“Certain-uncertain” inconsistency within the basic experimental procedures of behavioral economics

Harin, Alexander

Modern University for the Humanities

17 August 2024

“Certain-uncertain” inconsistency within the basic experimental procedures of behavioral economics

Alexander Harin
Modern University for the Humanities
aaharin@gmail.com

Old problems of the mathematical description of the economical behavior of a man are briefly reviewed. They are the comparison of choices of a man between uncertain and sure games and the radically different behavior of a man in different domains. The proposed solution of the problems consists in the purely mathematical method and models and is briefly reviewed in the Appendix.

In the present paper the main attention is paid to the analysis of the experimental support of this possible solution.

The generally accepted random incentive experimental procedures are discussed. A “certain-uncertain” inconsistency between the certain type of the choices and the uncertain type of the incentives is revealed and analyzed.

1. Introduction. The old problems of economics

There are a number of problems related to the mathematical description of the behavior of a man. Examples of these are the underweighting of high and the overweighting of low probabilities, the Allais paradox (Allais, 1953), risk aversion, loss aversion, equity premium puzzle, fourfold pattern of risk preferences, etc.

1.1. Comparison of choices of a man between uncertain and sure games

One of the problems of this mathematical description is a comparison of choices of a man between uncertain and sure games.

The essence of this problem consists in biases of choices of people (subjects) for the uncertain and sure games in contradiction with the predictions of the theory of probability. This problem is crucial and well-known. It is most important in behavioral economics in prospect and utility theories and also in psychology, decision theory, and the social sciences. It is pointed out in a wealth of works.

For example, we see in Kahneman and Tversky (1979) (pp. 265):
"PROBLEM 1: Choose between
A: 2,500 with probability .33, //
2,400 with probability .66, //
0 with probability .01;
B: 2,400 with certainty.
N = 72 [18] [82]."
My note. We see the clear inconsistency: 18% for A, that is less than 82% for B (for 72 trials) in opposition with the expectations 2,500×.33+2,400×.66 = 2,409 for A, that is more than 2.400 for B.

For example, we see in Starmer and Sugden (1991) (pp. 974):
"... a choice between two lotteries *R'* (for "riskier") and *S'* (for "safer"). *R'* gave a 0.2 chance of winning £10.00 and a 0.75 chance of winning £7.00 (with the residual 0.05 chance of winning nothing); *S'* gave £7.00 for sure."
My note. £10.00×0.2+£7.00×0.75 = £7.25 for R' and £7.00 for S'. Here the expectations are £7.25 for R' that is more than £7.00 for S', but the results were 13 choices for R' that is less than 27 choices for S'.

1.2. Radically different behavior of a man in different domains

An additional problem (that is probably a lot harder than the previous one) is, moreover, the radically different behavior of a man in different domains.

Thaler (2016) (pp. 1581–1582) wrote (the **boldface type** is my own):
"Kahneman and Tversky's research documented numerous choices that **violate any sensible definition of rational**. ... subjects were **risk averse in the domain of gains** but **risk seeking in the domain of losses**."
My note: at high probabilities.

For example, the data in Kahneman and Tversky (1979) (pp. 268) in Table 1 can be represented as:
Problem 3: (4,000 at 0.80) > (3,000 at 1.00) leads to choices [20%] < [80%].
Problem 3': (-4,000 at 0.80) < (-3,000 at 1.00) leads to choices [92%] > [8%].
My note. These data lead to the undoubted deduction of the clear inconsistency between the behavior of subjects in the domains of gains and losses.

The above problems were stated first in Allais (1953) and have not been solved at present (see, e.g., Kahneman and Thaler, 2006, Thaler, 2016).

2. Purely logical question

2.1. Qualitative difference between certain and uncertain outcomes

As was mentioned above, one of the problems of behavioral economics is to compare certain (sure, guaranteed) and uncertain (probable) outcomes. The essence of these problems consists in biases between choices of people for the certain and uncertain outcomes in contrast to the predictions of the theory of probability.

Some (little or big) qualitative difference between certain and uncertain outcomes is intuitively understood. Certain and uncertain outcomes can be moreover understood as opposite ones in some cases.

In the content of this article, it should be stressed that this qualitative difference takes place in any case regardless of the particular numerical results or values of such outcomes. It should be stressed especially that the particular numerical results (or values) of such outcomes can be very close to each other, but the qualitative difference remains true.

What does create the qualitative difference between the certain and uncertain outcomes? The basis for a certain outcome is some guarantee that this outcome will come. This guarantee should be supported by some additive efforts.

So the origin of this qualitative difference is based on some additive efforts that are intended to guarantee the certain outcomes.

2.2. Qualitative difference versus the quantitative equalities

Let us consider schematically (see fig. 1) the probability weighting (or Prelec's) curve (see, e.g., Kahneman and Tversky, 1979, Prelec, 1998).

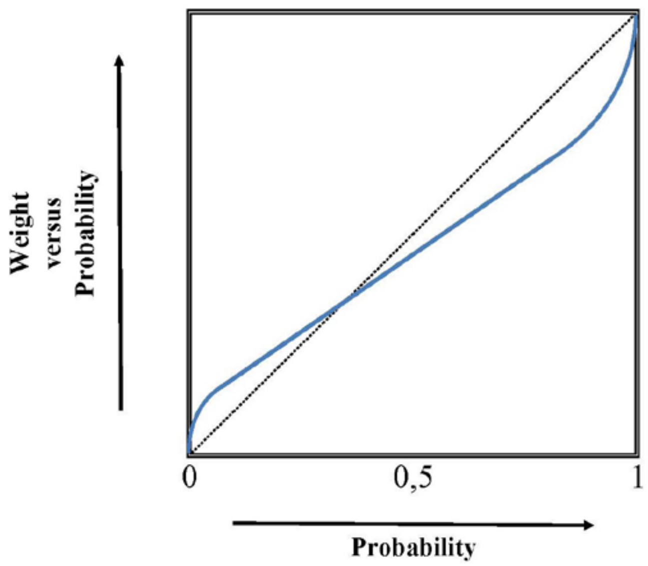

Fig. 1. Schematic sketch of the decision (or probability) weighting curve

This curve displays the weight of the reaction of subjects (people) versus probability in experiments with probable (non-guaranteed) outcomes at probabilities from 0 to 1.

It is an inverse *S*-shaped function $\pi(p)$, that is first concave $\pi(p)>p$ at low probabilities and then convex $\pi(p)<p$ at high probabilities. Two features can be mentioned that should attract our attention.

1. The first feature. The curve deviates from the straight line $\pi(p)=p$ of the predictions of the probability theory near the boundaries of the probability scale.

