# Factor Models as "Explanatory UniÖers" versus "Explanatory Ideals" of Empirical Regularities of Stock Returns

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# Factor Models as "Explanatory Unifiers" versus "Explanatory Ideals" of Empirical Regularities of Stock Returns

Phoebe Koundouri * Nikolaos Kourogenis†‡ Nikitas Pittis†

Panagiotis Samartzis†

**Abstract**

In this paper we investigate whether the empirical regularities of stock returns are independent of each other or whether any one of them implies all the others. If such a regularity exists, it is called "fundamental" and is usually thought of as a "deductive explanation" of the others. We demonstrate that such a fundamental regularity of stock returns is the one represented by the single factor model with a stochastically persistent beta coefficient (SFM-AR). Indeed, this regularity alone entails all the usual regularities of stock returns, including conditional heteroskedasticity, leptokurtosis aggregational Gaussianity and aggregational Independence. Hence, SFM-AR may be thought of as an "explanatory unifier" of the empirical regularities of stock returns. However, since the theoretical origins of SFM-AR are weak, its explanatory status falls short of meeting the standards of the "ideal explanatory text".

*JEL Classification*: C18, G10, C22, G11, G12

*Keywords:* empirical regularities, stock returns, single factor model, autoregressive beta, statistical explanation.

*Department of International and European Economic Studies, Athens University of Economics and Business

†Department of Banking and Financial Management, University of Piraeus.

‡*Correspondence to:* Nikolaos Kourogenis, Department of Banking and Financial Management, University of Piraeus, 80 Karaoli and Dimitriou str., 18534 Piraeus, Greece. Email(1): nkourogenis@yahoo.com Email(2): nkourog@unipi.gr. Tel: 00302104142142. Fax: 00302104142341.

# 1 Introduction

Can statistical models be explanatory, and - if yes - in what sense? During the first half of the twentieth century, the majority of scientists and philosophers, under the dominant influence of logical positivism, gave to this question a negative reply. Bunge (1979) summarizes this negative attitude as follows: "It is a widespread opinion that statistical statements are purely descriptive, that they are in need of being explained, without being entitled to perform an explanatory function." (1979, pp. 302). This attitude begun to change gradually since the early 1940's, when many philosophers came to realize that the so-called "stochastic revolution" in the field of theoretical physics should somehow be accounted for by any acceptable theory of scientific explanation. Rescher (1962) comments on this change of attitude as follows: "Only since the 1940's, with the fading influence of logical positivism, so heavily imbued with nineteenth-century conceptions, have statistical explanations come to be recognized as deserving not only a measure of acceptance but almost a place of prominence." (1962, pp. 202). Indeed, since 1948, the year in which the seminal work of Hempel and Oppenheim on scientific explanation was published, a lot of effort has been directed towards analyzing the nature and the structure of scientific explanation in general and statistical or probabilistic explanation in particular.

Statisticians have also attempted to analyze the concept of statistical explanation from a more pragmatic viewpoint, namely in relation to the origins and functions of statistical models. To this end, Box and his co-authors, in a series of papers, classified statistical models in two broad categories, the first including so-called *empirical* or *interpolatory* models, and the second *explanatory* or *mechanistic* ones (see Box and Hunter 1965, and Box and Draper 1987). Lehmann (1990) summarizes the main differences between these two types as follows: "Empirical models are used as a guide to *action*, often based on forecasts of what to expect from future observations. In contrast, explanatory models embody the search for the basic mechanism underlying the process being studied; they constitute an effort to achieve *understanding*" (Lehmann 1990, pp. 163). Indeed, a major theme in the philosophy of science since the early sixties has been the definition of statistical (or probabilistic) explanation, the analysis of its logical structure and the introduction of alternative sets of criteria for explanatory adequacy.

Concerning the object of statistical explanation, it can be either a specific event, or a statistical regularity. For example, an event that an interested investor might wish to "explain" is the following: "The price of the Citigroup stock on 22/02/11 fell by more than 4%". Note that the event to be explained has to be well defined spatiotemporally. On the other hand, an empirical regularity refers to a class of similar events occured in different points of time or space. For example, a well-known regularity of stock returns is their tendency to be distributed symmetrically around a constant mean. A statistical explanation of a single event differs in some important respects from that of a statistical regularity. The most important one is that the explanation of a single event usually takes the form of an inductive argument, whereas that of a statistical regularity retains its deductive structure. In this paper we shall focus exclusively on the alternative models of explanation for empirical regularities. For a thorough discussion of the specific features of explanations of single events and how these features differ from those akin to regularities, the interested reader may be referred to the classic papers of Hempel (1965), Salmon (1970), or more recently Fetzer (1993).

The existing literature, mainly in the philosophy of science, has put forward more than one sense in which an explanation of a regularity is defined. As a result, more than one models of statistical explanation have been proposed. In most of these models, deducibility from a "covering law" is the key concept through which statistical explanation is achieved. Redhead (1990) comments on this issue as follows: "So what do probabilistic or statistical explanations achieve? Well, they enable us to deduce and hence to explain the limiting relative frequencies with which events of a given kind turn up in a long-run repetition of the set-up producing the phenomenon." (1990, pp. 137). In other models, however, deducibility is just one among several others conditions that must be met by an adequate explanation. The main aim of this paper is to summarize these models, analyze their relevance for the empirical regularities of stock returns and identify alternative sets of criteria with respect to which adequate explanations of these regularities may be produced.

