An efficient Shapley value for games with fuzzy characteristic function

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Abstract

We consider cooperative games where the characteristic function is valued in the space of the fuzzy numbers. By using different fuzzy calculation methods to transform the game into a crisp cooperative one, we define and characterize an efficient extension of the Shapley value. This solution is a relevant member of a wider family of more general, fuzzy calculation method dependent extensions of the Shapley value.

Keywords: Cooperative game; Shapley value; fuzzy set; efficiency.

1 Introduction

As it often happens in the real world, data could be affected by uncertainty. Uncertainty can be introduced by a stochastic model, by interval analysis or by fuzzy environment. In the case of non-cooperative games, we may have uncertainty in the final payoffs. See for example Mallozzi and Vidal-Puga (2023) and references herein. In case of cooperative games with transferable utilities (TU games), we may have uncertainty on the coalitions as well as on the characteristic function values.

In this paper we deal with TU games. Mares (2001) and Mares and Vlach (2004) introduced cooperative games with fuzzy characteristic function defining the fuzzy counterparts of supperadditivity, convexity, core, and the Shapley value. Later, Yu and Zhang

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(2010) investigated a new class of fuzzy games with fuzzy coalitions and a fuzzy characteristic function and, by using the Hukuhara difference and the Choquet integral, gave the explicit form of the Shapley value for the considered class of fuzzy games. For a survey on the Shapley value of cooperative games under fuzzy settings see Borkotokey and Mesiar (2014).

A different type of solution to a cooperative game with fuzzy characteristic function has been given by Gallardo and Jiménez-Losada (2020): they defined a value in which the players' payoffs are given by real numbers and it is obtained from the classic Shapley value. A relevant characteristic of Gallardo and Jiménez-Losada (2020)'s value is that it does not satisfy efficiency, i.e., the sum of the players payoffs does not (in general) sum up the worth of the grand coalition, even in superadditive games.

In this paper, we also consider cooperative games where the characteristic function is valued in the space of the fuzzy numbers. Then, we introduce the definition of *efficient Shapley value* extending the usual definition of the Shapley value for a crisp cooperative game. The Efficiency property is an important property of this notion. We compare and discuss the new concept with Gallardo and Jiménez-Losada (2020)'s value in a fuzzy environment.

The paper is organized as follows. In Section 2, we present the model and recall some preliminaries. In Section 3, we define and characterize the efficient Shapley value and a more general family of efficient extensions of the Shapley value. In Section 4, we present the conclusions.

2 The model

2.1 Cooperative games

A *(crisp) cooperative game* is a pair (N, v) where $N = \{1, \dots, n\}$ is the set of players and $v : 2^N \to \mathbb{R}$ is a characteristic function that assigns to each subset $S \subseteq N$ a worth $v(S)$ satisfying $v(\emptyset) = 0$. The characteristic function describes how much collective payoff a set of players can gain by forming a coalition. Set N is called the *grand coalition*. The set of all characteristic functions on N is denoted by $\mathcal{G}^N$.

Several solution concepts have been introduced in the literature. The Shapley value (Shapley, 1953) $Sh(v)$ of $(N, v) \in \mathcal{G}^N$ is defined as

$$Sh_i(v) = \sum_{S, i \notin S} n_S \cdot (v(S \cup \{i\}) - v(S)) \tag{1}$$

for each $i \in N$ where

$$n_S = \frac{|S|!(n-1-|S|)!}{n!}$$

for each $S \subseteq N$.

2.2 Fuzzy sets

In this subsection we recall some basic concepts of fuzzy sets (refer to Cunlin and Qiang (2011); Mares (2001); Zadeh (1978)).

A *fuzzy number* $\tilde{a}$ is determined by a mapping $\mu_{\tilde{a}} : \mathbb{R} \to [0,1]$, called the *membership function*, satisfying that there exist four (not necessarily distinct) real numbers $\tilde{a}_0^- \leq \tilde{a}_1^- \leq \tilde{a}_1^+ \leq \tilde{a}_0^+$ such that

- $\mu_{\tilde{a}}(x) = 0$ for all $x < \tilde{a}_0^-$ and all $x > \tilde{a}_0^+$
- $\mu_{\tilde{a}}(x)$ is (weakly) increasing and right-continuous on the interval (a_0^-, a_1^-)
- $\mu_{\tilde{a}}(x) = 1$ for all $x \in (a_1^-, a_1^+)$
- $\mu_{\tilde{a}}(x)$ is (weakly) decreasing and left-continuous on the interval (a_1^+, a_0^+).

Under these conditions, the following sets are well-defined closed intervals:

$$[\tilde{a}]_\alpha = [\tilde{a}_\alpha^-, \tilde{a}_\alpha^+] = \{x \in \mathbb{R} : \mu_{\tilde{a}}(x) \geq \alpha\}$$

for all $\alpha \in (0,1]$, and

$$[\tilde{a}]_0 = [\tilde{a}_0^-, \tilde{a}_0^+] = cl\{x \in \mathbb{R} : \mu_{\tilde{a}}(x) > 0\}.$$

where clA is the clousure of set A.

We also define the *center* of a fuzzy number $\tilde{a}$ as

$$\langle \tilde{a} \rangle = \frac{\tilde{a}_1^- + \tilde{a}_1^+}{2} \in \mathbb{R}.$$

Let $\mathbb{F}$ denote the set of all fuzzy numbers. The set of real numbers can be embedded into $\mathbb{F}$. In particular, any $p \in \mathbb{R}$ can be expressed as a fuzzy number with membership function $\mu_p(x) = \mathbb{1}_{x=p}$ for all $x \in \mathbb{R}$.

A *triangular fuzzy number* is a particular case of fuzzy number $\tilde{a} \in \mathbb{F}$ where $a_1^- = a_1^+$ and $\mu_{\tilde{a}}(x)$ is a linear function increasing on the interval (a_0^-, a_1^-) and decreasing on (a_1^+, a_0^+). We denote such a triangular fuzzy number as $(\langle \tilde{a} \rangle, a_1^- - a_0^-, a_0^+ - a_1^+)$.