2. The second feature. At the end-points of the boundaries the curve tends to the straight line $\pi(p)=p$ of the predictions of the probability theory and touches it exactly at the boundaries. So $\pi(0)=0$ and $\pi(1)=1$.

Thus we see that the curve tends to the quantitative equalities at the end-points while the qualitative difference remains true. This fact was mentioned also in Kahneman and Tversky (1979) (pp. 282-283).

2.3. Purely logical question

2.3.1. Should they cause?

The above features (especially the second one) can lead one to ask:

Should qualitatively different situations cause the same quantitative result?

One can easily give the evident answer “**No**” to this evident question.

2.3.2. Can they cause?

If one asks:

Can qualitatively different situations lead to the same quantitative results?

then one can evidently answer: Yes. They can. But they can only under some special conditions. The discussed quantitative results nevertheless aren’t the special ones, they are widespread and well established.

If there are the facts that qualitatively different situations lead to the same quantitative results, then, at least, three evident consequences can be derived from these facts.

1. The situations are not qualitatively different.

There is evidently little likelihood of this consequence.

2. The quantitative results are not equal to each other (are not the same).

There is little likelihood of this consequence, because a plenty of skilled and independent research groups have repeated these results.

3. The situations have, in a sense, very little influence on the results.

Therefore one can come to a question:

2.3.3. Why do they cause?

As the result, we come to a purely logical question:

Why do the different qualitative situations
cause the same quantitative result
at the end-points of the probability weighting curve?

Note that, strictly speaking, one should name the question as “purely logical” only in that part of it which do not concern with experiments and other parts those cannot be named as “purely logical”.

3. Purely mathematical restrictions

3.1. One possible way to solve the problems

One (and the first possible) way to solve the above problems is widely discussed, e.g., in Schoemaker and Hershey (1992), Hey and Orme (1994), Chay et al (2005), Butler and Loomes (2007). The essence of this way consists in a proper attention to noise, uncertainty, imprecision, and other reasons that might cause dispersion, scattering, spread of the data.

3.2. Another possible way. The Aczél-Luce question whether $W(1)=1$

Another possible way to solve the problems of prospect and utility theories is to consider the vicinities of the boundaries of the probability scale, e.g. at *p~1* (see Aczél and Luce, 2007).

Aczél and Luce (2007) emphasized a fundamental question: whether $W(1)=1$ or, equivalently, $\pi(1)=1$ (whether the Prelec's weighting function (see Prelec, 1998) $W(1)$ is equal to 1 at $p=1$). From now on, I refer to this question as the Aczél-Luce question (or, in short, as Luce question).

Additional definitions

There are a number of the evidences of the qualitative difference between subjects' treatment of the probabilities of probable and certain outcomes (see, e.g., Kahneman and Tversky, 1979, McCord and de Neufville, 1986, Halevy, 2008). Therefore, in a general case, one should distinguish between the values of the probability weighting function of the certain outcome and of the limit of the probability weighting function of the uncertain outcome when the probability of this uncertain outcome tends to 1.

Let us additionally define or specify a value $W_{Certain}$ of the probability weighting function $W(p)$ for the certain outcome. $W_{Certain}$ may be assumed to be equal to 1 or else values of $W(p)$ may be normalized by $W_{Certain}$.

Let us additionally specify a value $W(1)$ as the limit of the probability weighting function $W(p)$ for the probable outcome when p tends to 1

$$W(1) \equiv \lim_{p \to 1} W(p) .$$

If $W(1)=W_{Certain}$, then $W(p)$ is continuous (at $p=1$). In the general case, this has not been proven. So, if W_{Unreal} is defined similarly for the impossible case,

$$W(p) = \begin{cases} W_{Unreal} & p = 0 \\ W(p) & p \in]0,1[\\ W_{Certain} & p = 1 \end{cases}$$

(see also Aczél and Luce, 2007) and $W(p)$ can be continuous or discontinuous.

3.3. Modification of the question. A synthesis of the two ways

Let us reformulate, modify the Aczél-Luce question whether $W(1)=1$ into the question whether $W(1)=W_{Certain}$ or whether $W(p)$ is continuous at $p=1$.

To answer to the question and to prove or disprove the continuity of $W(p)$ at $p=1$ one should determine and measure the difference

$$W_{Certain} - W(1) = ?$$

Note, that this does not put a question whether $W_{Certain}$ is equal to 1.

The answer $W(1)\neq W_{Certain}$ to the modified Aczél-Luce question means that the function can have a discontinuity at $p=1$. This is not only a quantitative but a qualitative, moreover, a topological feature also. So, the answer to the question can qualitatively change the situation in the prospect and utility theories, at least in their mathematical aspects.

In any case, one may suppose that a synthesis of these two possible ways can be of some interest. This idea of the synthesis turned out to be useful indeed. It has successfully explained, at least partially or qualitatively, the underweighting of high and the overweighting of low probabilities, risk aversion, and some other problems (see, e.g., Harin, 2012a, b). There are also works providing experimental support of this synthesis (see, e.g., Steingrimsson and Luce, 2007, Harin, 2014).

A purely mathematical research (see, e.g., Harin, 2012, 2022) combines, synthesizes the two above-mentioned ways. That is, it considers the dispersion of data (or the influence of the dispersion of the data) near the boundaries of the probability scale.

3.4. Existence theorem and its consequences

A purely mathematical theorem was proved, e.g., in Harin (2022). It states that near every boundary of the probability scale there is some non-zero restriction on the expectation of a real-valued random variable under the condition that its variance is non-zero. These restrictions may be considered as some forbidden zones for the expectation (see the Appendix of the present article).

The theorem gives the answer to the above logical question (strictly speaking, it cancels this question). Indeed, the theorem states that the probability weighting curve cannot tend to the straight line $\pi(p)=p$ exactly at the end-points of the probability scale. In particular, the expectations cannot take values within this forbidden zone and are therefore biased from the boundaries of the probability scale to its middle. So the considered particular qualitatively different situations should not and cannot lead to the same quantitative results.

The theorem allows moreover to explain the above problems, at least partially. Namely, an applied mathematical method of biases of expectations, qualitative mathematical models and special qualitative mathematical model are proposed in Harin (2022). The special qualitative mathematical model explains in particular the above problems. The method, general models and special model are briefly reviewed in the Appendix of the present article.

4. Pure mathematical issue

4.1. Preliminary

As have been mentioned in the preceding subsection, the theorem gives the answer to the above logical question. Indeed, the theorem states that, due to the forbidden zones, the probability weighting curve cannot tend to the straight line $\pi(p)=p$ exactly at the end-points of the probability scale.

Nevertheless the experiments are seemed to contradict the above logic and are seemed to don't manifest the predictions of the theorem.

For example, the experiments at p~1, in particular the tendency of the experimental (Prelec's) probability weighting curve to $\pi(p)=p$ (see, e.g., Bruhin et al, 2010, Abdellaoui et al, 2011), are seemed to do not support the theorem.