Before we analyze the aforementioned models of explanation of statistical regularities in some detail, let us first focus on the object of explanation itself, namely the concept of statistical regularity. In the minds of many philosophers of science, a statistical regularity is nothing but the type of the empirical distribution of a set of observations produced by a "long-run repetition" of the underlying chance mechanism. Implicit in this definition is the interpretation of the empirical distribution by a corresponding theoretical one, usually that producing the best fit. However, the aforementioned definition is appropriate only for the case of independent and identically distributed (*iid*) observations, such those produced in controlled experiments. In the case of non-experimental data, such those obtained in the field of economics, the concept of statistical regularity must be augmented to accommodate the presence of temporal dependence and time heterogeneity in the stochastic process $\{Y_t\}_{t\in\mathbb{Z}}$ that generated the available data. For example, in the case of a dependent but stationary process $\{Y_t\}_{t\in\mathbb{Z}}$, one regularity may take the form of the unconditional (stationary) distribution, $D(y)$, of $Y_t$, in the spirit of the Redhead definition, cited above. However, another regularity may be defined in terms of the type of the conditional distribution $D_t(y \mid \mathcal{F}_{t-1})$ where $\mathcal{F}_{t-1}$ is the information set available at $t-1$.

The preceding discussion suggest that a *set of statistical regularities* may be defined in terms of a *set of probabilistic properties* of $\{Y_t\}_{t\in\mathbb{Z}}$. These properties may be classified in three major categories, namely distributional, temporal dependence and time heterogeneity ones. The procedure of characterizing a statistical regularity may take the following two steps: First, a statistical regularity, $S_1$, is diagnosed. Second, a probabilistic property, $P_1$, of the underlying process that is capable of producing $S_1$ is identified. We may refer to $P_1$ as the theoretical counterpart of $S_1$. Once $P_1$ is found, it replaces $S_1$ as the object to be explained (explanandum). It must be noted a given regularity may be described by more than one probabilistic properties, in which case additional criteria pertaining to "empirical adequacy" are likely to be employed.

Hempel and Oppenheim (1948) and Hempel (1965) define the concept of statistical explanation of an empirical regularity, $S_2$, in terms of “deducibility” or “nomic expectability” of $S_2$ from a broader or more comprehensive empirical regularity, $S_1$. Explanation is achieved through *derivation*. Specifically, $S_2$ is thought to be explained if it can be deduced from a set of “explanans” that consists of $S_1$ and (possibly) some antecedent conditions. In this case, $S_2$ may be thought of as a *manifestation* of the more fundamental regularity $S_1$. It must be noted that “deducibility” of $S_2$ from $S_1$ amounts to a mathematical proof of the type $P_1 \Rightarrow P_2$. This in turn implies that the extent to which $S_2$ is explained by $S_1$ depends on the *choice* of probabilistic descriptions $P_1$ and $P_2$ of $S_1$ and $S_2$, respectively. For example, if $S_2$ were (chosen to be) described by $P_2'$ instead of $P_2$, then a proposition of the form $P_1 \Rightarrow P_2'$ might not be valid. In such a case, $S_2$ is not explained by $S_1$. This feature introduces some ambiguity to the extent that a given regularity is explained by a broader regularity, since the necessary deducibility relationship may be obtained under one probabilistic interpretation but may fail under another.

The selection of the relevant theoretical interpretation at any given time depends on the “background theory” that prevails at this particular time. Historically, alternative theoretical interpretations have been used at different points in time in order to explain the same empirical regularity (see, for example, Brewer and Lambert, 1993). From now on, when we say that a regularity, say, $S_2$ is explained by a broader regularity, say, $S_1$ or that $S_1$ entails or implies $S_2$, we shall mean that there exist (at least) two corresponding descriptions $P_1$ and $P_2$ (in the sense defined above) such that $P_1 \Rightarrow P_2$. In such a case, we may say that $S_1 \longrightarrow S_2$.

The model of explanation mentioned above is usually referred to as the Deductive-Statistical (DS) model of explanation which “...is used to explain a statistical regularity by showing that it follows with necessity from one or more statistical laws (and initial conditions in some cases).” (Salmon 1984, p. 295). Kinoshita (1990) comments on the issue of regularity explanation as follows: "A regularity explanation does not *amplify* the nature of a particular regularity, but rather *orients* the regularity relative to other regularities. Regularity explanations show a regularity to be reasonable or proper by showing it to be a special case of one or more (more comprehensive) regularities." (Kinoshita, 1990, p. 301). Specifically, an explanation of $S_2$ orients this regularity within a complex hierarchy of regularities by showing that $S_2$ is a special case or manifestation of $S_1$. Friedman (1974) defines explanation in terms of unification or conceptual economy. If the number of empirical regularities that have to be assumed as "brute" is minimized, then our understanding of the phenomenon is increased. For example, if $S_1$ and $S_2$ are two different sets of regularities then the case in which $S_1$ implies $S_2$ achieves a higher order of understanding than the case in which $S_1$ and $S_2$ are independent. Friedman (1974, p. 15) argues: "I claim that this is the crucial property of scientific theories we are looking for; this is the essence of scientific explanation - science increases our understanding of the world by reducing the total number of independent phenomena that we have to accept as ultimate or given. A world with fewer independent phenomena is, other things being equal, more comprehensible than with more".

A more demanding model of statistical explanation, the so-called Deductive-Nomological-Probabilistic (DNP) model of explanation was put forward by Railton (1977, 1981). In the context of this model a regularity is explained in terms of the *mechanism* that produced this regularity. "The goal of understanding the world is a theoretical goal, and if the world is a machine - a vast arrangement of nomic connections - then our theory ought to give us some insight into the structure and workings of the mechanism, above and beyond the capability of predicting and controlling its outcomes." (1978, pp. 208). A regularity $S_1$ does not explain the regularity $S_2$ unless $S_1$ is backed up with "an account of the mechanism(s) at work". In other words, $S_1$ in itself cannot form the basis for a satisfactory explanation of the explanadum event, unless $S_1$ is "derivable from our theory without appeal to particular facts." (1978, pp. 215). The DNP model does not reject the idea that "deducibility from a law" is an important feature of explanation.

However, it puts forward the view that "discovering the mechanism that produced that law" is another, equally important, condition for obtaining a satisfactory explanation of an empirical regularity. It must be noted that Railton's account of probabilistic explanation allows partial or incomplete explanations to qualify as adequate. This case arises when the relevant theory gives only a partial (rather than full) account of the statistical model that implies $S_1$. In such a case, the offered explanation does not illuminate all the explanatory text but only parts of it, that is, it furnishes explanatory information. In the limiting case, the corresponding explanation is called "ideal" explanation.