The sum of two fuzzy numbers $\tilde{a}, \tilde{b} \in \mathbb{F}$, denoted as $\tilde{a} + \tilde{b}$, is given by the membership function

$$\mu_{\tilde{a}+\tilde{b}}(x) = \sup_{y \in \mathbb{R}} \{\min\{\mu_{\tilde{a}}(y), \mu_{\tilde{b}}(x-y)\}\}$$

or, equivalently,

$$\left[\tilde{a} + \tilde{b}\right]_\alpha = \left[\tilde{a}_\alpha^- + \tilde{b}_\alpha^-, \tilde{a}_\alpha^+ + \tilde{b}_\alpha^+\right]$$

for all $\alpha \in [0, 1]$.

Lemma 2.1 *Given $\tilde{a}_1, \ldots, \tilde{a}_k \in \mathbb{F}$ with $\mu_{\tilde{a}_i}(0) = 1$ for all $i = 1, \ldots, k$, we have*

$$\mu_{\sum_{i=1}^k \tilde{a}_i}(0) = 1.$$

Proof. We proceed by induction on k. For $k = 1$, the result holds by assumption. Assume now the result holds for less than k fuzzy numbers. Let $\tilde{b} = \tilde{a}_1 + \cdots + \tilde{a}_k$ and $\tilde{c} = \tilde{a}_1 + \cdots + \tilde{a}_{k-1}$. By the induction hypothesis,

$$\begin{aligned}
\mu_{\tilde{b}}(0) &= \mu_{\tilde{c}+\tilde{a}_k}(0) \\
&= \sup_{y \in \mathbb{R}} \{\min\{\mu_{\tilde{c}}(y), \mu_{\tilde{a}_k}(-y)\}\} \\
&\geq \min\{\mu_{\tilde{c}}(0), \mu_{\tilde{a}_k}(0)\} = 1.
\end{aligned}$$

■

The difference of two fuzzy numbers $\tilde{a}, \tilde{b} \in \mathbb{F}$, denoted as $\tilde{a} - \tilde{b}$, is given by the membership function

$$\mu_{\tilde{a}-\tilde{b}}(x) = \sup_{y \in \mathbb{R}} \{\min\{\mu_{\tilde{a}}(y), \mu_{\tilde{b}}(x+y)\}\}$$

or, equivalently,

$$\left[\tilde{a} - \tilde{b}\right]_\alpha = \left[\tilde{a}_\alpha^- - \tilde{b}_\alpha^+, \tilde{a}_\alpha^+ - \tilde{b}_\alpha^-\right]$$

for all $\alpha \in [0, 1]$.

The product of two fuzzy numbers $\tilde{a}, \tilde{b} \in \mathbb{F}$, denoted as $\tilde{a} \cdot \tilde{b}$, is given by the membership function

$$\mu_{\tilde{a}\cdot\tilde{b}}(x) = \sup_{y,z \in \mathbb{R}: yz = x} \{\min\{\mu_{\tilde{a}}(y), \mu_{\tilde{b}}(z)\}\}$$

or, equivalently,

$$\left[\tilde{a} \cdot \tilde{b}\right]_\alpha = \left[\min\{a_\alpha^- b_\alpha^-, a_\alpha^- b_\alpha^+, a_\alpha^+ b_\alpha^-, a_\alpha^+ b_\alpha^+\}, \max\{a_\alpha^- b_\alpha^-, a_\alpha^- b_\alpha^+, a_\alpha^+ b_\alpha^-, a_\alpha^+ b_\alpha^+\}\right]$$

for all $\alpha \in [0, 1]$.

In particular, given $p \in \mathbb{R}$ and $\tilde{a} \in \mathbb{F}$, their product is given by the membership function

$$\mu_{p\cdot\tilde{a}}(x) = \begin{cases} \mu_a\left(\frac{x}{p}\right) & \text{if } p \neq 0 \\ \mathbb{1}_{x=0} & \text{if } p = 0 \end{cases}$$

or, equivalently,

$$[p \cdot \tilde{a}]_\alpha = \left[\min\{pa_\alpha^-, pa_\alpha^+\}, \max\{pa_\alpha^-, pa_\alpha^+\}\right] = \begin{cases} [pa_\alpha^-, pa_\alpha^+] & \text{if } p \geq 0 \\ [pa_\alpha^+, pa_\alpha^-] & \text{if } p < 0 \end{cases}$$

for all $\alpha \in [0, 1]$.

For the particular case $p = -1$, we denote $-\tilde{a} = -1 \cdot \tilde{a}$, and so

$$\mu_{-\tilde{a}}(x) = \mu_a(-x)$$

or, equivalently,

$$[-\tilde{a}]_\alpha = \left[-a_\alpha^+, -a_\alpha^-\right]$$

for all $\alpha \in [0, 1]$.

Notice that the sum of $\tilde{a}$ and $-\tilde{a}$ is, in general, not zero, but a fuzzy number centered at zero and with symmetric tails. We call these numbers 0-symmetric, i.e.,

Definition 2.1 *We say that a fuzzy number $\tilde{a} \in \mathbb{F}$ is* 0-symmetric *if $\mu_{\tilde{a}}(x) = \mu_{\tilde{a}}(-x)$ for all $x \in \mathbb{R}$.*

We should then work with the following equivalence class: We say that two fuzzy numbers are *uncertain-equivalent*, or *u-equivalent* for short, if their difference is 0-symmetric. Notice that this is an equivalence relation because $\tilde{a} = -\tilde{a}$ for all 0-symmetric number $\tilde{a}$.

Each fuzzy number can be then represented as the sum of a unique canonical fuzzy number and a unique 0-symmetric number. Notice that all real numbers are canonical, and 0 is the only 0-symmetric canonical number. We denote as $\tilde{\kappa}(\tilde{a})$ the unique canonical fuzzy number associated to a fuzzy number $\tilde{a}$, and as $\tilde{\rho}(\tilde{a})$ the unique 0-symmetric number such that $\tilde{a}$ can be uniquely written as the sum:

$$\tilde{a} = \tilde{\kappa}(\tilde{a}) + \tilde{\rho}(\tilde{a}).$$

Obviously, $\tilde{a}$ and $\tilde{\kappa}(\tilde{a})$ are u-equivalent.

