Periodically homogeneous Markov chains: The discrete state space case

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Abstract

A unified theory of periodically homogeneous Markov chains on countable state spaces with periodically time-varying transition probabilities is introduced. The finite-dimensional probability distributions of these time-periodic chains are first studied and their correspondence with the marginal distributions and transition probabilities is shown. Then, the concepts of periodic stability/regularity and limiting behaviors are proposed. The communicability and classification of states necessary for establishing periodic stability are then examined. In particular, periodic irreducibility and the main solidarity/class properties are presented, namely periodic recurrence, periodic positive recurrence, periodic transience, and periodic aperiodicity. Furthermore, sufficient conditions for periodic stochastic stability of time-periodic Markov chains are derived. Finally, various applications to some operations research models and time series analysis are considered. In particular, periodic Markov decision processes, periodic integer-valued time series models, and periodic Markov-switching time series models are examined.

Keywords Time-periodic Markov chains, Harris periodic ergodicity, periodic irreducibility, periodic recurrence, periodic stability, periodic invariant distributions, pe-

*An earlier version of this manuscript was completed in 2009.

riodic integer-valued time series models, Markov-switching periodic models, periodic Markov decision process.

1 Introduction

Most theories, models, methods, and applications concerning Markov chains are dedicated to the homogeneous case in which the transition probabilities are time-invariant. However, many random phenomena evolve in a non-homogeneous Markov way with rather time-varying transition probabilities. Notable examples can be found in applied probability (Seneta, 1980; Gray, 2001), time series analysis (Bittanti and Colaneri, 2009; Hurd and Miamee, 2007), and operations research (e.g. White, 1993).

A particularly important case of non-homogeneity appears when the Markov chain has periodic time-varying transition probabilities. We call this type of chains *periodically homogeneous* or *time-periodic* as opposed to the term "state-periodic" known for homogeneous Markov chains. Time-periodic Markov chains are the basis of many periodic statistical models (Franses, 1996; Ghysels and Osborn, 2001; Franses and Paap, 2004; Hipel and McLeod, 2005), namely Markov-switching periodic autoregressive models (Ghysels et *al*, 1998; Ghysels, 2000; Bac et *al*, 2001), periodic Markov decision processes (Carton, 1963; Riis, 1965; Veugen et *al*, 1983; Jacobson et *al*, 2003), and periodic integer-valued time series models (e.g. Monteiro et *al*, 2010; Aknouche et *al*, 2018) to name a few. Although a periodically homogeneous Markov chain is a special case of inhomogeneous Markov chains whose theory is well established (Seneta, 1980), it seems that a specific theory for the time-varying periodic case is needed. In fact, while there are many scattered results indirectly addressing some aspects of time-periodic Markov chains (e.g. Riis, 1965; Veugen et *al*, 1983; Jacobson et *al*, 2003; Bittanti and Colaneri, 2009), a general proper theory for time-periodic Markov chains which parallel that of homogeneous Markov chains seem to be missing.

The aim of this work is to implement a specific theory for finite/countable state-space Markov chains whose transition probabilities are time-periodic with period $S \geq 1$. The case

$S = 1$, degenerates into homogeneous Markov chains. Many elements of this theory for time-periodic chains are similar to those existing of homogeneous Markov chains, but some aspects that do not appear in the homogeneous case are worth showing. In particular, any time-periodic chain can be partitioned into S interdependent homogenous sub-chains each of which is relative to a channel (or season) representing the rest of division of any time by the period S. Thus, the well-known theory of homogeneous Markov chains can non-trivially be translated through these S homogeneous sub-chains to time-periodic Markov chains.

The rest of this paper is described as follows. Section 2 explores the finite-dimensional distributions of a periodically homogeneous Markov chain. The connection of these distributions with the transition probabilities, the marginal distributions, and a periodic version of Chapman-Kolmogorov identities are obtained. Section 3 introduces the main stability concepts for time-periodic Markov chains. In Section 4, periodic irreducibility and the main solidarity properties, namely periodic recurrence, periodic positive recurrence, periodic null recurrence, periodic transience, and periodic aperiodicity, are studied. Section 5 establishes the main periodic stability theorems. Finally, Section 6 provides some applications of the proposed theory to some famous probability models from operations research and time series analysis, namely periodic Markov decision processes, periodic integer-valued time series models, and periodic Markov regime-witching ARMA (autoregressive moving average), GARCH (generalized autoregressive conditional heteroskedastic), and positive conditional mean models.

2 Distributions of time-periodic Markov chains: transition probabilities and marginal distributions

Denote by $\mathbb{N} = \{0, 1, ...\}$ and $\mathbb{N}^* = \{1, 2, ...\}$ the sets of nonnegative integers and positive integers, respectively.

Definition 2.1 *A stochastic process* $(X_t, t \in \mathbb{N})$ *defined on a probability space* $(\Omega, \mathcal{F}, P)$ *and valued in a finite/countable state-space* $E = \{1, ..., K\}$ *(*K *can be infinite) is called a* *periodically homogeneous (or time-periodic) Markov chain of period $S \in \mathbb{N}^*$ if:*

i) $(X_t, t \in \mathbb{N})$ is a Markov chain, i.e. for every $t \in \mathbb{N}$ and every $i, j, i_j \in E$ $(0 \leq j \leq t-1)$

$$P(X_{t+1} = j | X_t = i, X_{t-1} = i_{t-1}, ..., X_0 = i_0) = P(X_{t+1} = j | X_t = i) := P_{ij}(t). \tag{2.1}$$

ii) The so-called transition probability, $P_{ij}(t)$, is periodic over t with period S, i.e. for all $t \in \mathbb{N}$

$$P_{ij}(t) = P_{ij}(t+S), \tag{2.2}$$

where S is the smallest positive integer satisfying (2.2).

To highlight the S-periodic homogeneity of the transition probabilities $P_{ij}(t)$, it is possible to write any integer $t \in \mathbb{N}$ according to its Euclidean division on S ($t = nS + v$, with $n \in \mathbb{N}$ and $0 \leq v \leq S-1$) and by the S-periodicity of $P_{ij}(t)$,

$$\begin{aligned} P_{ij}(v) &= P(X_{v+1} = j | X_v = i) \qquad (2.3) \\ &= P(X_{nS+v+1} = j | X_{nS+v} = i), \quad n \in \mathbb{N}, \quad 0 \leq v \leq S-1. \end{aligned}$$

Thus, $P_{ij}(v)$ represents the probability of transition from a state i at a time multiple of S modulo v, to a state j at the next time. It is therefore a one-step transition probability starting from a time whose division remainder on S is equal to v. In other words, $P_{ij}(v)$ is the (one-step) transition probability from i to j along the channel v, where by channel v $(0 \leq v \leq S-1)$, it is meant the set of numbers $\{v, S+v, 2S+v, 3S+v, ...\}$ whose rest of division on S is equal to v. For $S = 1$, the chain given by (2.1)-(2.2) is simply homogeneous where v is omitted in $P_{ij}(v)$, so that $P_{ij}(v) = P_{ij}$ for each v.

It is also possible to represent these (one-step) transition probabilities matrixly via S matrices $P(v) = (P(v))_{ij}$ $(0 \leq v \leq S-1)$ with

$$(P(v))_{ij} = P_{ij}(v).$$

These matrices are stochastic in the sense that their elements are nonnegative and sum to unity over each row. It happens that the state space $E = \bigcup_{t \in \mathbb{N}} E_t$ is such that E_t varies periodically over time. Thus, we can consider two cases of non-homogeneity:

- *Non-homogeneity with regard to probabilities*: The state space $E = E_t$ (for all $t \in \mathbb{N}$) is time-invariant and only the transition probabilities which vary periodically in time.

- *Non-homogeneity with regard to the state-space*: Not only the transition probabilities are periodically time-varying but the state-space E_t is S-periodic in t, meaning that $E_t = E_{t+S}$ for all nonnegative integer t. So, in this case, the chain has a system of S state spaces $(E_v)_{0\le v\le S-1}$ each of which is relative to a channel v and whose union is denoted by $E = \bigcup_{v=0}^{S-1} E_v$. This can lead to rectangular (non-square) one-step transition matrices. The following example shows this situation.

Example 2.1 i) Assume the evolution of the states of volatility (high (H), calm (C)) of a certain financial asset each half-day can be represented through a periodically homogeneous Markov chain with period $S = 2$. The data is twice daily and the observation of the volatility started on the morning of a certain day the channel of which is $v = 0$. The transition matrices are assumed to be

$$P(0) = \begin{array}{c} \\ H \\ C \end{array} \begin{array}{c} \begin{array}{cc} H & C \end{array} \\ \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix} \end{array} \qquad P(1) = \begin{array}{c} \\ H \\ C \end{array} \begin{array}{c} \begin{array}{cc} H & C \end{array} \\ \begin{pmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{pmatrix} \end{array}.$$

For instance, the transition probability from state C to state H, starting in the morning is $0.3 = P_{C,H}(0)$ whereas the transition probability for the same states from the afternoon is $0.1 = P_{C,H}(1)$.

ii) After many years, a weather service has judged that the weather next season only depends on the current season. The possible states from one season to another vary dependently on the current season. In Winter, the possible states are: Cold (F) and Very Cold (TF) so $E_0 = \{F, TF\}$. In Spring, the possible states are: Cold (F), Medium (M) and Hot (C) with $E_1 = \{F, M, C\}$. In Summer, the states are: Hot (C) and Very Hot (TC), and therefore $E_2 = \{C, TC\}$, while in Autumn, the possible states are the same as in spring, $E_1 = \{F, M, C\}$. The data are quarterly and the period S is equal to 4. It is assumed that the observation of the phenomenon began from a certain Winter for which the channel takes the value $v = 0$. The transition probabilities were established for each season.

$$P(0) = \begin{array}{c} \\ F \\ TF \end{array} \begin{array}{c} \begin{array}{ccc} F & M & C \end{array} \\ \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.7 & 0.25 & 0.05 \end{pmatrix} \end{array} \qquad P(1) = \begin{array}{c} \\ F \\ M \\ C \end{array} \begin{array}{c} \begin{array}{cc} C & TC \end{array} \\ \begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \\ 0.3 & 0.7 \end{pmatrix} \end{array}$$

$$P(2) = \begin{array}{c} \\ C \\ TC \end{array} \begin{array}{c} \begin{array}{ccc} F & M & C \end{array} \\ \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.3 & 0.3 \end{pmatrix} \end{array} \qquad P(3) = \begin{array}{c} \\ F \\ M \\ C \end{array} \begin{array}{c} \begin{array}{cc} F & TF \end{array} \\ \begin{pmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix} \end{array}.$$

□

Example i) is a case of a non-homogeneity with respect to the transition probabilities. For each channel, the states are the same. Example ii) shows, however, that the state-space itself can vary. This is the case of non-homogeneity with respect to the state-space.

As for the homogeneous case, the so-called weak Markov property (2.1) easily extends as follows

$$P(X_{t+1} = j_{t+1}, X_{t+2} = j_{t+2}, ..., X_{t+m} = j_{t+m} | X_t = i_t, X_{t-1} = i_{t-1}, ..., X_0 = i_0) = \\ P(X_{t+1} = j_{t+1}, X_{t+2} = j_{t+2}, ..., X_{t+m} = j_{t+m} | X_t = i_t)$$

for each $i_t, j_t \in E$. More generally,

$$P(X_{t+1} = j_{t+1}, X_{t+2} = j_{t+2}, ..., | X_t = i_t, X_{t-1} = i_{t-1}, ..., X_0 = i_0) = \\ P(X_{t+1} = j_{t+1}, X_{t+2} = j_{t+2}, ... | X_t = i_t),$$

roughly meaning that "*the future is conditionally independent of the past knowing the present*".

2.1 Finite-dimensional distributions

By the Markov property, the probability structure of the chain $(X_t, t \in \mathbb{N})$ which is represented by the finite-dimensional distributions

$$\{P_{t_1,t_2,...,t_n}(i_1, i_2, ..., i_n), \quad n \in \mathbb{N}^*, t_j \in \mathbb{N}, i_j \in E, \ 1 \leq j \leq n\},$$

where $P_{t_1,t_2,...,t_n}(i_1, i_2, ..., i_n) := P(X_{t_1} = i_1, X_{t_2} = i_2, ..., X_{t_n} = i_n)$, can be expressed in a much simpler way.

Proposition 2.1 *The finite-dimensional distributions*

$$\{P_{t_1,t_2,...,t_n}(i_1, i_2, ..., i_n),\ n \in \mathbb{N}^*,\ t_j \in \mathbb{N},\ i_j \in E,\ 1 \leq j \leq n\},$$

are entirely determined by:

i) The initial marginal distribution $\pi_j(0) = P(X_0 = j)$, *and*

ii) the S *one-step transition probability matrices* $P(v)$, $(0 \leq v \leq S-1)$.

Proof First, it is clear that the finite-dimensional distributions

$$\{P_{t_1,t_2,...,t_n}(i_1, i_2, ..., i_n),\quad n \in \mathbb{N}^*, t_j \in \mathbb{N}, i_j \in E, 1 \leq j \leq n\}$$

are entirely determined by the probabilities

$$\{P_{0,1,...,n}(i_0, i_1, ..., i_n),\quad n \in \mathbb{N}^*, i_j \in E, 0 \leq j \leq n\}$$

and vice versa. Second,

$$\begin{aligned} & P_{0,1,...,n}(i_0, i_1, ..., i_n) := P(X_0 = i_0, X_1 = i_1, ..., X_n = i_n) \\ & \quad = P(X_0 = i_0) P(X_1 = i_1 | X_0 = i_0) \cdots P(X_n = i_n | X_{n-1} = i_{n-1}, ..., X_0 = i_0) \\ & \quad = P(X_0 = i_0) P(X_1 = i_1 | X_0 = i_0) \cdots P(X_n = i_n | X_{n-1} = i_{n-1}) \\ & \quad = \pi_{i_0}(0) P_{i_0 i_1}(0) P_{i_1 i_2}(1) \cdots P_{i_{n-1} i_n}(n-1), \end{aligned}$$

establishing the result. □

2.2 n-step transition probabilities

Similarly to the homogeneous case, it is also possible to define the n-step transition probability, $P_{ij}^{(n)}(v)$, from a state i (at a time v modulo S) to a state j after n steps:

$$\begin{aligned} P_{ij}^{(n)}(v) &= P(X_{v+n} = j | X_v = i) \\ P_{ij}^{(1)}(v) &= P_{ij}(v) \\ P_{ij}^{(0)}(v) &= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

By induction on $n \in \mathbb{N}$, it is easily seen that this transition probability, being a function of v, is periodic with period S, i.e.

$$P_{ij}^{(n)}(kS+v) = P_{ij}^{(n)}(v), \text{ for all } k \in \mathbb{N}.$$

2.3 Non-homogeneous Chapman-Kolmogorov equations

The Chapman-Kolmogorov equations, known for homogeneous Markov chains, can be easily adapted to the non-homogeneous case and, in particular, to the periodically homogeneous case.

Proposition 2.2 *(Non-homogeneous Chapman-Kolmogorov equations)*
For every $i, j \in E$, $n, m \in \mathbb{N}$, and $v \in \{0, ..., S-1\}$,

$$\begin{aligned} P_{ij}^{(n+m)}(v) &= \sum_{k \in E} P_{ik}^{(n)}(v) P_{kj}^{(m)}(v+n) && (2.4a) \\ &= \sum_{k \in E} P_{ik}^{(m)}(v) P_{kj}^{(n)}(v+m). && (2.4b) \end{aligned}$$

Proof Just write

$$\begin{aligned} P_{ij}^{(n+m)}(v) &= P(X_{v+n+m} = j | X_v = i) \\ &= \sum_{k \in E} P(X_{v+n+m} = j, X_{v+n} = k | X_v = i) \\ &= \sum_{k \in E} P(X_{v+n+m} = j, X_{v+m} = k | X_v = i). \end{aligned}$$

□

Remark 2.1 *i) An important case in which the periodicity can be exploited appears when n or m are multiples of S. Equations (2.4) are simplified as follows*

$$P_{ij}^{(nS+m)}(v) = \sum_{k \in E} P_{ik}^{(nS)}(v+m) P_{kj}^{(m)}(v).$$

ii) For $m = 1$ and $n = 1$,

$$P_{ij}^{(2)}(v) = \sum_{k \in E} P_{ik}(v) P_{kj}(v+1).$$

In matrix form, this writes as follows

$$P^{(2)}(v) = P(v)\, P(v+1)\,.$$

Likewise

$$P^{(3)}(v) = P(v)\, P(v+1)\, P(v+2)\,,$$

and so on, for arbitrary m,

$$P^{(m)}(v) = P(v)\, P(v+1)\cdots P(v+m-1)\,.$$

iii) When m *is a multiple of* S *(say* $m = nS$*), the* S*-periodicity of the transition matrices yields*

$$P^{(nS)}(v) = (P(v)\, P(v+1)\cdots P(v+S-1))^n$$

and more generally, for $l \in \{1, ..., S-1\}$,

$$P^{(nS+l)}(v) = (P(v)\, P(v+1)\cdots P(v+S-1))^n\, P(v)\, P(v+1)\cdots P(v+l-1)\,. \quad (2.5)$$

iv) The matrix $\mathbb{P}_v = \prod_{k=0}^{S-1} P(v+k)$ *plays an important role in the theory of periodically homogeneous Markov chains and can be considered as the analog of the transition matrix for homogeneous Markov chains. It is often called the "monodromy" matrix and its elements are the* S*-step transition probabilities starting from times that are multiples of* S *modulo* v.

Example 2.1 (Continued) i) The monodromy matrices are

$$\begin{aligned}\mathbb{P}(0) &= \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}\begin{pmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{pmatrix} = \begin{pmatrix} 0.52 & 0.48 \\ 0.31 & 0.69 \end{pmatrix} \\ \mathbb{P}(1) &= \begin{pmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{pmatrix}\begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix} = \begin{pmatrix} 0.54 & 0.46 \\ 0.33 & 0.67 \end{pmatrix}.\end{aligned}$$

ii) The corresponding monodromy matrices are

$$\begin{aligned}
\mathbb{P}(0) &= \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.7 & 0.25 & 0.05 \end{pmatrix} \begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \\ 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.3 & 0.3 \end{pmatrix} \begin{pmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix} \\
&= \begin{pmatrix} 0.3556 & 0.6444 \\ 0.352 & 0.648 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(1) &= \begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \\ 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.3 & 0.3 \end{pmatrix} \begin{pmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix} \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.7 & 0.25 & 0.05 \end{pmatrix} \\
&= \begin{pmatrix} 0.6304 & 0.2674 & 0.1022 \\ 0.628 & 0.268 & 0.104 \\ 0.6264 & 0.2684 & 0.1052 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(2) &= \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.3 & 0.3 \end{pmatrix} \begin{pmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix} \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.7 & 0.25 & 0.05 \end{pmatrix} \begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \\ 0.3 & 0.7 \end{pmatrix} \\
&= \begin{pmatrix} 0.6694 & 0.3306 \\ 0.6658 & 0.3342 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(3) &= \begin{pmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix} \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.7 & 0.25 & 0.05 \end{pmatrix} \begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \\ 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.3 & 0.3 \end{pmatrix} \\
&= \begin{pmatrix} 0.4682 & 0.3 & 0.2318 \\ 0.4664 & 0.3 & 0.2336 \\ 0.4646 & 0.3 & 0.2354 \end{pmatrix}.
\end{aligned}$$

2.4 Marginal distributions

We saw above that the initial distribution $\pi_j(0) = P(X_0 = j)$ and the transition probabilities $P_{ij}(v)$ characterize all finite-dimensional distributions of the time-periodic chain and, in particular, all marginal distributions

$$\pi(nS+v) := (\pi_1(nS+v), \pi_2(nS+v), ..., \pi_K(nS+v)), \ \ 0 \leq v \leq S-1, \, n \in \mathbb{N},$$

where

$$\pi_j(nS+v) = P(X_{nS+v} = j), \quad 1 \leq j \leq K.$$

These marginal probabilities can be written in terms of the transition probabilities as follows

$$\begin{aligned} \pi_j(nS+v) &= \sum_{k \in E} P(X_{nS+v} = j, X_v = k) \\ &= \sum_{k \in E} \pi_k(v) P_{kj}^{(nS)}(v). \end{aligned} \tag{2.6}$$

In matrix form, (2.6) reduces to

$$\pi(nS+v) = \pi(v) P^{(nS)}(v), \qquad 0 \leq v \leq S-1,$$

and using (2.4), this leads to

$$\pi(nS+v) = \pi(v) (\mathbb{P}(v))^n, \qquad 0 \leq v \leq S-1. \tag{2.7}$$

From (2.7), the proof of the following result is obvious.

Proposition 2.3 *The marginal distributions* $(\pi(t))_{t \in \mathbb{N}}$ *are entirely determined by the* S *initial distributions* $(\pi(v))_{0 \leq v \leq S-1}$ *and the monodromy matrices* $(\mathbb{P}(v))_{0 \leq v \leq S-1}$.

It is worth noting that the S initial distributions $(\pi(v))_{0 \leq v \leq S-1}$ are themselves entirely determined by the initial distribution $\pi(0)$ and the transition matrices $(P(v))_{0 \leq v \leq S-1}$, hence Proposition 2.1.