The DNP model of explanation seems to subscribe to the so-called "principle of micro-reduction" according to which "the properties of wholes are explained in terms of the properties of their parts" (Brittan, 1970, pp. 447). Put differently, $S_1$ does not explain $S_2$ unless $S_1$ itself is reduced to (and explained in terms of) a micro-theory about the component parts that give rise to $S_1$. In contrast, the DS model of explanation does not place such severe conditions of explanatory adequacy. A necessary and sufficient condition for the explanation of $S_2$ by $S_1$ is the nomic deducibility of the former from the latter, namely $S_1 \longrightarrow S_2$. In the context of DS, the origins of $S_1$ are allowed to be left unexplored. Salmon (1990) argues that DS explanations, of the type $S_1 \longrightarrow S_2$ are complete. "If one wants an explanation of a law that entered into the first explanation ($S_1$, in our case) it can be supplied by deriving that law from more general laws or theories. *The result is another explanation.* The fact that a second explanation of this sort can be given does nothing to impugn the credentials of the first explanation" (pp. 156, emphasis added).

As far as stock returns are concerned there are currently certain statistical regularities which are widely recognized as "stylized facts", since they appear to be common across many different markets, assets and time periods. These regularities may be classified in two broad categories: The first category, hereafter "Regularities of Type-I" (RT-I), refers to the individual temporal behavior of each returns series $R_{i,t}$, $i = 1, 2, ..., N$. The most important regularities in RT-I are the following: (i) Fast Mean Reversion ($FMR$), that is, the tendency of stock returns to revert to an average value quite rapidly. Put differently, the degree of persistence of returns' deviation from this average value is very low, if not zero. (ii) Volatility Clustering ($VC$), that is, the fact that "large (price) changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes" (Mandelbrot 1963, p. 418). (iii) Empirical Leptokurtosis ($EL$), namely, the empirical distribution of stock returns are characterized by heavy tails with positive excess kurtosis. (iv) Empirical Aggregational Gaussianity ($EAG$), that is, the fact that the degree of leptokurtosis in the empirical distributions tends to diminish as the return horizon increases. (v) Empirical Aggregational Independence ($EAI$), that is, the observation that the volatility clustering effects tend to disappear as the returns horizon increases or equivalently, as we move from higher to lower frequencies (e.g. from daily to quarterly observations).

The probabilistic interpretations/descriptions of these empirical regularities have taken the following forms. $FMR$ is described by assuming that the stochastic sequence $\{R_{i,t}\}$ is martingale difference ($MD$). $VC$ is usually interpreted as "dynamic conditional heteroskedasticity" ($DCH$) which is a specific type of non-linear temporal dependence of $\{R_{i,t}\}$. $DCH$ may be further described, for example, by a parametric $GARCH(p,q)$ model. $EL$ has a natural interpretation in terms of theoretical leptokurtosis ($TL$) of the (stationary) distributions of the random variables $R_{i,t}$. $EAG$ is interpreted as a tendency of the aggregate random variables, $R_{i,\tau}(k) = \sum_{l=1}^{k} R_{i,t-k+l}$, to converge in law to the Normal distribution as the returns' horizon $k$ increases ($AG$). Finally, $EAI$ is described by the probabilistic property that the random variables $R_{i,\tau}(k)$ and $R_{i,s}(k)$, $\tau \neq s$, tend to be independent as the returns horizon, $k$, increases ($AI$). From the very beginning of the $DCH$ interpretation of $VC$, the following relationship was proved to hold: $CH \Rightarrow TL$ (see, for example, Engle 1982, Bollerslev 1986). Moreover, within the $GARCH(p,q)$ model, and under some parametric restrictions ensuring asymptotic independence (mixing) and finiteness of the unconditional variance it is quite easy to show that $CH \Rightarrow AG$ and $CH \Rightarrow AI$. To this end, Drost and Nijman (1993) and, more recently, Meddahi and

Renault (2004) have proved $AG$ and $AI$ within the weak $GARCH$ model, by showing that the weak GARCH coefficients tend to zero under temporal aggregation. Finally, there is no implied relationship between $MD$ and $DCH$. These two probabilistic properties are independent of each other.

By mapping the aforementioned theoretical results back to empirical world, we may say that $VC \longrightarrow \{EL, EAG, EAI\}$ although $FMR \not\leftrightarrow VC$. This in turn implies that there is no regularity in the set $\mathcal{R} = \{FMR, VC, EL, EAG, EAI\}$ that can be thought of as "fundamental" or "unifying", that is, a regularity in terms of which all the other regularities in $\mathcal{R}$ are explained. Nonetheless, $VC$ seems to possess some quite significant unifying properties.

The second category, hereafter referred to as "Regularities of Type-II" (RT-II) refers to the joint temporal behavior of all return series or to the joint behavior between each returns series $R_{i,t}$ and another factor (or factors). The most important regularity in RT-II, referred to as empirical contemporaneous correlation ($CC$), is the fact that the stock prices tend to move together over time. In other words, stock returns appear to be contemporaneously correlated . Roll and Ross (1980) refer to the common variability of stock returns as "the single most widely-acknowledged empirical regularity" (1980, p. 1073). An obvious way to describe $CC$ is to assume that the correlation matrix $\Sigma$ of the (stationary) random vector $[R_{1,t}, R_{2,t}, ..., R_{N,t}]^\top$ is non-diagonal. This assumption will be referred to as theoretical contemporaneous correlation ($TCC$).