2.3 Cooperative games with fuzzy environment

A cooperative game with fuzzy environment is a pair $(N, \tilde{v})$ where $N = \{1, \ldots, n\}$ is the set of the players and $\tilde{v} : 2^N \to \mathbb{F}$ is the (fuzzy) characteristic function such that $\tilde{v}(\emptyset) = 0$. The set of all (fuzzy) characteristic functions on N is denoted by $\tilde{\mathcal{G}}^N$.

A *fuzzy numbers calculation method* is a function $M : \mathbb{F} \to \mathbb{R}$ such that $M(x) = x$ for all $x \in \mathbb{R}$. Given $(N, \tilde{v}) \in \tilde{\mathcal{G}}$, one can consider solutions of the cooperative crisp game $\left(N, \tilde{v}^M\right)$ where the characteristic function is given as $\tilde{v}^M(S) = M(\tilde{v}(S))$ for all $S \subseteq N$. This procedure allows a defuzzification of a cooperative fuzzy game.

Classical examples of fuzzy numbers calculation methods are the *Center of Gravity* (CoG), the *Bisector of Area* (BoA), the Maxima Methods and others (see Sivanandam and Deepa (2019)).

Given $\tilde{a} \in \mathbb{F}$ with membership function $\mu_{\tilde{a}}$, the CoG is defined as the real number

$$CoG(\tilde{a}) = \frac{\int_{\mathbb{R}} x\mu_{\tilde{a}}(x)\ dx}{\int_{\mathbb{R}} \mu_{\tilde{a}}(x)\ dx}$$

while the BoA as the real number $BoA(\tilde{a})$ such that

$$\int_{-\infty}^{BoA(\tilde{a})} \mu_{\tilde{a}}(x)\ dx = \int_{BoA(\tilde{a})}^{+\infty} \mu_{\tilde{a}}(x)\ dx.$$

We consider in the following one of the Maxima Methods: if $\mu_{\tilde{a}}(x) = 1$ for all $x \in (a_1^-, a_1^+)$, we call

- *First of Maxima* (FoM) the real number $FoM(\tilde{a}) = a_1^-$.
- *Last of Maxima* (LoM) the real number $LoM(\tilde{a}) = a_1^+$.
- Mean of Maxima (MoM) the real number $MoM(\tilde{a}) = \frac{a_1^- + a_1^+}{2}$.

The mean of maxima coincides with the center of the fuzzy number, i.e., $MoM(\tilde{a}) = \langle \tilde{a} \rangle$.

Example 2.1 *Consider the fuzzy number* $\tilde{a}$ *given by*

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x}{3} & \text{if } x \in [0,3] \\ 1 & \text{if } x \in [3,4] \\ 0 & \text{otherwise.} \end{cases}$$

This fuzzy number is depicted in Figure 1 with its respective $CoG(\tilde{a}) = \frac{13}{5}$, $BoA(\tilde{a}) = \sqrt{\frac{15}{2}}$, $FoM(\tilde{a}) = 3$, $LoM(\tilde{a}) = 4$, *and* $MoM(\tilde{a}) = \frac{7}{2}$.

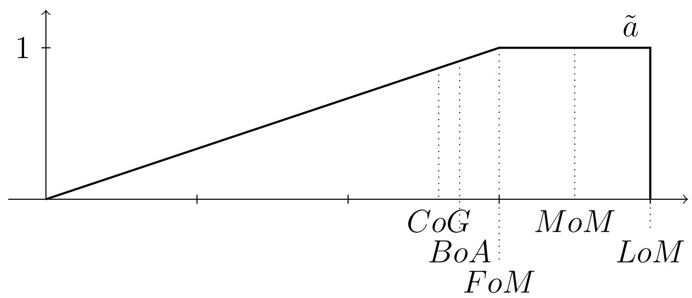

Figure 1: Some fuzzy numbers calculation methods.

We say that a fuzzy numbers calculation method M is *additive* if $M(\tilde{a}+\tilde{b}) = M(\tilde{a}) + M(\tilde{b})$ for all $\tilde{a}, \tilde{b} \in \mathbb{F}$. In particular, FoM, LoM, and MoM are additive.

We say that a fuzzy numbers calculation method M is 0-*maximal* if $M(\tilde{a}) = 0$ implies $\mu_{\tilde{a}}(0) = 1$. In particular, FoM, LoM and MoM are 0-maximal, whereas CoG and BoA are not.

3 Cooperative values with fuzzy environment

A natural extension of the Shapley value to cooperative games with fuzzy environments is due to Gallardo and Jiménez-Losada (2020) and named the *fuzzy Shapley value*:

$$Sh_i(\tilde{v}) := \sum_{S\subseteq N, i\notin S} n_S \cdot (\tilde{v}(S\cup\{i\}) - \tilde{v}(S)) \tag{2}$$

for all $(N, \tilde{v}) \in \tilde{\mathcal{G}}^N$ and $i \in N$. In the following $\Psi : \tilde{\mathcal{G}}^N \to \mathbb{F}^N$ denotes a generic solution for cooperative games with fuzzy environment.

The natural extension of the Efficiency property to cooperative games with fuzzy environment is:

Efficiency For each $(N, \tilde{v}) \in \tilde{\mathcal{G}}^N$,

$$\sum_{i\in N} \Psi_i(\tilde{v}) = \tilde{v}(N).$$

We cannot expect, in general, the fuzzy Shapley value to be efficient, as next example shows.

Example 3.1 *Consider $n = 2$ and the fuzzy cooperative game $(N, \tilde{v})$ given by $\tilde{v}(\{1\}) = 0$, $\tilde{v}(\{2\}) = (1, 1, 2)$ and $\tilde{v}(N) = (3, t, 0)$ for a real parameter $t \geq 0$. See top picture in Figure 2. In this game, $Sh(\tilde{v}) = \left(\left(1, 1+\frac{t}{2}, \frac{1}{2}\right), \left(2, \frac{1}{2}+\frac{t}{2}, 1\right)\right)$ is not efficient since $Sh_1(\tilde{v}) + Sh_2(\tilde{v}) = \left(3, \frac{3}{2}+t, \frac{3}{2}\right)$. See middle picture in Figure 2. Note that for $t \geq 0$, $Sh_1(\tilde{v}) + Sh_2(\tilde{v})$ is u-equivalent to $\tilde{v}(N)$.*

Gallardo and Jiménez-Losada (2020) characterize the fuzzy Shapley value using a property of Central Efficiency:

Central Efficiency For each $(N, \tilde{v}) \in \tilde{\mathcal{G}}^N$, there exists a 0-symmetric fuzzy number $\tilde{d}_{\tilde{a}}$ such that

$$\sum_{i\in N} \Psi_i(\tilde{v}) = \tilde{v}(N) + \tilde{d}_{\tilde{a}}.$$

Gallardo and Jiménez-Losada (2020) argue that it "would not be reasonable to require that the players' payoffs sum up to $v(N)$." (page 103). Their reasoning is that efficiency would lead to provide no uncertainty when $v(N) \in \mathbb{R}$, even when that uncertainty is present in other coalitions.

