule of capital accumulation is satisfied, i.e., by assuming that saving is equal to the capital income (the savings rate is equal to the capital income share). However, this equation takes account of both business cycles and growth along an equilibrium path. In the plane $\{y(k,T); k\}$, the steady growth at a constant rate is described by the straight line defined in Eq (2), and the business cycle is represented by departure of the economy from the equilibrium point along the production function, which is described by Eq (10).

In the absence of fluctuations in the rate of economic growth and in the MPK, when the economy moves along an equilibrium path, the capital-to-labor ratio is proportional to the level of wage in accordance with Eq (3), and Eq (10) reduces to Eq (4), governing steady growth at a constant rate.

So, the production function magnitude and the equilibrium point spacing can be represented by the level of wage instead of by capital-to-labor ratio both if the economy is steadily growing at a constant rate (4) and for the more general case of economic growth, which can be represented as the sum of cyclical and growth components (10). It is quite reasonable because in an actual economy the level of wage shows some rigidity; it fluctuates weakly during the course of a business cycle as compared to the capital-to-labor ratio. Moreover, the level of wage is linearly independent from factors, which govern economy during business cycle fluctuations (investment and hours worked).

In an actual economy, data on the nominal average productivity of labor $(Y \times P)/L$ and the level of wage W demonstrate conclusively the excellent statistical correlation (e.g., the data in Figure 2 and Table II give a correlation coefficient of 0.79; however, the dependence of both these nominal quantities mainly on inflation is clearly evident in this figure). Variations in the real wage level and the real productivity of labor also show a definite statistical correlation between them (Figure 3, Table II: a correlation coefficient of approximately 0.50).

The way of finding the answer to the question of whether the cause of an increase in the average labor productivity is the capital-to-labor ratio or the level of wage is not evident. On the one hand, an advance in the productivity of labor without a corresponding growth in the capital-to-labor ratio is impossible, but on the other hand, in accordance with the neoclassical growth model, an increase in the level of wage when labor shortages occur gives the business motivation for increasing the level of technology in order to increase the labor productivity.

The theory of Cooley and Prescott [7] for the real business cycle states that technology shocks produce impacts on labor productivity, and a rise in the level of wage consequently occurs as consequence of increase of the labor productivity when there is a labor force shortage.

However, Gomme, Kydland, and Rupert [8] have found two anomalies that have plagued all household production models - the positive correlation between investments by the business and household sectors, and household investment's leading business investment over the business cycle. This contradiction disappears if an increase in the level of wage and consequently an increase in the wages occur first, because demand for labor exceeds supply, and an increase in the labor income compels the business sector to increase investment to adequately increase labor productivity. In practice (Figure 3), on the other hand, the changes in the real labor productivity often outpace the changes in the real wage level (in 1957, 1970, and 1980). This occurs because we deal with a self-consistent process when changes in one factor (e.g., in the capital-to-labor ratio) result in changes in other factors (the wage level and labor productivity) that in turn affect the first factor.

So the level of technology in the intensive production function may be considered reasonable to express in terms of the level of wage, in accordance with Eq (4) for growth along an equilibrium path and Eq (10) and (11) for the more general case of economic growth, which can be represented as the sum of cyclical and growth components. This does not mean that the technological progress does not matter. This means that technological progress provides an economy with the choice of technology based on the cost and efficiency. And since in the finally developed economy there is a shortage of labor, the level of wage tends to increase, which in turn causes the application of the more and more expensive machines, and consequently a rise in the fixed capital per worker. Such scheme explains a smooth increase of labor productivity despite the fact that the level of technology often changes in leaps and bounds.

IV. Conclusion

In this paper the neoclassical growth model is extended for the general case of economic growth, which can be represented as the sum of cyclical and growth components. During business cycle growth rate and MPK values have experienced short-run fluctuations, but long-run averaged values g and $r = \tau$ are assumed to vary weakly. Savings rate is considered to be a constant also. The general formulation of golden rule of capital accumulation is considered to be satisfied, i.e. saving is equal to the capital income (the savings rate is equal to the capital income share). The consequence of fulfilling last condition is equity of long-term averaged values of the growth rate g and the MPK= r. Then the g constancy is not independent condition for presented model, but is consequence of the general formulation of the golden rule of capital accumulation fulfillment and the long-term averaged MPK constancy. Consequently, the fact that the average growth rate of the real U.S. GDP constitutes approximately 3%, takes on the nature of objective laws if human beings inhere some (about 3%) long-term interest rate.