Another empirical regularity in RT-II is based on the observation that stock returns tend to respond to (unanticipated) changes in one or more variables, such as the market portfolio or certain macroeconomic variables/factors. This regularity will be referred to as "common factor" regularity ($CF$). Sharpe (1964) refers implicitly to this regularity as follows: "it is common practice for investment counselors to accept a lower expected return from defensive securities (those which respond little to changes in the economy) than they require from aggressive securities (which exhibit significant response)" (1964, p. 442). In the simplest case of a single factor, $X_t$, the probabilistic interpretation of

$CF$ takes the form of a linear regression model, $R_{i,t} = a_i + \beta_i M_t + u_{i,t}$, where $M_t = X_t - E(X_t \mid \mathcal{F}_{t-1})$. The error term, $u_{i,t}$, is assumed to be a zero-mean i.i.d. process with finite variance, satisfying the condition $E(u_{i,t} \mid M_t) = 0$, for $i = 1, 2, ..., N$. The slope coefficient, $\beta_i$, is interpreted as a measure of the systematic risk of the stock $i$, and is usually referred to as the "beta coefficient", or simply the "beta" of the stock $i$. In the context of this model, the returns on all the existing assets, $i = 1, 2, ..., N$, are related only through $M_t$. This assumption amounts to the covariance matrix, $\Sigma_u$, of $u_{i,t}$, $i = 1, 2, ..., N$, being diagonal, that is, $Cov(u_{j,t}, u_{i,t}) = 0$ for $j \neq i$. Following the relevant literature, this particular description of $CF$ will be referred to as the single factor model $(SFM)$. It is well known that the following relationship holds, $SFM \Rightarrow TCC$, which is re-interpreted as $CF \longrightarrow CC$. This last relationship is taken to imply that the observed positive correlations among stock returns are explained exclusively by the presence of a common factor causing all stock returns simultaneously. The relationship $CF \longrightarrow CC$ means that between $CF$ and $CC$, the first is the fundamental regularity which explains the second.

Initially, an implicit assumption in the $CF$ regularity mentioned above, was that the degree of response of each stock to changes in the factor was constant over time. Therefore, the corresponding description of this degree of response in the context of $SFM$ took the form of a time-invariant beta. However, more detailed statistical analysis showed that the estimates of beta were not constant over time. As a result the $CF$ regularity was replaced by a more general regularity according to which stock returns tend to respond to changes in the factor with the degree of this response changing over time in an unpredictable fashion. Some researchers took the view that the variation in the degree of response is random (see, for example, Blume 1971, 1975, Fabozzi and Francis 1977) whereas others believed that this variation exhibits signs of temporal persistence (see, for example, Fisher and Kamin 1985, Sunder 1980, Bos and Newbold 1984, Collins, Ledolter and Rayburn 1987, Andersen et al. 2005, and Jostova and Philipov 2005). As a result, the original $CF$ regularity was replaced by either the "random-variation" regularity or the "persistent-variation" one, referred to as $CF-R$ and $CF-P$, respectively. These new regularities may be probabilistically described by a single factor model in which the stochastic sequence $\{\beta_{i,t}\}$ is assumed to be a first-order autoregressive process, $\beta_{i,t}=\varphi_i\beta_{i,t-1}+\varepsilon_{i,t}$. The resulting SFM model accommodates both the $CF-R$ and $CF-P$ regularities in the form of the cases $\varphi=0$ and $\varphi\neq 0$, respectively. From now on, the $SFM$ model with $\varphi=0$ and $\varphi\neq 0$ will be referred to as $SFM-R$ and $SFM-AR$, respectively.

The questions we try to answer in this paper are the following: Are the $CF-R$ or $CF-P$ regularities of type II independent of the set $\mathcal{R}$ of regularities of type-I? Or is it the case that one type of regularities implies the other, in which case the first type is more fundamental than the second? Put differently, is the regularity implied by $SFM-R$ or $SFM-AR$ the most fundamental regularity of stock returns? Moreover, if the answer to this question is affirmative, what are the implications for the explanatory status of $SFM-R$ or $SFM-AR$? Does $SFM-R$ or $SFM-AR$ enjoy, in a certain sense, a higher degree of explanatory adequacy than other models for stock returns, such as, for example, $GARCH$ models? Moreover, even if $SFM-R$ or $SFM-AR$ is explanatory adequate in relative terms (that is compared to other models), is it also explanatory sufficient in absolute terms? Put differently, does the explanatory status of $SFM-R$ or $SFM-AR$ satisfy the criteria of the DNP model of statistical explanation?

The answers to these questions may be summarized as follows: (i) The most important result of the paper is that $SFM-AR\Rightarrow\{MD,DCH,TL,AG,AI\}$, which is re-interpreted as $CF-P\longrightarrow\{FMR,VC,EL,EAG,EAI\}$. This result implies that the only type of regularity regarding stock returns, that should be assumed as brute is the following: $CF-P=$ "stock returns respond to unanticipated changes in the risk factor, with the degrees of response (betas) evolving over time in a stochastically persistent fashion". This type of regularity alone entails all the other aforementioned regularities of stock returns. (ii) Although $SFM-AR$ is explanatory adequate in the DS sense, it fails to meet the standards of DNP. This is due to the fact that none of the existing asset pricing theories can serve as a description of the chance mechanism that gives rise to $SFM - AR$. (iii) The relevant theory that comes closer to achieving the explanatory ideal is a specific version of the Capital Asset Pricing Model (CAPM).

The remainder of this paper is organized as follows: Section 2 defines the $SFM - AR$ model and summarizes existing results showing that this model exhibits the theoretical properties of martingale difference, dynamic conditional heteroskedasticity and leptokurtosis. Section 3 proves that an invariance principle holds for the properly standardized sequence of partial sums of SFM-AR returns. This, in turn, implies that Aggregational Gaussianity holds for SFM-AR returns and that two sequential long-horizon returns tend to be independent as the return horizon increases (Aggregational Independence). Section 4 discusses which (if any) of the existing asset pricing theories may serve as a description of the chance mechanism that gives rise to $SFM - AR$, thus satisfying the conditions of DNP explanatory adequacy and producing the ideal explanatory text. Section 4 concludes the paper.