One can see that an issue arises:

4.2. Pure mathematical issue

So a pure mathematical issue arises:

Why the mathematics
has not been manifested and confirmed by the experiments?

Note that, strictly speaking, one should name the issue as "purely mathematical" only in that part of it which do not concern with experiments and other parts those cannot be named as "purely mathematical".

So the theorem need experimental confirmation.

5. Analysis of a typical detail of the experiments

5.1. Random, uncertain incentives

Let us analyze a feature of the abovementioned experiments. Let us consider some typical descriptions of the prospect and utility experiments. One can see in the literature (the **boldface type** is my own):

Loewenstein and Thaler (1989), page 188: “The students … were told that the experimenter would select and implement one of their choices **at random**.”

Tversky and Thaler (1990), page 206: “The subjects are told that one of these pairs will be selected **at random** at the end of the session, and that they will play one of these bets.”

Kahneman et al, (1991), page 195: “One of the four market trials would subsequently be selected **at random** and only the trades made on this trial would be executed”. Page 197: “One of the accepted offers (including the original endowment) was selected **at random** at the end of the experiment to determine the subject's payment.”

Vossler et al, (2012), page 158: “Participants are instructed that one of the 12 choice sets will be **randomly** chosen at the end of the experiment, each with equal probability.”

Baltussen et al (2012), page 424: “In the WRIS treatment, subjects play the game ten times, one of which for real payment. In the BRIS treatment, subjects play the game only once with a one-in-ten chance of real payment.”; page 425: “In both RIS treatments, a **ten-sided die** was thrown individually by each subject to determine her payment.”

Such a procedure can be seen not only in the field of the prospect and utility theories but also in other fields of the economics, see, e.g., Larkin and Leider (2012), page 193: “Subjects made fifteen choices between a lottery and a fixed payment. … Subjects were paid for one **randomly** selected decision”.

5.2. Certain outcomes

So, subjects are stimulated by random incentives. This is a well-known feature of the experiments including in the field of the prospect and utility theories. But let us consider this feature more closely. One can see a fine detail in the literature (the **boldface type** and **<u>underlining</u>** are my own):

Andreoni and Sprenger (2012), page 3365: "One choice for each subject was selected for payment by drawing a numbered card **at random**. Subjects were told to treat each decision as if it were to determine their payments." and page 3366: "Section I provided a testable hypothesis for behavior across **<u>certain</u>** and uncertain intertemporal settings."

Von Gaudecker et al, (2011), page 669: "Additionally, for one in every ten participants in these two treatments, one lottery was **randomly** selected and played out, and the payoff of that lottery was paid out." and page 667: "the probabilities of the high payoff in each option vary from 25 percent to **<u>100 percent</u>**".

Holt and Laury (2002), page 1646: "… subjects began by indicating a preference, Option A or Option B, for each of the ten paired lottery choices in Table 1, with the understanding that one of these choices would be selected **at random** ex post and played to determine the earnings for the option selected." and "Even the most risk-averse person should switch over by decision 10 in the bottom row, since Option B yields a **<u>sure</u>** payoff of $3.85 in that case" and page 1645: TABLE 1 "Option A: **<u>10/10</u>** of $2.00, 0/10 of $1.60. Option B: **<u>10/10</u>** of $3.85, 0/10 of $0.10"

Harrison et al (2005), page 898: "We undertook a new series of experiments that build closely on the basic design features of HL," ("HL" means here Holt and Laury, 2002) "but allow an identification of the extent to which the apparent scale effects on risk aversion are actually order effects." and from TABLE 1: **<u>Probability=1</u>** Payoff=$2 Probability=0 Payoff=$1.60 **<u>Probability=1</u>** Payoff=$0.10.

Bruhin et al, (2010), page 1380: "At the end of the experiment, one row number of one decision sheet was **randomly** selected for each subject, and the subject's choice in that row determined her payment." and "The subjects had to indicate in each row of the decision sheet whether they preferred the lottery or the **<u>certain</u>** payoff." and page 1378: "We elicited **<u>certainty equivalents</u>** for a large number of two-outcome lotteries."

Abdellaoui et al (2011), page 704: "Each series involved a choice between a prospect and an ascending range of **<u>sure</u>** payments, … At the beginning of the experiment, each subject was told that one of his choices would be **randomly** drawn and then played for real."

Starmer and Sugden (1991), page 974: "subjects in groups B and C knew that they were taking part in a **random**-lottery experiment in which questions 21 and 22 had equal chances of being for real." and "One problem, which we shall call P', required a choice between two lotteries R' (for "riskier") and S' (for "**<u>safer</u>**"). R' gave a 0.2 chance of winning ₤10.00 and a 0.75 chance of winning ₤7.00 (with the residual 0.05 chance of winning nothing); S' gave ₤7.00 **<u>for sure</u>**."

So, the random incentive procedures are used not only in the probable but in the certain situations too. Let us consider this detail more closely.

6. “Certain-uncertain” inconsistency of the random-lottery incentive systems

6.1. Inconsistency between the uncertain incentives and certain outcomes

First, let us note that the stimulation by the random payment of one from two or more alternatives may be named as a random, uncertain stimulation. One may name it also as the stimulation by an uncertain incentive.

Further, let us consider the stimulation by this uncertain incentive for uncertain and certain choices.

Suppose, that subjects choose an uncertain choice, that is the choice, which probability is strictly less than 1 (and strictly more than 0). In this case, the choice and the incentive are of the same type.

Suppose, that the subjects choose a certain choice, that is the choice, which probability is strictly equal to 1 (or strictly equal to 0). In this case, the choice and the incentive are of the qualitatively different types. The choice is certain but the incentive is uncertain.

Hence, there is an evident inconsistency between the certain type of the choice and the uncertain type of the incentive.

One may name this problem as a “certain-uncertain” inconsistency.

This “certain-uncertain” inconsistency is evident but is not still mentioned in the literature (see, e.g., Andreoni and Sprenger, 2012, Vossler et al, 2012, Baltussen et al, 2012, Harin, 2005-2024). The inconsistency was revealed (discovered) in the report Harin (2014).

6.2. Role of the incentives

Incentives are widely discussed in economics (see, e.g., Starmer and Sugden, 1991, Fehr and Falk, 2002, Holt and Laury, 2002, Baltussen et al, 2012, Larkin and Leider, 2012). Do incentives influence the choice of the subjects in the prospect and utility theories and models?

The correct answer to this question needs a special research. However, one may be sure, that if incentives would not have any significant influence on the choice of the subjects, then there would be no reason to use such incentives.

So, one may not exclude that an incentive can influence a choice of a subject, at least partially.