For presented model, the production function is analytically derived in Section III and takes the form of the Cobb-Douglas function (10) and (11). Moreover, the Cobb-Douglas equation is shown to be valid only when the general formulation of the golden rule of capital accumulation is satisfied. In this function, the savings rate, s, is the exponent of capital, the level of technology corresponds to the level of wage raised to the power of (1-s).

Despite all these simplifications, the relation derived shows generally good agreement with data. It can be seen (Figure 4) that the calculations and the real data are in a remarkable agreement. The correlation coefficient between nominal U.S. GDP changes and calculated one is equal to 0.91.

During long term growth path we can average the deviations from equilibrium growth in the MPK and in the rate of economic growth. Then the economy will move along an equilibrium path, and Cobb-Douglas Eq (10) reduces to Eq (2) and (4), which are particular cases of Eq (10). Although Eq (2) which governs the steadily growing at a constant rate economy is very similar to the Harrod-Domar condition [1,2], this equation has an entirely different meaning. In contrast, the Harrod-Domar condition is a relation between the growth rate of total output, g, and the savings rate, s, with a priori constant capital-output ratio. Whereas Eq (2) determines the limit for the capital-output ratio when the mean values of g and s are constant with time because their oscillations are averaged in the course of business cycles.

The level of technology in the production function may be considered reasonable to express in terms of the level of wage, in accordance with Eq (4) for growth along an equilibrium path and Eq (10) and (11) for the general case of economic growth, which can be represented as the sum of cyclical and growth components. This does not mean that the technological progress does not matter. It does mean that technological progress provides an economy with the choice of technology based on the cost and efficiency. And since in the developed economy there is a shortage of labor, the level of wage tends to increase, which in turn causes the application of the more and more expensive machines, and consequently a rise in the fixed capital per worker.

Appendix A

Let the price of any type of fixed capital at the end of an interval, i, under consideration be designated by A_i. Suppose that an annual increase in this fixed capital is proportional to total output with the constant of proportionality s_A, the total output $Y \times P$ grows at a constant rate of g, and its depreciation rate is equal to δ_A. Then A_{i+1} is given by

$$A_{i+1} = A_i + s_A \times P \times Y_{i+1} - \delta_A \times A_i$$

where P is a price index.

Further, we calculate the fixed capital-output ratio, $A_n/(P \times Y_n)$. If this ratio converges for large n, then the limit X is given by

$$X = \frac{A_n}{Y_n \times P} \cong \frac{A_{n+1}}{Y_{n+1} \times P} = \frac{A_n + s_A \times Y_{n+1} \times P - \delta_A \times A_n}{Y_{n+1} \times P} =$$

$$= s_A + \frac{A_n \times (1-\delta_A)}{Y_n \times (1+g) \times P} = s_A + X \times \frac{1-\delta_A}{1+g},$$

and hence

$$X = s_A + X \times \frac{1-\delta_A}{1+g}$$

Solving this relation for X gives

$$X = \frac{s_A \times (1+g)}{g+\delta_A} \cong \frac{s_A}{g+\delta_A}.$$

Thus, we have

$$\frac{A}{Y \times P} = \frac{s_A}{g+\delta_A} \quad \text{(A1)}$$

Appendix B

A society is assumed to be comprised of a household sector and a business sector (the government and the external world are not taken into account) and saving is equal to investment therefore. Equating the aggregate income and the expenditures yields the well-known equation applicable to production on an enlarged scale:

$$W \times L + (r+\delta) \times K = C + I$$

where C is households' consumption expenditures, $W \times L$ is wages (labor income), and $(r+\delta) \times K$ is the capital income.

At the fictitious early beginning, when the accumulation of a stock of capital was absent, the income of labor was equal to consumption expenditures automatically, $W \times L = C$. In a capitalist economy, the aggregate income increases by the value added by capital, $(r+\delta) \times K$. The aggregate expenditures are correspondingly increased by the value of a gross investment, I.