## 2 The Single Factor Model with Autoregressive Beta (SFM-AR)

Let us consider a market with $n$ assets (stocks) and let $R_{i,t}$ be the one-period continuously compounded return on an individual stock, defined as $R_{i,t} = p_{i,t} - p_{i,t-1}$, where $p_{i,t}$ is the natural logarithm of the price of the particular stock. Following the discussion of the previous section, we assume that $R_{i,t}$ is related to a single factor, $M_t$, via the following relationship:

$$R_{i,t} = a_i + (\beta_i + \beta_{i,t})M_t + u_{i,t},\ i = 1, 2, ..., n \tag{1}$$

where $a_i$ and $\beta_i$ are real numbers, and $u_{i,t}$, $\beta_{i,t}$, are zero-mean sequences of random variables whose exact properties will be defined below. Equation (1) can be written in vector form as follows

$$\mathbf{R}_t = \boldsymbol{\alpha} + M_t(\boldsymbol{\beta} + \boldsymbol{\beta}_t) + \mathbf{u}_t, \tag{2}$$

where $\mathbf{R}_t' = [R_{1,t}, R_{2,t}, \ldots, R_{n,t}]$, $\boldsymbol{\alpha}' = [a_1, a_2, \ldots, a_n]$, $\boldsymbol{\beta}' = \left[\beta_{1,t}, \beta_{2,t}, \ldots, \beta_{n,t}\right]$ and $\mathbf{u}_t' = [u_{1,t}, u_{2,t}, \ldots u_{n,t}]$.

**Assumption M:** $\beta_{i,t}$ follows a zero-mean AR(1) process,

$$\beta_{i,t} = \varphi_i \beta_{i,t-1} + \varepsilon_{i,t}, \ \ |\varphi_i| < 1, \, 1 \leq i \leq n \tag{3}$$

and

$$\begin{bmatrix} \mathbf{u}_t \\ M_t \\ \boldsymbol{\varepsilon}_t \end{bmatrix} \sim NIID \left( \mathbf{0}, \begin{bmatrix} \Sigma_u & 0 & 0 \\ 0 & \sigma_m^2 & 0 \\ 0 & 0 & \Sigma_\varepsilon \end{bmatrix} \right)$$

where $\boldsymbol{\varepsilon}_t = [\varepsilon_{1,t}, \varepsilon_{2,t}, \ldots, \varepsilon_{n,t}]'$, $\Sigma_u = diag\left\{\sigma_{u_1}^2, \sigma_{u_2}^2, \ldots, \sigma_{u_n}^2\right\}$, $\Sigma_\varepsilon = (\sigma_{i,j})_{1 \leq i,j \leq n}$ and $E_{t-1}[M_t] = 0$, where $E_{t-1}[\cdot] = E[\cdot \mid \mathcal{F}_{t-1}]$ and $\mathcal{F}_{t-1}$ denotes the information set that is generated by all the random variables under consideration up to time $t-1$.

Let

$$\Sigma_\beta := Var(\boldsymbol{\beta}_t) = E\left[\boldsymbol{\beta}_t \boldsymbol{\beta}_t'\right] = \left( \frac{\sigma_{i,j}}{1 - \varphi_i \varphi_j} \right)_{1 \leq i,j \leq n} .$$

Then, for $i = j$, $1 \leq i \leq n$, the diagonal elements of $\Sigma_\beta$, are given by:

$$\sigma_{\beta_i}^2 = Var\left(\beta_i\right) = \frac{\sigma_{i,i}}{1 - \varphi_i^2} \ .$$

Note that under assumption **M**, equation (2) implies that $\mathbf{R}_t$ is a strictly stationary process with finite second moments. Equation (3) can be also written in vector form as

$$\boldsymbol{\beta}_t = \boldsymbol{\Phi} \boldsymbol{\beta}_{t-1} + \boldsymbol{\varepsilon}_t \ ,$$

where $\boldsymbol{\Phi}$=$diag\left\{\varphi_1, \varphi_2, \ldots, \varphi_n\right\}$.

Koundouri et. al (2014) show that the $SFM - AR$ model defined above exhibit the properties of martingale difference, conditional heteroskedasticity and leptokurtosis. In the rest of this section, we prove that $SFM - AR$ implies asymptotic Gaussianity and asymptotic independence as well. To do this, we must analyze the temporal aggregation properties of the $SFM - AR$ process.

In order to simplify notation, let us assume that $n = 1$ and define the $k$-period return $p_t - p_{t-k}$, where $p_t$ is the logarithm of the stock price at time $t$. Since we study non-overlapping returns, the series of $k$-period returns under consideration will be of the form $\{\ldots, p_{t-k} - p_{t-2k}, p_t - p_{t-k}, p_{t+k} - p_t, \ldots\}$. For this reason, we introduce a new index, denoted by $\tau$, which represents the $k$-period interval, in terms of $t$. More specifically, if $t$ and $\tau$ correspond to the same moment in time, then $\tau + 1$ will coincide with $t + k$. In other words, one unit in terms of $\tau$ corresponds to $k$ units in terms of $t$. This change of index allows us to denote the $k$-period returns by

$$R_\tau(k) = p_t - p_{t-k} = \sum_{i=1}^{k} R_{t-k+i} \ .$$

Respectively, for the $k$-period return at lag 1, we use the notation

$$R_{\tau-1}(k) = p_{t-k} - p_{t-2k} = \sum_{i=1}^{k} R_{t-2k+i}$$

and so forth. In the subsequent paragraphs we will make use of the notation "$\tau - l$" and "$\tau + l$", instead of "$t - lk$" and "$t + lk$", where $l \geq 0$.

