The inconsistency if the production function is a homogeneous degree CES function, solving the problem and the presentation of the Modern Universal Growth Theory; The foundation of economic growth theory

de la Fonteijne, Marcel R.

DLF Macro Economic Research

31 August 2024

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Marcel R. de la Fonteijne

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*Technical Progress, Growth Model, Maximum Profit Condition, Production Functions, General Technological Progress, Capital-Labor mix, Elasticity of Substitution, Normalized CES Functions, inconsistency, homogeneous **ν** degree CES production function, Total Factor Productivity, DSGE Model, Solow Model, Hicks, Harrod*

Marcel R. de la Fonteijne[1]

Delft, Augustus 31, 2024

Abstract

The use of a homogeneous degree ν CES production function in a simple economic model under conditions of maximum profit leads to an inconsistency. This paper identifies the root of this problem and provides a solution. Building on this, we propose an improved formulation of the Modern Universal Growth Theory, without focusing on all the difference with Solow, Harrod, Hicks, Uzawa and others, eliminating the errors and limitations inherent in earlier models like those developed by Solow in the 1960s. We conclude that approximate 40 % of the existing theory on economic growth is now rendered invalid.

Keywords: Technical Progress, Growth Model, Maximum Profit Condition, Production Functions, General Technological Progress, Capital-Labor mix, Elasticity of Substitution, Normalized CES Functions, inconsistency, homogeneous degree ν CES production function, Total Factor Productivity, DSGE Model, Solow Model, Hicks, Harrod

JEL Classification E00 · E20 · E23 · E24

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Content

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Contact information:

- Marcel R. de la Fonteijne
- DLF Macro Economic Research
- Montgomerylaan 218
- 2625 PV Delft
- The Netherlands
- +31621885101
- marcel@dlfeconomics.com
- https://dlfeconomics.com

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Citation: De la Fonteijne, Marcel R., 2024, The inconsistency if the production function is a homogeneous degree ν CES function, solving the problem and the presentation of the Modern Universal Growth Theory, DLF Macro Economic Research, Delft, The Netherlands, *ISBN* 9789082554564

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1 Introduction

Homogeneous CES production functions of degree ν are used in economic theory and, in some cases, in practice. However, the most commonly applied form in practice is the first-degree homogeneous function ($\nu = 1$). This first-degree homogeneity implies constant returns to scale, meaning that if all inputs are multiplied by a certain factor, the output increases by the same factor. This is the case, for example, with the Cobb-Douglas production function, which is frequently used in empirical economic research.

Homogeneous functions of degrees other than $\nu = 1$ are used less often, but they can be relevant in specific theoretical models. For example:

- **Homogeneous functions of degree** $\nu < 1$: These are used to model decreasing returns to scale, where a proportional increase in all inputs leads to a less-than-proportional increase in output.
- **Homogeneous functions of degree** $\nu > 1$: These model increasing returns to scale, where a proportional increase in all inputs results in a more-than-proportional increase in output.

While these different forms are theoretically important for modeling various economic phenomena, they are less frequently applied in practice than first-degree homogeneous functions. This is because constant returns to scale is a common assumption in empirical studies, as it simplifies analysis and often fits observable data well.

We take it one step further by claiming that you should always use a homogeneous order $\nu = 1$ production function, because the use of a homogeneous degree ν production function in a simple economic model under conditions of maximum profit leads to an inconsistency. This paper identifies the root of this problem and provides a solution. Building on this, we propose an improved formulation of the Modern Universal Growth Theory, without focusing on all the differences with Solow, Harrod, Hicks, Uzawa and others, which makes it easier to understand. You can find a detailed description of all the differences in De la Fonteijne (2018), where we explain the why and how of eliminating the errors and limitations inherent in earlier models like those developed by Solow in the 1960s.

Finally, we will end with some conclusions.

Overall, we conclude that this new approach to representing economic growth through the MUGT necessitates a thorough reconsideration of all existing theory on the topic, as approximate a 40% of it is now rendered invalid.