The values of the rate of economic growth, g, the savings rate, s, are assumed to vary weakly, hence Eq (1) holds. Substituting Eq (1) into last equation and taking into account that saving S are equal to investment I.

$$\frac{W^* \times L - C^*}{Y^* \times P} = s \times \frac{g-r}{g+\delta}. \quad \text{(B1)}$$

or

$$\frac{w^* - c^*}{y^*} = s \times \frac{g - r}{g + \delta}. \qquad \text{(B2)}$$

To begin the accumulation a stock of capital, the consumption expenditures must be less than the wages, and the left-hand and right-hand sides of Eq (B1) and (B2) must be greater than zero.

Next, we analyze the behavior of the left-hand and right-hand sides of these Equations over long enough periods when the economy has a sufficient stock of capital and in a state of stable long-run growth. If $W \times L - C \geq 0$, then this means that society continues to accumulate capital without consuming the capital income and only partially consuming wages. This case does not conform to saving logic because the saving is now made in order to consume more later on. This 'later on' does not ensue ever if both sides of Eq (B1) and (B2) are greater than zero, although the rate of growth ($g \geq r$) remains high. Here $r = \tau$, is the long-term interest rate which is considered to be a constant inherent to the internal nature of human beings.

Similarly, if $W \times L - C \leq 0$, then society relaxes and consumes not only wages but also the capital income produced by capital, and $g \leq r$, i.e., the rate of economic growth is lower than the MPK. This scenario may happen, and society has to submit oneself to the lower output rates. The United States manages to keep up this rate by decreasing saving, but only by sharply increasing its liabilities to foreign nations.

A conclusion may be reached regarding the equivalency of the saving and the capital income in an economy. Such a condition has the same corollary as the golden rule of capital accumulation derived by Phelps [4]: the growth rate of such an economy is equal to the MPK. However, we have arrived at this equality in a different way. Here, it is a corollary to the equality between the saving and the capital income and not a criterion for the savings rate to be optimal in order to maximize consumption.

The above-mentioned considerations may be illustrated in another way. If the value added by capital ($r \times K$) in excess of the capital depreciation were not consumed but invested (or correspondingly, the wages were consumed completely and both sides of Eq (B1) were equal to zero), then the capital would grow exponentially with time, $\exp(r \times t)$, while for an economy growing along an equilibrium path, according to Eq (1), the capital grows proportionally to total output, i.e., proportionally to $\exp(g \times t)$.

If society would choose to consume more and invest less and consequently the saving would be less than the capital income, then the growth rate would be less than the MPK, $g \leq r$. And the reverse is true, if investment were high, then the rate of growth would be higher than the MPK, $g \geq r$. Symmetry considerations also lead to the capital income = investment (saving) parity over a very long period, $g = r$.

Appendix C

Data Sources

All the numerical data presented in this paper are the author's calculations performed by using the data from NIPA Tables that may be obtained on the World Wide Web on: http://www.bea.gov/ (GDP and investment in Table 1.1.5, lines 1 and 5; the price index in Table 1.1.4, line 1 for the GDP and line 2 for the personal consumption expenditures; the wage accruals in Table 6.6A, line 1; the hours worked in Table 6.9B, line 1). The author has also used the data that may be obtained on http://www.federalreserve.gov/, Flow of Funds Accounts of the United States (wages in Table F.7, line 3; gross saving in Table F.8, line 1; the capital equal to the sum of business tangible assets in Table B.102, line 2 and in Table B.103, line 2). The "interest rate current year" means r have been obtained on http://www.federalreserve.gov/, "US government securities/ Treasury constant maturities/ Nominal; Maturity 1 year; Monthly”, TCMNOMY1. The correlations presented in Table 2 have been calculated using the interest rates values, and not their changes.

The labor cost share is the ratio of wages to GDP. The average labor productivity is the ratio of GDP to the hours worked. Real GDP is the ratio of GDP to GDP price index. The real wage is the ratio of the wage level to the personal consumption expenditures price index. The real average labor productivity is the ratio of real GDP to the hours worked. The tables and figures show the increment of a variable divided by this variable value that it has in the previous year. The ratio of the change in investment to GDP is the increment of the investment divided by GDP value in the previous year. The magnitude of the changes in GDP shown in Figure 4 has been computed by (12) for $s = 0.3$. It is equal to the 0.7*th* power of the product ($\Delta W/W \times \Delta L/L$) and to the 0.3*th* power of ($\Delta K/K$), where ($\Delta W/W$) designates the increment of the wage level divided by the wage level value that it has in the previous year, and the designations for the rest of the variables are given in a similar fashion.