6.3. Random-lottery incentive systems as the generally accepted experimental procedure systems

The discussed random incentive procedures are usually referred to as the random-lottery incentive systems (or the random lottery incentive systems or random incentive systems (RIS), etc.).

Baltussen et al (2012), page 219: “Savage (1954, p. 29) credits W. Allen Wallis for first proposing the RIS.”

Page 420: “RISs are known under several names, including random lottery incentive system (Starmer and Sugden 1991), random lottery selection method (Holt 1986), random problem selection procedure (Beattie and Loomes 1997), and random round payoff mechanism (Lee 2008). The different names apply to particular types of experiments (risky choice or social dilemma), rewards (lotteries or outcomes), or tasks (composite or single-choice)”

The random-lottery incentive systems are the well-known mechanisms in experiments in the prospect and utility theories and models.

Starmer and Sugden (1991), page 971: “This is the random lottery procedure. This incentive system has several attractive features. It allows the experimenter to collect a considerable amount of data from each subject, thus economizing on the costs of recruiting subjects and allowing tests that compare a subject's responses to two or more tasks. At the same time, it avoids the problem of reference-point and wealth effects that would be created if subjects were paid according to their performances on each of a number of tasks. (A subject's response to one task might be affected by the amount he or she had won on a previous task.)”

Andreoni and Sprenger (2012), page 3365: “random-lottery mechanism, which is widely used in experimental economics”

Starmer (2000), page 371: “although most experiments involve real — usually monetary — incentives, the most common reward mechanism is the random lottery incentive system. In experiments with this design, subjects are rewarded according to their response to one task which is randomly selected at the end of the experiment.”

Wakker (2007): “the random-lottery incentive system has become the almost exclusively used incentive system for individual choice, and numerous studies have used and tested it. It is used by people well recognized in experimental economics … Without it, real incentives for individual choice are no longer well possible.”

Baltussen et al (2012), page 419: “If a subject performs multiple tasks in an experiment where each task is for real, then income and portfolio effects will arise (Cho and Luce 1995; Cox and Epstein 1989). The RIS is the only incentive system known today that can avoid such effects. In addition, for a given research budget and with the face values of the monetary amounts kept the same, RISs allow for a larger number of observations.”

So, the random-lottery incentive systems are the prevailing and generally accepted experimental procedures in the prospect and utility theories and models. The tests of the Aczél-Luce questions are often connected with these systems.

7. To consider certain outcomes as uncertain

So, one may conclude:

1) The random-lottery incentive systems are widespread in the prospect and utility theories and models. They are moreover the generally accepted experimental procedure systems.

2) The essence, in particular the incentives of these random-lottery incentive systems, corresponds to the random, uncertain character of their name.

3) These incentives are used for all the outcomes including the certain ones. This causes the evident inconsistency between the uncertain incentives and certain outcomes. One may name this inconsistency as the "certain-uncertain" inconsistency.

4) The specific "certain-uncertain" inconsistency of the random-lottery incentive systems is a natural feature of these systems. But it has still not been considered as it deserves.

In many works (see, e.g., von Gaudecker et al, 2011, Vossler et al, 2012, Harin, 2005-2023) one can find the elaborated researches of correctness of the random-lottery incentive systems. But the author of the present article has found no mention about what one can name the "certain-uncertain" inconsistency.

5) So, the (direct and non-direct) tests of the modified Luce question those have been performed to date (those, maybe, have not been analyzed thoroughly enough) can lead to the misleading opinion that, at p~1, the probable outcomes are quantitatively the same as the certain ones. That is, it can lead to the misleading opinion that the probability weighting function $W(p) \rightarrow W_{Certain}$, when $p \rightarrow 1$.

But there are a number of the evidences of the qualitative difference between the probable and certain outcomes (see, e.g., Kahneman and Tversky, 1979, McCord and de Neufville, 1986, Halevy, 2008). The existence theorem support this qualitative difference and the answer $W(1) \neq W_{Certain}$ to this question.

6) Because of this evident "certain-uncertain" inconsistency, the deductions from the random-lottery incentive experiments, those include the certain outcomes, cannot be unquestionably correct.

In particular the existing random-lottery incentive systems seems at present unable to determine the quantitative difference between certain and probable outcomes at p~1.

So, these deductions need an additional proof or an amendment.

7) In any case, additional independent analyses and/or researches of the "certain-uncertain" inconsistency are needed.

8) A following feature should be stressed: the uncertain incentives can call the certain outcomes into question.

That is, the uncertain incentives of the generally accepted experimental procedure systems can mislead the subjects to consider (to some extent, indirectly or directly, unconsciously or consciously) the certain outcomes, that are stimulated by these uncertain incentives, as some uncertain outcomes.

So, very approximately, the generally accepted experimental procedure systems can mislead the subjects to consider certain outcomes as uncertain.

8. Confirmations of the theorem

Let us consider some of the many confirmations of the occurrence of the discussed forbidden zones and of usefulness of them and of SQMM in behavioral economics. Let us also find adequate answers to the logical question and mathematical issue.

8.1. Practical everyday life evidences of the occurrence of the considered forbidden zones

8.1.1. Boat and waves

Consider a calm or mirror-like sea and a small rigid boat or any other small rigid floating body at rest in the sea. Suppose that this boat or body rests right against (or is constantly touching) a rigid moorage wall. As long as the sea is calm, the expectation of its side can touch the wall.

Suppose there is a heavy sea. Consider a small rigid boat or any other small rigid floating body which oscillates on the waves in the heavy sea. Suppose that this boat or body oscillates on the waves near this rigid moorage wall.

When the boat is oscillated by sea waves, then its side oscillates also (both up–down and left–right) and it can touch the wall only in the (nearest) extremity of the oscillations. Hence the expectation of the side cannot touch the wall. Hence the expectation of the side is biased away from the wall.

So, one can say that, in the presence of waves, a forbidden zone exists between the expectation of the side and the wall.

This forbidden zone biases the expectation away from the wall. The width of the forbidden zone is roughly one-half of the amplitude of the oscillations.

8.1.2. Washing machine, drill

Consider a washing machine or drill (or any other rigid body) that can vibrate when it works. Suppose its rigid (and not sharp, thick) side (or some rigid limiter of the movement of its side) is located near a rigid surface or wall.

If the machine or drill is at rest, then the expectation of this side can be located right against (be constantly touching) the wall.

If the machine or drill vibrates, then the expectation of this side is biased and kept away from the wall due to the vibrations.

So, in the presence of the vibrations, a forbidden zone exists between the expectation of the side of the rigid body and the rigid wall. This is evidently true for any rigid body near any rigid surface or wall.