2 The equations of the economic system

The system under consideration consists of the following equations:

The production function expressed in its base point (Y_0, K_0, L_0) with parameters α, γ and ν

$$Y = Y_0 \left[\alpha \left(\frac{K}{K_0}\right)^{\gamma} + (1-\alpha)\left(\frac{L}{L_0}\right)^{\gamma}\right]^{\nu/\gamma} \tag{1}$$

where ν is expressing that the function is a homogeneous degree ν production function, α is the capital-labor mix and σ is the elasticity of substitution

$$\sigma = \frac{1}{1-\gamma} \tag{2}$$

$$Y = C + I \tag{3}$$

$$C = c_1 Y \tag{4}$$

$$\dot{K} = I - \delta K \tag{5}$$

From these equations we can solve the equilibrium values.

Additionally, we have the equations with the wages

$$Y \;=\; wL \;+\; (r+\delta)K \tag{6}$$

The derivatives are

$$\frac{\partial Y}{\partial K} = \frac{\nu \alpha Y}{\alpha\left(\frac{K}{K_0}\right)^{\gamma} + (1-\alpha)\left(\frac{L}{L_0}\right)^{\gamma}} \left(\frac{K}{K_0}\right)^{\gamma-1} \frac{1}{K_0} \tag{7}$$

$$\frac{\partial Y}{\partial L} = \frac{\nu (1-\alpha) Y}{\alpha\left(\frac{K}{K_0}\right)^{\gamma} + (1-\alpha)\left(\frac{L}{L_0}\right)^{\gamma}} \left(\frac{L}{L_0}\right)^{\gamma-1} \frac{1}{L_0} \tag{8}$$

In the process of maximum profit, we require

$$\left(\frac{\partial Y}{\partial K}\right)_0 = \frac{\nu \alpha Y_0}{K_0} = r + \delta \tag{9}$$

and

$$\left(\frac{\partial Y}{\partial L}\right)_0 = \frac{\nu (1-\alpha) Y_0}{L_0} = w \tag{10}$$

3 The inconsistency when using a homogeneous degree ν CES function

The good thing is that the introduction of ν will allow all mathematical possible functions around the basepoint.

However, using equation 9 and 10 in equation 6 will lead to an inconsistency.

Notice, that the only way of avoiding this inconsistency is by putting $\nu = 1$, that is why

we used equation 10 as an alternative for equation 9 in all earlier articles on homogeneous degree 1 CES production functions. In fact, one of the two equation gives redundant information, and for $\nu = 1$ is, fortunately, not conflicting. It will however, restrict the solution space, because the derivatives are dependent.

If $\nu \neq 1$ then equation 9, 10 and 6 are conflicting. You either have to choose equation 9 or 10. We will use equation 9.

We now have to calculate w straight forward from equations 9 and 6 resulting in

$$w = \frac{Y}{L} - (r + \delta)\frac{K}{L} = \frac{Y_0}{L_0}(1 - \nu\alpha) \qquad (11)$$

As you can see, this is not the same as equation 10.

In total this solves the system. Realize, however that no longer the standard maximum profit condition does apply.

4 An alternative way of avoiding the inconsistency

Another way of dealing with the inconsistency is splitting the process in two parts. First, we assume that there is no technical progress at all, then it is quite reasonable that the production function is a homogeneous degree 1 function. On the other hand, the factor ν in equation 7 and 8 is very useful, as we explained, to make the derivatives independent of each other. It will allow all mathematical growth paths possible. Therefore, alternatively, we introduce a growth factor ξ_{TFP}, (total factor productivity), equal to ν. In contradiction of its misleading name, the factor ξ_{TFP} expresses only technical progress due to innovations, education, labor improvement, capital improvement, etc. and not through capital increase.

Because the production function is homogeneous degree 1, we can write the production function in the intensive form

$$y = \xi_{TFP} y_0 \left[\alpha_1 \left(\frac{k}{k_0}\right)^{\gamma} + (1 - \alpha_1)\right]^{1/\gamma} \qquad (12)$$

where ξ_{TFP} expresses the factor of total factor productivity, which is the increase of productivity by technical progress only, expressed by moving from point $(k_{0,} y_0)$ to point $(k_{1,} y_1) = (k_{0,}\ \xi_{TFP} y_0)$. Even α and γ may change due to technical progress.

5 The Modern Universal Growth Theory (MUGT) and a Balanced Growth Path (BGP)

In general, we can write

$$y = \xi_{TFP} f(k) \quad (13)$$

It is this equation that we present as the Modern Universal Growth theory. The factor ξ_{TFP} might be a function of capital K, labor L, time t, education, research and/or other determinants of technical growth progress.