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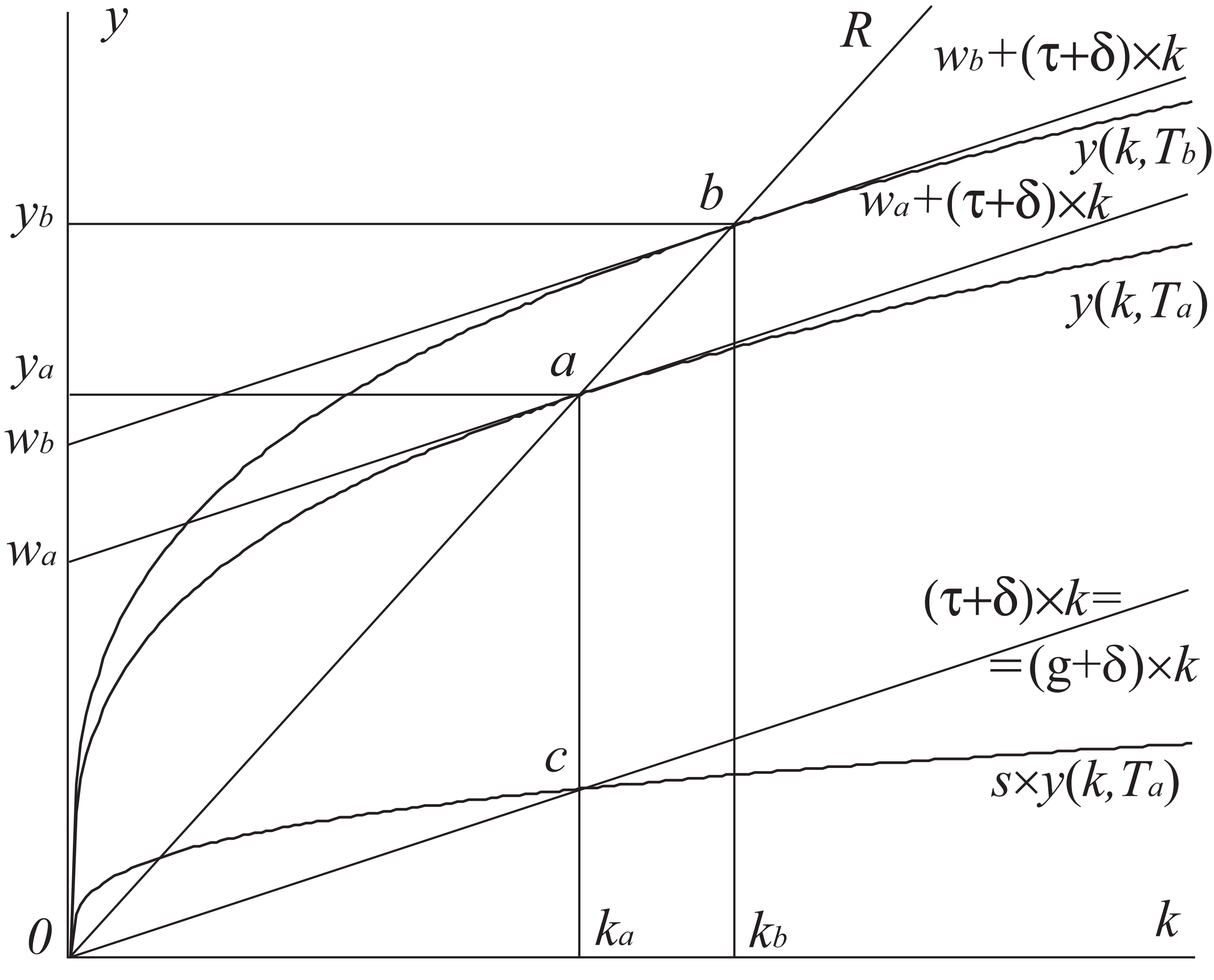

Figure 1. Two intensive production functions $y(k,T_a)$ and $y(k,T_b)$ of different magnitudes due to variations in the level of technology.