Remark 2.2 *For ranks not necessarily having the same division remainder by* S, *(2.7) extends to*

$$\pi(nS+v) = \pi(r) P^{(nS+v-r)}(v), \qquad 0 \leq v, r \leq S-1.$$

In particular,

$$\pi\left(v+1\right)=\pi\left(0\right)P\left(0\right)P\left(1\right)\cdots P\left(v\right),\qquad 0\leq v\leq S-1$$

or also

$$\pi_j(v)=\sum_{i_0\in E}\cdots\sum_{i_{v-1}\in E}\pi_{i_0}(0)P_{i_0i_1}(0)P_{i_1i_2}(1)\cdots P_{i_{v-1},j}(v-1),\ j\in E. \tag{2.8}$$

2.5 Connection with homogeneous Markov chains

2.5.1 Dimensionality augmentation approach (connection with S-variate homogeneous Markov chains)

It is well known that any dynamical system with S-periodic coefficients (which can be a differential or difference equation with periodic coefficients, a *periodically stationary process*, etc.) can be cast in a S-variate system with constant (matrix) coefficients by means of an appropriate transformation (S-variate time-invariant differential or difference equations, S-variate stationary processes, etc.). This transformation most often consists of stacking the members of the initial time-periodic process into successive vectors each of which is associated with a quotient of the integer division by S. This dimensionality augmentation technique has been known since Gladyshev (1961) who studied a *periodically correlated* process through a *multivariate covariance stationary* process whose covariance structure is a function of that of the original time-periodic process (see also Meyer and Burris, 1975; Pagano, 1978; Aknouche, 2007-2015). A similar but quite different approach is that proposed by Veugen et *al* (1983) for periodic Markov decision processes. Although in what follows, we link a S-periodically homogeneous Markov chain and its corresponding homogeneous S-variate chain, we prefer, like Floquet (1883), to develop a theory specific to periodically homogeneous Markov chains rather than going through S-variate homogeneous Markov chains. The reasons are:

i) The probability transition matrix of the augmented homogeneous chain has a more complex structure as will be seen below.

ii) The results obtained for the dimensionality-augmented homogeneous matrix are not easily interpretable for the original periodically homogeneous Markov chain.

iii) In the case of countably infinite state spaces, the manipulation of infinite matrices is formidable.

iv) The complexity is even more pronounced in the case of augmented homogeneous transition kernels corresponding to uncountable state-space Markov chains.

Assuming that $(X_n, n \in \mathbb{N})$ is periodically homogeneous with a finite state-space E, define on $(\Omega, \mathcal{F}, P)$ the augmented process $(\underline{X}_n, n \in \mathbb{N})$ valued in E^S as follows

$$\underline{X}_n = (X_{nS+S-1}, X_{nS+S-2}, ..., X_{nS+1}, X_{nS})'. \tag{2.9}$$

Proposition 2.4 *The process $(\underline{X}_n, n \in \mathbb{N})$ defined by (2.9) is a homogeneous Markov chain with a multivariate initial distribution $\underline{\pi}_0$ and a transition probability $\underline{P} = \left(\underline{P}_{\underline{i},\underline{j}}\right)_{\underline{i},\underline{j} \in E^S}$ both given in terms of $(P(v))_{0 \le v \le S-1}$, where*

$$\begin{aligned} \underline{\pi}_0\left(\underline{j}\right) &= \left(\pi_{j_1}(0), \pi_{j_2}(1), ..., \pi_{j_S}(S-1)\right)' & (2.10a) \\ \underline{P}_{\underline{i},\underline{j}} &= P_{i_S j_1}(S-1) P_{j_1 j_2}(0) P_{j_2 j_3}(1) \cdots P_{j_{S-2} j_{S-1}}(S-3) P_{j_{S-1} j_S}(S-2), & (2.10b) \end{aligned}$$

$\underline{i} = (i_1, i_2, ..., i_S)'$, *and* $\underline{j} = (j_1, j_2, ..., j_S)' \in E^S$.

Proof By the Markov property of the chain $(X_n, n \in \mathbb{N})$, it can be seen that $(\underline{X}_n, n \in \mathbb{N})$ is also a Markov chain on E^S, whose transition matrix is given by

$$\begin{aligned} \underline{P}_{\underline{i},\underline{j}} &= P\left(\underline{X}_{n+1} = \underline{j} / \underline{X}_n = \underline{i}\right) \\ &= P\left(X_{(n+1)S+S-1} = j_S, ..., X_{(n+1)S} = j_1 | X_{nS+S-1} = i_S, ..., X_{nS} = i_1\right) \\ &= P_{i_S j_1}(nS+S-1) P_{j_1 j_2}(nS+S) P_{j_2 j_3}(nS+S+1) \cdots \\ &\quad \times P_{j_{S-2} j_{S-1}}(nS+S-3) P_{j_{S-1} j_S}(nS+S-2), \end{aligned}$$

where by the S-periodicity of $(P(v))_{0 \le v \le S-1}$, we find (2.10b). The correspondence (2.10a) is trivial. $\square$

Consider the following example.

Example 2.2 i) Let the chain $(X_n, n \in \mathbb{N})$ be 2-periodically homogeneous and valued in $E = \{1, 2\}$ with a system of transition probabilities

$$P(0) = P(2) = \begin{pmatrix} P_{11}(2) & P_{12}(2) \\ P_{21}(2) & P_{22}(2) \end{pmatrix} \text{ and } P(1) = \begin{pmatrix} P_{11}(1) & P_{12}(1) \\ P_{21}(1) & P_{22}(1) \end{pmatrix}.$$

The corresponding 2-variate homogeneous Markov chain $(\underline{X}_n, n \in \mathbb{N})$ is by Proposition 2.4 defined on

$$E^2 = \{(1,1), (1,2), (2,1), (2,2)\}$$

with a transition probability given from $(2.10b)$ by

$$\underline{P} = \begin{array}{c} \\ (1,1) \\ (1,2) \\ (2,1) \\ (2,2) \end{array} \begin{array}{c} \begin{array}{cccc} (1,1) & (1,2) & (2,1) & (2,2) \end{array} \\ \begin{pmatrix} P_{11}(1)\,P_{11}(2) & P_{12}(1)\,P_{21}(2) & P_{11}(1)\,P_{12}(2) & P_{12}(1)\,P_{22}(2) \\ P_{21}(1)\,P_{11}(2) & P_{22}(1)\,P_{21}(2) & P_{21}(1)\,P_{12}(2) & P_{22}(1)\,P_{22}(2) \\ P_{11}(1)\,P_{11}(2) & P_{12}(1)\,P_{21}(2) & P_{11}(1)\,P_{12}(2) & P_{12}(1)\,P_{22}(2) \\ P_{21}(1)\,P_{11}(2) & P_{22}(1)\,P_{21}(2) & P_{21}(1)\,P_{12}(2) & P_{22}(1)\,P_{22}(2) \end{pmatrix} \end{array}.$$

ii) Now consider a 3-periodically homogeneous Markov chain $(X_n, n \in \mathbb{N})$ valued in $E = \{1, 2\}$ with a system of transition probabilities

$$\begin{aligned} P(1) &= \begin{pmatrix} P_{11}(1) & P_{12}(1) \\ P_{21}(1) & P_{22}(1) \end{pmatrix}, P(2) = \begin{pmatrix} P_{11}(2) & P_{12}(2) \\ P_{21}(2) & P_{22}(2) \end{pmatrix}, \\ P(3) &= P(0) = \begin{pmatrix} P_{11}(3) & P_{12}(3) \\ P_{21}(3) & P_{22}(3) \end{pmatrix}. \end{aligned}$$

The corresponding 3-variate homogeneous Markov chain $(\underline{X}_n, n \in \mathbb{N})$ is defined on $E^3 = \{(1,1,1), (1,1,2), (1,2,1), (1,2,2), (2,1,1), (2,1,2), (2,2,1), (2,2,2)\}$ with transition probabilities

$$\underline{P} = \left(\begin{array}{cccc}
P_{11}(2)P_{11}(0)P_{11}(1) & P_{11}(2)P_{11}(0)P_{12}(1) & P_{11}(2)P_{12}(0)P_{21}(1) & P_{11}(2)P_{12}(0)P_{22}(1) \\
P_{21}(2)P_{11}(0)P_{11}(1) & P_{21}(2)P_{11}(0)P_{12}(1) & P_{21}(2)P_{12}(0)P_{21}(1) & P_{21}(2)P_{12}(0)P_{22}(1) \\
P_{11}(2)P_{11}(0)P_{11}(1) & P_{11}(2)P_{11}(0)P_{12}(1) & P_{11}(2)P_{12}(0)P_{21}(1) & P_{11}(2)P_{12}(0)P_{22}(1) \\
P_{21}(2)P_{11}(0)P_{11}(1) & P_{21}(2)P_{11}(0)P_{12}(1) & P_{21}(2)P_{12}(0)P_{21}(1) & P_{21}(2)P_{12}(0)P_{22}(1) \\
P_{11}(2)P_{11}(0)P_{11}(1) & P_{11}(2)P_{11}(0)P_{12}(1) & P_{11}(2)P_{12}(0)P_{21}(1) & P_{11}(2)P_{12}(0)P_{22}(1) \\
P_{21}(2)P_{11}(0)P_{11}(1) & P_{21}(2)P_{11}(0)P_{12}(1) & P_{21}(2)P_{12}(0)P_{21}(1) & P_{21}(2)P_{12}(0)P_{22}(1) \\
P_{11}(2)P_{11}(0)P_{11}(1) & P_{11}(2)P_{11}(0)P_{12}(1) & P_{11}(2)P_{12}(0)P_{21}(1) & P_{11}(2)P_{12}(0)P_{22}(1) \\
P_{21}(2)P_{11}(0)P_{11}(1) & P_{21}(2)P_{11}(0)P_{12}(1) & P_{21}(2)P_{12}(0)P_{21}(1) & P_{21}(2)P_{12}(0)P_{22}(1)
\end{array} \right.$$

$$\left. \begin{array}{cccc}
P_{12}(2)P_{21}(0)P_{11}(1) & P_{12}(2)P_{21}(0)P_{12}(1) & P_{12}(2)P_{22}(0)P_{21}(1) & P_{12}(2)P_{22}(0)P_{22}(1) \\
P_{22}(2)P_{21}(0)P_{11}(1) & P_{22}(2)P_{21}(0)P_{12}(1) & P_{22}(2)P_{22}(0)P_{21}(1) & P_{22}(2)P_{22}(0)P_{22}(1) \\
P_{12}(2)P_{21}(0)P_{11}(1) & P_{12}(2)P_{21}(0)P_{12}(1) & P_{12}(2)P_{22}(0)P_{21}(1) & P_{12}(2)P_{22}(0)P_{22}(1) \\
P_{22}(2)P_{21}(0)P_{11}(1) & P_{22}(2)P_{21}(0)P_{12}(1) & P_{22}(2)P_{22}(0)P_{21}(1) & P_{22}(2)P_{22}(0)P_{22}(1) \\
P_{12}(2)P_{21}(0)P_{11}(1) & P_{12}(2)P_{21}(0)P_{12}(1) & P_{12}(2)P_{22}(0)P_{21}(1) & P_{12}(2)P_{22}(0)P_{22}(1) \\
P_{22}(2)P_{21}(0)P_{11}(1) & P_{22}(2)P_{21}(0)P_{12}(1) & P_{22}(2)P_{22}(0)P_{21}(1) & P_{22}(2)P_{22}(0)P_{22}(1) \\
P_{12}(2)P_{21}(0)P_{11}(1) & P_{12}(2)P_{21}(0)P_{12}(1) & P_{12}(2)P_{22}(0)P_{21}(1) & P_{12}(2)P_{22}(0)P_{22}(1) \\
P_{22}(2)P_{21}(0)P_{11}(1) & P_{22}(2)P_{21}(0)P_{12}(1) & P_{22}(2)P_{22}(0)P_{21}(1) & P_{22}(2)P_{22}(0)P_{22}(1)
\end{array} \right).$$

iii) We finally consider a 2-periodically homogeneous Markov chain $(X_n, n \in \mathbb{N})$ valued in $E = \{1, 2, 3\}$ with a transition probability system

$$P(1) = \begin{pmatrix} P_{11}(1) & P_{12}(1) & P_{13}(1) \\ P_{21}(1) & P_{22}(1) & P_{23}(1) \\ P_{31}(1) & P_{32}(1) & P_{23}(1) \end{pmatrix}, \; P(2) = \begin{pmatrix} P_{11}(2) & P_{12}(2) & P_{13}(2) \\ P_{21}(2) & P_{22}(2) & P_{23}(2) \\ P_{31}(2) & P_{32}(2) & P_{23}(2) \end{pmatrix}.$$

The corresponding homogeneous 2-variate Markov chain $(\underline{X}_n, n \in \mathbb{N})$ is then defined over $E^2 = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$ with a transition probability matrix

$$\underline{P} = \left(\begin{array}{llll} P_{11}(1)\,P_{11}(2) & P_{11}(1)\,P_{12}(2) & P_{11}(1)\,P_{13}(2) & P_{12}(1)\,P_{21}(2) \\ P_{21}(1)\,P_{11}(2) & P_{21}(1)\,P_{12}(2) & P_{21}(1)\,P_{13}(2) & P_{22}(1)\,P_{21}(2) \\ P_{31}(1)\,P_{11}(2) & P_{31}(1)\,P_{12}(2) & P_{31}(1)\,P_{13}(2) & P_{32}(1)\,P_{21}(2) \\ P_{11}(1)\,P_{11}(2) & P_{11}(1)\,P_{12}(2) & P_{11}(1)\,P_{13}(2) & P_{12}(1)\,P_{21}(2) \\ P_{21}(1)\,P_{11}(2) & P_{21}(1)\,P_{12}(2) & P_{21}(1)\,P_{13}(2) & P_{22}(1)\,P_{21}(2) \\ P_{31}(1)\,P_{11}(2) & P_{31}(1)\,P_{12}(2) & P_{31}(1)\,P_{13}(2) & P_{32}(1)\,P_{21}(2) \\ P_{11}(1)\,P_{11}(2) & P_{11}(1)\,P_{12}(2) & P_{11}(1)\,P_{13}(2) & P_{12}(1)\,P_{21}(2) \\ P_{21}(1)\,P_{11}(2) & P_{21}(1)\,P_{12}(2) & P_{21}(1)\,P_{13}(2) & P_{22}(1)\,P_{21}(2) \\ P_{31}(1)\,P_{11}(2) & P_{31}(1)\,P_{12}(2) & P_{31}(1)\,P_{13}(2) & P_{32}(1)\,P_{21}(2) \end{array} \right.$$

$$\left. \begin{array}{lllll} P_{12}(1)\,P_{22}(2) & P_{12}(1)\,P_{23}(2) & P_{13}(1)\,P_{31}(2) & P_{13}(1)\,P_{32}(2) & P_{13}(1)\,P_{33}(2) \\ P_{22}(1)\,P_{22}(2) & P_{22}(1)\,P_{23}(2) & P_{23}(1)\,P_{31}(2) & P_{23}(1)\,P_{32}(2) & P_{23}(1)\,P_{33}(2) \\ P_{32}(1)\,P_{22}(2) & P_{32}(1)\,P_{23}(2) & P_{33}(1)\,P_{31}(2) & P_{33}(1)\,P_{32}(2) & P_{33}(1)\,P_{33}(2) \\ P_{12}(1)\,P_{22}(2) & P_{12}(1)\,P_{23}(2) & P_{13}(1)\,P_{31}(2) & P_{13}(1)\,P_{32}(2) & P_{13}(1)\,P_{33}(2) \\ P_{22}(1)\,P_{22}(2) & P_{22}(1)\,P_{23}(2) & P_{23}(1)\,P_{31}(2) & P_{23}(1)\,P_{32}(2) & P_{23}(1)\,P_{33}(2) \\ P_{32}(1)\,P_{22}(2) & P_{32}(1)\,P_{23}(2) & P_{33}(1)\,P_{31}(2) & P_{33}(1)\,P_{32}(2) & P_{33}(1)\,P_{33}(2) \\ P_{12}(1)\,P_{22}(2) & P_{12}(1)\,P_{23}(2) & P_{13}(1)\,P_{31}(2) & P_{13}(1)\,P_{32}(2) & P_{13}(1)\,P_{33}(2) \\ P_{22}(1)\,P_{22}(2) & P_{22}(1)\,P_{23}(2) & P_{23}(1)\,P_{31}(2) & P_{23}(1)\,P_{32}(2) & P_{23}(1)\,P_{33}(2) \\ P_{32}(1)\,P_{22}(2) & P_{32}(1)\,P_{23}(2) & P_{33}(1)\,P_{31}(2) & P_{33}(1)\,P_{32}(2) & P_{33}(1)\,P_{33}(2) \end{array} \right).$$

Remark 2.3 *i) In Example 2.2 i), the elements of the hyper-matrix* $\underline{P}$ *are the terms of the elements (which are sums of terms) of the monodromy matrix* $\mathbb{P}(1) = P(1)\,P(2)$ *rearranged according to a certain order, and are also the terms of the matrix* $\mathbb{P}(2) = P(2)\,P(1)$ *according to another order.*

ii) The matrix $\underline{P}$ *consists of a block of* S *rows and* K^S *columns repeated* K^{S-1} *times.*

iii) We conjecture that when $E_t = E$ *for all* t*, the hyper-matrix* $\underline{P}$ *and the monodromy matrix* $\mathbb{P}(v)$ *for a certain* v *have the same eigenvalues. By the property of invariance of eigenvalues by circular permutations of product matrices, it can be concluded that* $\underline{P}$ *et* $\mathbb{P}(v)$ *have the same eigenvalues for each* $v \in \{0, 1, ..., S-1\}$*.*

Remark 2.4 *The inverse problem consisting in finding a S-periodically homogeneous Markov chain from a S-variate homogeneous Markov chain has no unique solution, unless the chain* $(X_n, n \in \mathbb{N})$ *is periodically stationary (see the definition in Section 3 below and the Appendix B) or, in an equivalent manner,* $(\underline{X}_n, n \in \mathbb{N})$ *is stationary (Gladyshev,* 1961*).*

Thus, from the link between the periodically homogeneous matrix and the corresponding homogeneous hyper-matrix, it appears that it is more judicious and even necessary to develop a specific theory for periodically homogeneous Markov chains (Floquet, 1883). However, in some special cases, it is possible to use augmented homogeneous representations.

2.5.2 Partitioning approach

Instead of the above dimensionality augmentation approach, it is possible to connect a periodically homogeneous Markov chain with S homogeneous Markov *sub-chains*, through the S monodromy matrices.

Let $(X_t, t \in \mathbb{N})$ be a periodically homogeneous Markov chain with transition probabilities $P_{ij}(v) := P(X_{v+1} = j | X_v = i)$ and monodromy probabilities $\mathbb{P}_{ij}(v) = P(X_{v+S} = j | X_v = i)$ $(0 \leq v \leq S-1,\ i, j \in E)$. For all $0 \leq v \leq S$, let $(X_n^{(v)}, n \in \mathbb{N})$ be the S sub-chains of $(X_t, t \in \mathbb{N})$ defined by

$$X_n^{(v)} = X_{nS+v},\ n \in \mathbb{N}. \tag{2.11}$$

Proposition 2.5 *The Markov chain* $(X_t, t \in \mathbb{N})$ *is S-periodically homogeneous with transition probabilities* $(P_{ij}(v))_v$ *if and only if each sub-chain* $\left(X_n^{(v)}, n \in \mathbb{N}\right)$ *is Markov homogeneous with a transition probability*

$$P\left(X_n^{(v)} = j | X_{n-1}^{(v)} = i\right) = P\left(X_{nS+v} = j | X_{(n-1)S+v} = i\right) = \mathbb{P}_{ij}(v),\quad 0 \leq v \leq S. \tag{2.12}$$

In the sequel, we will mainly use this approach. Note that the correspondence (2.11) does not mean that the study of a periodically Markov chain is trivial since the monodromy (homogeneous) matrices in (2.12) are interdependent. There are actually many properties that appear for time-periodic Markov chains and not for homogeneous Markov chains.

2.6 Independent and periodically distributed chains

The simplest case of time-periodic Markov chains is that of independent and S-periodically (ipd_S) distributed sequences. A sequence $(X_t, t \in \mathbb{N})$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and valued in a finite/countable set $E = \{0, 1, ..., K\}$ (K could be infinite) is said to be ipd_S if $(X_t, t \in \mathbb{N})$ is independent and $X_t \overset{d}{=} X_{t+S}$ for all t such that $t + S \in \mathbb{N}$. When $S = 1$, an idp_1 sequence is just independent and identically distributed (iid).

Following the dimensionality augmentation view, $(X_t, t \in \mathbb{N})$ is ipd_S if and only if $(\mathbf{X}_n, n \in \mathbb{N})$ is iid, where $\mathbf{X}_n = (X_{nS}, X_{nS+1}, ..., X_{nS+S-1})'$. Following the partitioning approach, $(X_t, t \in \mathbb{N})$ is ipd_S if and only if $(X_{nS+v}, n \in \mathbb{N})$ is iid for all $0 \leq v \leq S - 1$.