# 3 Aggregational Gaussianity and Independence

Before proceeding to the next theorem, which proves the Aggregational Gaussianity of stock returns under $SFM - AR$, we can first observe that since $R_\tau(k) - ka$ is a martingale difference process,

$$Var(R_\tau(k)) = kVar(R_t) = k\left(\left(\beta^2 + \sigma_\beta^2\right)\sigma_m^2 + \sigma_u^2\right) \ . \tag{4}$$

Next theorem proves that the sequence of weighted sums of returns, as described by

$SFM-AR$, satisfies an invariance principle. To this end, let us fix some $t_0 \in \mathbb{Z}$, set

$$S_{t_0,k} = \sum_{i=1}^{k} \left(R_{t_0+i} - E\left[R_{t_0+i}\right]\right)$$

and for $0 \leq r \leq 1$, define

$$W_k(r) = \sum_{i=1}^{[rk]} \left(R_{t_0+i} - E\left[R_{t_0+i}\right]\right),$$

where for $r < 1/k$, $W_k(r) := 0$. Then, we have the following theorem :

**Theorem 1** *Under Assumption **M**,*

$$\frac{1}{\sqrt{Var(R_\tau(k))}} W_k \xrightarrow{D} W, \text{ as } k \to \infty,$$

*where $W$ is a standard Brownian motion and "$\xrightarrow{D}$" denotes the usual weak convergence on the real line.*

**Proof.** See Appendix D. ■

**Remark 2** *Note that $R_\tau(k)$ does not have a well defined limit as $k \to \infty$. This fact, does not allow us to obtain any conclusion with respect to the independence between $R_\tau(k)$ and $R_{\tau-1}(k)$ as $k \to \infty$, since the definition of asymptotically independent random sequences requires that they are stochastically bounded. On the other hand, Theorem 1 implies that*

$$K_\tau(k) := \left(R_\tau(k) - E\left[R_\tau(k)\right]\right) / \sqrt{Var(R_\tau(k))} \xrightarrow{d} N(0,1) \text{ as } k \to \infty, \tag{5}$$

*where by $N(0,1)$ we denote the standard Gaussian distribution. By virtue of (4), we can re-write (5) as follows:*

$$\left(R_\tau(k) - E\left[R_\tau(k)\right]\right) / \sqrt{k} \xrightarrow{d} N\left(0, \left(\beta^2 + \sigma_\beta^2\right)\sigma_m^2 + \sigma_u^2\right) \tag{6}$$

*as $k \to \infty$. The left hand sides in (5) and (6) provide us with sequences (of $k$) with well defined limits. Theorem 1 implies that for every $\tau$, $K_{\tau-1}(k)$ and $K_\tau(k)$ are asymptotically independent as $k \to \infty$. In other words, this proves the asymptotic independence between the de-meaned and properly standardized long-horizon returns $K_{\tau-l}(k)$ and $K_\tau(k)$, for every $l \neq 0$, as the return horizon, $k$, tends to infinity.*

# 4 Does SFM-AR provides an ideal explanatory text for the empirical regularities of stock returns?

In the previous two sections we showed that $SFM-AR \Rightarrow \{MD, DCH, TL, AG, AI\}$. Is this all that we require in order to claim that $\{MD, DCH, TL, AG, AI\}$ have actually been explained by $SFM-AR$? Or do we need to explore the origins of $SFM-AR$ itself? In other words, do we also have to answer the question of "where does $SFM-AR$ come from?" In the context of the DS model, the answer is "no". In contrast, in the context of DNP, the answer is "yes". According to DNP, $SFM-AR$ in itself cannot form the sole basis for a satisfactory explanation of $\{MD, DCH, TL, AG, AI\}$ unless $SFM-AR$ is derivable from a theory on the causal mechanism at work. Put differently, DNP requires an account of the explanatory web that gave rise to $SFM-AR$.

The explanatory web mentioned above is what Railton defines as "an ideal explanatory text." The full derivation of $SFM-AR$ from its elementary parts constitutes the full ideal text relevant to $\{MD, DCH, TL, AG, AI\}$. In other words, full understanding of $\{MD, DCH, TL, AG, AI\}$ requires the full ideal DNP text, which gives deeper insights into the details of the process that ends up in the emergence of the $SFM-AR$ regularity. Is the requirement of obtaining such an ideal text a very strict one? Railton himself asks this question: "Is it preposterous to suggest that any such ideal could exist for scientific explanation and understanding? Has anyone ever attempted or even wanted to construct an ideal causal or probabilistic text?" (1981, pp. 246-247). Railton answers his own question as follows: "It is not preposterous if we recognize that the actual ideal is not to *produce* such texts, but to have the ability (in principle) to produce arbitrary parts of them." (1981, pp. 246-247). Put differently, the absence of the full text does not imply complete lack of understanding of the observed regularities. As Psillos (2002) puts it, the ideal DNP text "is more of a regulative ideal than what, in practice, we need and should strive for. In practice, what we (or the scientists) need and should strive for is "explanatory information" relevant to the explanandum. Such information, if indeed it is information relevant to the explanandum, will be part of the ideal DNP text. By producing such parts, no matter how underdeveloped and incomplete they may be, scientists understand why a certain explanandum happens. Finding more and more bits of the ideal texts, we move closer to the ideal of a full understanding" (2002, p. 260). Railton (1981) himself refers to the ideal DNP text as "a yardstick for proffered explanations of chance phenomena" and also allows for these proffered explanations to take various forms and "still be successful in virtue of communicating information about the relevant ideal text" (1981, pp. 246-247).

The preceding discussion generates naturally the following question: Is there any ideal explanatory text concerning $SFM - AR$? In order to answer this question, we must examine whether $SFM - AR$ can be derived deductively from a certain theory, accounting for the chance mechanism at work. If such a theory exists, then the regularity represented by $SFM - AR$ would be reduced and explained in terms of the properties of the component parts postulated by the given theory. To this end, we shall first examine whether an ideal explanation for the simple, constant-beta $SFM$ model can be produced. If this attempt turns out to be successful, then we shall examine under what additional conditions the purported explanation can be adapted in order to explain $SFM - AR$.