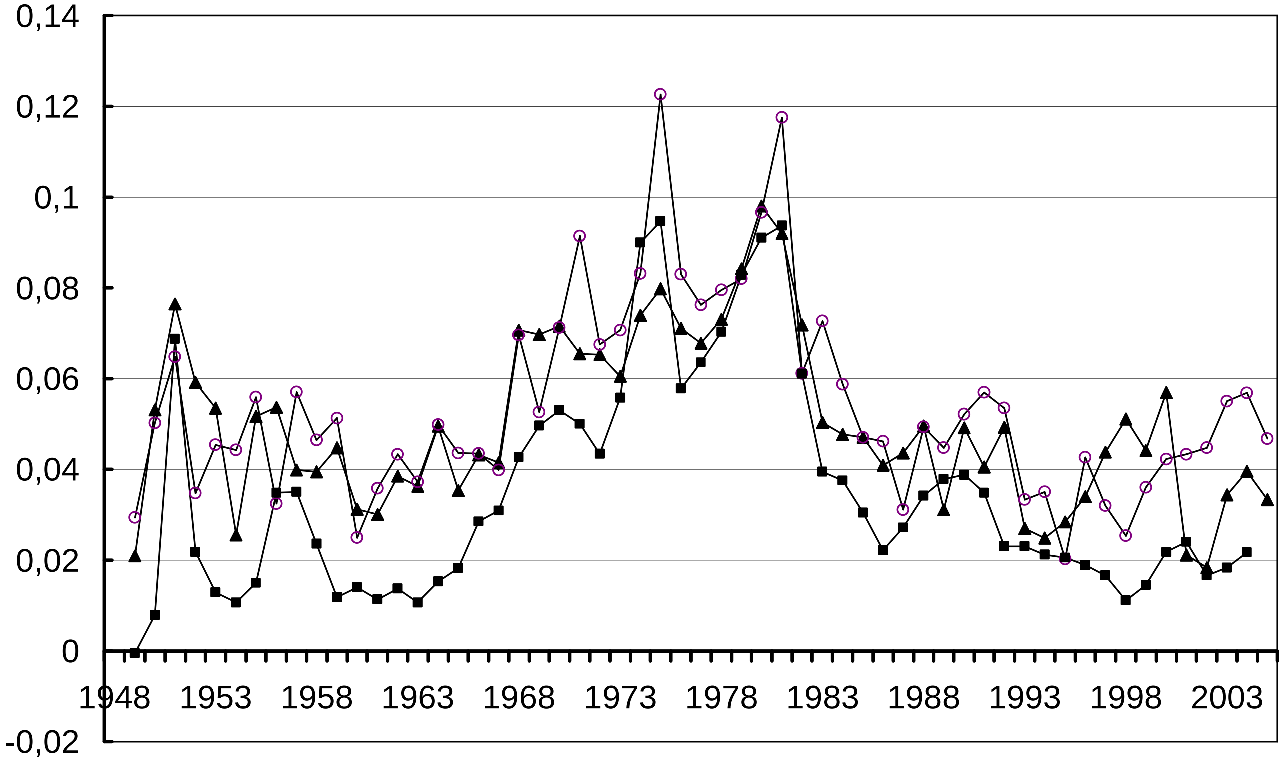

Figure 2. Changes in the wage level (▲), the average productivity of labor (○), and GDP price index (■) in the U.S. economy. For the data source, see Appendix B.