The transition probability of an ipd_S sequence is given by

$$P_{ij}(t) = P(X_{t+1} = j | X_t = i) = P(X_{t+1} = j) = \pi_j(t+1) \text{ for all } i, j \in E.$$

Hence the one-step transition matrix has the form

$$P(v) = \begin{pmatrix} \pi_1(v+1) & \pi_2(v+1) & \cdots & \pi_K(v+1) \\ \pi_1(v+1) & \pi_2(v+1) & \cdots & \pi_K(v+1) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_1(v+1) & \pi_2(v+1) & \cdots & \pi_K(v+1) \end{pmatrix}, 0 \leq v \leq S-1$$

with identical lines. Thanks to the latter form, the S-step monodromy matrix $\mathbb{P}(v)$ is equal to $P(v - S + 1)$, the last matrix in the factor

$$\begin{aligned} \mathbb{P}(v) &: = P(v) P(v-1) \cdots P(v - S + 1) = P(v - S + 1) = P(v+1) \\ &= \begin{pmatrix} \pi_1(v+1) & \pi_2(v+1) & \cdots & \pi_K(v+1) \\ \pi_1(v+1) & \pi_2(v+1) & \cdots & \pi_K(v+1) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_1(v+1) & \pi_2(v+1) & \cdots & \pi_K(v+1) \end{pmatrix}, 0 \leq v \leq S-1. \end{aligned} \tag{2.13}$$

2.7 The existence problem and stochastic recurrence equations

A periodically homogeneous Markov chain can always be represented via a stochastic recurrence equation.

Theorem 2.1 *Let $(X_t, t \in \mathbb{N})$ be a E-valued random process defined by means of the following recurrence relation*

$$X_{t+1} = f_t\left(X_t, \varepsilon_{t+1}\right), \quad t \in \mathbb{N}, \tag{2.14}$$

where E is a countable set, $(\varepsilon_t, t \in \mathbb{N})$ is an ipd_S sequence valued in a countable set F, and $(f_t)_t$ is a S-periodic sequence of real functions (i.e. $f_{t+S} = f_S$ for all t). Assume that X_0 and $(\varepsilon_t, t \in \mathbb{N})$ are independent. Then, $(X_t, t \in \mathbb{N})$ is a periodically homogeneous Markov chain.

Proof Iterating (2.14), it follows that there exists a sequence of functions (h_t) such that

$$X_t = h_t\left(X_0, \varepsilon_1, \varepsilon_2, ..., \varepsilon_t\right).$$

By (2.14) and the ipd_S property of $(\varepsilon_t, t \in \mathbb{N})$,

$$\begin{aligned} P\left(X_{t+1} = j | X_t = i\right) &= P\left(f_t\left(i, \varepsilon_{t+1}\right) | h_t\left(X_0, \varepsilon_1, \varepsilon_2, ..., \varepsilon_t\right)\right) \\ &= P\left(f_t\left(i, \varepsilon_{t+1}\right)\right) \end{aligned}$$

and

$$\begin{aligned} P\left(X_{t+1} = j | X_t = i, X_{t-1} = i_{t-1}, ..., X_0 = i_0\right) &= P\left(f_t\left(i, \varepsilon_{t+1}\right) | X_t = i, X_{t-1} = i_{t-1}, ..., X_0 = i_0\right) \\ &= P\left(f_t\left(i, \varepsilon_{t+1}\right)\right). \end{aligned}$$

This shows that the chain $(X_t, t \in \mathbb{N})$ is periodically homogeneous with an initial distribution $\pi_j\left(0\right) = P\left(X_0 = j\right)$ and a transition probability matrix

$$\begin{aligned} P_{ij}\left(v\right) &= P\left(f_{nS+v}\left(i, \varepsilon_{nS+v+1}\right)\right) \\ &= P\left(f_v\left(i, \varepsilon_{v+1}\right)\right). \ \square \end{aligned}$$

Remark 2.5 *In the above theorem, the independence of $(\varepsilon_t, t \in \mathbb{N})$ can be relaxed under a broader assumption. It is possible to just assume that ε_{t+1} is conditionally independent of $X_0, X_1, ..., X_{t-1}, \varepsilon_1, \varepsilon_2, ..., \varepsilon_t$ given X_t, i.e.*

$$P\left(\varepsilon_{t+1} = j | X_t = i, ..., X_0 = i_0, \varepsilon_1 = l_1, ..., \varepsilon_t = l_t\right) = P\left(\varepsilon_{t+1} = j | X_t = i\right).$$

Under the latter assumption, it can be easily seen that any process satisfying (2.14) *is a periodically homogeneous Markov chain with a transition probability*

$$\begin{aligned} P_{ij}(v) &= P\left(f_{nS+v}\left(i,\varepsilon_{nS+v+1}\right)|X_{nS+v}\right) \\ &= P\left(f_v\left(i,\varepsilon_{v+1}\right)|X_v\right). \end{aligned}$$

The inverse problem which consists of showing that, for any system of S stochastic matrices $(P(v))_{0\leq v\leq S-1}$, there exists a periodically homogeneous Markov chain having these matrices as a transition matrix system, is a simple instance of the Kolmogorov existence theorem (see e.g. Breiman, 1968; Aknouche, 2008).

Theorem 2.2 *For any system of stochastic matrices* $(P(v))_{0\leq v\leq S-1}$ *of dimension* $K\times K$ *(*K *can even be infinite), corresponds a unique periodically homogeneous Markov chain* $(X_t, t\in\mathbb{N})$ *valued in* $E=\{1,2,...,K\}$ *and admitting the system* $(P(v))_{0\leq v\leq S-1}$ *as transition probability matrices. Furthermore, the chain* $(X_t, t\in\mathbb{N})$ *is a solution to a recurrence equation of the form* (2.14)*.*

2.8 Stopping time along a channel and the strong Markov property

The strong Markov property valid for homogeneous Markov chains remains true for any periodic chain even if the latter is not time-homogeneous.

Definition 2.2 (Stopping time along a channel) *Let* $(X_t, t\in\mathbb{N})$ *be a periodically* S*-homogeneous Markov chain defined on a probability space* $(\Omega,\mathcal{F},P)$*. For all* $v\in\{0,...S-1\}$*, a random time* Υ_v *defined on* $(\Omega,\mathcal{F},P)$ *and valued in* $\mathbb{N}\cup\{\infty\}$ *is called a stopping-time along the channel* v *if the event* $\{\Upsilon_v=v+nS\}$ *belongs to the* σ*-algebra* $\mathcal{F}_n^{(v)}:=\sigma\{X_{v+kS}, k\leq n\}$ for all $n\in\mathbb{N}$.

Example 2.3 Let $(X_t, t\in\mathbb{N})$ be a 2-periodic random walk starting at the origin with

$$\begin{array}{l} X_{nS+v}=\xi_v+\xi_{S+v}+...+\xi_{nS+v},\ n\in\mathbb{N}^* \\ X_v=0 \end{array},\ 0\leq v\leq S-1$$

where the so-called step-size sequence $(\xi_t, t \in \mathbb{N})$ is *ipd*$_2$ and is valued in $\{-1, 1\}$ with

$$P(\xi_0 = 1) = \tfrac{1}{2} \text{ and } P(\xi_1 = 1) = \tfrac{1}{3}.$$

Let $\Upsilon_v = \min\{n \in \mathbb{N}^* : X_{nS+v} = 0\}$. Then

$$\begin{aligned} \{\Upsilon_v = n\} &= \left\{X_v \neq 0, X_{S+v} \neq 0, ..., X_{(n-1)S+v} \neq 0, X_{nS+v} = 0\right\} \\ &\in \mathcal{F}_n^{(v)}. \end{aligned}$$

Definition 2.3 (Strong Markov property) *Let $(\Upsilon_v)_{0 \le v \le S-1}$ be S stopping times corresponding to a S-periodically homogeneous Markov chain $(X_t, t \in \mathbb{N})$.*

i) *The chain $(X_t, t \in \mathbb{N})$ is said to have the strong Markov property over the channel $v \in \{0, ..., S-1\}$ if*

$$\begin{aligned} P(X_{\Upsilon_v S+v} = i_{\Upsilon_v} | X_{(\Upsilon_v-1)S+v} = i_{\Upsilon_v-1}, X_{(\Upsilon_v-2)S+v} = i_{\Upsilon_v-2}, ..., X_v = i_0) \\ = P(X_{\Upsilon_v S+v} = j | X_{(\Upsilon_v-1)S+v} = i_{\Upsilon_v-1}) \end{aligned} \quad (2.15)$$

for all $i_k \in \mathbb{N}$ $(k = \Upsilon_v, \Upsilon_v - 1, ..., 0)$.

ii) *The chain $(X_t, t \in \mathbb{N})$ is said to have the strong Markov property if* (2.15) *is satisfied for all $v \in \{0, ..., S-1\}$.*

Proposition 2.6 *Every time-periodic Markov chain satisfies the strong Markov property.*

Proof Let $(X_t, t \in \mathbb{N})$ be a S-periodically homogeneous Markov chain and denote by $(\Upsilon_v)_{0 \le v \le S-1}$ the S corresponding stopping times. By periodic homogeneity of the chain $(X_t, t \in \mathbb{N})$ we have for all $v \in \{0, ..., S-1\}$

$$\begin{aligned} P\left(X_{(\Upsilon_v+n)S+v} = i_n | X_{(\Upsilon_v+n-1)S+v} = i_{n-1}, ..., X_{S\Upsilon_v+v} = i_0\right) \\ = P(X_{nS+v} = i_n | X_{(n-1)S+v} = i_{n-1}, ..., X_v = i_0) \\ = P(X_{nS+v} = i_n | X_{(n-1)S+v} = i_{n-1}) \\ = P(X_{\Upsilon_v S+v} = i_n | X_{(\Upsilon_v-1)S+v} = i_{n-1})), \end{aligned}$$

establishing the result. □

3 Periodic stochastic stability

3.1 The problem

Although the conditional (transition) probabilities $(P_{ij}(v) = P(X_{v+1} = j | X_v = i))_v$ are S-periodic over v, it is not necessarily the case for unconditional (marginal) probabilities $\pi_j(t) = P(X_t = j)$. Except for a particular choice of the initial distribution $\pi(0)$ and a suitable form of the matrices $(P(v))_{0\leq v\leq S-1}$, the distributions $\pi(v)$, $\pi(v+S)$, $\pi(v+2S), ...$ are in general not equal. However, when the transition probabilities $(P(v))_{0\leq v\leq S-1}$ satisfy certain suitable properties, even if the distributions for small ranks t are not periodic, there exists a quite large rank t_0, above which

$$\pi(v+t) \simeq \pi(v+(t+1)S) \simeq \pi(v+(t+2)S) \simeq \ldots,$$

for all $t \geq t_0$, where the latter approximation is understood in the sense that there exist S probability distributions $\left(\pi^{(v)}\right)_{0\leq v\leq S-1}$ such that

$$\lim_{n\to\infty} \pi(nS+v) = \pi^{(v)}, \quad (0 \leq v \leq S-1), \tag{3.1}$$

where the vector equality (3.1) is component-wise. Equality (3.1) means that from a certain quite large rank t_0, the corresponding marginal distributions $(\pi(t))_{t\geq t_0}$ are more and more approximately periodic with period S, regardless of the initial distribution $\pi(0)$. This reflects a certain *periodic stochastic stability.* We then say that the chain has moved to a periodically stable regime, periodically steady state, periodically stationary regime, or also periodically permanent regime.

3.2 Periodically regular/stable Markov chains

Equality (3.1) also translates in terms of conditional probabilities as follows

$$\lim_{n\to\infty} P_{ij}^{(nS)}(v) = \pi_j^{(v)}, \; 0 \leq v \leq S-1, \, i \in E, \tag{3.2}$$

meaning that whatever the initial state i from which the chain starts, the nS-step transition probability tends to a limit $\pi_j^{(v)}$ (as $n \to \infty$) which depends only on the arrival state j. In matrix form, (3.2) writes as follows

$$\lim_{n\to\infty} P^{(nS)}(v) = \Pi^{(v)}, \;\; 0 \le v \le S-1$$

i.e.

$$\lim_{n\to\infty} (\mathbb{P}(v))^n = \Pi^{(v)}, \quad 0 \le v \le S-1, \tag{3.3}$$

where $\Pi^{(v)}$ is a stochastic limiting matrix of the form

$$\Pi^{(v)} = \begin{pmatrix} \pi_1^{(v)} & \pi_2^{(v)} & \cdots & \pi_K^{(v)} \\ \pi_1^{(v)} & \pi_2^{(v)} & \cdots & \pi_K^{(v)} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_1^{(v)} & \pi_2^{(v)} & \cdots & \pi_K^{(v)} \end{pmatrix} = \begin{pmatrix} \pi^{(v)} \\ \pi^{(v)} \\ \vdots \\ \pi^{(v)} \end{pmatrix}.$$

This leads to the following definition.

Definition 3.1 i) *A periodically homogeneous Markov chain is said to be regular/stable along a channel* v_0 *(or* v_0*-regular) if* (3.2) *is satisfied for* $v = v_0$.

ii) *A periodically homogeneous Markov chain is said to be periodically regular if* (3.2) *is satisfied for every* $v \in \{0, ..., S-1\}$.

Since the eigenvalues of a (square) matrix product are invariant under a circular permutation of the product factors (see e.g. Bittanti and Colaneri, 2009 for a proof), the eigenvalues of monodromy matrices $(\mathbb{P}_v)_{0\le v\le S-1}$ are all the same for every $v \in \{0, ..., S-1\}$. This supposes that periodic homogeneity is only with regards to the transition probabilities and thus $E_t = E$ is time-invariant. Consequently the asymptotic behavior of the matrices $(\mathbb{P}_v^n)_{0\le v\le S-1}$ (as $n \to \infty$) is the same across v. *Thus, if a periodically homogeneous Markov chain is regular along a certain channel* $v_0 \in \{0, ..., S-1\}$ *then it is regular along any other channel, and hence is periodically regular.*

3.3 Periodically stationary (invariant) distributions

The distributions $\left(\pi^{(v)}\right)_{0\le v\le S-1}$ when exist are called *periodically stationary* (or *periodically invariant*). In the case of finite time-periodic chains, they are determined as follows.

Proposition 3.1 *When the limit* $\lim_{n\to\infty} P^{(nS)}(v)$ *exists for each* v *and is equal to* $\Pi^{(v)}$*, the lines* $\left(\pi^{(v)}\right)_{0\le v\le S-1}$ *of this matrix necessarily satisfy the following system of equations*

$$\begin{aligned} \pi^{(v-1)}P(v-1) &= \pi^{(v)}, \quad 1\le v\le S-1 \\ \pi^{(S-1)}P(S-1) &= \pi^{(0)} \\ \pi^{(v)}\mathbf{1} &= 1, \quad 0\le v\le S-1 \end{aligned} \tag{3.4a}$$

or in terms of a monodromy transition matrix

$$\begin{cases} \pi^{(v)}\mathbb{P}(v) = \pi^{(v)} \\ \pi^{(v)}\mathbf{1} = 1, \end{cases} \quad 0\le v\le S-1, \tag{3.4b}$$

where $\mathbf{1}$ *is a* K*-vector whose elements are equal to unity.*

Proof The periodic Chapman-Kolmogorov equations (see Remark 2.1, i)) and the S-periodicity of $P_{kj}^{(S)}(v)$ over v together yield

$$P_{ij}^{(nS+S)}(v) = \sum_{k\in E} P_{ik}^{(nS)}(v)\, P_{kj}^{(S)}(v).$$

Taking the limit as $n\to\infty$ in both sides of the latter equality (while assuming the limit exists and is finite) gives

$$\lim_{n\to\infty} P_{ij}^{((n+1)S)}(v) = \sum_{k\in E} \lim_{n\to\infty} P_{ik}^{(nS)}(v)\, P_{kj}^{(S)}(v)$$

i.e.

$$\pi_j^{(v)} = \sum_{k\in E} \pi_k^{(v)} P_{kj}^{(S)}(v),\ j\in E,$$

which, in matrix form, is exactly (3.4). □

If the initial distribution $\pi(0)$ and the transition probabilities $(P(v))_{0\le v\le S-1}$ are such that

$$\pi(v) = \pi^{(v)}, \quad 0\le v\le S-1,$$

then by (3.4) and the law of total probabilities,

$$\begin{aligned} \pi_j(v+S) &= \sum_{k\in E} P(X_{v+S}=j, X_v=k) \\ &= \sum_{k\in E} \pi_k(v)\, P_{kj}^{(S)}(v) \\ &= \sum_{k\in E} \pi_k^{(v)} P_{kj}^{(S)}(v) = \pi_j^{(v)} = \pi_j(v),\ j\in E. \end{aligned}$$

By doing the same for $\pi_j(v+2S)$ and so on, it follows that for each $0 \leq v \leq S-1$,

$$\pi(v) = \pi(v+S) = \pi(v+2S) = \ldots$$

reflecting the periodic stability (or periodic stationarity) from the beginning.

3.4 Periodically stationary Markov chains

Thus, when

$$\pi(v) = \pi^{(v)}, \text{ for all } 0 \leq v \leq S-1,$$

i.e. when the chain is initialized from its S periodically stationary distributions, it can be seen that the process $(X_t, t \in \mathbb{N})$ is strictly periodically stationary (hence the name) in the sense that the finite-dimensional distributions

$$\left\{P_{0,1,\ldots,n}(i_0, i_1, \ldots, i_n), \quad n \in \mathbb{N}^*,\ i_j \in E,\ 0 \leq j \leq n\right\},$$

are invariant under a translation multiple of S. Indeed,

$$\begin{aligned}
P_{0,1,\ldots,n}(i_0, i_1, \ldots, i_n) &= \pi_{i_0}(0) P_{i_0 i_1}(0) P_{i_1 i_2}(1) \cdots P_{i_{n-1} i_n}(n-1) \\
&= \pi_{i_0}^{(0)} P_{i_0 i_1}(0) P_{i_1 i_2}(1) \cdots P_{i_{n-1} i_n}(n-1) \\
&= \pi_{i_0}^{(S)} P_{i_0 i_1}(S) P_{i_1 i_2}(S+1) \cdots P_{i_{n-1} i_n}(S+n-1) \\
&= P_{S,S+1,\ldots,S+n}(i_0, i_1, \ldots, i_n).
\end{aligned}$$

A strictly periodically stationary Markov chain with a finite state space is also second-order periodically stationary (or periodically correlated) since, being finite, its first two moments $\mu_t := E(X_t)$ and $\widetilde{\gamma}_h^{(t)} = E(X_t X_{t+h})$ are also finite and are, thanks to the strict periodic stationarity property, S-periodic over time.

The expressions of the first two moments for a finite Markov chain valued in $E = \{1, 2, \ldots, K\}$ are given as

$$\begin{aligned}
\mu_v &:= E(X_{nS+v}) = \sum_{k=1}^{K} k \pi_k^{(v)} \\
&= \pi^{(v)} 1_{1,K}
\end{aligned}$$

and

$$\begin{aligned}\widetilde{\gamma}_h^{(v)} &:= E\left(X_{nS+v}X_{nS+v+h}\right) = \sum_{k=1}^{K}\sum_{l=1}^{K} klP\left(X_{nS+v}=k, X_{nS+v+h}=l\right) \\ &= \sum_{k=1}^{K}\sum_{l=1}^{K} kl\pi_k^{(v)}P_{kl}^{(h)}\left(v\right) = \pi_{1,K}^{(v)}\mathbf{1}_{1,K}P^{(h)}\left(v\right)1_{1,K},\end{aligned}$$

where $1_{1,K} = (1,2,...,K)'$ and $\mathbf{1}_{1,K} = diag\left(1,2,...,K\right)$ stands for the diagonal matrix formed by the elements $1,...,K$ in this order. Thus the autocovariance function is given by

$$\begin{aligned}\gamma_h^{(v)} &:= Cov\left(X_{nS+v}, X_{nS+v+h}\right) \\ &= \pi_{1,K}^{(v)}\mathbf{1}_{1,K}P_v^{(h)}1_{1,K} - \pi^{(v)}\mathbf{1}_{1,K}\pi^{(v+h)}1_{1,K}.\end{aligned}$$

Remark 3.1 *The existence and uniqueness of a system of periodically stationary distributions* $\left(\pi^{(v)}\right)_{0\le v\le S-1}$ *and the limit of the marginal distributions to that system, independently of the initial marginal distribution, depend on certain structural properties of the time-periodic Markov chain. These properties parallel and extend the properties known for homogeneous Markov chains, namely irreducibility, recurrence, positive recurrence, and state-periodicity. We will see that for* (3.1) *and* (3.2) *to be satisfied, it is sufficient that the periodically homogeneous chain is periodically irreducible, periodically positive recurrent, and periodically aperiodic. These properties will be defined in the next section.*

When the periodic aperiodicity is not satisfied, the convergence relations (3.1) and (3.2) will only be satisfied in Cesàro sense, i.e.

$$\lim_{n\to\infty}\tfrac{1}{n}\sum_{k=1}^{n}\pi\left(kS+v\right) = \pi^{(v)},\ \ 0\le v\le S-1 \tag{3.5}$$

and

$$\lim_{n\to\infty}\tfrac{1}{n}\sum_{k=1}^{n}P_{ij}^{(kS)}\left(v\right) = \pi_j^{(v)},\ \ 0\le v\le S-1. \tag{3.6}$$

It is suitable in certain cases to replace the traditional element-wise convergence (3.2) by the convergence in *total variation norm*

$$\left\|P_{i,.}^{(nS)}\left(v\right) - \pi^{(v)}\right\| = \tfrac{1}{2}\sum_{j\in E}\left|P_{ij}^{(nS)}\left(v\right) - \pi_j^{(v)}\right|, \tag{3.7}$$

where the factor $\frac{1}{2}$ in the right-hand side of the last equality is put so that the norm is between 0 and 1. This type of convergence, which is defined as

$$\lim_{n\to\infty} \left\| P_{i,\cdot}^{(nS)}(v) - \pi^{(v)} \right\| = 0, \quad (0 \leq v \leq S-1), \ i \in E, \tag{3.8}$$

is useful since it can easily be extended to the case of periodically homogeneous Markov chains with uncountable state-spaces. If (3.8) is satisfied, we continue to say that the chain is periodically regular/stable or simply *Harris periodically ergodic.*

Definition 3.2 (Harris-periodic ergodicity)

i) *A periodically homogeneous Markov chain is called Harris ergodic along a certain channel* $v_0 \in \{0, ..., S-1\}$ *or* v_0*-Harris ergodic (or also* v_0*-stable/*v_0*-regular) if* (3.8) *is satisfied for* $v = v_0$.

ii) *A periodically homogeneous Markov chain is called Harris periodically ergodic if it is* v*-Harris ergodic along each channel* $v \in \{0, ..., S-1\}$.

iii) *A periodically homogeneous Markov chain is called geometrically Harris ergodic along a channel* $v_0 \in \{0, ..., S-1\}$ *or geometrically* v_0*-Harris ergodic (or also geometrically* v_0*-stable/*v_0*-regular) if there exists* $\rho \in (0,1)$ *such that*

$$\lim_{n\to\infty} \rho^{-n} \left\| P_{i,\cdot}^{(nS)}(v_0) - \pi^{(v_0)} \right\| = 0 \text{ for all } i \in E. \tag{3.9}$$

iv) *A periodically homogeneous Markov chain is called geometrically Harris periodically ergodic if it is geometrically* v*-Harris ergodic along each* $v \in \{0, ..., S-1\}$.