We argue that sharing no uncertainty when it is not present in the grand coalition can be perfectly reasonable and even advisable. Assume, for example, that one of the players, say player 1, is an investment partner that provides insurance. Hence, uncertainty is removed wherever player 1 is present, i.e., $\tilde{v}(S \cup \{1\}) \in \mathbb{R}$ for all $S \subseteq N \setminus \{i\}$. It is then reasonable that any coalition containing player 1 (including the grand coalition) should share a payoff allocation without uncertainty.

We present the following alternative extension of the Shapley value.

Given a cooperative game with fuzzy numbers $(N, \tilde{v})$, we define the cooperative game $(N, \langle \tilde{v} \rangle)$ as $\langle \tilde{v} \rangle (S) = \langle \tilde{v}(S) \rangle$ for all $S \subseteq N$.

Definition 3.1 *We define the efficient Shapley value as*

$$\tilde{Sh}_i(\tilde{v}) = Sh_i(\langle \tilde{v} \rangle) + \frac{1}{n} \cdot (\tilde{v}(N) - \langle \tilde{v}(N) \rangle)$$

or, equivalently,

$$\tilde{Sh}_i(\tilde{v}) = Sh_i\left(\tilde{v}^{MoM}\right) + \frac{1}{n} \cdot \left(\tilde{v}(N) - \tilde{v}^{MoM}(N)\right)$$

for each $(N, \tilde{v}) \in \tilde{\mathcal{G}}^N$ *and each* $i \in N$.

This value assigns to each player a payoff with two parts. Firstly, the Shapley value of the center, where uncertainty has been removed. Secondly, an equal division of the uncertainty of the grand coalition, given by $\tilde{v}(N) - \langle \tilde{v}(N) \rangle$.

Example 3.2 *Consider* $n = 2$ *and the fuzzy cooperative game* $\tilde{v}$ *defined in Example 3.1. In this game,* $\tilde{Sh}(\tilde{v}) = \left(\left(1, \frac{t}{2}, 0\right), \left(2, \frac{t}{2}, 0\right)\right)$ *and efficiency is satisfied. See bottom picture in Figure 2.*

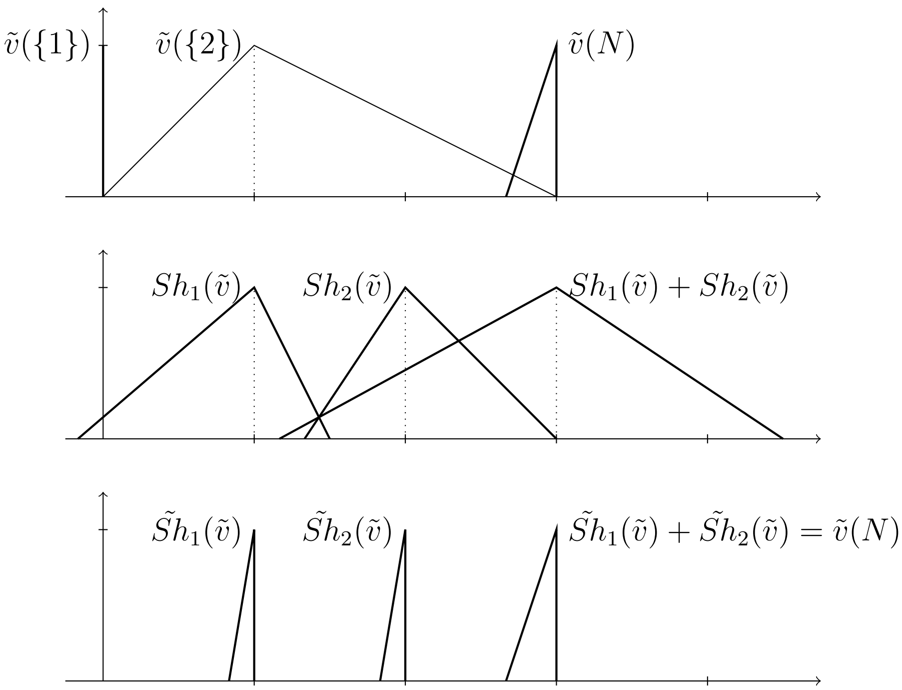

Figure 2: Fuzzy cooperative game (top) in Example 3.1 for $t = \frac{1}{3}$, its fuzzy Shapley value (middle), and its efficient Shapley value (bottom).

Equal division is a basic principle in allocation problems. In crisp cooperative games, it results in the application of the equal division value (axiomatized in van den Brink (2007)). Another options are the equal surplus division, which results in the application of the equal surplus division value (axiomatized in van den Brink and Funaki (2009)) and the proportional share, which result in the application of values such as the proportional value (Ortmann, 2000), the proportional Shapley value (Béal et al., 2018; Besner, 2019) and the proper Shapley value (Vorob'ev and Liapounov, 1998; van den Brink et al., 2015). However, only the equal division value assures a non-negative payoff when the worth of the grand coalition is also non-negative. Hence, neither the equal surplus division nor the proportional principle can be applied to uncertainty.[1]

We will use the following properties:

Additivity For each $(N, \tilde{v}), (N, \tilde{w}) \in \tilde{\mathcal{G}}^N$, $\Psi(\tilde{v} + \tilde{w}) = \Psi(\tilde{v}) + \Psi(\tilde{w})$, where $(N, \tilde{v} + \tilde{w}) \in \tilde{\mathcal{G}}^N$ is defined as $(\tilde{v} + \tilde{w})(S) = \tilde{v}(S) + \tilde{w}(S)$ for all $S \subseteq N$.