8.2. Numerical examples of the forbidden zones in behavioral economics

8.2.1. Practical numerical example. First domain. Gains

Suppose that the parameters of the special practical qualitative mathematical model (see the Appendix) for the gains are: the presupposed bias for the choices for the uncertain game is equal to \$2, and for the sure game it is equal to \$1.

The typical examples (see, e.g., Correa et al, 2013, Zhang and Siemsen, 2019) can be simplified to the special qualitative situations similar to that of Harin (2022) and Kumar and Goyal (2015). The numeration corresponds to the Appendix.

Imagine that you face the following pair of concurrent games (a sure game and an uncertain game). Choose between:

A) A sure gain of \$99.

B) A 99% chance to gain \$100 and a 1% chance to gain or lose nothing.

In the ideal case, without taking into account the dispersion of the data, the expectations μ_{sure} and μ_{uncert} are equal to each other: $\mu_{sure} = \$99 \times 100\% = \$99$ and $\mu_{uncert} = \$100 \times 99\% = \$99$. So, in the ideal case we have

$$\$99 = \$99,$$

that is, the uncertain and sure games are equally preferable.

In the real case, one should take into account some dispersion of the data, and hence the minimal non-zero variance (3) caused by this dispersion, and the forbidden zones (4) caused by this variance, at least for the uncertain games.

One can consider the real case of a non-zero variance of the data, the corresponding forbidden zones, and presupposed biases.

The biases are $\Delta_{ch\text{-}\mu.uncert}$ = \$2 and $\Delta_{ch\text{-}\mu.sure}$ = \$1. So we have $\mu_{sure} - \Delta_{ch\text{-}\mu.sure}$ = \$99×100% - \$1 = \$98 and $\mu_{uncert} - \Delta_{ch\text{-}\mu.uncert}$ = \$100×99% - \$2 = \$97. The expectation μ_{uncert} is biased more than μ_{sure} and

$$\$98 > \$97.$$

We see the clear and evident difference between the resulting expectations (with their biases caused by the forbidden zones of the theorem) and its correspondence with the salient and unequivocal choices of the subjects.

8.2.2. Practical numerical example. Second domain. Losses

The case of gains has been explained many times, and in a lot of ways. But a uniform explanation for both gains and losses, without any additional suppositions (as, e.g., in Correa et al, 2013), had not been nevertheless recognized by the author of the present article (see a slightly similar work Arnold et al, 1983).

SQMM (Special Qualitative Mathematical Model, see the Appendix) turns out to be useful for such a uniform explanation.

The case of losses was considered under the same suppositions as for the case of gains.

Imagine that you face the following pair of concurrent games (a sure game and an uncertain game). Choose between:

A) A sure loss of -$99.

B) A 99% chance to lose -$100 and a 1% chance to lose or gain nothing.

In the ideal case, for the sure game $\mu_{sure} = -\$99 \times 100\% = -\$99$ and for the uncertain one μ_{uncert} = -$100×99% = -$99. So they are exactly equal to each other:

$$-\$99 = -\$99 .$$

Therefore the both choices (games) should be equally preferable.

Consider the real case. The forbidden zone biases the expectation from the boundary of the interval to its middle (see also, e.g., Kahneman and Thaler, 2006, Madansky, 1959, Zhang and Siemsen 2019). Therefore, at high probabilities, the biases are subtracted from the absolute values for both cases, gains and losses. That is, due to the opposite signs of the values for gains and losses, the bias is subtracted for the gains and added for the losses.

Note. This is not a supposition but a rigorous conclusion. Hence the conditions of the SQMM are naturally uniform for more than one domain.

The forbidden zones were considered under the same suppositions as for the gains, that is for the same, uniform parameters.

The biases are $\Delta_{ch\text{-}\mu.uncert}$ = $2 and $\Delta_{ch\text{-}\mu.sure}$ = $1. So we have $\mu_{sure} + \Delta_{ch\text{-}\mu.sure}$ = -$99×100% + $1 = -$98 and $\mu_{uncert} + \Delta_{ch\text{-}\mu.uncert}$ = -$100×99% + $2 = -$97. The expectation μ_{uncert} is biased more than μ_{sure} and

$$-\$98 < -\$97 .$$

The expectation for the uncertain game is biased more than that for the sure one, as was also the case for the gains, but here the bias increases the preferability of the uncertain loss and it is (due to the obvious difference between the resulting expectations) more preferable than the sure one.

So here is the clear difference between the resulting expectations and its correspondence with the salient preferences and choices. So the SQMM provides the explanation for the domain of losses as well. Moreover, this explanation is uniform for the both domains of gains and losses.

9. Answers to the question and issue

Let us find adequate answers to the logical question and mathematical issue.

9.1. Logical question

The purely logical question was:

Why do the different qualitative situations cause the same quantitative result at the end-points of the probability weighting curve?

The above analysis and confirmations lead to the statement that “the same quantitative result” is at least under question. The theorem states that it is moreover impossible.

The discussed different qualitative situations don’t (and cannot) cause the same quantitative result. Feasibly this is due to some mistake of the interpretation of the results of the experiments.

So the purely logical question is cancelled.

9.2. Mathematical issue

The pure mathematical issue was:

Why the mathematics has not been manifested and confirmed by the experiments?

The above analysis and confirmations lead to the statement that the mathematics is confirmed by everyday experience and (at correct interpretation of the experiments) by the experiments in behavioral economics.

So the pure mathematical issue is answered.

10. Conclusions

The old problems of behavioral economics and a new approach to their possible solution are briefly reviewed. The problems are concerned with the mathematical description of the behavior of a man. The first problem of this mathematical description is the comparison of choices of a man between uncertain and sure games. The second is the radically different behavior of a man in different domains.

The basis of the possible solution of these problems is the purely mathematical theorem of the existence of some forbidden zones for the expectations of any real-valued random variable. These forbidden zones are located near the boundaries of the probability scale. The consequences of the theorem are the hypothesis of presupposed biases in the behavior and decisions of peoples, applied mathematical method of biases of expectations (MMBE), general and special (SQMM) qualitative mathematical models.

In the present paper the main attention is paid to the analysis of the experimental support of this possible solution.

The basic and prevailing random incentive experimental procedures are analyzed and the "certain-uncertain" inconsistency between the certain type of the outcomes and the uncertain type of the incentives is revealed.

This inconsistency can lead (in the minds of peoples) to the (hidden) misleading understanding of the certain outcomes as some uncertain ones. This misleading understanding can explain the incorrect interpretation of the experiments at very high and very low probabilities where the forbidden zones must bias the behavior and choices of peoples.

The confirmations of the occurrence of the discussed forbidden zones and usefulness of them and of SQMM are considered as well.

References

Abdellaoui, M., A. Baillon, L. Placido, and P. P. Wakker, “The Rich Domain of Uncertainty: Source Functions and Their Experimental Implementation,” *American Economic Review*, **101** (2011), 695-723.