4 Classification of states and properties of solidarity

A fundamental concept related to periodically homogeneous Markov chains is *periodic stochastic stability* which can be summarized in the following three questions:

i) (Existence) Do the system of periodically stationary distributions $\left(\pi^{(v)}\right)_{0\leq v\leq S-1}$ exist?

ii) (Uniqueness) If so, is $\left(\pi^{(v)}\right)_{0\leq v\leq S-1}$ unique?

iii) (Convergence) Do the marginal distributions $(\pi(nS+v))_{v,n}$ converge to the periodically stationary distributions $\left(\pi^{(v)}\right)_{0\leq v\leq S-1}$ in the following sense

$$\lim_{n\to\infty}\pi(nS+v)=\pi^{(v)},\ 0\leq v\leq S-1$$

(which is equivalent to the fact that the relation (3.3) holds)?

Answering these three questions amounts to studying certain structural properties of the chain (called solidarity, contagion, class) which will be defined and analyzed in this Section. In addition to being auxiliary tools to answer the three theoretical questions above, these solidarity properties are important in themselves and allow to answer the following practical questions.

vi) If the chain visits a state i, along a channel $v\in\{0,1,....,S-1\}$, can it reach any other state $j\in E$ in a finite number of steps multiple of S? If so, is this true for each $v\in\{0,1,....,S-1\}$? If not, is there a subset $\mathcal{C}\subset E$ having this property (i.e. when the chain is in $\mathcal{C}$ along a channel $v\in\{0,1,....,S-1\}$, can it reach any other state $j\in\mathcal{C}$ in a finite number of steps multiple of S)?

v) If the chain is at a state i along a channel $v\in\{0,1,....,S-1\}$, "how many" times does it passes by another state j in a number of steps multiple of S? Is this number finite? Infinite? What is the average number of passages through a state j given that the chain started at the state i? What is the first return time to a state i along a channel $v\in\{0,1,....,S-1\}$? What is the corresponding average time? Is it finite? Infinite?

To formalize these questions, consider the following random variables.

- The number $\eta_j(v)$ of visits of the state j along the channel v is given by

$$\eta_j(v)=\sum_{n=1}^{\infty}1_{[X_{nS+v}=j]}, \tag{4.1}$$

where $1_{[.]}$ denotes the indicator function.

- The time $\tau_j(v)$ of first passage by j, along the channel v, is defined by

$$\begin{aligned}\tau_j(v) &= \min\{n\geq 1: X_{nS+v}=j\} \\ &\in [1,\infty]\end{aligned} \tag{4.2}$$

(if the chain never visits the state j, along the channel v, then by convention $\tau_j(v) = \infty$).

The following related variables which will be used in studying the structure of the time-periodic chain are considered.

- The expected number of visits to state j along the channel v given that the chain started from the state i at time v ($0 \leq v \leq S-1$):

$$\begin{aligned} E\left(\eta_j(v)\,|X_v = i\right) &= E\left(1_{[X_{nS+v}=j]}|X_v = i\right) \\ &= E\left(\sum_{n=0}^{\infty} 1_{[X_{nS+v}=j|X_v=i]}\right) \\ &= \sum_{n=0}^{\infty} P_{ij}^{(nS)}(v)\,. \end{aligned} \tag{4.3}$$

- The expected time of passage by j along the channel v given $X_v = i$,

$$m_{ij}(v) := E\left(\tau_j(v)\,|X_v = i\right). \tag{4.4}$$

When $i = j$, $m_{ii}(v)$ is simply called the expected time of return to state i along the channel v.

- The probability that the passage time by j along the channel v is finite given $X_v = i$:

$$L_{ij}(v) := P\left(\tau_j(v) < \infty|X_v = i\right). \tag{4.5a}$$

Note that (4.5a) can be expressed slightly differently (see also Karlin and Taylor, 1975). Define

$$f_{ij}^{(nS)}(v) := P\left(X_{nS+v} = j, X_{mS+v} \neq j,\, m = 1, ..., n-1|X_v = i\right) \tag{4.5b}$$

($n \in \mathbb{N}$) to be the probability of the first passage by the state j along the channel v at time nS, given $X_v = i$. Obviously, $f_{ij}^{(0)}(v) = 0$ and $f_{ij}^{(S)}(v) = P_{ij}^{(S)}(v)$. Then $L_{ij}(v)$ can be expressed as

$$L_{ij}(v) = \sum_{n=1}^{\infty} f_{ij}^{(nS)}(v)\,. \tag{4.5c}$$

When $i = j$, $f_{ii}^{(nS)}(v)$ is simply called the probability of the first return to state i along the channel v, at the nSth transition. On a final note,

$$m_{ii}(v) = E\left(\tau_i(v)\,|X_v = i\right) = \sum_{n=1}^{\infty} n f_{ij}^{(nS)}(v)\,. \tag{4.5d}$$

4.1 Communication and classification of states

4.1.1 Communication

Due to the inhomogeneity of the time-periodic Markov chain and in particular to its periodicity in terms of both dimensionality and structure, it is only possible to study the accessibility from one state to another along a given channel. Thus, we extend the notions of accessibility, communication, and irreducibility, known in the homogeneous case, to the case of periodically homogeneous Markov chains, by studying each of these properties along each channel $v \in \{0, ..., S-1\}$.

Definition 4.1 *i) A state j is said to be accessible from a state i along the channel v, which writes*

$$i \overset{v}{\rightsquigarrow} j,$$

if there exists $n := n_v > 0$ such that $P_{ij}^{(nS)}(v) > 0$.

ii) If $i \overset{v}{\rightsquigarrow} j$ and $j \overset{v}{\rightsquigarrow} i$ then i communicates with j along the channel v, and we write

$$i \overset{v}{\leftrightsquigarrow} j.$$

In other words, there exist $n = n_v > 0$ and $m = m_v > 0$ such that $P_{ij}^{(nS)}(v) > 0$ and $P_{ji}^{(mS)}(v) > 0$.

There are other characterizations of communication, expressed in terms of $L_{ij}(v)$ and $E_i\left(\eta_j(v)\right)$ defined, respectively, by (4.5) and (4.3).

Proposition 4.1 *The following assertions are equivalent.*

i) $i \overset{v}{\leftrightsquigarrow} j$.

ii) $L_{ij}(v) > 0$ and $L_{ji}(v) > 0$.

iii) $\sum\limits_{n=0}^{\infty} P_{ij}^{(nS)}(v) > 0$ and $\sum\limits_{n=0}^{\infty} P_{ji}^{(nS)}(v) > 0$.

Proof The fact that i) is equivalent to iii) is obvious. Now, in view of (4.5b), if i) holds then ii) follows. Finally, iii) implies ii). □

Proposition 4.2 *The relation "$\overset{v}{\leftrightsquigarrow}$" on E is an equivalence relation for each $v \in \{0, ..., S-1\}$.*

Proof The relation "$\overset{v}{\leftrightsquigarrow}$" is reflexive on E since by convention $P_{ii}^{(0)}(v) = 1 > 0$ for every $i \in E$ and $v \in \{0, ..., S-1\}$. It is symmetric by definition. To show transitivity, note that if for $i, j,\ k \in E$, $P_{ij}^{(nS)}(v) > 0$ and $P_{jk}^{(mS)}(v) > 0$ then by Chapman-Kolmogorov equations (cf. Remark 2.1, i))

$$P_{ik}^{((n+m)S)}(v) \geq P_{ij}^{(nS)}(v)\, P_{jk}^{(mS)}(v) > 0,$$

meaning that $i \overset{v}{\rightsquigarrow} k$. A similar argument shows that $k \overset{v}{\rightsquigarrow} i$. □

4.1.2 Classification of states and associated v-graphs

The equivalence relation "$\overset{v}{\leftrightsquigarrow}$" induces p $(p \geq 1)$ equivalence classes $(\mathcal{C}_k(v),\ k = 1, ..., p)$ on $E = \bigcup_{v=0}^{S-1} E_v$ such that

$$\mathcal{C}_k(v) = \left\{ j \in E_v : j \overset{v}{\leftrightsquigarrow} k \right\}.$$

Remark 4.1 *By classification of states of a periodically homogeneous Markov chain, we mean highlighting the classes of communication for each channel* $v \in \{0, ..., S-1\}$.

It is possible to represent a (finite) periodically homogeneous Markov chain by means of a system of S oriented graphs $(G(v), U(v))_{0 \leq v \leq S-1}$ with

$$G(v) = E_v$$

and

$$(i, j) \in U(v) \qquad \text{if and only if} \qquad P_{ij}(v) > 0.$$

However, these graphs do not directly reflect the periodic communication between states, especially in the case where the chain is not homogeneous with respect to the state-space, where the matrices are even not square. The above graphs are called "one-step transition graphs". To better represent communication, we instead use the monodromy matrices $(\mathbb{P}(v))_{0 \leq v \leq S-1}$. Thus, a periodically homogeneous Markov chain can also be represented with a system of S oriented graphs $((\mathbb{G}(v), \mathbb{U}(v)))_{0 \leq v \leq S-1}$ such that

$$\mathbb{G}(v) = E_v$$

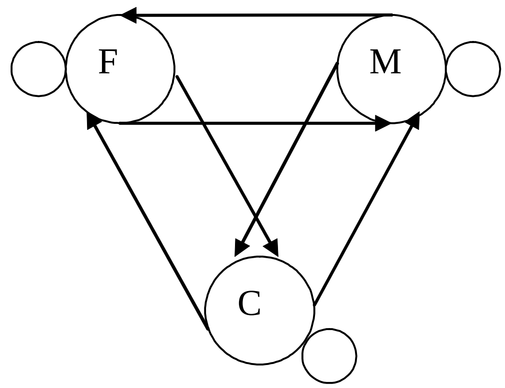
and

$$(i, j) \in \mathbb{U}(v) \quad \text{if and only if} \quad \mathbb{P}_{ij}(v) > 0.$$

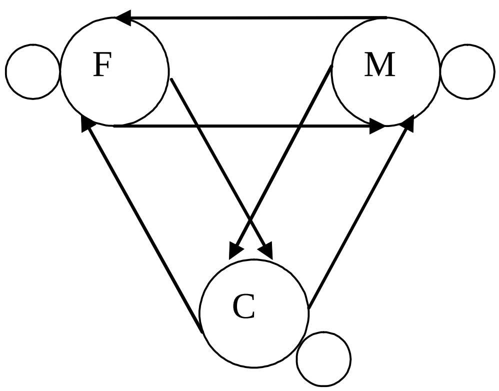
We call $(\mathbb{G}(v), \mathbb{U}(v))$ the S-step transition graph along the channel v, or v-graph. From now on, only the S-step graphs are considered. From the S-step v-graph, a v-communication class is assimilated to a "*strongly connected component*".

Example 2.1 (Continued) Let us return to Example 2.1, ii). The S-step graphs associated with the chain are represented as follows.

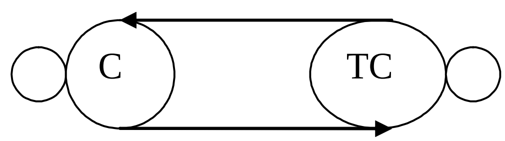
$\mathbb{G}(0)$



$\mathbb{G}(1)$



$\mathbb{G}(2)$



$\mathbb{G}(3)$

4.2 Periodic irreducibility

A remarkable communication property arises when the chain has only one equivalence class for the relation $\overset{v}{\leftrightsquigarrow}$. In this case, all states communicate along the channel v.

Definition 4.2 *A periodically homogeneous Markov chain is said to be irreducible along* *a channel v or simply v-irreducible if the relation $\overset{v}{\leftrightsquigarrow}$ induces a single equivalence class $\mathcal{C}(v)$ which is E_v itself. If the chain is not v-irreducible, then it is called v-reducible.*

When E is finite, there exists another characterization of v-irreducibility, rather based on the monodromy transition matrices (see, e.g. Cox and Miller, 1965 and Cinlar, 1975 in the case of homogeneous Markov chains).

Theorem 4.1 *The following assertions are equivalent:*
i) The chain is v-reducible.
ii) There is a way to number the states so that the corresponding monodromy matrix $\mathbb{P}(v)$ can be written in the following form

$$\mathbb{P}(v) = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where A, B, and C are non-null matrices with compatible dimensions.

Remark 4.2 *A remarkable (if not surprising) result regarding the communication of periodically homogeneous Markov chains is the following: A periodically homogeneous chain can be irreducible along some channel $v \in \{0, ..., S-1\}$ and reducible along some other channel $v' \neq v$.*

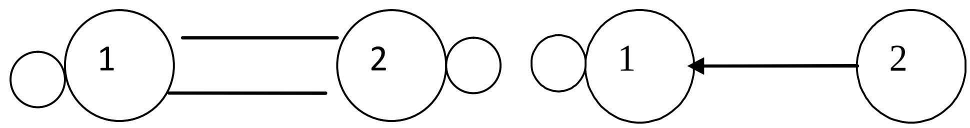
Example 4.1 Let $S = 2$, $E_1 = E_2 = E = \{1, 2\}$,

$$P(0) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } P(1) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}.$$

Then the monodromy transition matrices of the chain are

$$\begin{aligned} \mathbb{P}(0) &= P(0)P(1) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \\ \mathbb{P}(1) &= P(1)P(2) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

The corresponding 2-step transition graphs are



$\mathbb{G}(0)$ $\mathbb{G}(1)$

and clearly show that the chain is 0-irreducible and 1-reducible.

From this, follows the property of *periodic irreducibility*.

Definition 4.3 *A periodically homogeneous Markov chain is said to be periodically irreducible if it is v-irreducible for every $v \in \{0, ..., S-1\}$.*

Remark 4.3 *If a time-periodic chain is initialized from its system of periodically stationary distributions (and hence it is strictly periodically stationary) then the concept of periodic irreducibility introduced here is synonymous with that of periodic ergodicity for strictly periodically stationary processes (cf. Appendix B). Thus, "periodic irreducibility" is a more general property which arises even for stochastic processes that are not necessarily periodically stationary. However, for a homogeneous Markov chain, the word "ergodic" is rather reserved to a stricter notion. A homogeneous Markov chain is called ergodic if it is irreducible, positive recurrent, and aperiodic (e.g. Karlin and Taylor,* 1975*). Some authors rightly qualify the latter ergodic property as regularity and we adopt this view here. Kemeny and Snell* (1976) *aptly employed the term "ergodicity" to refer to the concept of irreducibility. Note finally that Harris-periodic ergodicity introduced in* (3.8)-(3.9) *is rather a "limiting" property (like the periodic regularity* (3.2)*) and not a "communication" property like "periodic irreducibility" and also "periodic ergodicity" for periodically stationary processes (cf. Aknouche,* 2008-2014*).*

4.3 Solidarity/contagion/class properties along a channel

We now study some class properties also called solidarity properties since the states in the same class along some channel share the same properties.

4.3.1 Periodic recurrence, periodic transience, periodic positive recurrence, and periodic null recurrence

Definition 4.4 i) *A state i is called recurrent along a channel v (or v-recurrent) if the expected number of returns to state i along the channel v is infinite, i.e.*

$$E(\eta_i(v)|X_v = i) = \infty.$$

ii) *If i is not v-recurrent then it is called transient along v or simply v-transient and hence $E(\eta_i(v)|X_v = i) < \infty$.*

In reference to (4.3), define for each $v \in \{0, ..., S-1\}$

$$E\left(\eta_j\left(v\right)|X_v = i\right) = \sum_{n=0}^{\infty} P_{ij}^{(nS)}(v).$$

Then v-recurrence is a solidarity property along v.

Proposition 4.3 *i) If a periodically homogeneous Markov chain is irreducible along some channel v, then for all $i, j \in E$ either $E\left(\eta_j\left(v\right)|X_v = i\right) = \infty$ or $E\left(\eta_j\left(v\right)|X_v = i\right) < \infty$.*

ii) *If a periodically homogeneous Markov chain is periodically irreducible, then for all $i, j \in E$ and all $v \in \{0, \cdots, S-1\}$, either $E\left(\eta_j\left(v\right)|X_v = i\right) = \infty$ or $E\left(\eta_j\left(v\right)|X_v = i\right) < \infty$.*

Proof i) If $\sum_{n=0}^{\infty} P_{ij}^{(nS)}(v) = \infty$ for some $i, j \in E$, then by v-irreducibility of the chain, $r \overset{v}{\rightsquigarrow} i$ and $j \overset{v}{\rightsquigarrow} s$ for all $r, s \in E_v$. So there exist positive integers k and l such that $P_{ij}^{(kS)}(v) > 0$ and $P_{ij}^{(lS)}(v) > 0$. Consequently,

$$\sum_{n=0}^{\infty} P_{rs}^{((k+l+n)S)}(v) > P_{rs}^{(kS)}(v) \left(\sum_{n=0}^{\infty} P_{ij}^{(nS)}(v) \right) P_{js}^{(lS)}(v), \tag{4.6}$$

so that $E\left(\eta_j\left(v\right)|X_v = i\right)$ and $E\left(\eta_s\left(v\right)|X_v = r\right)$ together converge or together diverge.

ii) For each $v \in \{0, \cdots, S-1\}$, the inequality (4.6) is satisfied. $\square$

Proposition 4.4 i) *If a periodically homogeneous Markov chain is v-irreducible, then either all states are v-recurrent or they are all v-transient.*

ii) *If a periodically homogeneous chain is periodically irreducible, then either all states are v-recurrent or they are all v-transient for all $v \in \{0, ..., S-1\}$.*

Proof i) If the chain is v-irreducible, then Proposition 4.3 implies that for each $i \in E$, either $E\left(\eta_j(v)|X_v = i\right) < \infty$ or $E\left(\eta_j(v)|X_v = i\right) = \infty$, i.e. all states are either v-recurrent or v-transient.

ii) A consequence of i). □

Proposition 4.4 simply states that if i and j are in the same class along a channel v, then i is v-recurrent if and only if j is v-recurrent. We thus speak of a v-recurrent (or v-transient) class rather than a v-recurrent state. A special case appears when a v-recurrent class contains a single element.

Definition 4.5 *A state i is called absorbing along a channel v (or v-absorbing) if the singleton $\{i\}$ is a v-recurrent class.*

It is possible to characterize v-recurrence and v-transience using the probabilities of return

$$L_{ii}(v) := P\left(\tau_i(v) < \infty | X_v = i\right)$$

along a channel v.