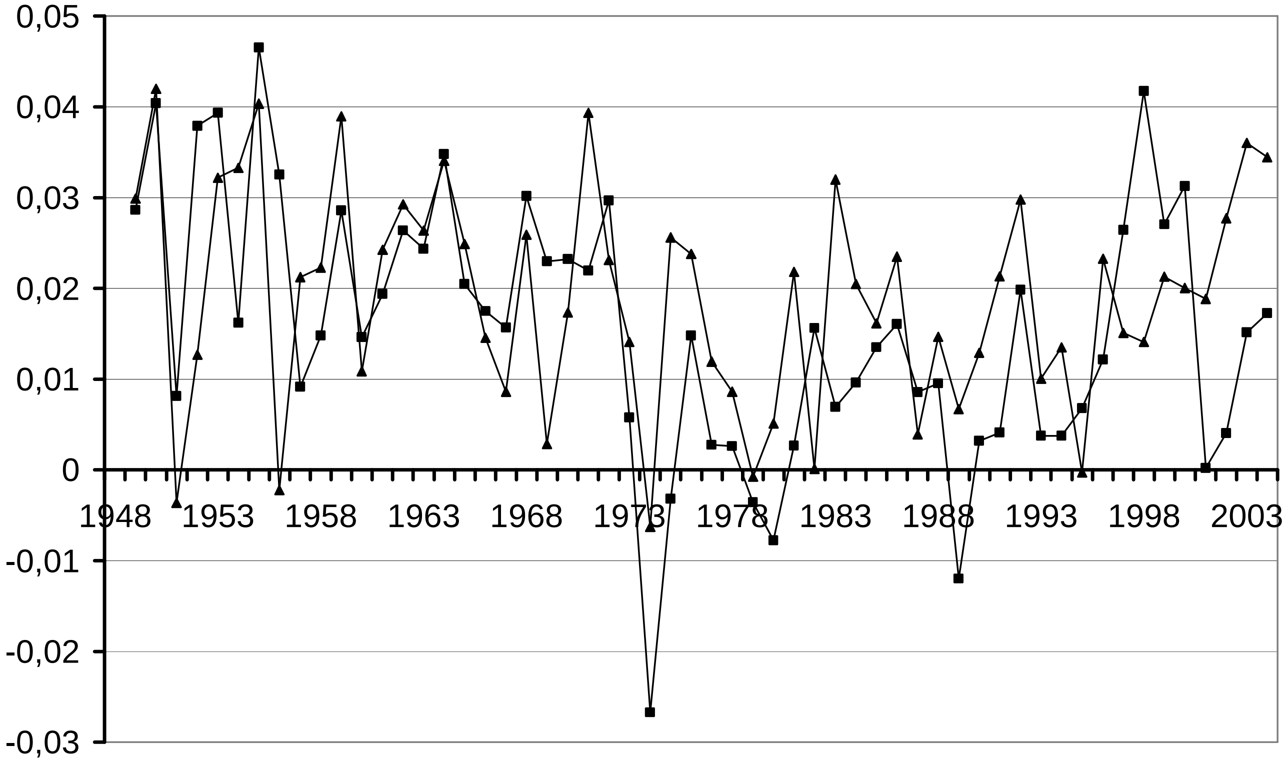

Figure 3. Changes in the real level of wage (■) and in the real average productivity of labor (▲) in the U.S. economy. For the data source, see Appendix B.