Aczél, J., and D. R. Luce, “A behavioral condition for Prelec’s weighting function on the positive line without assuming *W*(1)=1,” *Journal of Mathematical Psychology*, **51** (2007), 126–129.

Aldashev, G., Carletti, T., and Righi, S., “Follies subdued: Informational efficiency under adaptive expectations and confirmatory bias,” *Journal of Economic Behavior & Organization*, **80** (2011), 110–121.

Allais, M., “Le comportement de l'homme rationnel devant le risque: critique des postulats et axiomes de l'école Américaine,” *Econometrica*, **21** (1953), 503–546.

Andreoni, J., and C. Sprenger, “Risk Preferences Are Not Time Preferences,” *American Economic Review*, **102** (2012) 3357-76.

Arnold, L., H. Crauel, and V. Wihstutz, “Stabilization of linear systems by noise,” *SIAM J. Control Optim*, **21** (1983), 451–461.

Baltussen, G., T. Post, M. J. van den Assem, and P. P. Wakker “Random Incentive Systems in a Dynamic Choice Experiment,” *Experimental Economics*, **15** (2012), 418–443.

Beattie, J., and G. Loomes, “The impact of incentives upon risky choice experiments,” *Journal of Risk and Uncertainty*, **14** (1997), 155–168.

Bruhin, A., H. Fehr-Duda, and T. Epper, “Risk and Rationality: Uncovering Heterogeneity in Probability Distortion,” Econometrica, **78** (2010), 1375-1412.

Butler, D., and G. Loomes, “Imprecision as an Account of the Preference Reversal Phenomenon,” *American Economic Review*, **97** (2007), 277-297.

Chay, K., P. McEwan, and M. Urquiola, “The Central Role of Noise in Evaluating Interventions that Use Test Scores to Rank Schools”, *American Economic Review*, **95** (2005), 1237-1258.

Cho, Y., and D. R. Luce, ‘Tests of hypotheses about certainty equivalents and joint receipt of gambles,” *Organizational Behavior and Human Decision Processes*, **64** (1995), 229–248.

Correa, M., Gonzalez-Sabate, L., and Serrano, I., “Home bias effect in the management literature,” *Scientometrics*, **95** (2013), 417–433.

Cox, J. C., and S. Epstein, “Preference reversals without the independence axiom,” *American Economic Review*, **79** (1989), 408–426.

Fehr, E., and A. Falk, “Psychological foundations of incentives,” *European Economic Review*, **46** (2002), 687-724.

Gaudecker von, Hans-Martin, Arthur van Soest, and Erik Wengstrom, “Heterogeneity in Risky Choice Behavior in a Broad Population.” *American Economic Review*, **101** (2011), 664-94.

Halevy, Y., “Strotz Meets Allais: Diminishing Impatience and the Certainty Effect,” *The American Economic Review*, **98** (2008), 1145-1162.

Harin, A., “Forbidden Zones for the Expectations of Data: New Mathematical Methods and Models for Behavioral Economics,” *Academic Journal of Applied Mathematical Sciences*, **8** (2022), 12-26.

Harin, A., “Macroscopic analogs of quantum-mechanical phenomena and auto-transformations of PDFs,” *THE 5TH INTERNATIONAL CONFERENCE ON STOCHASTIC METHODS (ICSM-5). Track B. Applications of Stochastic Methods. B.2*, Moscow, 2020. See also: Harin, A., “Macroscopic analogs of quantum-mechanical phenomena and auto-transformations of functions,” MPRA Paper, 2020, No. 104188, [Online] Available at: https://mpra.ub.uni-muenchen.de/104188/1/MPRA_paper_104188.pdf

Harin, A., “Forbidden zones for the expectation. New mathematical results for behavioral and social sciences,” MPRA Paper, 2018, No. 86650, [Online] Available_at: https://mpra.ub.uni-muenchen.de/86650/1/MPRA_paper_86650.pdf

Harin, A., “The random-lottery incentive system. Can *p~1* experiments deductions be correct?” *16th conference on the Foundations of Utility and Risk*, Rotterdam, 2014. See also: Harin, A., “Problems of utility and prospect theories. A ”certain-uncertain” inconsistency of the random-lottery incentive system,” MPRA Paper, 2014, No. 55706, [Online] Available_at: https://mpra.ub.uni-muenchen.de/55706/1/MPRA_paper_55706.pdf

Harin, A., “Data dispersion in economics (II) – Inevitability and Consequences of Restrictions,” *Review of Economics & Finance*, **2**(3) (2012b), 24-36.

Harin, A., “Data dispersion in economics (I) – Possibility of Restrictions,” *Review of Economics & Finance*, **2**(2) (2012a), 59-70.

Harin, A., “Principle of uncertain future, examples of its application in economics, potentials of its applications in theories of complex systems, in set theory, probability theory and logic”, *Seventh International Scientific School "Modelling and Analysis of Safety and Risk in Complex Systems"*, Saint-Petersburg, 2007. See also: Harin, A., “Principle of uncertain future and utility,” MPRA Paper, 2007, No. 1959, [Online] Available_at: https://mpra.ub.uni-muenchen.de/1959/1/MPRA_paper_1959.pdf

Harin, A., “New Economics Papers. Utility Models & Prospect Theory”, (2005-2024) [Online] Available at: http://econpapers.repec.org/scripts/nep.pf?list=nepupt

Harrison, G. W., E. Johnson, M. Mcinnes, and E. Rutström, “Risk Aversion and Incentive Effects: Comment,” *American Economic Review*, **95** (2005), 897-901.

Hey, J., and C. Orme, “Investigating Generalizations of Expected Utility Theory Using Experimental Data,” *Econometrica*, **62** (1994), 1291-1326.

Holt, C. A., “Preference reversals and the independence axiom,” *American Economic Review*, **76** (1986), 508–515.

Holt, C. A., S. K. Laury, “Risk Aversion and Incentive Effects” *The American Economic Review*, **92** (2002), 1644-1655.

Kahneman, D., J. L. Knetsch, and R. H. Thaler, “Anomalies: The Endowment Effect, Loss Aversion, and Status Quo Bias,” *The Journal of Economic Perspectives*, **5** (1991), 193-206.

Kahneman, D., and R. H. Thaler, “Anomalies: Utility Maximization and Experienced Utility,” *Journal of Economic Perspectives*, **20** (2006), 221-234.

Kahneman, D., and A. Tversky, “Prospect Theory: An Analysis of Decision under Risk,” *Econometrica*, **47** (1979), 263-291.

Kumar, D. and Goyal, N., "Behavioural biases in investment decision making – a systematic literature review." *Qualitative Research in Financial Markets*, **7** (2015), 88–108.