For $n \geq 1$, consider the event $\{X_{nS+v} = k\}$ which is the union of pairwise disjoint events defined by $\{X_{nS+v} = k, \tau_k(v) = j\}$, $j = 1, \cdots, n$. For $n \geq 1$ and $i \in E$,

$$\begin{aligned} P_{ik}^{(nS)}(v) &= P(\tau_k(v) = n | X_v = i) + \sum_{j=1}^{n-1} P(X_{nS+v} = k, \tau_k(v) = j | X_v = i) \\ &= P(\tau_k(v) = n | X_v = i) + \sum_{j=1}^{n-1} P(\tau_k(v) = j | X_v = i) P_{kk}^{((n-j)S)}(v). \end{aligned} \tag{4.7}$$

Define the probability generating functions

$$U_{ik}^{(z)}(v) = \sum_{n=1}^{\infty} P_{ik}^{(nS)}(v) z^n, \ |z| < 1 \tag{4.8}$$

$$L_{ik}^{(z)}(v) = \sum_{n=1}^{\infty} P(\tau_k(v) = n | X_v = i) z^n, \ |z| < 1. \tag{4.9}$$

Multiplying (4.7) by z^n and summing over n, gives

$$U_{ik}^{(z)}(v) = L_{ik}^{(z)}(v) L_{ik}^{(z)}(v) + L_{ik}^{(z)}(v) U_{ii}^{(z)}(v). \tag{4.10}$$

Proposition 4.5 *The following assertions are equivalent.*

i) i *is* v*-recurrent.*

ii) $\sum_{n=0}^{\infty} P_{ii}^{(nS)}(v) = \infty$.

iii) $L_{ii}(v) := P(\tau_i(v) < \infty | X_v = i) = \sum_{n=1}^{\infty} f_{ii}^{(nS)}(v) = 1$.

Proof From (4.10) with $i = k$,

$$U_{kk}^{(z)}(v) = \frac{L_{kk}^{(z)}(v)}{1 - L_{kk}^{(z)}(v)}$$

so when z tend to 1, it follows that $L_{kk}(v) = 1$ is equivalent to $U_{kk}(v) = \sum_{i \in E} P_{ii}^{(nS)}(v) = \infty$. □

Proposition 4.6 i) *If a periodically homogeneous Markov chain is* v*-irreducible, then for all* $i, j \in E$*, either* $L_{ij}(v) = 1$ *or* $L_{ij}(v) < 1$.

ii) *If a periodically homogeneous Markov chain is periodically irreducible, then for all* $i, j \in E$ *and* $v \in \{0, \cdots, S-1\}$*, either* $L_{ij}(v) = 1$ *or* $L_{ij}(v) < 1$.

Proof i) If the chain is v-irreducible, then Propositions 4.3-4.5 imply that for every $i \in E$ either $L_{ii}(v) = 1$ or $L_{ii}(v) < 1$. Assume that $L_{ii}(v) = 1$ for each $i \in E$. If there exist $i, j \in E$ such that $L_{ij}(v) < 1$, then since the chain is v-irreducible, it follows that $E\left(\eta_j(v) | X_v = i\right) > 0$ and there exists $n \geq 1$ such that

$$P(X_{nS+v} = j, \tau_j(v) > n | X_v = j) > 0,$$

so

$$\sum_{n=1}^{\infty} P(X_{nS+v} = j, \tau_j(v) > n | X_v = j) = L_{jj}(v) < 1.$$

Hence, $L_{ii}(v) = 1$ for each i entails $L_{ij}(v) = 1$ for each $i, j \in E$.

i) The same argument shows that for every $v \in \{0, \cdots, S-1\}$, if $L_{ii}(v) = 1$ for all $i \in E$ then $L_{ij}(v) = 1$ for all $j \in E$. □

Remark 4.4 *i) The probability of first return to state* i *along the channel* v *at the* nS*th transition* $f_{ii}^{(nS)}(v)$ *(see (4.5b)) can be expressed recursively as*

$$f_{ii}^{(nS)}(v) = \sum_{k=0}^{n} f_{ii}^{(kS)}(v) P_{ii}^{((n-k)S)}, \quad n \geq 1$$

See also Karlin and Taylor (1975, p. 62) when $S = 1$.

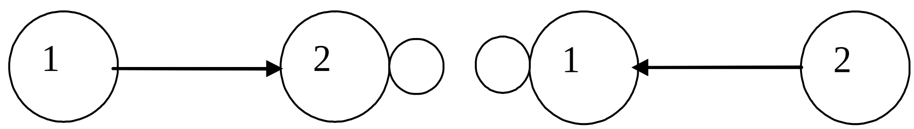
ii) A state i can be recurrent along a channel v and transient along another channel v'. Consider a 2-periodically homogeneous Markov chain valued in $E = \{1, 2\}$ with transition matrices

$$P(0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } P(1) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

The corresponding monodromy matrices take the form

$$\begin{aligned} \mathbb{P}(0) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \\ \mathbb{P}(1) &= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

and from the 2-step transition graphs



$\mathbb{G}(0)$ $\mathbb{G}(1)$

it is clear that the state 1 is 0-transient and 1-recurrent.

iii) For a periodically irreducible Markov chain, however, v-recurrence becomes a solidarity property along all channels $\{0, ..., S-1\}$ provided $E_t = E$ for all $t \in \mathbb{N}$. In other words:

Theorem 4.2 *A periodically irreducible Markov chain with a time-invariant state-space is recurrent along some v if and only if it is recurrent along all $v' \in \{0, 1, ..., S-1\}$.*

Proof Since the chain is periodically irreducible, it follows that for each $i, j \in E$ and $v \in \{0, ..., S-1\}$, $P_{ij}^{nS}(v) > 0$ and $P_{ij}^{mS}(v) > 0$ for some $n, m \in \mathbb{N}$. If a state i is v-recurrent and v'-transient then $\sum_{n=0}^{\infty} P_{ii}^{(nS)}(v) = \infty$ and $\sum_{n=0}^{\infty} P_{ii}^{(nS)}(v') < \infty$. Assume without loss of generality that $v' = v + 1$. Then the Chapman-Kolmogorov equations entail $P_{ii}^{(nS)}(v') =$

$\sum_{k \in E} P_{ik}^{(S-1)}(v') P_{ki}^{(nS+1-S)}(v)$ and hence $\sum_{n=0}^{\infty} P_{ii}^{(nS)}(v') \geq \sum_{n=0}^{\infty} P_{ii}^{(S-1)}(v') P_{ii}^{(nS+1-S)}(v)$. Repeating the same argument as many times as necessary, it follows that $\infty > \sum_{n=0}^{\infty} P_{ii}^{(nS)}(v') \geq \sum_{m=0}^{\infty} P_{ii}^{(mS)}(v) = \infty$, which is a contradiction. □

Thus, as for periodic irreducibility, we speak of periodic recurrence.

Definition 4.6 *A periodically irreducible Markov chain is called periodically recurrent if it is v-recurrent for every $v \in \{0, 1, ..., S-1\}$. Similarly, it is called periodically transient if it is v-transient for every $v \in \{0, 1, ..., S-1\}$.*

Remark 4.5 *i) When E is finite, checking whether a chain is v-recurrent can be simplified using the corresponding S-step v-graph: A communication class is v-recurrent if its correspondence in the S-step v-graph is closed, i.e. the exterior degree of the class in the reduced v-graph is null.*
ii) When E is countable, checking v-recurrence or v-transience is more involved. However, in the case of periodic irreducibility, there exist recurrence criteria based on stochastic Lyapunov (drift) functions as in the homogeneous case (see Karlin and Taylor, 1975*, Theorem 4.2 of Chapter 3).*

Theorem 4.3 *Let $(\mathbb{P}(v))_{0 \leq v \leq S-1}$ be a system of monodromy matrices of a periodically irreducible Markov chain valued in $\mathbb{N}$.*
i) The chain is periodically recurrent if for every v $(0 \leq v \leq S-1)$, there exists a sequence of positive functions $(V_j(v))_{j \in \mathbb{N}}$ satisfying $V_j(v) \to \infty$ as $j \to \infty$ and a finite positive integer $N(v)$ such that

$$\sum_{j=0}^{\infty} \mathbb{P}_{ij}(v) V_j(v) \leq V_i(v), \qquad i > N(v).$$

ii) The chain is periodically transient if and only if for all v $(0 \leq v \leq S-1)$ there exists a bounded non-constant function $V^{(v)}(j)$ $(j \in \mathbb{N})$ and a finite positive integer $N(v)$ such that

$$\sum_{j=0}^{\infty} \mathbb{P}_{ij}(v) V_j(v) = V_i(v), \quad i > N(v).$$

Proof Similar to that of Theorem 4.2 of Karlin and Taylor (1975, Chapter 3). □

Another important solidarity property is that of positive recurrence along a channel. When i is v-recurrent, then (in view of Proposition 4.5, ii)) $\sum_{i \in E} P_{ii}^{(nS)}(v) = \infty$, which implies that either $P_{ii}^{(nS)}(v) \to \pi_i(v) > 0$ or $P_{ii}^{(nS)}(v) \to \pi_i(v) = 0$ as $n \to \infty$. In the former case, the state i is called positive recurrent along v, or just v-positive recurrent. In the latter, i is called null recurrent along the channel v, or simply v-null recurrent. Based on the fact that $\pi_i(v) = \frac{1}{m_{ii}(v)}$ as will be shown in Theorem 5.1 below, another characterization of positive recurrence is given by the following definition.

Definition 4.7 *A state i is said to be positive recurrent along the channel v (or v-positive recurrent) if the average time to return to i is finite, i.e.*

$$m_{ii}(v) = E(\tau_i(v) | X_v = i) < \infty.$$

Otherwise, i.e. if $m_{ii}(v) = \infty$ then it is called v-null recurrent.

There is an obvious relationship between v-positive recurrence and v-recurrence.

Proposition 4.7 *If a state i is v-positive recurrent then it is v-recurrent.*

Proof Just write down the definitions while using Proposition 4.5, iii). □

v-Positive recurrence is also a solidarity/class property.

Proposition 4.8 *If $i \overset{v}{\leftrightsquigarrow} j$ then*

i is v-positive recurrent if and only j is.

As for v-recurrence, v-positive recurrence for a periodically irreducible chain becomes a property of solidarity across all channels $v \in \{0, 1, ..., S-1\}$.

Theorem 4.4 *A periodically irreducible Markov chain with a time-invariant state-space is positive recurrent along some channel v, if and only if it is positive recurrent along any other channel $v' \in \{0, 1, ..., S-1\}$.*

Proof The result follows using the same argument as in the proof of Theorem 4.2. □

Like periodic irreducibility and periodic recurrence, we speak of periodic positive recurrence.

Definition 4.8 *A periodically irreducible Markov chain is said to be periodically positive recurrent if it is v-positive recurrent for each $v \in \{0, ..., S-1\}$.*

Thus, in a periodically irreducible Markov chain, all the states share the same properties.

Proposition 4.9 *i) For a periodically irreducible Markov chain all states are either periodically null recurrent, periodically positive recurrent or periodically transient.*

ii) For a finite v-irreducible Markov chain all states are v-positive recurrent.

iii) For a finite periodically irreducible Markov chain all states are periodically positive recurrent.

Thus for finite time-periodic chains, periodic irreducibility implies periodic positive recurrence.

4.3.2 Periodic periodicity and periodic aperiodicity

Definition 4.9 *i) The period of a state i along a channel v is a positive integer $d_i(v) \in \mathbb{N}^* \cup \{\infty\}$ satisfying*

$$d_i(v) = gcd \left\{ n \geq 1 : P_{ii}^{(nS)}(v) > 0 \right\}.$$

ii) If $d_i(v) = 1$, the state i is called v-aperiodic.

iii) If i is not accessible from itself, then by convention $d_i(v) = \infty$.

Thus, in the context of periodically homogeneous Markov chains, there are two types of periodicity: periodicity regarding time (time-periodicity) and periodicity regarding states (state-periodicity).

Remark 4.6 *A state i can have different periods along different channels. For example, if a time-periodic chain is defined with a system of one-step transition probabilities*

$$P(0) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } P(1) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix},$$

then the monodromy matrices are

$$\mathbb{P}(0) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\mathbb{P}(1) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

Hence the period of the state 2 along the channel 0 is $d_2(0) = 1$ *and the period of the same state 2 along the channel 1 is* $d_2(1) = \infty$. $\square$

This leads to the following definition.

Definition 4.10 *The period* d_i *of a state* i *along all channels is the number* $d_i \in \mathbb{N}^* \cup \{\infty\}$ *given by*

$$d_i = \gcd\{n \geq 1 : P_{ii}^{(nS)}(v) > 0,\ 0 \leq v \leq S-1\},$$

with the convention that $gcd\{n, \infty\} = \infty$ *for* $n \geq 1$.

Naturally, Definition 4.10 implies that $d_i = d_i(v)$ for each $0 \leq v \leq S-1$. Note that v-periodicity is also a class property.

Proposition 4.10 i) *If* i *and* j *belongs to the same communication class along a channel* v, *then* $d_i(v) = d_j(v)$.

ii) *If a state* i *has a period* d_i *and* $i \overset{v}{\leftrightsquigarrow} j$ *for all* v, *then* $d_j = d_i$.

Proof i) Since $i \overset{v}{\leftrightsquigarrow} j$, there exist two positive integers n_v and m_v such that $P_{ij}^{(n_vS)}(v) > 0$ and $P_{ji}^{(m_vS)}(v) > 0$. By the Chapman-Kolmogorov equations,

$$P_{ii}^{((n_v+m_v)S)}(v) \geq P_{ij}^{(n_vS)}(v) P_{ji}^{(m_vS)}(v) > 0, \tag{4.11}$$

showing that $n_v + m_v$ is a multiple of $d_i(v)$. Let k be a positive integer that is not a multiple of $d_i(v)$. Then $k + n_v + m_v$ is not a multiple of $d_i(v)$ so

$$P_{ij}^{(n_vS)}(v) P_{jj}^{(kS)}(v) P_{ji}^{(m_vS)}(v) \leq P_{ii}^{((n_v+m_v+k)S)}(v) = 0. \tag{4.12}$$

Hence $P_{jj}^{(kS)}(v) = 0$, implying that $d_j(v) \geq d_i(v)$. The same argument, interchanging i and j in (4.12), yields $d_i(v) \geq d_j(v)$.

ii) Since $i \overset{v}{\leftrightsquigarrow} j$ for each $0 \leq v \leq S-1$, there exist n_v and m_v such that $P_{ki}^{(nS)}(v) > 0$ and $P_{jk}^{(mS)}(v) > 0$. Using the same argument as i), it follows that $d_i = d_j$. □

Definition 4.11 *A periodically irreducible Markov chain is said to be periodically d-periodic if it is d_i-periodic for each $i \in E$ with $d = d_i$ for all $i \in E$. If $d = 1$ the chain is called periodically aperiodic.*

For each non-empty subset $D \subset E$, denote by $P_{i,D}^{(nS)}(v) := \sum_{j \in D} P_{ij}^{(nS)}(v)$.

Proposition 4.11 i) *If a periodically homogeneous Markov chain is irreducible along a channel v and $d(v)$ is the period of all states in E along the channel v, then there exists a partition $D_1, ..., D_{d(v)}$ of E ($E = \bigcup_{l=1}^{d(v)} D_l$) such that $P_{i,D_{l+1}}^{(S)}(v) = 1$ for each $i \in D_l$, $l =$* $1, ..., d(v)$.

ii) *If a periodically homogeneous Markov chain is periodically irreducible and $(d(v))_{0 \leq v \leq S-1}$ are the periods of the states of E along all channels, then for each v there exists a partition $D_1, \cdots, D_{d(v)}$ of E such that result i) is satisfied.*

Proof i) Let $i, j \in E$. Since the chain is v-irreducible, there exist n_v and m_v such that $P_{ji}^{(n_v S)}(v) > 0$ and $P_{ij}^{(m_v S)}(v) > 0$. The Chapman-Kolmogorov equations entail $P_{ii}^{((n_v+m_v)S)}(v)$ > 0 and hence $n_v + m_v = ld(v)$ for some $l \geq 1$. Since $P_{ij}^{(m_v S)}(v) > 0$, it follows that m_v can be written as $m_v = ld(v) + r$ for some $l \in \mathbb{N}$, where $r \in \{1, ..., d(v)\}$. For each $r \in \{1, ..., d(v)\}$, set

$$D_r = \{j \in E : P_{ij}^{((ld(v)+r)S)}(v) > 0\},\ l \in \mathbb{N}.$$

Note that $i \in D_{d(v)}$ and $P_{i,D_1^c}^{(S)}(v) = 0$ together imply that $P_{i,D_1}^{(S)}(v) = 1$, where D_1^c is the complement of the set D_1. Likewise, it is easily seen that for each $j \in D_1$, $P_{j,D_2^c}^{(S)}(v) = 0$ so that $P_{j,D_2}^{(S)}(v) = 1$, and so on.

ii) If the chain is periodically irreducible, then it is irreducible along each channel $v \in \{0, ..., S-1\}$ and thus the property i) is satisfied for each $v \in \{0, ..., S-1\}$. □

5 Convergence to periodically stationary distributions and ergodic theorems

In this Section we will answer the questions of periodic stochastic stability raised in Section 3, namely the existence and uniqueness of the system of periodically stationary distributions and the convergence of the marginal distributions to this system independently of the initial distribution. The approach adopted for periodically homogeneous Markov chains consists in generalizing known proof techniques in the homogeneous case where three major approaches can be distinguished.

i) The approach based on the Perron-Frobenius theorem (e.g. Cox and Miller, 1965; Cinlar, 1975, Appendix; Seneta, 1980).

ii) The approach based on the coupling method (e.g. Lindvall, 1982; Norris, 2002).

iii) The approach based on the discrete renewal theorem (e.g. Karlin and Taylor, 1975, Theorem 1.1 and Theorem 1.2 of Chapter 3; Cinlar, 1975, Theorem 5.2.3 and (2.28) of Chapter 9).

The first approach can be recommended for finite Markov chains. However, it is of limited interest for countably infinite Markov chains and seems difficult to adapt in the inhomogeneous Markov case (see Seneta, 1980, Chapters 5, 6, and 7). The second approach seems the most elegant one because it is easily generalizable to the case of Markov chains with a general state space (Meyn and Tweedie, 2009). However, the third approach appears to be the simplest one in our context, which we adopt here.

The following results show that v-recurrence and v-aperiodicity together imply the existence of $\lim_{n\to\infty} \mathbb{P}_{ij}^{n}(v)$ independently of the state i while v-positive recurrence guarantees the positivity and hence the uniqueness of $\lim_{n\to\infty} \mathbb{P}_{ij}^{n}(v)$. Finally, periodic irreducibility together with the above two properties ensure the existence and uniqueness of the above limit along all channels and all states, and hence the existence and uniqueness of the system of periodically stationary distributions.

Theorem 5.1 *i) Let $(X_t, t \in \mathbb{N})$ be a v-irreducible, v-aperiodic, and v-recurrent S-periodically* *homogeneous Markov chain* $(0 \le v \le S-1)$*. Then for each* $i \in E$

$$\lim_{n\to\infty} P_{ii}^{(nS)}(v) = \frac{1}{\sum\limits_{n=0}^{\infty} n f_{ii}^{(nS)}(v)} = \frac{1}{m_{ii}(v)}. \tag{5.1}$$

ii) If $(X_t, t \in \mathbb{N})$ *is periodically irreducible, periodically aperiodic, and periodically recurrent, then* (5.1) *holds for each* $i \in E$ *and each* $v \in \{0, ..., S-1\}$*.*

Proof i) In view of Remark 4.3, i), the probability $P_{ii}^{(nS)}(v)$ satisfies the following renewal equation

$$P_{ii}^{(nS)}(v) = \sum_{k=0}^{n} f_{ii}^{(kS)}(v)\, P_{ii}^{((n-k)S)} + 1_{[n=0]}, \quad n \in \mathbb{N}. \tag{5.2}$$

Applying Theorem A.1 (see Appendix A) with $u_n = P_{ii}^{(nS)}(v)$, $a_n = f_{ii}^{(nS)}(v)$, and $b_n = 1_{[n=0]}$ gives

$$\lim_{n\to\infty} P_{ii}^{(nS)}(v) = \frac{\sum\limits_{k=0}^{\infty} b_k}{\sum\limits_{n=0}^{\infty} n a_n} = \frac{1}{m_{ii}(v)}$$

where $m_{ii}(v) = \sum\limits_{n=0}^{\infty} n f_{ii}^{(nS)}(v)$ is the expected time of return to state i along the channel v.

ii) The result is a simple consequence of i). □

The following Theorem shows that under the same conditions of Theorem 5.1, the limit $\lim\limits_{n\to\infty} P_{ij}^{(nS)}(v)$ exists independently of the initial state i.

Theorem 5.2 *i) Under the same conditions of Theorem 5.1, i),*

$$\lim_{n\to\infty} P_{ij}^{(nS)}(v) = \lim_{n\to\infty} P_{jj}^{(nS)}(v) \quad \textit{for every } i, j \in E. \tag{5.3}$$

ii) Under the same conditions of Theorem 5.1, ii), (5.3) *holds for all* $0 \le v \le S-1$*.*

Proof i) From Remark 4.3, i), it follows that

$$P_{ij}^{(nS)}(v) = \sum_{k=0}^{n} f_{ij}^{(kS)}(v)\, P_{jj}^{((n-k)S)} \text{ for } i \ne j,\, n \ge 0. \tag{5.4}$$

Hence, applying Theorem A.1, ii) with $y_n = P_{ij}^{(nS)}(v)$, $a_n = f_{ij}^{(nS)}(v)$, and $x_n = P_{jj}^{(nS)}$ gives the desired result.

ii) The result is an obvious consequence of i). □

For a finite time-periodic chain, periodic irreducibility implies that all states are periodically positive recurrent. If, in addition, the chain is periodically aperiodic then $P_{ij}^{(nS)}(v) = \pi_j(v) > 0$ exists and is positive of all $v \in \{0, ..., S-1\}$, and all $i, j \in E$. Thus, the system of periodically stationary distributions $(\pi_j(v))_{0 \le v \le S-1, j \in E}$ is uniquely determined by (3.4).