[1] Notice that this restriction may vanish in certain classes of games, such as those uncertain-monotonic, i.e., games in which uncertainty never decreases (or never increases) with the size of the coalition, once a suitable notion of uncertainty of a fuzzy number is considered.

Symmetry For each $(N, \tilde{v}) \in \tilde{\mathcal{G}}^N$, if $i, j \in N$ satisfy $\tilde{v}(S \cup \{i\}) = \tilde{v}(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$, then $\Psi_i(\tilde{v}) = \Psi_j(\tilde{v})$.

Weak null player For each $(N, \tilde{v}) \in \tilde{\mathcal{G}}^N$, if $i \in N$ is such that $\tilde{v}(S \cup \{i\}) = \tilde{v}(S)$ for all $S \subseteq N \setminus \{i\}$, then $\mu_{\Psi_i(\tilde{v})}(0) = 1$.

Additivity and Symmetry are, as Efficiency, natural extensions of the respective properties used in crisp games to characterize the Shapley value. About Weak null player, it extends the Null player property used in crisp games to characterize the Shapley value. Another extension is the property defined in Gallardo and Jiménez-Losada (2020) and also named Null player. Weak null player is a weak version of Gallardo and Jiménez-Losada (2020)'s Null player in the general setting. That's it, any rule satisfying Gallardo and Jiménez-Losada (2020)'s property will also satisfy Weak null player. However, they both generalize the property of Null Player in crisp games.

The last property is also defined by Gallardo and Jiménez-Losada (2020):

Zero solution Given $(N, \tilde{v}) \in \tilde{\mathcal{G}}^N$, if $\mu_{\tilde{v}(S)}(0) = 1$ for all $S \subseteq N$, then $\mu_{\Psi_i(\tilde{v})}(0) = 1$ for all $i \in N$.

Zero solution says that "if it is possible (at the maximum possibility level) that the payments of all the coalitions in a game are equal to zero, then it is possible (at the maximum possibility level) that the payoffs of all the players in the game are equal to zero" (Gallardo and Jiménez-Losada, 2020, page 103).

Zero solution generalizes the property for crisp games that says that the solution in the null game (game (N, v) such that $v(S) = 0$ for all $S \subseteq N$) is the null payoff allocation ($\Psi_i(v) = 0$ for all $i \in N$).

Theorem 3.1 *The efficient Shapley value $\tilde{Sh}$ is the only solution in the set of fuzzy cooperative games satisfying Efficiency, Additivity, Symmetry, Weak null player, and Zero solution.*

Proof. We first check that the efficient Shapley value satisfies these properties. Under the definition of the efficient Shapley value and Theorem 1 in Gallardo and Jiménez-Losada (2020), it is straightforward to check that the efficient Shapley value satisfies Additivity, Symmetry, and Zero solution. The next step is to check Efficiency. Given $(N, \tilde{v}) \in \tilde{\mathcal{G}}^N$,

$$\begin{aligned}\sum_{i \in N} \tilde{Sh}_i(\tilde{v}) &= \sum_{i \in N} Sh_i(\langle \tilde{v} \rangle) + (\tilde{v}(N) - \langle \tilde{v}(N) \rangle) \\ &= \langle \tilde{v} \rangle (N) + (\tilde{v}(N) - \langle \tilde{v}(N) \rangle) = \tilde{v}(N).\end{aligned}$$

The next step is to check Weak null player. Given $(N,\tilde{v}) \in \tilde{\mathcal{G}}^N$ and $i \in N$ such that, for all $S \subseteq N \setminus \{i\}$, $\tilde{v}(S \cup \{i\}) = \tilde{v}(S)$, it is straightforward to check that that player i is a null player in the crisp game $(N, \langle \tilde{v} \rangle)$ given by $\langle v \rangle(S) = \langle \tilde{v}(S) \rangle$ for all $S \subseteq N$. Under the null player property of the Shapley value:

$$\left\langle \tilde{Sh}_i(\tilde{v}) \right\rangle = Sh_i(\langle \tilde{v} \rangle) + \frac{1}{n} \cdot \langle \tilde{v}(N) - \langle \tilde{v}(N) \rangle \rangle = 0$$

and hence $\mu_{\tilde{Sh}_i(\tilde{v})}(0) = 1$.

We now check uniqueness. Let Ψ be a rule satisfying these properties. Given $\tilde{v} \in \tilde{\mathcal{G}}^N$, we have $\tilde{v} = \langle \tilde{v} \rangle + \tilde{u}_N + \tilde{w}$ where, for each $S \subseteq N$,

$$\tilde{u}_N(S) = \begin{cases} 0 & \text{if } S \neq N \\ \tilde{v}(N) - \langle \tilde{v}(N) \rangle & \text{if } S = N. \end{cases}$$

$$\tilde{w}(S) = \begin{cases} \tilde{v}(S) - \langle \tilde{v}(S) \rangle & \text{if } S \neq N \\ 0 & \text{if } S = N. \end{cases}$$

Since $\tilde{v} = \langle \tilde{v} \rangle + \tilde{u}_N + \tilde{w}$, it is enough to prove that $\Psi(\langle \tilde{v} \rangle)$, $\Psi(\tilde{u}_N)$, and $\Psi(\tilde{w})$ are unique (and hence their sum coincides with the efficient Shapley value).

1. $\Psi(\langle \tilde{v} \rangle)$ is unique by Efficiency, Additivity, Symmetry and Weak null player (equivalent to Null player in crisp games). The proof is equivalent to that of Shapley (1953) taking into account the additivity of MoM and that there is no uncertainty in $\tilde{v}(N)$.

2. All players $i, j \in N$ satisfy $\tilde{u}_N(S \cup \{i\}) = 0 = \tilde{u}_N(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$. Under Symmetry, there exists some $\tilde{a} \in \mathbb{F}$ such that $\Psi_i(\tilde{u}_N) = \tilde{a}$ for all $i \in N$. Under Efficiency, $\tilde{v}(N) = \sum_{i \in N} \Psi_i(\tilde{u}_N) = n \cdot \tilde{a}$ and hence $\tilde{a} = \frac{\tilde{v}(N)}{n}$, so that $\Psi_i(\tilde{u}_N) = \frac{\tilde{v}(N)}{n}$ is unique for each $i \in N$.