Larkin, I., and S. Leider, "Incentive Schem+es, Sorting, and Behavioral Biases of Employees: Experimental Evidence," *American Economic Journal: Microeconomics*, **4** (2012), 184-214.

Lee, J., "The effect of the background risk in a simple chance improving decision model," *Journal of Risk and Uncertainty*, **36** (2008), 19–41.

Loewenstein, G., and R. H. Thaler, "Anomalies. Intertemporal Choice," *Journal of Economic Perspectives*, **3** (1989), 181–193.

Madansky, A., "Bounds on the expectation of a convex function of a multivariate random variable". *Ann. Math. Stat*., 30 (1959), 743–746.

McCord, M., and R. de Neufville, "Lottery Equivalents: Reduction of the Certainty Effect Problem in Utility Assessment," *Management Science*, **32** (1986), 56-60.

Meier, S. and Sprenger, C., "Present-biased preferences and credit card borrowing," *American Economic Journal: Applied Economics*, **2** (2010), 193–210.

Prelec, D., "The Probability Weighting Function," *Econometrica*, **66** (1998), 497-527.

Schoemaker, P., and J. Hershey, "Utility measurement: Signal, noise, and bias," *Organizational Behavior and Human Decision Processes*, **52** (1992), 397-424.

Starmer, C., "Developments in Non-Expected Utility Theory: The Hunt for a Descriptive Theory of Choice under Risk," *Journal of Economic Literature*, **38** (2000), 332-382.

Starmer, C., and R. Sugden, "Does the Random-Lottery Incentive System Elicit True Preferences? An Experimental Investigation," *American Economic Review*, **81** (1991), 971–78.

Steingrimsson, R., and R. D. Luce, Empirical evaluation of a model of global psychophysical judgments: IV. Forms for the weighting function. *Journal of Mathematical Psychology*, **51** (2007), 29–44.

Thaler, R., "Behavioral Economics: Past, Present, and Future," *Am. Econ. Rev.*, **106** (2016), 1577–1600.

Tversky, A., and R. H. Thaler, "Anomalies: Preference Reversals," *The Journal of Economic Perspectives*, **4** (1990), 201-211.

Vossler, C. A., M. Doyon, and D. Rondeau, "Truth in Consequentiality: Theory and Field Evidence on Discrete Choice Experiments," *American Economic Journal: Microeconomics*, **4** (2012), 145-171.

Wakker, P. P., "Message to referees who want to embark on yet another discussion of the random-lottery incentive system for individual choice," 2007. http://people.few.eur.nl/wakker/miscella/debates/randomlinc.htm

Zhang, Y., E. Siemsen. "A meta-analysis of newsvendor experiments: Revisiting the pull-to-center asymmetry", *Production and Operations Management*, **28** (2019), 140–156.

A. Appendix. A new applied mathematical method and models for behavioral economics

This Appendix summarizes briefly some of the results of Harin (2022). The notation and numbering of expressions are maintained.

A.1. Existence theorem in a little more detail

A set $\{X_i\}$, $i = 1, \dots, n$, of real-valued random variables X_i whose values lie within an interval $[a, b]$ was considered in, e.g., Harin (2020), Harin (2022). Under the normalizing condition

$$\sum_{k=1}^{K} p_X(x_k) + \int_{-\infty}^{+\infty} f_X(x)dx = \sum_{x_k \in [a,b]} p_X(x_k) + \int_{a}^{b} f_X(x)dx = 1, \qquad (1)$$

where K is the number of the discrete value(s) of X, $p_X(x_k)$ is the probability mass function and $f_X(x)$ is the probability density function, the expectation and variance of X, and their relations were analyzed.

A proof was made that, for the variance σ^2 of any real-valued random variable X_i with range $[a, b]$ and expectation μ,

$$\sigma^2 \le (\mu - a)(b - \mu). \qquad (2)$$

Suppose a set $\{X_i\}$, $i = 1, \dots, n$, of random variables X_i whose values lie within an interval $[a, b]$. If $0 < (b\text{-}a) < \infty$ and there exists a forbidden zone (or a lower bound) of a non-zero width $\sigma^2_{\min}$ for the variances σ_i^2 of X_i, such that for all i,

$$\sigma^2{}_i \ge \sigma^2{}_{\min} > 0. \qquad (3)$$

The theorem states: certain forbidden zones (or boundary bounds, or restrictions) of a non-zero width r_μ exist for the expectations μ_i of each X_i such that

$$a < (a + r_\mu) \le \mu_i \le (b - r_\mu) < b. \qquad (4)$$

The proof uses inequalities (2) and (3) to obtain inequalities including

$$\mu_i \ge a + \frac{\sigma^2{}_i}{b-a} \ge a + \frac{\sigma^2{}_{\min}}{b-a},$$

to obtain restrictions r_μ or boundary bounds or forbidden zones of the width

$$r_\mu \equiv \frac{\sigma^2{}_{\min}}{b-a} \le \frac{\sigma^2{}_i}{b-a}, \qquad (5)$$

and inequalities

$$a < \left(a + \frac{\sigma^2{}_{\min}}{b-a}\right) \le \mu_i \le \left(b - \frac{\sigma^2{}_{\min}}{b-a}\right) < b \qquad (6)$$

those are equivalent to (4).

A.2. Hypothesis of presupposed biases

The above practical examples of section 6 evidently illustrate possible forbidden zones of the theorem. Similar examples are widely disseminated in real life. Due to this dissemination, subjects (people) can keep in mind the feasibility of such forbidden zones and the biases of the expectations caused by the zones. This can influence the behavior and choices of the subjects.

In consequence of this consideration, I propose a statement that can be named as a behavioral hypothesis of presupposed biases:

"People, as economic subjects, behave and decide (at least to a considerable degree) as if there were some presupposed (hypothetical) biases of the expectations for games."

Note. This hypothesis can be supported by the reason that such biases may be proposed and tested even from a purely formal point of view.

This hypothesis can be found in hidden forms in the literature or derived from it (see, e.g., Kumar and Goyal, 2015, Correa et al, 2013, Schoemaker and Hershey, 1992, etc.) in this particular field, and in an explicit form in neighboring fields (see, e.g., Aldashev et al, 2011, Meier and Sprenger, 2010). Nevertheless one should state it here in an explicit form and emphasize it.

A.3. Applied mathematical method of presupposed biases of expectations

A.3.1. Propositions

Two main propositions can be suggested for a mathematical method of solution of the above problems.

Proposition 1. *Subjects behave and decide (at least to a considerable degree) as if there were some presupposed biases of the expectations for games*.

Due to this proposition, the method (approach) can be called an Applied Mathematical Method of Presupposed Biases of Expectations, or AMMPBE, or shortly MMBE. The MMBE is to explain not only the objective situations but also and mainly the subjective behavior and choices of subjects.