When the chain is infinite, the following result extends (3.4).

Theorem 5.3 *i) For a v-irreducible, v-aperiodic, and v-positive recurrent time-periodic Markov chain $(X_t, t \in \mathbb{N})$ valued in $\mathbb{N}$,*

$$\lim_{n\to\infty} P_{jj}^{(nS)}(v) = \pi_j^{(v)}, \tag{5.5}$$

where the $\left(\pi_j^{(v)}\right)_{j\in\mathbb{N}}$ are uniquely determined from the equations

$$\pi_j^{(v)} = \sum_{i=0}^{\infty} \pi_i^{(v)} P_{ij}^{(S)}(v), \tag{5.6a}$$

$$\sum_{j=0}^{\infty} \pi_j^{(v)} = 1, \quad \pi_j^{(v)} > 0. \tag{5.6b}$$

ii) If $(X_t, t \in \mathbb{N})$ is periodically irreducible, periodically aperiodic, and periodically positive recurrent with states in $\mathbb{N}$ then (5.5) and (5.6) are satisfied for every $v \in \{0, ..., S-1\}$.

Proof i) Given positive integers n and K, we have $\sum\limits_{j=0}^{\infty} P_{ij}^{(n)}(v) = 1 \ge \sum\limits_{j=0}^{K} P_{ij}^{(n)}(v)$. Taking the limit as $n \to \infty$ while using Theorem 5.1, we obtain $\sum\limits_{j=0}^{K} \pi_j^{(v)} \le 1$ for every K and hence $\sum\limits_{j=0}^{\infty} \pi_j^{(v)} \le 1$. On the other hand, the Chapman-Kolmogorov equations yield

$$\begin{aligned} P_{ij}^{(nS+S)}(v) &= \sum_{k=0}^{\infty} P_{ik}^{(nS)}(v) P_{kj}^{(S)}(v) \\ &\ge \sum_{k=0}^{K} P_{ik}^{(nS)}(v) P_{kj}^{(S)}(v), \end{aligned} \tag{5.7}$$

and by taking the limit as $n \to \infty$,

$$\pi_j^{(v)} \ge \sum_{k=0}^{K} \pi_k^{(v)} P_{kj}^{(S)}(v) \text{ for every } K$$

so that

$$\pi_j^{(v)} \geq \sum_{k=0}^{\infty} \pi_k^{(v)} P_{kj}^{(S)}(v). \tag{5.8}$$

Multiplying both sides of (5.8) by $P_{ji}^{(S)}(v)$ and summing over j gives

$$\pi_j^{(v)} \geq \sum_{k=0}^{\infty} \pi_k^{(v)} P_{kj}^{(2S)}(v)$$

and repeating this argument yields

$$\pi_j^{(v)} \geq \sum_{k=0}^{\infty} \pi_k^{(v)} P_{kj}^{(nS)}(v) \text{ for each } n. \tag{5.9}$$

If the strict inequality in (5.9) holds, i.e.

$$\sum_{k=0}^{\infty} \pi_k^{(v)} P_{kj}^{(nS)}(v) < \pi_j^{(v)}$$

then

$$\begin{aligned} \sum_{j=0}^{\infty} \pi_j^{(v)} &> \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \pi_k^{(v)} P_{kj}^{(nS)}(v) \\ &= \sum_{k=0}^{\infty} \pi_k^{(v)} \sum_{j=0}^{\infty} P_{kj}^{(nS)}(v) \\ &= \sum_{k=0}^{\infty} \pi_k^{(v)} \end{aligned}$$

leading to a contradiction. Hence (5.9) becomes

$$\pi_j^{(v)} = \sum_{k=0}^{\infty} \pi_k^{(v)} P_{kj}^{(nS)}(v) \text{ for each } n. \tag{5.10}$$

Taking $n \to \infty$ in (5.10) and using the convergence of $\sum_{k=0}^{\infty} \pi_k^{(v)}$ and the uniform boundedness of $(P_{kj}^{(nS)}(v))_n$ we get

$$\begin{aligned} \pi_j^{(v)} &= \sum_{k=0}^{\infty} \pi_k^{(v)} \lim_{n \to \infty} P_{kj}^{(nS)}(v) \\ &= \pi_j^{(v)} \sum_{k=0}^{\infty} \pi_k^{(v)}. \end{aligned} \tag{5.11}$$

Since the chain is v-irreducible and v-positive recurrent, it follows that $\pi_j^{(v)} > 0$ for all j, so (5.11) entails $\sum_{k=0}^{\infty} \pi_k^{(v)} = 1$, proving (5.6$b$). The proof of the existence of $(\pi_j^{(v)})$ in (5.6b) is thus completed while taking $n \to \infty$ in (5.7), giving $\pi_j^{(v)} = \sum_{i=0}^{\infty} \pi_i^{(v)} P_{ij}^{(S)}(v)$.

To prove the uniqueness of $(\pi_j^{(v)})_j$, let $(\mu_j^{(v)})_j$ be a sequence of real numbers satisfying (5.6). Then

$$\begin{aligned} \mu_j^{(v)} &= \sum_{i=0}^{\infty} \mu_i^{(v)} P_{ij}^{(S)}(v) \\ &= \sum_{i=0}^{\infty} \mu_i^{(v)} \lim_{n\to\infty} P_{ij}^{(nS)}(v) \\ &= \pi_j^{(v)} \sum_{i=0}^{\infty} \mu_i^{(v)} \\ &= \pi_j^{(v)} \text{ for all } j. \end{aligned}$$

ii) A consequence of i). □

The limiting results introduced so far concern the distributions of the time-periodic chain (ensemble average) and not its sample paths (sample average). We now examine the behavior of the sequence of sample means of the time-periodic chain. We first consider the case where the chain is initialized from its periodically stationary distributions (i.e. the chain is strictly periodically stationary) and is periodically irreducible (i.e. periodically ergodic in the sense of Appendix B).

Let $\xi_j^{(n)}(v) = \frac{1}{n} \sum_{k=1}^{n} 1_{[X_{v+kS}=j]}$ be the mean number of visits of state j along the channel v up to time $nS+v$. If the chain $(X_t, t \in \mathbb{N})$ is periodically stationary then

$$\begin{aligned} E\left(\xi_j^{(n)}(v)\right) &= \frac{1}{n} \sum_{k=1}^{n} P\left(X_{kS+v} = j\right) \\ &= \pi_j^{(v)} \text{ for all } v \text{ and } j. \end{aligned} \tag{5.12}$$

If, in addition, $(X_t, t \in \mathbb{N})$ is periodically irreducible then the point-wise ergodic theorem (e.g. Aknouche, 2008) entails

$$\frac{1}{n} \sum_{k=1}^{n} 1_{[X_{v+kS}=j]} \underset{n\to\infty}{\overset{a.s.}{\to}} E\left(1_{[X_{v+kS}=j]}\right) = P\left(X_{v+kS} = j\right) = \pi_j^{(v)}. \tag{5.13}$$

An extension of (5.13) is given by the following result.

Theorem 5.4 *For a S-periodically irreducible and periodically stationary time-periodic Markov chain $(X_t, t \in \mathbb{N})$ valued in $\mathbb{N}$,*

$$\frac{1}{n}\sum_{k=1}^{n} f\left(X_k\right) \underset{n\to\infty}{\overset{a.s.}{\to}} \frac{1}{S}\sum_{v=1}^{S}\sum_{j=0}^{\infty} f\left(j\right)\pi_j^{(v)}, \tag{5.14}$$

where f is a measurable bounded real-valued function and $\underset{n\to\infty}{\overset{a.s.}{\to}}$ denotes the almost sure convergence as $\to \infty$.

Proof For every $n \in \mathbb{N}^*$, setting $n = v + mS$ $(1 \leq v \leq S)$, we have

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n} f\left(X_k\right) &= \frac{1}{n}\sum_{k=1}^{n}\sum_{j=0}^{\infty} f\left(j\right) 1_{[X_k=j]} \\ &= \frac{1}{mS+v}\sum_{k=1}^{m}\sum_{v=1}^{S}\sum_{j=0}^{\infty} f\left(j\right) 1_{[X_{kS+v}=j]} + \frac{1}{mS+v}\sum_{v'=1}^{v}\sum_{j=0}^{\infty} f\left(j\right) 1_{[X_{kS+v}=j]} \\ &= \frac{mS}{mS+v}\sum_{j=0}^{\infty}\frac{1}{S}\sum_{v=1}^{S}\frac{1}{m}\sum_{k=1}^{m} f\left(j\right) 1_{[X_{kS+v}=j]} + \frac{1}{mS+v}\sum_{v'=1}^{v}\sum_{j=0}^{\infty} f\left(j\right) 1_{[X_{kS+v}=j]}. \end{aligned} \tag{5.15}$$

Letting $n \to \infty$ and $m \to \infty$ so $\frac{mS}{mS+v} \to 1$, we obtain

$$\frac{1}{mS+v}\sum_{v'=1}^{v}\sum_{j=0}^{\infty} f\left(j\right) 1_{[X_{kS+v}=j]} \underset{n\to\infty}{\overset{a.s.}{\to}} 0.$$

By periodic stationarity and periodic ergodicity of $(X_t, t \in \mathbb{N})$ and hence of $\left(f\left(X_t\right), t \in \mathbb{N}\right)$ (cf. Appendix B), the ergodic theorem yields

$$\frac{1}{m}\sum_{k=1}^{m} f\left(j\right) 1_{[X_{kS+v}=j]} \underset{n\to\infty}{\overset{a.s.}{\to}} E\left(f\left(j\right) 1_{[X_{kS+v}=j]}\right) = f\left(j\right)\pi_j^{(v)}. \tag{5.16}$$

Hence in view of (5.15) and (5.16), it follows that

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n} f\left(X_k\right) &= \sum_{j=0}^{\infty}\frac{1}{S}\sum_{v=1}^{S}\frac{1}{m}\sum_{k=1}^{m} f\left(j\right) 1_{[X_{kS+v}=j]} \\ &\underset{n\to\infty}{\overset{a.s.}{\to}} \frac{1}{S}\sum_{v=1}^{S}\sum_{j=0}^{\infty} f\left(j\right)\pi_j^{(v)}, \end{aligned}$$

which proves the desired result. □

Theorem 5.4 does not require periodic aperiodicity and periodic positive recurrence since the chain has already been assumed to be periodically stationary and periodically irreducible (hence periodically ergodic). However, when the chain is not periodically stationary, periodic irreducibility is no longer synonymous with periodic ergodicity (in the sense of Appendix B) and the point-wise ergodic theorem can no longer be applicable. In this situation, periodic aperiodicity and positive periodic recurrence are required to obtain a variant of Theorem 5.4 where convergence is instead valid for the means of the sample means.

Theorem 5.5 *If* $(X_t, t \in \mathbb{N})$ *is periodically irreducible, periodically aperiodic, and periodically positive recurrent with states in* $\mathbb{N}$ *then*

$$\tfrac{1}{n}\sum_{k=1}^{n} E\left(f\left(X_k\right)\right) \to \tfrac{1}{S}\sum_{v=1}^{S}\sum_{j=0}^{\infty} f\left(j\right)\pi_j^{(v)} \text{ as } n \to \infty$$

where f *is a measurable bounded real-valued function.*

Proof Using the same arguments in the proof of Theorem 5.4 (see (5.15)) we have

$$\tfrac{1}{n}\sum_{k=1}^{n} f\left(X_k\right) = \sum_{j=0}^{\infty}\tfrac{1}{S}\sum_{v=1}^{S}\tfrac{1}{m}\sum_{k=1}^{m} f\left(j\right) 1_{[X_{kS+v}=j]} + R_n, \tag{5.17}$$

where R_n is a term satisfying $E\left(R_n\right) \to 0$ as $n \to \infty$.

Now, periodic irreducibility, periodic aperiodicity, and periodic positive recurrence together imply

$$\begin{aligned}\lim_{m\to\infty} E\left(\tfrac{1}{m}\sum_{k=1}^{m} f\left(j\right) 1_{[X_{kS+v}=j]}\right) &= \lim_{m\to\infty}\tfrac{1}{m}\sum_{k=1}^{m} f\left(j\right) E\left(1_{[X_{kS+v}=j]}\right)\\ &= \lim_{m\to\infty}\tfrac{1}{m}\sum_{k=1}^{m} f\left(j\right) P\left(X_{kS+v}=j\right)\\ &= \lim_{m\to\infty}\tfrac{1}{m}\sum_{k=1}^{m} f\left(j\right)\pi_j^{(v)}\\ &= f\left(j\right)\pi_j^{(v)},\end{aligned}$$

so

$$\lim_{m\to\infty} E\left(\frac{1}{n}\sum_{k=1}^{n} f(X_k)\right) = \frac{1}{S}\sum_{v=1}^{S}\sum_{j=0}^{\infty} f(j)\,\pi_j^{(v)}$$

and thus $\frac{1}{n}\sum\limits_{k=1}^{n} Ef(X_k)$ converges to $\frac{1}{S}\sum\limits_{v=1}^{S}\sum\limits_{j=0}^{\infty} f(j)\,\pi_j^{(v)}$ as $n\to\infty$. □

6 Applications

6.1 Periodic Markov decision process

A periodic Markov decision process (PMDP in short) is a stochastic process $((X_t, D_t, C_t), t \in \mathbb{N})$ defined on a probability space $(\Omega, \mathcal{F}, P)$ where the triple (X_t, D_t, C_t) is described as follows:

i) X_t is a random variable valued in a finite set E_t $(t\in\mathbb{N})$ and represents the state of a system of interest at time t. The whole state-space of the system is $E = \bigcup\limits_{t\in\mathbb{N}} E_t = \{1, ..., K\}$ and is supposed to be finite.

ii) D_t $(t\in\mathbb{N})$ is a decision rule specifying at each time t which action a to be taken in a finite set of actions $A_i(t)$ when $X_t = i$ for some $i \in E_t$. The action space is denoted by $A = \bigcup\limits_{t\in\mathbb{N}, i\in E_t} A_i(t)$ and its number of elements by $|A|$. Thus, D_t is a A_t-valued measurable function of X_t.

iii) C_t $(t\in\mathbb{N})$ is a random variable representing the cost (or reward) incurred by choosing an action $a\in A$ when the system moves from a certain state at time t to another state at the following time. Hence, C_t is a measurable function of X_t, X_{t+1} and D_t. Given $X_t = i$, $X_{t+1} = j$, and $D_t = a$, the value of the reward C_t at time t is denoted by $C_{ij}(t,a)$, i.e.

$$C_{ij}(t,a) = C_t | \left[X_t = i, X_{t+1} = j, D_t = a\right]. \tag{6.1a}$$

Likewise, denote by

$$C_{ij}^{(n)}(t,a) = C_t | \left[X_t = i, X_{t+n} = j, D_t = a\right] \tag{6.1b}$$

the cost incurred by choosing an action $a\in A$ when the system moves from state i at time t to state j at time $t+n$.

We further assume that the model is a periodically homogeneous Markov chain in the sense that:

a) $(X_t, t \in \mathbb{N})$ is a finite S-periodically homogeneous, periodically regular, and periodically stationary Markov chain with transition probabilities

$$P_{ij}(t,a) := P(X_{t+1} = j | X_t = i, D_t = a) := P_{ij}(t + kS, a), \; k \in \mathbb{N} \tag{6.2a}$$

and S-periodically stationary marginal distribution

$$P(X_{t+1} = i | D_t = a) = \pi_i(t,a), \tag{6.2b}$$

where $(\pi_i(t,a))_{i,t}$ is S-periodic in t and satisfies (3.4) for each $a \in A$. Since $(X_t, t \in \mathbb{N})$ is finite, its periodic regularity is ensured by just assuming that it is periodically irreducible and periodically aperiodic.

b) The parameters $A_i(t)$ and $C_{ij}(t,a)$ are S-periodic over $t \in \mathbb{N}$. So letting $t = kS + v$ ($v \in \{0, ..., S-1\}$, $k \in \mathbb{N}$), E_t, $A_i(t)$, $C_{ij}(t,a)$, and $\pi_i(t,a)$ can be rewritten as E_v, $A_i(v)$, $C_{ij}(v,a)$, and $\pi_i(v,a)$, respectively. In particular, $C_{ij}^{(S)}(v,a) := \mathbb{C}_{ij}(v,a)$ denotes the cost incurred by choosing an action $a \in A$ when the system moves from the state i at time $nS+v$ to the state j at time $(n+1)S+v$.

A policy $\mathbf{\Delta}$ is a sequence of decision rules and is nothing but the decision process $\mathbf{\Delta} := (D_t, t \in \mathbb{N})$. By periodic homogeneity of the model, $\mathbf{\Delta}$ can be described by a $S \times K$ matrix such that $\mathbf{\Delta}(v,i)$ denotes the action to be taken when the system is in the state i at a time multiple of S modulo $v \in \{0, ..., S-1\}$. A policy $\mathbf{\Delta}$ is called *pure* when the probability at each time to take a given action is either 0 or 1. If not, it is called *randomized*. A pure policy is called *periodically stationary* if whenever the system is in the state i at the time period $kS + v$ ($0 \le v \le S-1$, $k \in \mathbb{N}$), the same action $a \in A_i(v)$ is taken independently of k. Obviously, the number of periodically stationary policies is finite and is equal to

$$\prod_{v=0}^{S-1} \prod_{i \in E_v} |A_i(v)|. \tag{6.3}$$

As long as a periodically stationary policy Δ is involved, the transition probability

$P_{ij}(t,a)$ in (6.2a) as well as the marginal distribution (6.2b) will be simply denoted by $P_{ij}(t,\Delta)$ and $\pi_i(t,\Delta)$, respectively.

The aim is to determine the optimal periodically stationary policy in the sense of a certain criterion. The most used criterion for non-periodic MDPs is to minimize the long-term expected cost per a time-unit. In the periodic context, this criterion takes the form

$$g(\Delta) = \tfrac{1}{S}\sum_{v=0}^{S-1}\sum_{i=1}^{K}\pi_i(v,\Delta)\mathcal{C}_i^{(S)}(v,\Delta), \tag{6.4}$$

where $\mathcal{C}_i^{(S)}(v,\Delta) = \sum_{j=1}^{K} P_{ij}^{(S)}(v,\Delta)C_{ij}^{(S)}(v,\Delta)$. In fact, $\mathcal{C}_i^{(S)}(v,\Delta)$ is interpreted as being the expected cost over the next S time-units if the system is in state i at a time multiple of S modulo v. Optimal policy determination is thus expressed through the following discrete optimization problem

$$\Delta^* = \arg\min_{\Delta\in\Pi} g(\Delta), \tag{6.5}$$

where Π is the set of all periodically stationary policies.

Problem (6.5) can be solved by enumerating all possible periodically stationary policies and choosing the one with minimum $g(\Delta)$. However, the number of periodically stationary policies can be extremely large and this solution is in general very time-consuming. Carton (1963) and Riis (1965) extended Howard's algorithm to solve (6.5). Other approaches using linear programming could also be used (cf. Aknouche and Kahoul, 2010).

On a final note, when the decision process $\mathbf{\Delta} := (D_t, t \in \mathbb{N})$ degenerates at a singleton $\{a\}$ for some action a, the PMDP process $((X_t, D_t, C_t), t \in \mathbb{N})$ reduces to $(X_t, t \in \mathbb{N})$ which is a periodically homogeneous Markov chain as defined in Section 2. Thus a PMDP is a strict extension of a time-periodic Markov chain.

6.2 Periodic integer-valued time series models

6.2.1 Periodic binomial AR(1)

A time-periodic extension of the first-order binomial integer-valued autoregressive (BINAR(1)) model proposed by McKenzie (1985) is defined as

$$\begin{aligned} X_t &= \alpha_t \circ X_{t-1} + \beta_t \circ (n - X_{t-1}), \quad t \in \mathbb{N}^* \\ &= \sum_{l=1}^{X_{t-1}} \xi_l^{(t-1)} + \sum_{l=1}^{n-X_{t-1}} \zeta_l^{(t-1)}, \end{aligned} \tag{6.6}$$

where X_0 is a random variable. The sequences $\left(\xi_l^{(t-1)}\right)$ and $\left(\zeta_l^{(t-1)}\right)$ are ipd_S Bernoulli distributed with S-periodic means (α_t) and (β_t), respectively, i.e. $E(\xi_l^{(t-1)}) = \alpha_t = \alpha_{t+S} \in (0,1)$ and $E(\zeta_l^{(t-1)}) = \beta_t = \beta_{t+S} \in (0,1)$ for all integer t. The symbol $\circ$ stands for the binomial random sum operator (Steutel and van Harn, 1978) defined as $\alpha_t \circ X_{t-1} := \sum_{l=1}^{X_{t-1}} \xi_l^{(t-1)}$, where $\left(\xi_l^{(t-1)}\right)$ are independent of X_{t-1}.