In the context of Arbitrage Pricing Theory (APT) put forward by Ross (1973), such a derivation is not available. In fact, the constant-beta SFM constitutes the starting point of the theory. Roll and Ross state explicitly this fact: "... the APT is based on a linear return generating process *as a first principle*..." (1980, pp. 1074). They motivate this "law" by appealing not to a theory but to observable facts, namely to the common variability of stock returns. This means that at least within APT the origins of $SFM$ (let alone $SFM-AR$) are left unexplored. As Roll and Ross admit "We do consider the basic underlying causes of the generating process of returns to be potentially important area of research, but we think it is an area that can be investigated separately from testing asset pricing theories" (1980, pp. 1077).

Apart from the absence of any derivation of $SFM$ from any "underlying causes" the standard APT does not shed much light on the identity of the factor $M_t$, or the factors $M_{1t}, M_{2t}, ..., M_{lt}$ in a multi-factor model ($MFM$). On this point Roll and Ross themselves raise the question "What are the common or systematic factors?" (1980, pp. 1077). In searching the identity of these factors, Roll and Ross argue as follows: "If there are only a few systematic components of risk, one would expect these to be related to fundamental economic aggregates, such as GNP, or to interest rates or weather (*although no causality is implied by such relations*)" (1980, pp. 1077, emphasis added). Roll and Ross seem to suggest that the systematic components of risk (the real causal factors) are likely to be non-identifiable. However, they suggest that the true factors are likely to be related to observable macroeconomic variables such as GNP. As a result, in empirical tests of APT, these macroeconomic variables can approximate the true systematic (causal) risk factors. In fact this is exactly what they do in Chen Roll and Ross (1986). However, one important question is raised at this point: What are the origins of the statistical, possibly non-causal, relationship between the true risk factors and macroeconomic variables? If the risk factors causally affected the macroeconomic variables, then we have a case in which the empirical version of $SFM$ employs symptomatic factors (the macroeconomic variables) rather than the true causal ones. However, this is fatal to any attempts to use $SFM$ in order to "explain" facts such as "returns of stock A at time $t$ was $x\%$". Indeed, explaining such facts by appealing to symptomatic factors is equivalent to explaining the storm by appealing to the drop in the barometer rather than to a fall in the atmospheric pressure. If the risk factors do not affect the macro variables then where does their statistical relationship come from? Are there any other set of variables (common causes) that jointly affect the risk factors and the macro variables? If yes, then this set of variables should play the role of the true risk factors in $MFM$ since their presence in $MFM$ would screen off the original set of "factors".

The preceding discussion suggests that our aim of deriving $SFM - AR$ (or at least $SFM$) from theoretical principles cannot be achieved in the context of APT. This means that alternative theories must be examined. Next section deals with this issue.

## 4.1 Capital Asset Pricing Model

One prime alternative candidate to APT is the Capital Asset Pricing Model (CAPM) of Sharpe (1964) and Litner (1965). Indeed, Roll and Ross admit that "elegant derivations of the CAPM equation have been concocted beginning from the first principles of utility theory" (1980, pp 1074). Although the theoretical details of CAPM are in general well known, a brief but careful outline of some aspects of this model are necessary in order to assess whether CAPM can be thought of as providing the missing explanatory web behind $SFM - AR$. To this end, we must distinguish between two alternative versions of CAPM, hereafter referred to as CAPM-D and CAPM-U. The first version produces the well-known CAPM theoretical result by making some specific assumptions on the joint distribution of stock returns, with the most common one being that of Gaussianity. The second version does not make any direct assumptions on the stochastic process that generates returns but instead, it assumes that investors' utility functions are quadratic. Both CAPM-D and CAPM-U conclude that the risk premium $E(R_i)-R_0$ for any asset $i$ is linearly related to its "beta", $b_i$, with the latter being defined as $\frac{Cov(R_i,R_M)}{Var(R_M)}$. This feature is the connecting link, or as Nagel (1961) puts it the "condition of connectability", between the theoretical mechanism described by CAPM and the empirical law represented by $SFM$.

Careful consideration of the "causal chain" leading to $SFM$ reveals some aspects which threaten the apparent derivation of $SFM$ from solely theoretical first principles.

More specifically, let us consider the two alternative versions CAPM-D and CAPM-U under which SFM is derived. Under CAPM-D, the joint distribution of returns $R_i$ of all the existing assets, $i = 1, 2, ..., n$ *is* Gaussian and stationary over time. This means that in the derivation of the statistical law $SFM$ for stock returns another statistical law, namely that of joint normality of returns, has been assumed. This creates a circularity in the arguments producing $SFM$ similar to the circularity of the APT arguments. Put it differently, both APT and CAPM derive $SFM$ not by appealing solely to theoretical principles but to empirical facts as well. As Roll and Ross put it "In both instances, the return generating process is taken as one of the primitive assumptions of the theory" (1980, pp. 1077). The basic problem can be stated as follows: If the returns generating process is one of the primitive assumptions of the theory then how could it be possible for the theory to yield a returns generating process other than the one that has already been assumed? In other words, how is it possible to "derive" the returns generating process from a theory since it has already been assumed to be a building block of that theory? What is required, instead, is to have a theory whose possible structure is the following: Investors' ex ante probabilistic beliefs (plus preferences, endowments etc) are specified; based on these beliefs or any other assumption not referring explicitly to the returns generating process an asset pricing relationship is derived; this relationship dictates investors' actions; as a result of their actions, ex post returns are generated; these ex post returns define the distribution of relative frequencies; this is the objective probability distribution of stock returns. In this scheme, no primitive assumption on the statistical law governing the generation of stock returns has been made; instead this law is derived from a theoretical account of the mechanism at work.