3. It is enough to prove that, for each $i \in N$, $\Psi_i(\tilde{w}) = 0$, i.e., $\mu_{\Psi_i(\tilde{w})}(x) = \mathbb{1}_{x=0}$ for all $x \in \mathbb{R}$, so that $\Psi(\tilde{w})$ is unique. For each coalition $S \subseteq N$, we have that $\mu_{\tilde{w}(S)}(0) = 1$. Under Zero Solution, $\mu_{\Psi_i(\tilde{w})}(0) = 1$ for all $i \in N$. Let $\tilde{a} = \sum_{i \in N} \Psi_i(\tilde{w})$. By Efficiency, $\tilde{a} = 0$ and hence $\mu_{\tilde{a}}(x) = \mathbb{1}_{x=0}$ for all $x \in \mathbb{R}$. Given $i \in N$, let $\tilde{b} =$

$\sum_{j\in N\setminus\{i\}} \Psi_j(\tilde{w})$. Under Lemma 2.1, $\mu_{\tilde{b}}(0) = 1$. Hence, for each $x \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{1}_{x=0} &= \mu_{\tilde{a}}(x) \\
&= \mu_{\tilde{b}+\Psi_i(\tilde{w})}(x) \\
&= \sup_{y\in\mathbb{R}} \left\{\min\{\mu_{\tilde{b}}(y), \mu_{\Psi_i(\tilde{w})}(x-y)\}\right\} \\
&\geq \min\{\mu_{\tilde{b}}(0), \mu_{\Psi_i(\tilde{w})}(x)\} \\
&= \min\{1, \mu_{\Psi_i(\tilde{w})}(x)\} \\
&= \mu_{\Psi_i(\tilde{w})}(x)
\end{aligned}$$

and thus $\Psi_i(\tilde{w}) = 0$.

■

The properties in Theorem 3.1 are independent.

- The fuzzy Shapley value (Gallardo and Jiménez-Losada, 2020) satisfies all the properties but Efficiency.

- The efficient fuzzy nucleolus given by

$$\tilde{Nu}_i(\tilde{v}) = Nu_i(\langle\tilde{v}\rangle) + \frac{1}{n}\cdot(\tilde{v}(N) - \langle\tilde{v}(N)\rangle)$$

  for each $(N, \tilde{v}) \in \tilde{\mathcal{G}}^N$ and $i \in N$, where $Nu(v)$ is the nucleolus (Schmeidler, 1969) of $(N, v) \in \mathcal{G}^N$, satisfies all the properties but Additivity.

- Fix $\omega \in \mathbb{R}^N_{++}$ vector of positive real numbers. The efficient fuzzy weighted Shapley value given by

$$\tilde{Sh}^{\omega}_i(\tilde{v}) = Sh^{\omega}_i(\langle\tilde{v}\rangle) + \frac{\omega_i}{\sum_{j\in N}\omega_j}\cdot(\tilde{v}(N) - \langle\tilde{v}(N)\rangle)$$

  for each $(N, \tilde{v}) \in \tilde{\mathcal{G}}^N$ and $i \in N$, where $Sh^{\omega}(v)$ is the weighted Shapley value (Kalai and Samet, 1987) of $(N, v) \in \mathcal{G}^N$ with weights given by ω, satisfies all the properties but Symmetry.

- The fuzzy egalitarian value given by

$$Eg_i(\tilde{v}) = \frac{\tilde{v}(N)}{n}$$

  for each $(N, \tilde{v}) \in \tilde{\mathcal{G}}^N$ and $i \in N$ satisfies all the properties but Weak null player.

- The First of Maxima Shapley value given as

$$Sh_i^{FoM}(\tilde{v}) = Sh_i\left(\tilde{v}^{FoM}\right) + \frac{1}{n} \cdot \left(\tilde{v}(N) - \tilde{v}^{FoM}(N)\right)$$

  for each $(N, \tilde{v}) \in \tilde{\mathcal{G}}^N$ and $i \in N$, satisfies all the properties but Zero solution.

In the last rule, the First of Maxima can be replaced by any other additive and 0-maximal fuzzy numbers calculation method.

Definition 3.2 *Given $M : \mathbb{F} \to \mathbb{R}$ an additive and 0-maximal fuzzy numbers calculation method, we define the M-Shapley value as*

$$Sh_i^M(\tilde{v}) = Sh_i\left(\tilde{v}^M\right) + \frac{1}{n} \cdot \left(\tilde{v}(N) - \tilde{v}^M(N)\right)$$

for each $(N, \tilde{v}) \in \tilde{\mathcal{G}}^N$ and $i \in N$.

Given $M : \mathbb{F} \to \mathbb{R}$ a fuzzy numbers calculation method, we consider the following property:

M-**solution** For each $(N, \tilde{v}) \in \tilde{\mathcal{G}}^N$, if $M(\tilde{v}(S)) = 0$ for all $S \subseteq N$, then $M(\Psi_i(\tilde{v})) = 0$ for all $i \in N$.

Analogously to Zero solution, and independently of the chosen M, M-solution also generalizes the property for crisp games that says that the solution in the null game is the null payoff allocation.

We then characterize the M-Shapley value as follows:

Theorem 3.2 *For any additive and 0-maximal fuzzy numbers calculation method M, the M-Shapley value Sh^M is the only solution in the set of fuzzy cooperative games satisfying Efficiency, Additivity, Symmetry, Weak null player, and M-solution.*

Proof. We first check that the M-Shapley value satisfies these properties.