Clearly, $(X_t, t \in \mathbb{N})$ is a periodically homogeneous Markov chain on the finite state space $E := \{0, ..., n\}$. Given $i, j \in E$, the one-step transition probabilities $P_{ij}(t) := P(X_{t+1} = j | X_t = i)$ have the form

$$\begin{aligned} P_{ij}(t) &= P\left(\sum_{l=1}^{X_t} \xi_l^{(t)} + \sum_{l=1}^{n-X_{t-1}} \zeta_l^{(t-1)} = j | X_t = i\right) \\ &= \sum_{k=\max(0,i+j-n)}^{\min(i,j)} \binom{i}{k}\binom{n-i}{j-k} \alpha_t^k (1-\alpha_t)^{i-k} \beta_t^{j-k} (1-\beta_t)^{n-i-j+k}. \end{aligned} \tag{6.7}$$

Since $P_{ij}(t) > 0$ for all $t \in \mathbb{N}$ and all $i, j \in \{0, ..., n\}$, so are the monodromy probabilities $\mathbb{P}_{ij}(v) := P(X_{v+S} = j | X_v = i)$ for all $i, j \in E$ and $v \in \{0, ..., S-1\}$. Consequently, the chain $(X_t, t \in \mathbb{N})$ is v-irreducible and v-aperiodic for each $v \in \{0, ..., S-1\}$ and thus is periodically irreducible and periodically aperiodic. This implies that

$$\lim_{n \to \infty} P_{ij}^{(nS)}(v) = \pi_j^{(v)} \text{ exists} \tag{6.8}$$

for all v and $i, j \in \{0, ..., n\}$. Since $(X_t, t \in \mathbb{N})$ is finite and periodically irreducible, all states are v-positive recurrent along each $v \in \{0, ..., S-1\}$ and therefore $(X_t, t \in \mathbb{N})$ is periodically positive recurrent, implying that $\pi_j^{(v)} > 0$ for every j and v. Consequently, $(X_t, t \in \mathbb{N})$ is periodically regular (stable) and therefore the limit in (6.8) is unique and positive. The system of periodically stationary distributions $\left(\pi_j^{(v)}, j \in \{0, ..., n\}\right)_{0 \leq v \leq S-1}$ thus satisfies

$$\pi_j^{(v)} = \sum_{k=0}^{n} \pi_k^{(v)} P_{kj}^{(S)}(v), \ 0 \leq v \leq S-1, \tag{6.9}$$

which is simply (3.4), where

$$P_{kj}^{(S)}(v) = \mathbb{P}_{kj}(v) := P\left(X_{v+S} = j | X_v = k\right)$$

can be obtained from (6.7) by iterating (6.6) S times.

6.2.2 Periodic integer-valued ARCH(1) model

The first-order periodic Poisson integer-valued ARCH(1) (PINARCH) process (Grunwald et *al*, 2000) is given by

$$X_t | X_{t-1} \sim \mathcal{P}\left(\omega_t + \alpha_t X_{t-1}\right), \ \ t \in \mathbb{N}^* \tag{6.10}$$

where X_0 is a given random variable, $\omega_t > 0$, $\alpha_t \geq 0$ and $\mathcal{P}(\lambda)$ stands for the Poisson distribution with mean $\lambda > 0$. It is assumed that (ω_t) and (α_t) are periodic with period $S \geq 1$ in the sense $\omega_t = \omega_{t+S}$ and $\alpha_t = \alpha_{t+S}$ for all t. When $S = 1$, model (6.10) reduces to the Poisson INARCH introduced by Grunwald et *al* (2000). It is also a particular instance of the Poisson INGARCH studied by Aknouche, Bendjeddou and Touche (2018) and Aknouche et *al* (2022b). Clearly, $(X_t, t \in \mathbb{N})$ is a time-periodic Markov chain with transition probabilities

$$\begin{aligned} P_{ij}(v) &= P\left(X_{v+1} = j | X_v = i\right) \\ &= e^{-(\omega_v + \alpha_v i)} \tfrac{(\omega_v + \alpha_v i)^j}{j!}, \ i, j \in \mathbb{N}, \ 0 \leq v \leq S-1. \end{aligned} \tag{6.11}$$

The monodromy transition probability has a complicated expression and we give it first for the case $S = 2$,

$$\begin{aligned} \mathbb{P}_{ij}(v) &= P\left(X_{v+2} = j | X_v = i\right) = \sum_{k=0}^{\infty} P_{ik}(v) P_{kj}(v+1) \\ &= \sum_{k=0}^{\infty} e^{-(\omega_v + \alpha_v i)} \tfrac{(\omega_v + \alpha_v i)^k}{k!} e^{-(\omega_{v+1} + \alpha_{v+1} k)} \tfrac{(\omega_{v+1} + \alpha_{v+1} k)^j}{j!}. \end{aligned} \tag{6.12}$$

Since $\omega_v > 0$, the chain $(X_t, t \in \mathbb{N})$ is periodically irreducible and periodically aperiodic. To show its periodic recurrence, we use Theorem 4.3. Taking $V_j(v) = j$ for all v as a Lyapunov function, it is clear that

$$V_j(v) \underset{j\to\infty}{\to} \infty.$$

In addition,

$$\begin{aligned}
\sum_{j=0}^{\infty} \mathbb{P}_{ij}(v)\, V_j(v) &= \sum_{k=0}^{\infty} e^{-(\omega_v+\alpha_v i)} \tfrac{(\omega_v+\alpha_v i)^k}{k!} e^{-(\omega_{v+1}+\alpha_{v+1}k)} (\omega_{v+1}+\alpha_{v+1}k) \sum_{j=1}^{\infty} \tfrac{(\omega_{v+1}+\alpha_{v+1}k)^{j-1}}{(j-1)!} \\
&= \sum_{k=0}^{\infty} e^{-(\omega_v+\alpha_v i)} \tfrac{(\omega_v+\alpha_v i)^k}{k!} (\omega_{v+1}+\alpha_{v+1}k) \\
&= \omega_{v+1} + \alpha_{v+1}(\omega_v+\alpha_v i) \sum_{k=0}^{\infty} e^{-(\omega_v+\alpha_v i)} \tfrac{(\omega_v+\alpha_v i)^{k-1}}{(k-1)!} \\
&= \omega_{v+1} + \alpha_{v+1}\omega_v + \alpha_{v+1}\alpha_v V_i(v).
\end{aligned}$$

If

$$\alpha_{v+1}\alpha_v < 1, \tag{6.13}$$

then

$$\sum_{j=0}^{\infty} \mathbb{P}_{ij}(v)\, V_j(v) < V_i(v)$$

as long as

$$i > \frac{\omega_{v+1} + \alpha_{v+1}\omega_v}{1 - \alpha_{v+1}\alpha_v} := N_i(v).$$

Therefore, the chain $(X_t, t \in \mathbb{N})$ is periodically recurrent under (6.13). For general S, using the same argument, simple but tedious computations show that under

$$\prod_{v=0}^{S-1} \alpha_v < 1, \tag{6.14}$$

$$\sum_{j=0}^{\infty} \mathbb{P}_{ij}(v)\, V_j(v) < V_i(v)$$

provided that

$$i > \frac{\sum_{j=1}^{S} \prod_{i=0}^{j-1} \alpha_{v+S-1-i}\omega_{v+S-j}}{1 - \prod_{v=0}^{S-1} \alpha_v} := N_i(v).$$

If the chain were periodically null recurrent, then $\lim_{n\to\infty} P_{ij}^{nS}(v) = \pi_j^{(v)} = 0$ for all $0 \leq v \leq S-1$ and $i, j \in \mathbb{N}$, so taking $S = 2$, the equations

$$\begin{aligned} \pi_j^{(v)} &= \sum_{k=0}^{\infty} e^{-(\omega_{v+1}+\alpha_{v+1}k)} \tfrac{(\omega_{v+1}+\alpha_{v+1}k)^j}{j!} \sum_{i=0}^{\infty} \pi_i^{(v)} e^{-(\omega_v+\alpha_v i)} \tfrac{(\omega_v+\alpha_v i)^k}{k!} \qquad (6.15) \\ \sum_{j=0}^{\infty} \pi_j^{(v)} &= 1, \ \pi_j^{(v)} > 0, \end{aligned}$$

would have no solution. Since $\omega_v > 0$ for every v, (6.15) has a unique solution and hence the chain is periodically regular.

Condition (6.14) coincides with the periodic ergodicity condition for the periodic INARCH(1) as proposed by Aknouche et *al* (2018).

6.2.3 Periodic integer-valued autoregressive (INAR(1)) model

The periodic INAR(1) ($\text{PINAR}_S(1)$) model proposed by Monteiro et *al* (2010) is given by

$$\begin{aligned} X_t &= \phi_t \circ X_{t-1} + \varepsilon_t, \ \ t \in \mathbb{N}^* \qquad (6.16) \\ &= \sum_{l=1}^{X_{t-1}} \xi_l^{(t-1)} + \varepsilon_t, \end{aligned}$$

where X_0 is a random variable, $(\varepsilon_t, \ t \in \mathbb{N})$ is a Poisson distributed ipd_S sequence with mean $\mu_t > 0$, and $(\xi_l^{(t-1)})_l$ is an ipd_S Bernoulli distributed sequence with S-periodic mean ϕ_t. It is assumed that $(\xi_l^{(t-1)})_l$ is independent of $(\varepsilon_t, \ t \in \mathbb{N})$ and X_{t-1}, and that μ_t is S-periodic in t. For every $i, j \in \mathbb{N}$, $P_{ij}(t) := P(X_{t+1} = j | X_t = i)$ has the form

$$P_{ij}(t) = \sum_{k=0}^{\min(i,j)} \tbinom{i}{k} \phi_t^k (1-\phi_t)^{i-k} e^{-\mu_t} \tfrac{\mu_t^{j-k}}{(j-k)!} > 0. \qquad (6.17)$$

Thus the chain is periodically irreducible and periodically aperiodic. In addition, Monteiro et *al* (2010) showed that for all $v \in \{0, ..., S-1\}$ the homogeneous chain $(X_{nS+v}, n \in \mathbb{N})$ is positive recurrent. So the chain $(X_t, t \in \mathbb{N})$ is periodically regular.

6.2.4 Periodic integer-valued RCA(1) model

Let $(\Phi_t, \ t \in \mathbb{Z})$ and $(\varepsilon_t, \ t \in \mathbb{Z})$ be mutually independent $\mathbb{N}$-valued ipd_S sequences where $E(\Phi_t) = \phi_t > 0$, $Var(\Phi_t) = \delta_t^2 \geq 0$, $E(\varepsilon_t) = \mu_t > 0$, and $Var(\varepsilon_t) = \sigma_t^2 > 0$ are all S-

periodic in t. A $\mathbb{N}$-valued process $(X_t,\ t \in \mathbb{Z})$ is said to be a periodic integer-valued random coefficient AR(1) model, in short PINRCA$_S$(1), if X_t admits the representation

$$X_t = \Phi_t X_{t-1} + \varepsilon_t, \quad t \in \mathbb{Z}. \tag{6.18a}$$

A $\mathbb{N}$-evolution version of (6.18a) is

$$X_t = \Phi_t X_{t-1} + \varepsilon_t, \quad t \in \mathbb{N}^* \tag{6.18b}$$

for a given integer-valued initial variable X_0. Model (6.18) is a time-periodic extension of the RMINAR model introduced by Aknouche et *al* (2023). The distribution of the input sequences $(\Phi_t, t \in \mathbb{N}^*)$ and $(\varepsilon_t, t \in \mathbb{N}^*)$ can be specified (binomial, Poisson, negative binomial, etc.). Note that the PINRCA(1) process $(X_t,\ t \in \mathbb{N})$ defined by (6.18) is a periodically homogeneous Markov chain with an initial distribution

$$\pi_j(0) = P(X_0 = j), \, j \in \mathbb{N}$$

and transition probabilities given by

$$\begin{aligned} P_{ij}(t) &= P(X_{t+1} = j | X_t = i) \\ &= \begin{cases} \sum\limits_{0 \le k \le j, \frac{j-k}{i} \in \mathbb{N}_0} P(\varepsilon_{t+1} = k) \, P\left(\Phi_{t+1} = \frac{j-k}{i}\right) & i > 0 \\ P(\varepsilon_{t+1} = j) & i = 0, \end{cases} \end{aligned} \tag{6.19}$$

where $t, i, j \in \mathbb{N}$. For instance, when Φ_t and ε_t are Poisson distributed so that $\delta_t^2 = \phi_t$ and $\sigma_t^2 = \mu_t$,

$$P_{ij}(t) = \begin{cases} \sum\limits_{0 \le k \le j, \frac{j-k}{i} \in \mathbb{N}} e^{-\mu_{t+1}} \frac{\mu_{t+1}^k}{k!} e^{-\phi_{t+1}} \frac{\phi_{t+1}^{\frac{j-k}{i}}}{\left(\frac{j-k}{i}\right)!} & i > 0 \\ e^{-\mu_{t+1}} \frac{\mu_{t+1}^j}{j!} & i = 0. \end{cases}$$

Equation (6.18b) can be rewritten as

$$X_{nS+v} = \mathbb{A}_{nS+v} X_{v+(n-1)S} + \boldsymbol{\varepsilon}_{nS+v}, \qquad n \in \mathbb{N}^*,\ 0 \le v \le S-1, \tag{6.20a}$$

where for each $v \in \{0, ..., S-1\}$,

$$\mathbb{A}_{nS+v} = \prod_{l=0}^{S-1} \Phi_{nS+v-l} \quad \text{and} \quad \boldsymbol{\varepsilon}_{nS+v} = \sum_{k=0}^{S-1} \prod_{l=0}^{k-1} \Phi_{nS+v-l} \varepsilon_{nS+v-k}. \tag{6.20b}$$

Hence the monodromy transition probabilities are given by

$$\begin{aligned}\mathbb{P}_{ij}(v) &= P(\mathbb{A}_{v+S}X_v + \boldsymbol{\varepsilon}_{v+S} = j|X_v = i) \\ &= \begin{cases} \sum\limits_{0\leq k\leq j, \frac{j-k}{i}\in\mathbb{N}_0} P(\boldsymbol{\varepsilon}_v = k)\, P\left(\prod\limits_{l=0}^{S-1}\Phi_{nS+v-l} = \frac{j-k}{i}\right), & i>0 \\ P(\boldsymbol{\varepsilon}_v = j), & i=0. \end{cases}\end{aligned} \quad (6.21)$$

Instead of studying (6.21), it is much easier to consider the asymptotic behavior of the solution of the (monodromy) recurrence equations (6.20). Since the sequence $((\mathbb{A}_{nS+v}, \boldsymbol{\varepsilon}_{nS+v}), n \in \mathbb{N})$ is *iid* for each $v \in \{0, ..., S-1\}$, the S monodromy equations (6.20) are standard stochastic recurrence equations with iid inputs (e.g. Aknouche and Guerbyenne, 2009; Bibi and Aknouche, 2010). We are interested in the convergence in distribution of X_{nS+v} (as $n \to \infty$) to a random variable $X^{(v)}$ $(0 \leq v \leq S-1)$, independently of the initial variable X_0, i.e.

$$X_{nS+v} \underset{n\to\infty}{\overset{d}{\to}} X^{(v)} \quad \forall X_0, \ \ \forall v \in \{0, ..., S-1\}, \quad (6.22)$$

where the limiting variables $\left(X^{(v)}\right)_v$, also called circular distributional fixed points of (6.18), satisfy the following circular system

$$\begin{aligned} &X^{(v)} \overset{d}{=} \Phi_v X^{(v-1)} + \varepsilon_v, \ \ \{1, ..., S-1\} \\ &X^{(0)} \overset{d}{=} \Phi_0 X^{(S-1)} + \varepsilon_0 \\ &X^{(v)} \text{ and } (\Phi_v, \varepsilon_v) \text{ are independent for each } v \in \{0, ..., S-1\}. \end{aligned} \quad (6.23)$$

The S circular distributional fixed points $\left\{X^{(v)}, 0 \leq v \leq S-1\right\}$ given by (6.23) also satisfy the following identities in distribution

$$\begin{aligned} &X^{(v)} \overset{d}{=} \mathbb{A}_v X^{(v)} + \boldsymbol{\varepsilon}_v \\ &X^{(v)} \textit{ and } (\mathbb{A}_v, \boldsymbol{\varepsilon}_v) \textit{ are independent} \end{aligned} \quad 0 \leq v \leq S-1, \quad (6.24)$$

meaning that $\left\{X^{(v)}, 0 \leq v \leq S-1\right\}$ are also distributional fixed points of the monodromy equation (6.20) in the standard sense (Goldie and Maller, 2000).

Iterating (6.20) S times gives

$$X_{nS+v} = \prod_{i=1}^{n} \mathbb{A}_{iS+v} X_v + \sum_{j=1}^{n} \prod_{i=j+1}^{n} \mathbb{A}_{iS+v} \boldsymbol{\varepsilon}_{jS+v}, \quad 0 \leq v \leq S-1, \quad (6.25)$$

where the S variables $\{X_v, 1 \leq v \leq S-1\}$ are such that $X_v = A_v X_{v-1} + B_v$, $1 \leq v \leq S-1$. To study the limit behavior of X_{nS+v} as $n \to \infty$, it is important to emphasize that from the iid property of the sequence $((\mathbb{A}_{nS+v}, \boldsymbol{\varepsilon}_{nS+v}), n \in \mathbb{N})$, $(0 \leq v \leq S-1)$ it follows that

$$\sum_{j=1}^{n} \prod_{i=j+1}^{n} \mathbb{A}_{iS+v} \boldsymbol{\varepsilon}_{jS+v} \stackrel{d}{=} \sum_{j=1}^{n} \prod_{i=1}^{j-1} \mathbb{A}_{iS+v} \boldsymbol{\varepsilon}_{jS+v}, \quad 0 \leq v \leq S-1. \tag{6.26}$$

In addition, if $(\Phi_{nS+v} = 0) > 0$ for all v, as happens for most usual discrete distributions such as Poisson, Binomial, negative binomial, etc., then Brandt's (1986) theorem (see also Vervaat, 1979) implies that under the condition

$$P(\mathbb{A}_{nS+v} = 0) > 0, \tag{6.27}$$

$$\sum_{j=1}^{n} \prod_{i=1}^{j-1} \mathbb{A}_{iS+v} \boldsymbol{\varepsilon}_{jS+v} \underset{n\to\infty}{\overset{a.s.}{\to}} \sum_{j=1}^{\infty} \prod_{i=1}^{j-1} \mathbb{A}_{iS+v} \boldsymbol{\varepsilon}_{jS+v}, \quad 0 \leq v \leq S-1, \tag{6.28}$$

where the infinite series in (6.28) converges absolutely almost surely. Taking

$$X^{(v)} \stackrel{d}{=} \sum_{j=1}^{\infty} \prod_{i=1}^{j-1} \mathbb{A}_{iS+v} \boldsymbol{\varepsilon}_{jS+v},$$

it follows that under (6.27),

$$X_{nS+v} \underset{n\to\infty}{\overset{a.s.}{\to}} X^{(v)}$$

for all $0 \leq v \leq S-1$, where $\left(X^{(v)}\right)_v$ satisfies (6.23) and (6.24). This indirectly shows that the Markov chain $(X_t, t \in \mathbb{N})$ is periodically regular. The distribution of the system $\left(X^{(v)}\right)_v$ is the periodically stationary (or periodically invariant) distribution of the Markov chain $(X_t,\ t \in \mathbb{N})$. Such a system of distributions exists provided condition (6.27) is satisfied. Finally, if the initial variables X_v are such that $X_v \stackrel{d}{=} X^{(v)}$ for each $v \in \{0, ..., S-1\}$, i.e. the time-periodic chain is initialized from its periodically invariant distributions, then $(X_t,\ t \in \mathbb{N})$ given by (6.18) is strictly periodically stationary and periodically ergodic.

6.3 Markov-switching periodic time series models

Markov-switching time series models have attracted increasing interest in recent years. These models can be broadly classified into two categories, namely Markov-switching conditional mean models (MS-ARMA models; e.g. Hamilton, 1989-1990, Francq and Zakoïan, 2001; Yao, 2001; Aknouche and Rabehi, 2010; Aknouche and Demmouche, 2019; Aknouche and Francq, 2022) and Markov-switching conditional variance models (MS-GARCH models; e.g. Cai, 1994; Hamilton and Susmel, 1994; Gray, 1996; Francq et *al*, 2001; Klaassen, 2002; Hass et *al*, 2004; Francq and Zakoian, 2005-2008-2019; Aknouche et *al*, 2024). Most existing Markov-switching formulations concern stationary and ergodic inputs depending on the states of a finite homogeneous Markov chain which is assumed to be regular and stationary.

This Subsection describes three Markov-switching periodic time series models, namely a Markov-Switching periodic ARMA (MS-PARMA) model, a Markov-Switching periodic GARCH (MS-PGARCH) model, and a more general Markov-Switching positive conditional mean model. For the three models, the inputs are S-periodically distributed and depend upon the (finite) states of a periodically homogeneous Markov chain which is assumed to be periodically regular and periodically stationary. Special cases of Markov-switching periodic autoregressive (MS-PAR) models have already been examined by Ghysels et *al* (1998), Ghysels (2000), and Bac et *al* (2001). The MS-PARMA model we present here was previously proposed by Aknouche et *al* (2008) and was named *Markov-mixture periodic ARMA*.