Proposition 2. *Explanation by the forbidden zones of the theorem.*

That is these biases (real biases or subjective reactions and choices of the subjects) can be explained (at least to a considerable degree) with the help of the forbidden zones of the theorem.

A.3.2. Notation

The real expectations for the games were denoted by

$$\mu_{sure} \quad \text{and} \quad \mu_{uncert} \equiv \mu_{uncertain} .$$

The presupposed biases (of the expectations) that are required to obtain the data corresponding to the choices of the subjects were denoted by

$$\Delta_{ch-\mu.uncert} \equiv \Delta_{choice-\mu.uncertain} \quad \text{and} \quad \Delta_{ch-\mu.sure} \equiv \Delta_{choice-\mu.sure} .$$

That is the resulting expectations (i.e., expectations including these biases) for the observed choices of the subjects can be written as $\mu_{uncert} + \Delta_{ch-\mu.uncert}$ for the uncertain games and as $\mu_{sure} + \Delta_{ch-\mu.sure}$ for the sure ones.

A.3.3. General relations

1. Condition for the MMBE. Due to the first main proposition, the method of biases of expectations can be useful only if these biases for the choices for the uncertain (see the third relation below) games are non-zero

$$| \Delta_{ch-\mu.uncert} | > 0 \quad \text{or} \quad \operatorname{sgn}(\Delta_{ch-\mu.uncert}) \neq 0 . \tag{7}$$

2. Forbidden zones as, at least, one of the origins of biases. The presupposed bias $\Delta_{ch\text{-}\mu.uncert}$ may be introduced and considered purely formally. The question is not only whether $\Delta_{ch\text{-}\mu.uncert}$ can explain the problems. Due to the above second proposition, $\Delta_{ch\text{-}\mu.uncert}$ itself should be explained by the forbidden zones of the theorem, at least partially.

First of all, the theorem should be applicable. Therefore some minimal variance $\sigma^2_{\min} > 0$ is required to exist and inequality $\sigma^2 \geq \sigma^2_{\min}$, is required to be true.

Further, the bias caused by the forbidden zone of the theorem was denoted by $\Delta_{theorem}$. The sign of the presupposed bias should coincide with that for the bias caused by the theorem

$$\operatorname{sgn}(\Delta_{ch-\mu.uncert}) = \operatorname{sgn}(\Delta_{theorem}) .$$

Then the conditions for the explanation can be written as $\Delta_{ch\text{-}\mu.uncert} \approx \Delta_{theorem}$ in the case when the forbidden zones are the main source of the biases. If these zones are only one of the essential sources of the biases, then these conditions can be represented as $\Delta_{ch\text{-}\mu.uncert} = O(\Delta_{theorem})$.

So the relations of the explanation by the theorem are

$$\sigma^2 \geq \sigma^2_{\min} > 0 \quad \text{and also}$$

$$\Delta_{ch-\mu.uncert} \approx \Delta_{theorem} \ \text{ or at least } \ \Delta_{ch-\mu.uncert} = O(\Delta_{theorem}) . \tag{8}$$

3. Biases for sure games. The above considerations about noise suppression and sure games emphasize the condition that the sure games are guaranteed by some guaranteeing efforts. Due to these efforts, the biases for the sure games can be suppressed and reduced. They can be moreover equal to zero.

Therefore the presupposed biases of the data for the sure games were assumed to be equal to zero or, more generally, are strictly less than the presupposed biases for the corresponding uncertain games.

So, the relation for the sure and uncertain games is

$$| \Delta_{ch-\mu.uncert} | > | \Delta_{ch-\mu.sure} | . \tag{9}$$

A.4. General and special qualitative mathematical models

A.4.1. General qualitative mathematical models

A *qualitative problem* was defined for the purposes of the models as the problem such that the sign of the difference between the resulting expectations for the choices of the subjects (people) for the uncertain and sure games is distinct from the sign of the difference between the real expectations for these games.

This type of problems was chosen as the example that is usual in experiments (see, e.g., Correa, 2013, Zhang and Siemsen, 2019, Schoemaker and Hershey, 1992). It can make manifest a qualitative representation of the problems and can be a background for further research. Such problems will be the item of the first stage of the MMBE.

The inalienable feature of the analyzed qualitative problems is the necessary change of the sign. There can be only three combinations of the signs: the expectation for the uncertain game (or outcome) can be greater than, less than, or equal to that for the sure game. So, the signs of their differences can be correspondingly positive, negative, or zero.

In other words, when the difference between the real expectations is, e.g., positive (that is, $\operatorname{sgn}(\mu_{uncert} - \mu_{sure}) > 0$), then, to obtain the observed data, the difference for the choices (resulting expectations) should be non-positive, that is, $\operatorname{sgn}(\mu_{uncert} + \Delta_{ch\text{-}\mu.uncert} - \mu_{sure} - \Delta_{ch\text{-}\mu.sure}) \leq 0$. When it is negative, then the difference for the choices should be non-negative. When the difference between the real expectations is equal to zero, then the difference for the choices should be undoubtedly positive or negative.

This feature can be represented by

$$\operatorname{sgn}(\mu_{uncert} - \mu_{sure}) \neq \operatorname{sgn}(\mu_{uncert} + \Delta_{ch-\mu.uncert} - \mu_{sure} - \Delta_{ch-\mu.sure}) . \qquad (10)$$

To overcome the real difference between the expectations for the uncertain and sure games, the absolute value of the presupposed bias for the uncertain game should be evidently not less than this real difference. That is

$$|\Delta_{ch-\mu.uncert}| \geq |\mu_{uncert} - \mu_{sure}| . \qquad (11)$$

So, relations (10) and (11) constitute an addition to the method. The sum of the method and addition can be named as a family of preliminary general qualitative mathematical models.

A.4.2. Special qualitative mathematical model

A preliminary estimate of Harin (2018) restricts applications of the general model by the secure upper bound (5) for the bias. One of the main questions for future research is to analyze the possible widths of the forbidden zones for various types of distributions.

Let us consider the qualitative problems under the special condition

$$\mu_{uncert} = \mu_{sure} \tag{12}$$

This special condition and relation assert that the expectations for the uncertain games are exactly equal to the expectations of the corresponding sure games. This is the well-known and important case of real experimental situations. Here (12) (keeping in mind (7)) substitutes (10) and (11).

Such a special situation enables avoiding the constraints of preliminary estimate Harin (2018) of the secure upper bound (5) for the bias, and making this special model less formal. The biases can be selected to be much less than the secure upper bound (5), and the suppositions will be simpler.

This Special Practical Qualitative Mathematical Model (SPQMM or shortly SQMM) can be considered as a first step of the first stage of the approach (method) MMBE.