We first check efficiency. Given $(N, \tilde{v}) \in \tilde{\mathcal{G}}^N$,

$$\begin{aligned}\sum_{i \in N} Sh_i^M(\tilde{v}) &= \sum_{i \in N} Sh_i(\tilde{v}^M) + \left(\tilde{v}(N) - \tilde{v}^M(N)\right) \\ &= \tilde{v}^M(N) + \left(\tilde{v}(N) - \tilde{v}^M(N)\right) = \tilde{v}(N).\end{aligned}$$

We now check Additivity. Given $\tilde{v}, \tilde{w} \in \tilde{\mathcal{G}}^N$, additivity of M implies

$$(\tilde{v} + \tilde{w})^M(S) = M((\tilde{v} + \tilde{w})(S)) = M(\tilde{v}(S)) + M(\tilde{w}(S)) = \tilde{v}^M(S) + \tilde{w}^M(S)$$

and hence

$$\begin{aligned} Sh_i^M(\tilde{v}+\tilde{w}) &= Sh_i\left((\tilde{v}+\tilde{w})^M\right) + \frac{1}{n}\left((\tilde{v}+\tilde{w})(N) - (\tilde{v}+\tilde{w})^M(N)\right) \\ &= Sh_i\left(\tilde{v}^M\right) + Sh_i\left(\tilde{w}^M\right) + \frac{1}{n}\left(\tilde{v}(N) + \tilde{w}(N) - (\tilde{v}^M(N) + \tilde{w}^M(N))\right) \\ &= Sh_i^M(\tilde{v}) + Sh_i^M(\tilde{w}). \end{aligned}$$

We now check Symmetry. Given $(N,\tilde{v}) \in \tilde{\mathcal{G}}^N$ and $i,j \in N$ such that $\tilde{v}(S\cup\{i\}) = \tilde{v}(S\cup\{j\})$ for all $S \subseteq N\setminus\{i,j\}$, then $\tilde{v}^M(S\cup\{i\}) = \tilde{v}^M(S\cup\{j\})$ for all $S \subseteq N\setminus\{i,j\}$ and symmetry of the Shapley value implies

$$\begin{aligned} Sh_i^M(\tilde{v}) &= Sh_i\left(\tilde{v}^M\right) + \frac{1}{n}\cdot\left(\tilde{v}(N) - \tilde{v}^M(N)\right) \\ &= Sh_j\left(\tilde{v}^M\right) + \frac{1}{n}\cdot\left(\tilde{v}(N) - \tilde{v}^M(N)\right) = Sh_j^M(\tilde{v}). \end{aligned}$$

The next step is to check Weak null player. Given $(N,\tilde{v}) \in \tilde{\mathcal{G}}^N$ and $i \in N$ such that, for all $S \subseteq N\setminus\{i\}$, $\tilde{v}(S\cup\{i\}) = \tilde{v}(S)$, it is straightforward to check that player i is a null player in the crisp game $(N,\tilde{v}^M)$. Under the additivity of M and null player property of the Shapley value:

$$M\left(Sh_i^M(\tilde{v})\right) = M\left(Sh_i\left(\tilde{v}^M\right)\right) + \frac{1}{n}\cdot M\left(\tilde{v}(N) - \tilde{v}^M(N)\right) = M(0) + 0 = 0$$

and, since M is 0-maximal, $\mu_{Sh_i^M(\tilde{v})}(0) = 1$.

The next step is to check M-solution. Let $(N,\tilde{v}) \in \tilde{\mathcal{G}}^N$ with $M(\tilde{v}(S)) = 0$ for all $S \subseteq N$. Hence $\tilde{v}^M(S) = 0$ for all $S \subseteq N$ and thus $Sh_i(\tilde{v}^M) = 0$ for all $i \in N$. Then,

$$M\left(Sh_i^M(\tilde{v})\right) = M\left(Sh_i\left(\tilde{v}^M\right)\right) + \frac{1}{n}\cdot M\left(\tilde{v}(N) - \tilde{v}^M(N)\right) = M(0) + 0 = 0$$

for all $i \in N$.

We now check uniqueness. Let Ψ be a rule satisfying these properties. Given $\tilde{v} \in \tilde{\mathcal{G}}^N$, we have $\tilde{v} = \tilde{v}^M + \tilde{u}_N^M + \tilde{w}^M$ where, for each $S \subseteq N$,

$$\tilde{u}_N^M(S) = \begin{cases} 0 & \text{if } S \neq N \\ \tilde{v}(N) - \tilde{v}^M(N) & \text{if } S = N. \end{cases}$$

$$\tilde{w}^M(S) = \begin{cases} \tilde{v}(S) - \tilde{v}^M(S) & \text{if } S \neq N \\ 0 & \text{if } S = N. \end{cases}$$

Since $\tilde{v} = \tilde{v}^M + \tilde{u}_N^M + \tilde{w}^M$, it is enough to prove that $\Psi(\tilde{v}^M)$, $\Psi(\tilde{u}_N^M)$, and $\Psi(\tilde{w}^M)$ are unique (and hence their sum coincides with the M-Shapley value).

1. $\Psi(\tilde{v}^M)$ is unique by Efficiency, Additivity, Symmetry and Weak null player (equivalent to Null player in crisp games). The proof is equivalent to that of Shapley (1953) taking into account the additivity of M and that there is no uncertainty in $\tilde{v}(N)$.

2. All players $i, j \in N$ satisfy $\tilde{u}_N^M(S \cup \{i\}) = 0 = \tilde{u}_N^M(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$. Under Symmetry, there exists some $\tilde{a} \in \mathbb{F}$ such that $\Psi_i(\tilde{u}_N^M) = \tilde{a}$ for all $i \in N$. Under Efficiency, $\tilde{v}(N) = \sum_{i \in N} \Psi_i(\tilde{u}_N^M) = n \cdot \tilde{a}$ and hence $\tilde{a} = \frac{\tilde{v}(N)}{n}$, so that $\Psi_i(\tilde{u}_N^M) = \frac{\tilde{v}(N)}{n}$ is unique for each $i \in N$.

3. For each coalition $S \subseteq N$, we have that $M(\tilde{w}^M(S)) = 0$. Under M-solution, $\Psi_i(\tilde{w}^M) = 0$ is unique for all $i \in N$.

■

The properties in Theorem 3.2 are independent. Fix M additive and 0-maximal fuzzy numbers calculation method.

- The null value Ψ^0 given by $\Psi_i^0(\tilde{v}) = 0$ for all $(N, \tilde{v}) \in \tilde{\mathcal{G}}^N$ and $i \in N$ satisfies all the properties but Efficiency.