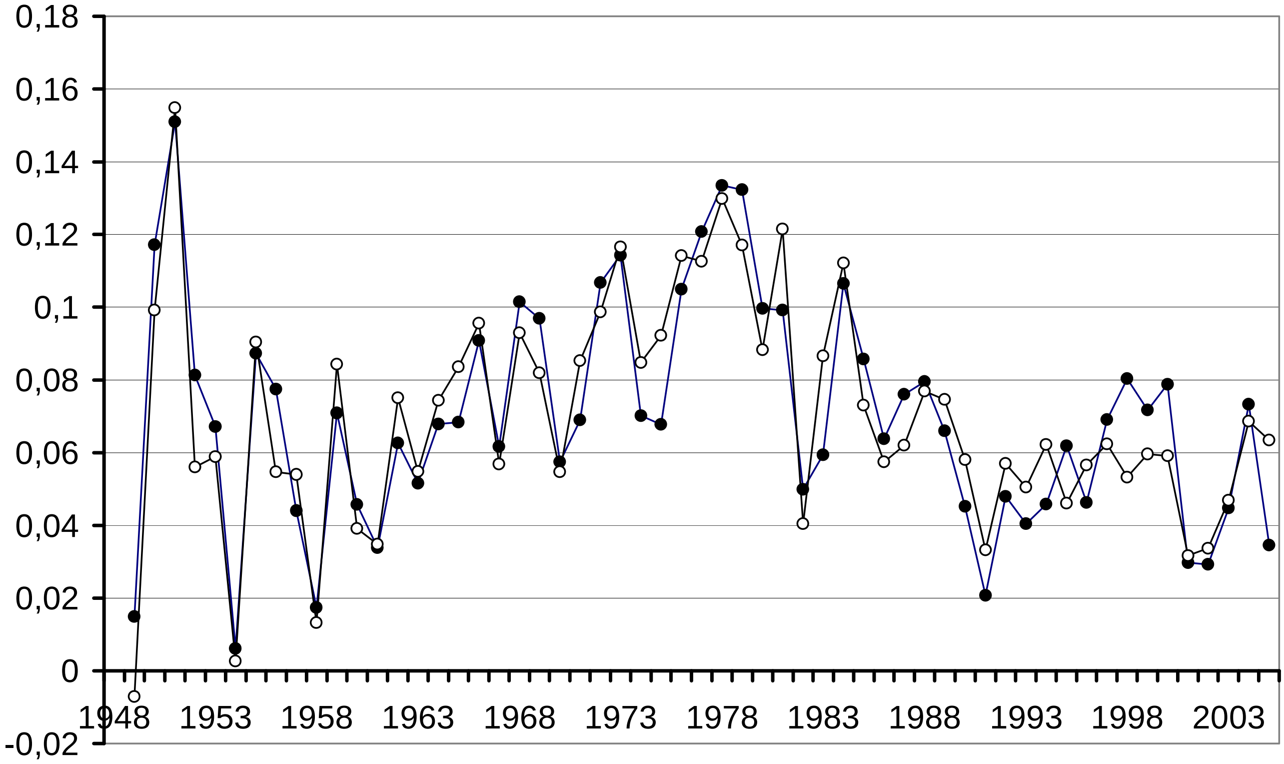

Figure 4. Changes in U.S. GDP (○) and changes in GDP calculated from the Cobb-Douglas equation with savings rate value s= 0.3 (●). For the data source, see Appendix B.