In view of the above objections let us consider CAPM-U. This case does not make any explicit assumption on the "objective" process generating returns and hence, it appears to be less vulnerable than CAPM-D to the circularity argument discussed above. However, the assumption of quadratic utilities imposes a strong and rather unrealistic restrictions on investors preferences. Moreover, as analyzed above, the theory is still in need of the assumption that all investors agree on the mean $\mu(R_p)$ and standard deviation $\sigma(R_p)$ of all the available candidate portfolios $p$. The way by which such homogeneity is achieved may be a process of statistical learning. However, many authors believe that the assumption of homogeneity of beliefs is actually imposed without any realistic justification. As Ross (1978) puts it "Given that such homogeneity is going to be imposed eventually, it would seem natural to begin the CAPM story with restrictions on distributions, rather than preferences" (1978, pp. 888). In other words, it seems "more natural" to obtain the key assumption of CAPM, namely that each investor's expected utility is a function of only $\mu(R_p)$ and $\sigma(R_p)$, by imposing restrictions on the returns generating process itself rather than on investors' preferences. As Ross puts it: "A theory that obtains strong implications for equilibrium asset prices from restrictions on perceived distributions and permits heterogeneity in preferences is surely to be preferred to one which obtains similar market implications, but imposes restrictions on preferences along with strong similarity of beliefs" (1978, pp. 888). However, withstanding the aforementioned criticisms, CAPM-U comes closer to fill some gaps of the ideal explanatory text than any other of the theoretical models proposed so far.

More recent work attempts to add pieces of the ideal explanatory text by investigating the theoretical origins not only of $SFM$ but of $SFM - AR$ itself from an alternative perspective. For example, Berk, Green and Naik (1999) suggest a theoretical model which implies that a firm's systematic risk and expected returns change through time in a predictable way as a result of temporal variations in firm's growth and investment opportunities. More specifically, this model illustrates how the stochastic behavior of systematic risk is driven by firm's value maximizing choices, with the latter exhibiting some degree of persistence. In a similar vein, Avramov and Chordia (2006) attribute some of the well-known anomalies of the empirical literature, such as the size and book-to-market effects to the persistent behavior of betas (see also Petkova and Zhang, 2005, Ang and Chen, 2007, and Zhang, 2005). Although, these studies do not satisfy the impossible task of providing the full account of the causal mechanism at work that produces $SFM -$

$AR$, they enhance our understanding of the possible origins of the persistent variation in systematic risk. In doing this, they convey relevant information for the explanandum.

## 5 Conclusions

Back in 1974, Friedman puts the very idea of scientific understanding as follows: "I claim that this is the crucial property of scientific theories we are looking for; this is the essence of scientific explanation - science increases our understanding of the world by reducing the total number of independent phenomena that we have to accept as ultimate or given. A world with fewer independent phenomena is, other things equal, more comprehensible than with more" (1974, p. 15). In the present paper we demonstrated that the statistical regularity of the systematic risk changing over time in an autoregressive fashion may serve as a fundamental unifier of the most well established empirical regularities of stock returns. These regularities include some of the most well known ones, such as unconditional leptokurtosis, conditional heteroskedasticity, aggregational gaussianity and aggregational independence. Moreover, whether the act of unification alone can be thought of as sufficient for explanation of the well established empirical regularities of stock returns depends on the model of explanation that one is willing to adopt. In the context of the Hempelian Deductive Statistical model of explanation, the fact that $SFM - AR$ deductively implies all the empirical regularities of interest is sufficient for the explanation of those regularities.

On the other hand, in the context of Railton's Deductive - Nomological - Probabilisitc model, the mere subsumption of the aforementioned empirical regularities under $SFM - AR$ is not sufficient unless $SFM - AR$ is backed up with an account of the chance mechanism at work that produced $SFM$. To this end, the existing asset pricing theories fail to give a full account of the process that results in $SFM - AR$. However, a specific version of CAPM succeeds in conveying explanatory relevant information which weaves the web of the "ideal explanatory text".

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# Appendix

**Proof of Theorem 1:**

For the proof we use the relatively recent invariance principle of Peligrad and Utev (2005), stated below:

**Theorem PU (Invariance Principle of Peligrad and Utev (2005)):** *Let* $\{X_i\}_{i\in\mathbb{Z}}$ *be a stationary sequence with* $E[X_0]=0$ *and* $E[X_0^2]<\infty$*. Assume that*

$$\sum_{n=1}^{\infty}\frac{\|E[S_n|\mathcal{F}_0]\|_2}{n^{3/2}}<\infty\ . \tag{7}$$

*Then,* $\left\{\max_{1\le k\le n} S_k^2/n\right\}_{n\ge 1}$ *is uniformly integrable and* $n^{-1/2}W_n\xrightarrow{D}\sqrt{\eta}W$*, where* $\eta$ *is a non-negative random variable with finite mean* $E[\eta]=\sigma^2$ *and independent of* $\{W(t)\}_{t\ge 0}$*. Moreover,* $\eta$ *is determined by the limit* $\lim_{n\to\infty}\left(E\left[S_n^2|\mathcal{I}\right]/n\right)=\eta$ in $L_1$ *, where* $\mathcal{I}$ *is the invariant sigma field. In particular,* $\lim_{n\to\infty}\left(E\left[S_n^2\right]/n\right)=\sigma^2$.

Assumption **M** implies that

$$E\left[R_{t_0+i}-E\left[R_{t_0+i}\right]|\mathcal{F}_{t_0}\right]=0\text{ a.e.}$$

Therefore $\|E[S_{t_0,k}|\mathcal{F}_{t_0}]\|_2=0$ and condition (7) is trivially satisfied. By virtue of the existence of finite second moments for all random variables involved, we can apply Theorem PU. From the joint normality of $\beta_t$, $M_t$ and $u_t$, we have that $\{R_t\}_{t\in\mathbb{Z}}$ is ergodic, hence the invariant $\sigma$-field is trivial. Applying, now, (4) we obtain

$$\eta=\lim_{n\to\infty}\frac{E\left[S_k^2\right]}{k}=\lim_{k\to\infty}\frac{Var\left(R_\tau(k)\right)}{k}=\beta^2\sigma_m^2+\sigma_u^2+\sigma_\beta^2\sigma_m^2\in\mathbb{R}.$$