In the Modern Universal Growth Theory (MUGT) a momentary production function is expressed by

$$y = \xi_{TFP} f(k)$$

where $f(k)$ is basis the production function and the ξ_{TFP} is the technical progress factor (Total Productivity Factor), which might be a function of capital K, labor L, time t, education, research and/or other determinants of technical growth progress.

If ξ_g is the total increase of productivity due to TFP and capital increase and we move from point $(k_{1,}\ y_1) = (k_{0,}\ \xi_{TFP} y_0)$ along the production function to point $(k_{2,}\ y_2) = (\ \xi_g k_{0,}\ \xi_g y_0)$ then the capital to income ratio β_2 will obviously remain the same as the original β_0 where we started with.

The capital share ks_2 will change to

$$ks_2 = \alpha_1 \left(\frac{\beta_2}{\beta_1}\right)^{\gamma} = \alpha_1 \left(\frac{\beta_0}{\beta_1}\right)^{\gamma} = \alpha_1\ \xi_{TFP}{}^{\gamma} \quad (14)$$

Definition: A Balanced Growth Path is a growing economy where the growth rate per capita of Capital g_k and Income g, are equal and constant (Jones, 2013), i.e., the Capital to Income ratio is kept constant (c_1 is constant).

It would, however, not harm if growth rates vary over time.

This definition does not restrict the characteristics of the production function.

The parameter α_2 is equal to the capital share if we express the production function in base point 2, i.e., $\alpha_2 = ks_2$. The result is that a continuously growing economy with the Capital to Income ratio constant, except for $\gamma = 0$, **always** will lead to a capital only or labor only economy if α_1 remains unchanged.

If we request, however, the **additional condition** that after growth the character of the production must remain unchanged, i.e., in the end, in case of a CES function, the capital-labor mix α and the elasticity of substitution remain unchanged, so requesting that $\alpha_2 = \alpha_0$, then α_1, during the growth of technical progress, should change to

$$\alpha_1 = \frac{\alpha_0}{\xi_{TFP}{}^{\gamma}} \quad (15)$$

This is the same result as stated in De la Fonteijne (2018).

It is important to realize that the considered production functions are CES functions and in combination with the MUGT serve as a second order approximation to an arbitrary production function around the base point (De la Fonteijne, 2023). In total, it allows us to generate all mathematically possible growth paths, even if it has no economic relevance. The capital-labor mix and the elasticity of substitution may vary over time.

Lemma: Within the MUGT concept it is possible to approximate with a CES function any arbitrary growth path with order 2 accuracy around its base point.

Equations 733 and 734 also show us why the Cobb-Douglas production function is so popular. After all, for $\gamma = 0$ ($\sigma = 1$, Cobb-Douglas) there is no needs for an additional change in the capital-labor mix α to make a balanced growth path (BGP) with unchanged characteristics of the production function possible. And for that the Cobb-Douglas production function is unique.

$$y = \xi_{TFP} y_0 \left(\frac{k}{k_0}\right)^{\alpha} \tag{16}$$

Lemma: Within the MUGT a Cobb-Douglas production function is the only CES function for which a BGP with unchanged characteristics of the production function is possible without the need of adapting the capital-labor mix α. $y = \xi_{TFP} y_0 \left(\frac{k}{k_0}\right)^{\alpha}$

Overall, we conclude that this new approach to representing economic growth through the MUGT necessitates a thorough reconsideration of all existing theory on the topic, as approximate a 40 % of it is now rendered invalid (Jones (2013), Acemoglu (2006)).

Acknowledgement

The research presented in this manuscript was conducted under the auspices of DLF Macro Economic Research, generously supported by private funding.

Literature

- Acemoglu, D., 2009, *Introduction to modern economic growth*, Princeton University Press
- De la Fonteijne, Marcel R., 2018, Why the Concept of Hicks, Harrod, Solow Neutral and even Non-Neutral Augmented Technical Progress is flawed, *MPRA*, https://mpra.ub.uni-muenchen.de/107730/
- De la Fonteijne, Marcel R., 2023, The Consequence of the Modern Universal Growth Theory (MUGT) with respect to homogeneous degree 1 CES functions, DLF Macro Economic Research, https://mpra.ub.uni-muenchen.de/118888/
- Jones, C.I., 2013, *Introduction to economic growth*, WW Norton & Co