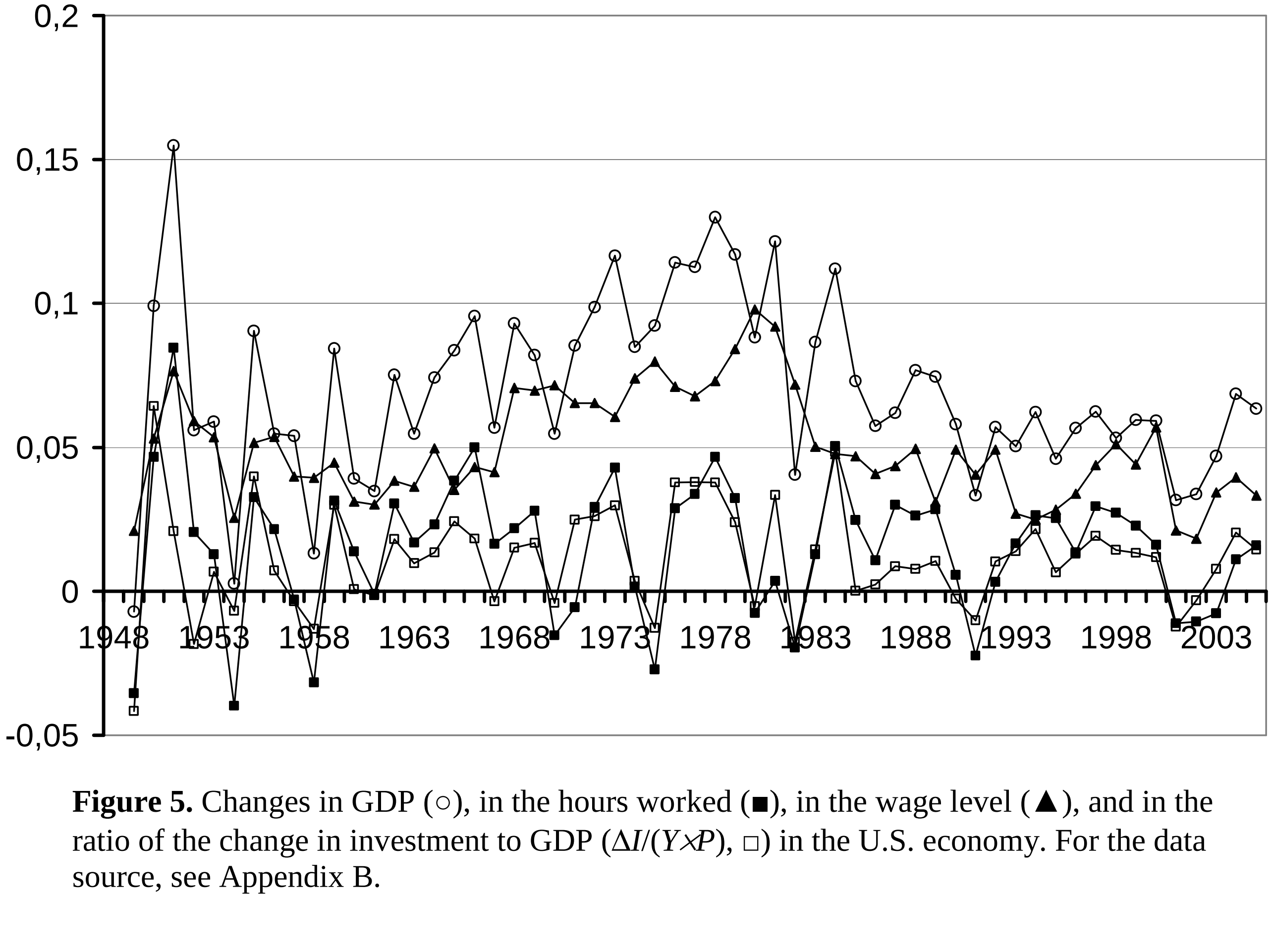

Figure 5. Changes in GDP (○), in the hours worked (■), in the wage level (▲), and in the ratio of the change in investment to GDP ($\Delta I/(Y\times P)$, □) in the U.S. economy. For the data source, see Appendix B.

Table I. Standard deviations of U.S. economy variables and variable changes. Sample period: 1948 – 2005

	σ	$\sigma/\bar{x}$
Private investment/GDP, I/(Y*P)	.014	.087
Labor cost share, W*L/(Y*P)	.010	.019
GDP change, Y*P	.031	
Hours worked change, L	.024	
Wage accruals change, W	.019	
Average labor productivity change, Y*P/L	.021	
GDP price index change, P	.024	
Investment change, I	.117	
GDP real change, Y	.025	
Capital/labor ratio change, K/L	.035	
Real wage change, W/P	.014	
Real average labor productivity change, Y/L	.012	
Real capital/labor change, K/(L*P)	.031	
Gross savings change, S	.095	

SOURCE. – See Appendix B.

NOTE. – σ =sample standard deviation of a variable; $\sigma/\bar{x}$ = relative standard deviation, the ratio of the sample standard deviation of the variable to its mean value.

Table II. Correlations of U.S. economy variable changes. Sample period: 1948 – 2005

	r	Y*P	L	W	Y*P/L	P	I	Y	K/L	W/P	Y/L
Y*P	.537										
L	.145	.725									
W	.658	.681	.172								
Y*P/L	.535	.603	-.112	.785							
P	.729	.612	.022	.837	.859						
I	.097	.702	.700	.230	.202	.062					
Y	-.114	.639	.872	.032	-.090	-.217	.812				
K/L	.247	.110	-.346	.411	.555	.439	-.069	-.294			
W/P	-.476	-.164	.144	-.123	-.404	-.610	.181	.393	-.152		
Y/L	-.491	-.157	-.234	-.281	.043	-.474	.232	.272	.099	.495	
K/(L*P)	-.366	-.529	-.580	-.281	-.095	-.373	-.295	-.292	.532	.349	.564
S	.246										

SOURCE. **–** See Appendix B.

NOTE. **–** Table 2 shows the correlations between changes in the economy variables except for the interest rate, r, for which interest rate values are used instead of their changes. The correlation coefficients are found at the intersection of the corresponding rows and columns. The designations in this table are the same as in Table 1, viz.,

GDP change, Y*P
Hours worked change, L
Wage accruals change, W
Average labor productivity change, Y*P/L
GDP price index change, P
Investment change, I
GDP real change, Y
Capital/labor ratio change, K/L
Real wage change, W/P
Real average labor productivity change, Y/L
Real capital/labor change, K/(L*P)
Gross savings change, S