- The M-nucleolus given by

$$Nu_i^M(\tilde{v}) = Nu_i\left(\tilde{v}^M\right) + \frac{1}{n} \cdot \left(\tilde{v}(N) - \tilde{v}^M(N)\right)$$

  for each $(N, \tilde{v}) \in \tilde{\mathcal{G}}^N$ and $i \in N$, satisfies all the properties but Additivity.

- Fix $\omega \in \mathbb{R}_{++}^N$ vector of positive scalars. The M-weighted Shapley value given by

$$Sh^{M\omega}(\tilde{v}) = Sh_i^\omega\left(\tilde{v}^M\right) + \frac{\omega_i}{\sum_{j \in N} \omega_j} \cdot \left(\tilde{v}(N) - \tilde{v}^M(N)\right)$$

  for each $(N, \tilde{v}) \in \tilde{\mathcal{G}}^N$ and $i \in N$, satisfies all the properties but Symmetry.

- The fuzzy egalitarian value satisfies all the properties but Weak null player.

- The M'-Shapley value, where $M' \neq M$ is also additive and 0-maximal, satisfies all the properties but M-solution.

Clearly, $\tilde{Sh} = Sh^{MoM}$. Hence, Theorem 3.2 allows us deduce a new characterization of the efficient Shapley value:

Corollary 3.1 *The efficient Shapley value is the only solution in the set of fuzzy cooperative games satisfying Efficiency, Additivity, Symmetry, Weak null player, and MoM-solution.*

Zero solution and MoM-solution are not related, i.e., there are values that satisfy Zero solution but not MoM-solution, and there are values that satisfy MoM-solution but not Zero solution. Hence, Theorem 3.1 is not deduced from Corollary 3.1, nor vice versa.

A value that satisfies Zero solution but not MoM-solution is

$$\Psi_i^1(\tilde{v}) = (\inf\{|x| : \mu_{\tilde{v}(\{i\})}(x) = 1\}, 1, 2)$$

for all $(N, \tilde{v}) \in \tilde{\mathcal{G}}^N$ and $i \in N$. A value that satisfies MoM-solution but not Zero solution is

$$\Psi_i^2(\tilde{v}) = \begin{cases} 1 \text{ if } \exists S \subseteq N : \langle \tilde{v}(S) \rangle \neq 0 \\ 0 \text{ otherwise} \end{cases}$$

for all $(N, \tilde{v}) \in \tilde{\mathcal{G}}^N$ and $i \in N$.

4 Conclusion

In games with fuzzy valued characteristic function, finding real valued solution concepts is a relevant tool to solve the uncertainty. In this article, we define a new value for cooperative games where the characteristic function takes values in the space of the fuzzy numbers. In particular, we introduce the definition of the efficient Shapley Value extending the usual definition of the Shapley value for crisp cooperative games. The proposed solution concept saves the efficiency property that, together with other relevant properties, characterize the solution. We illustrate the difference of the new concept with a previous generalization of the value in a fuzzy environment by means of an examples. We also show that the efficient Shapley value is a particular member of a wider family of efficient generalizations of the Shapley value. We also provide a characterization of this family of rules. A future work is to extend this analysis to cooperative games with a graph structure and fuzzy environment.

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References

Béal, S., Ferrières, S., Rémila, E., and Solal, P. (2018). The proportional Shapley value and applications. *Games and Economic Behavior*, 108:93–112. Special Issue in Honor of Lloyd Shapley: Seven Topics in Game Theory.

Besner, M. (2019). Axiomatizations of the proportional Shapley value. *Theory and Decision*, 86:161–183.

Borkotokey, S. and Mesiar, R. (2014). The Shapley value of cooperative games under fuzzy setting: A survey. *International Journal of General Systems*, 43(1):75–95.

Cunlin, L. and Qiang, Z. (2011). Nash equilibrium strategy for fuzzy non-cooperative games. *Fuzzy Sets and Systems*, 176(1):46–55.

Gallardo, J. and Jiménez-Losada, A. (2020). A characterization of the Shapley value for cooperative games with fuzzy characteristic function. *Fuzzy Sets and Systems*, 398:98–111.

Kalai, E. and Samet, D. (1987). On weighted Shapley values. *International Journal of Game Theory*, 16(3):205–222.

Mallozzi, L. and Vidal-Puga, J. (2023). Equilibrium and dominance in fuzzy games. *Fuzzy Sets and Systems*, 458:94–107.

Mares, M. (2001). *Fuzzy cooperative games*. Physica-Verlag, Heidelberg.

Mares, M. and Vlach, M. (2004). Fuzzy classes of cooperative games with trasferable utilities. *Scientiae Mathematicae Japonicae Online*, 10:197–206.

Ortmann, K. (2000). The proportional value for positive cooperative games. *Mathematical Methods of Operations Research*, 51:235–248.

Schmeidler, D. (1969). The nucleolus of a characteristic function game. *SIAM Journal of Applied Mathematics*, 17(6):1163–1170.

Shapley, L. S. (1953). A value for n-person games. In Kuhn, H. and Tucker, A., editors, *Contributions to the theory of games*, volume II of *Annals of Mathematics Studies*, pages 307–317. Princeton University Press, Princeton NJ.

Sivanandam, S. N. and Deepa, S. N. (2019). *Principles of Soft Computing*. Wiley, third edition.

van den Brink, R. (2007). Null or nullifying players: The difference between the Shapley value and equal division solutions. *Journal of Economic Theory*, 136(1):767–775.

van den Brink, R. and Funaki, Y. (2009). Axiomatizations of a class of equal surplus sharing solutions for TU-games. *Theory and Decision*, 67:303–340.

van den Brink, R., Levínský, R., and Zelený, M. (2015). On proper Shapley values for monotone TU-games. *International Journal of Game Theory*, 44(2):449–471.

Vorob'ev, N. and Liapounov, A. (1998). The proper Shapley value. In Petrosyan, L. and Mazalov, M., editors, *Game Theory and Applications*, volume IV, pages 155–159. Nova Science, New York.

Yu, X. and Zhang, Q. (2010). An extension of cooperative fuzzy games. *Fuzzy Sets and Systems*, 161(11):1614–1634. Theme: Decision Systems.

Zadeh, L. (1978). Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems*, 1:3–28.