6.3.1 A Markov-switching periodic ARMA model

A real-valued random sequence $(Y_t, t \in \mathbb{Z})$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is said to be a Markov-switching periodic ARMA (MS-PARMA) with period $S \geq 1$ and orders $p, q \in \mathbb{N}$ if it is a solution to the following equation

$$Y_t = \phi_{0t}(X_t) + \sum_{i=1}^{p} \phi_{it}(X_t) Y_{t-i} + \varepsilon_t - \sum_{j=1}^{q} \theta_{jt}(X_t) \varepsilon_{t-j}, \ \ t \in \mathbb{Z}, \tag{6.29}$$

where $(\varepsilon_t, t \in \mathbb{Z})$ is a sequence of ipd_S random variables with mean zero and variance $Var(\varepsilon_t) = \sigma_t^2(X_t) > 0$. The parameters $\phi_{it}(X_t)$, $\theta_{jt}(X_t)$, and $\sigma_t^2(X_t)$ are S-periodic in t in the sense $\phi_{i,t}(X_t) = \phi_{i,t+S}(X_t)$ for all t and so on. When $q = 0$, the MS-PARMA model reduces to the Markov-switching periodic autoregressive (MS-PAR) model proposed by Ghysels et *al* (1998) and Ghysels (2000), and subsequently studied by Bac et *al* (2001). For $S = 1$, the model (6.29) reduces to the MS-ARMA specification studied by Francq and Zakoian (2001).

The sequence $(X_t, t \in \mathbb{N})$ is assumed to be a finite periodically homogenous Markov chain with state-space $E = \{1, ..., K\}$ and transition probability matrices $(P(v))_{0 \leq v \leq S-1}$ given by

$$\begin{aligned} P_{ij}(v) &= P(X_{nS+v+1} = j | X_{nS+v} = i) \\ &= P(X_{v+1} = j | X_v = i), \quad n \in \mathbb{N},\ i, j \in E,\ 0 \leq v \leq S-1. \\ P_{ij}^{(l)}(t) &= P(X_{t+l} = j | X_t = i). \end{aligned}$$

We assume further that $(X_t, t \in \mathbb{N})$ is periodically irreducible and periodically aperiodic. Since the chain $(X_t, t \in \mathbb{N})$ is finite, it is therefore periodically positive recurrent and hence periodically regular (or periodically Harris-ergodic). In addition, we assume that $(X_t, t \in \mathbb{N})$ is periodically stationary in the sense $P(X_v = j) := \pi_j(v) = \pi_j^{(v)}$ for $0 \leq v \leq S-1$, where the system of periodically invariant distributions $\left(\pi^{(v)}\right)_{0 \leq v \leq S-1}$ uniquely satisfies (3.4). Finally, the sequence $(X_t, t \in \mathbb{N})$ figuring in (6.29) is a (strictly) periodically stationary $\mathbb{Z}$-version of $(X_t, t \in \mathbb{N})$ (cf. Aknouche, 2008). Due to the moving-average part in (6.29), the MS-PARMA model is characterized by path dependence, making the estimation of the model quite involving.

A fundamental question concerning model (6.29) is to search for conditions ensuring the existence of periodically stationary and periodically ergodic solutions with finite logarithmic or higher-order moments. For this, the model (6.29) can be rewritten in the following Markov vector form

$$Z_t = A_t(X_t) Z_{t-1} + B_t(X_t), \tag{6.30}$$

where

$$\begin{aligned} Z_t &= (Y_t, ..., Y_{t-p+1}, \varepsilon_t, ..., \varepsilon_{t-q+1})'_{(p+q)\times 1} \\ B_t(X_t) &= (\phi_{0t}(X_t) + \varepsilon_t, 0, ..., 0, \varepsilon_t, 0, ..., 0)'_{(p+q)\times 1}, \end{aligned}$$

and

$$A_t(X_t) = \begin{pmatrix} \phi_{1t}(X_t) & \cdots & \phi_{p-1,t}(X_t) & \phi_{pt}(X_t) & \theta_{1t}(X_t) & \cdots & \theta_{q-1,t}(X_t) & \theta_{qt}(X_t) \\ 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

Note that the sequence of matrices $((A_t, B_t), t \in \mathbb{Z})$ is strictly periodically stationary and periodically ergodic by model's assumptions. Let $\mathcal{M}^r$ be the space of square real matrices of dimension $r := p + q$ and $\|.\|$ be an arbitrary operator norm in $\mathcal{M}^r$. The largest Lyapunov exponent for the sequence of matrices $(A_t,\ t \in \mathbb{Z})$ is given by (e.g. Aknouche, 2008; Aknouche, Al-Eid and Demouche, 2018)

$$\begin{aligned} \gamma^S(A) &= \inf_{n\in\mathbb{N}^*} \tfrac{1}{n} E\left\{\log \|A_{nS}(X_{nS}) A_{nS-1}(X_{nS-1}) ... A_1(X_1)\|\right\} \\ &= \inf_{n\in\mathbb{N}^*} \tfrac{S}{n} E\left\{\log \|A_n(X_n) A_{n-1}(X_{n-1}) ... A_1(X_1)\|\right\}. \end{aligned} \tag{6.31}$$

Proposition 6.1 *A sufficient condition for model* (6.29) *to have a causal strictly periodically stationary solution given by*

$$Z_t = \sum_{j=0}^{\infty} \prod_{i=0}^{j-1} A_{t-i}(X_{t-i}) B_{t-j}(X_{t-j}), \quad t \in \mathbb{Z}, \tag{6.32}$$

where the latter series converges absolutely a.s., is that

$$\gamma^S(A) < 0. \tag{6.33}$$

Furthermore, this solution is unique and periodically ergodic.

Proof By model's assumption we first have

$$\begin{aligned} \sum_{v=1}^{S} E\left(\log^+ \|A_v(X_v)\|\right) &\leq \sum_{v=1}^{S} E\left(\|A_v(X_v)\|\right) < \infty \\ \sum_{v=1}^{S} E\left(\log^+ \|B_v(X_v)\|\right) &\leq \sum_{v=1}^{S} E\left(\|B_v(X_v)\|\right) < \infty. \end{aligned}$$

Therefore, (6.33) implies

$$\left(\left\|\prod_{i=0}^{j-1} A_{t-i}\left(X_{t-i}\right)\right\|\left\|B_{t-j}\left(X_{t-j}\right)\right\|\right)^{1/j} = \exp\left(\tfrac{1}{j}\log\left\|\prod_{i=0}^{j-1} A_{t-i}\left(X_{t-i}\right)\right\| + \tfrac{1}{j}\log\left\|B_{t-j}\left(X_{t-j}\right)\right\|\right)$$
$$\underset{j\to\infty}{\to} \exp\left\{\tfrac{1}{S}\gamma^{S}\right\} < 1, \text{ a.s.}$$

Therefore, the Cauchy rule implies that the series (6.32) converges absolutely a.s., and the process given by (6.32) is a solution to (6.30). This solution is strictly periodically stationary and periodically ergodic in view of the periodicity of the model's coefficients and the periodic homogeneity of the chain (X_t). □

We now give a sufficient condition for the existence of a strictly periodically stationary solution (6.29) satisfying $E\left(Y_v^2\right) < \infty$, $0 \leq v \leq S-1$. Let $\otimes$ be the Kronecker product, $\rho(A)$ be the spectral radius of the matrix A (i.e. the maximum absolute eigenvalues of A), $vec(A)$ be the column vector stacking operator for the matrix A, and $A^{\otimes m}$ be the product $A \otimes A \otimes \cdots \otimes A$ of m factors. Define the matrix $\Sigma_v^{(d)}$ ($d \in \mathbb{N}^*$, $0 \leq v \leq S-1$) by

$$\Sigma_v^{(d)} = \begin{pmatrix} P_{11}(v)\,A_v(1)^{\otimes d} & P_{21}(v)\,A_v(1)^{\otimes d} & \ldots & P_{K1}(v)\,A_v(1)^{\otimes d} \\ P_{12}(v)\,A_v(2)^{\otimes d} & P_{22}(v)\,A_v(2)^{\otimes d} & \ldots & P_{K2}(v)\,A_v(2)^{\otimes d} \\ \vdots & \vdots & \ddots & \vdots \\ P_{1K}(v)\,A_v(K)^{\otimes d} & P_{2K}(v)\,A_v(K)^{\otimes d} & \ldots & P_{KK}(v)\,A_v(K)^{\otimes d} \end{pmatrix}.$$

Proposition 6.2 *A sufficient condition for model* (6.29) *to have a causal strictly periodically stationary and periodically ergodic solution* $(Y_t, t \in \mathbb{Z})$ *with a finite second moment* $E\left(Y_v^2\right)$*, for all* $0 \leq v \leq S-1$*, is that*

$$\rho\left(\prod_{v=0}^{S-1} \Sigma_{S-v}^{(2)}\right) < 1. \tag{6.34}$$

Moreover, the solution is unique and is given by (6.32)*, where the series in* (6.32) *converges in mean square.*

Proof Iterating (6.30) yields

$$\begin{aligned} Z_t &= \sum_{k=1}^{m} \prod_{i=0}^{k-1} A_{t-i}\left(X_{t-i}\right) B_{t-j}\left(X_{t-j}\right) + \prod_{i=0}^{m-1} A_{t-i}\left(X_{t-i}\right) B_{t-m+1}\left(X_{t-m+1}\right) Z_{t-m} \\ &= \sum_{k=1}^{m} \Pi_{t,k} + \prod_{i=0}^{m-1} \Pi_{t,m} Z_{t-m}, \end{aligned}$$

where $\Pi_{t,k} = \prod_{i=0}^{k-1} A_{t-i}\left(X_{t-i}\right) B_{t-j}\left(X_{t-j}\right)$. The result thus follows if

$$E(\|\Pi_{t,k}\|^2) \leq c\delta^k, \text{ for all } k \geq 1, \tag{6.35}$$

where $c > 0$, and $\delta \in (0,1)$. For every $t \in \mathbb{Z}$, define the sequence of $r \times 1$-vectors $(W_{t,k})_{k\geq 0}$ by

$$W_{t,k} = A_t\left(X_t\right) W_{t-1,k-1} \; k \geq 1$$

for some fixed $W_{t-k,0}$, and let $V_{t,k}\left(i\right)$ be a $r^2 \times 1$-vector defined for all $i = 1, ..., K$ by

$$V_{t,k}\left(i\right) = E\left(vec\left(W_{t,k}W'_{t,k}\right) 1_{[X_t=i]}\right).$$

Then

$$\begin{aligned} V_{t,k}\left(i\right) &= \sum_{j=1}^{K} E\left(vec\left(A_t\left(X_t\right) W_{t-1,k-1} W'_{t-1,k-1} A_t\left(X_t\right)\right) 1_{[X_t=i, X_{t-1}=j]}\right) \\ &= \sum_{j=1}^{K} P_{ji}\left(t-1\right) A_t\left(i\right)^{\otimes 2} V_{t-1,k-1}\left(j\right) \end{aligned}$$

so that

$$V_{t,k} = \Sigma^{(2)}_{t-1} V_{t-1,k-1}, \tag{6.36}$$

where $V_{t,k} = \left(V'_{t,k}\left(1\right), ..., V'_{t,k}\left(K\right)\right)'$ is $Kr^2 \times 1$-vector. Iterating (6.36) gives

$$V_{t,k} = \Sigma^{(2)}_{t-1}\Sigma^{(2)}_{t-2} \cdots \Sigma^{(2)}_{t-k} V_{t,0}, \; k \geq 1.$$

By the S-periodicity of the model coefficients and the S-periodic homogeneity of the chain (X_t), it can be seen that under (6.34),

$$\left\|\Sigma^{(2)}_{t-1}\Sigma^{(2)}_{t-2} \cdots \Sigma^{(2)}_{t-k}\right\| = \left\|\Sigma^{(2)}_{t-1} \cdots \Sigma^{(2)}_{t-S}\right\|^{l_k} \left\|\Sigma^{(2)}_{t-S-1} \cdots \Sigma^{(2)}_{t-S-\tau_k}\right\| \underset{k\to\infty}{\rightarrow} 0$$

at an exponential rate, where $k = l_k S + \tau_k$ is written in terms of its Euclidian division by S. From the multiplicity of the norm, we finally get

$$E(\|\Pi_{t,k}\|^2) \leq \|V_{t,k}\| \leq \left\| \Sigma_{t-1}^{(2)} \Sigma_{t-2}^{(2)} \cdots \Sigma_{t-k}^{(2)} \right\| \|V_{t,0}\| \leq c\delta^k,$$

proving the result. □

Following the same lines of the proof of Proposition 6.2, it can be seen that a sufficient condition for the existence of a strictly periodically stationary and periodically ergodic solution of (6.29) satisfying $E\left(Y_v^d\right) < \infty$ ($d \in \mathbb{N}^*$, $0 \leq v \leq S-1$), is that

$$\rho\left(\prod_{v=0}^{S-1} \Sigma_{S-v}^{(r)}\right) < 1,$$

provided that $E\left(|\varepsilon_t|^d\right) < \infty$.

6.3.2 A Markov-switching periodic GARCH model

Consider a Markov Switching periodic GARCH (MS-PGARCH) process defined as

$$\begin{aligned} Y_t &= \sigma_t \eta_t \\ \sigma_t^2 &= \omega_t\left(X_t\right) + \sum_{i=1}^{q} \alpha_{it}\left(X_t\right) Y_{t-i}^2 + \sum_{j=1}^{p} \beta_{jt}\left(X_t\right) \sigma_{t-j}^2,\ t \in \mathbb{Z} \end{aligned} \tag{6.37}$$

where $(\eta_t, t \in \mathbb{Z})$ is an ipd_S sequence of mean zero and unit variance, and (X_t) is a time-periodic finite Markov chain satisfying the same assumptions as the above MS-PARMA model. It is assumed that $(\eta_t, t \in \mathbb{Z})$ and $(X_t, t \in \mathbb{Z})$ are independent, and for all $1 \leq l \leq K$, the functions $\omega_t(l) > 0$, $\alpha_{it}(l) \geq 0$, and $\beta_{jt}(l) \geq 0$ are S-periodic in t.

Model (6.37) is a Markov Switching extension of the periodic GARCH (PGARCH) model of Bollerslev and Ghysels (1996). It is also a time-periodic extension of the MS-GARCH of Francq and Zakoian (2005, 2008) and thus is also characterized by path dependence.

To study the properties of the model, consider its Markov vectorial form

$$Z_t = C_t(X_t) Z_{t-1} + D_t(X_t), \tag{6.38}$$

where

$$Z_t = \left(Y_t^2, ..., Y_{t-p+1}^2, \sigma_t^2, ..., \sigma_{t-q+1}^2\right)'_{(p+q)\times 1}$$
$$D_t(X_t) = \left(\omega_t(X_t)\eta_t^2, 0, ..., 0, \omega_t(X_t), 0, ..., 0\right)'_{(p+q)\times 1},$$

and

$$C_t(X_t) = \begin{pmatrix} \alpha_{1t}(X_t)\eta_t^2 & \cdots & \alpha_{qt}(X_t)\eta_t^2 & \beta_{1t}(X_t)\eta_t^2 & \cdots & \eta_t^2\beta_{pt}(X_t) \\ 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \alpha_{1t}(X_t) & \cdots & \alpha_{qt}(X_t) & \beta_{1t}(X_t) & \cdots & \beta_{pt}(X_t) \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Let $\gamma^S(C)$ be the top Lyapunov exponent given by (6.31) while replacing the sequence $(A_t,\ t \in \mathbb{Z})$ by $(C_t,\ t \in \mathbb{Z})$.

Proposition 6.3 *If*

$$\gamma^S(C) < 0$$

then model (6.37) *admits a unique causal strictly periodically stationary and periodically ergodic solution given by*

$$Z_t = \sum_{j=0}^{\infty} \prod_{i=0}^{j-1} C_{t-i}(X_{t-i}) D_{t-j}(X_{t-j}), \quad t \in \mathbb{Z}, \tag{6.39}$$

where the latter series converges absolutely a.s.

Proof The proof is similar to that of Proposition 6.1. □

We now derive a sufficient condition for the existence of a strictly periodically stationary and periodically ergodic solution with a finite second moment. Let $\widetilde{C}_t(X_t) = E(C_t(X_t)|X_t)$

be the matrix obtained by replacing η_t^2 by 1 in $C_t(X_t)$ and set

$$\Gamma_v^{(d)} = \begin{pmatrix} P_{11}(v)\widetilde{C}_v(1)^{\otimes d} & P_{21}(v)\widetilde{C}_v(1)^{\otimes d} & \dots & P_{K1}(v)\widetilde{C}_v(1)^{\otimes d} \\ P_{12}(v)\widetilde{C}_v(2)^{\otimes d} & P_{22}(v)\widetilde{C}_v(2)^{\otimes d} & \dots & P_{K2}(v)\widetilde{C}_v(2)^{\otimes d} \\ \vdots & \vdots & \ddots & \vdots \\ P_{1K}(v)\widetilde{C}_v(K)^{\otimes d} & P_{2K}(v)\widetilde{C}_v(K)^{\otimes d} & \dots & P_{KK}(v)\widetilde{C}_v(K)^{\otimes d} \end{pmatrix}, \ d \in \mathbb{N}^*.$$

Proposition 6.4 *If*

$$\rho\left(\prod_{v=0}^{S-1}\Gamma_{S-v}^{(2)}\right) < 1, \tag{6.40}$$

then model (6.37) *has a unique causal strictly periodically stationary and periodically ergodic solution* $(Y_t, t \in \mathbb{Z})$ *given by* (6.39) *with* $E(Y_v^2) < \infty$ $(0 \leq v \leq S-1)$*. Moreover, the series in* (6.39) *converges in mean square.*

Proof The proof is similar to that of Proposition 6.2. □

It can be seen that a sufficient condition for the existence of a periodically stationary solution with $E\left(Y_t^{2d}\right) < \infty$ is that $E\left(\eta_t^{2d}\right) < \infty$ and

$$\rho\left(\prod_{v=0}^{S-1}\Gamma_{S-v}^{(2d)}\right) < 1.$$

Remark 6.1 *i) The second-order periodic stationarity condition* (6.40) *can be further simplified by using a simpler matrix instead of* $\Gamma_v^{(2)}$*. For* $1 \leq l \leq r = \max(p,q)$*, define the* $K \times K$ *matrix* $N_t^{(l)}$ *as*

$$N_t^{(l)}(i,j) = P_{ji}^{(l)}(t-l)\left(\alpha_{lt}(i) + \beta_{lt}(i)\right), \tag{6.41a}$$

where $P_{ij}^{(l)}(t) = P(X_{t+l} = j | X_t = i)$ *is the* l*-step transition probability. For example,*

$$N_t^{(1)} = \begin{pmatrix} P_{11}(t-1)(\alpha_{1t}(1)+\beta_{1t}(1)) & \cdots & P_{K1}(t-1)(\alpha_{1t}(1)+\beta_{1t}(1)) \\ \vdots & \ddots & \vdots \\ P_{1K}(t-1)(\alpha_{1t}(K)+\beta_{1t}(K)) & \cdots & P_{KK}(t-1)(\alpha_{1t}(K)+\beta_{1t}(K)) \end{pmatrix}. \tag{6.41b}$$

Define the companion block-matrix Λ_t *as*

$$\Lambda_t = \begin{pmatrix} N_t^{(1)} & N_{t-1}^{(2)} & \cdots & N_{t-r+2}^{(r-1)} & N_{t-r+1}^{(r)} \\ I_K & 0_{K\times K} & \cdots & 0_{K\times K} & 0_{K\times K} \\ 0_{K\times K} & I_K & \cdots & 0_{K\times K} & 0_{K\times K} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{K\times K} & 0_{K\times K} & \cdots & I_K & 0_{K\times K} \end{pmatrix}, \tag{6.41c}$$

where $0_{K\times K}$ *and* I_K *stand for respectively the* $K \times K$ *zero matrix and the* $K \times K$ *identity matrix.*

Note that

$$E\left(\sigma_t^2 | X_t\right) = E\left(Y_t | X_t\right) \tag{6.42}$$

and

$$\pi_j(t) E\left(\sigma_t^2 | X_t = j\right) = \sum_{k=1}^{K} P_{kj}^{(l)}(t-l)\, \pi_k(t-l)\, E\left(\sigma_{t-l}^2 | X_{t-l} = k\right), \quad 1 \leq l \leq r \tag{6.43}$$

which is a time-varying extension of Lemma 3 in Francq and Zakoian (2005)*. Hence from* (6.37) *we obtain*

$$\begin{aligned} &\pi_j(t) E\left(\sigma_t^2 | X_t = j\right) = \\ &\quad \pi_j(t)\,\omega_t(j) + \sum_{l=1}^{r}\sum_{k=1}^{K} P_{kj}^{(l)}(t-l)\left(\alpha_{lt}(j) + \beta_{lt}(j)\right)\pi_k(t-l)\, E\left(\sigma_{t-l}^2 | X_{t-l} = k\right). \end{aligned} \tag{6.44a}$$

The latter equality can be embedded as

$$\mathbf{h}_t = \mathbf{d}_t + \Lambda_t \mathbf{h}_{t-1}, \tag{6.44b}$$

where

$$\begin{aligned} h_{jt} &= \pi_j(t) E\left(\sigma_t^2 | X_t = j\right),\ h_t = (h_{1t}, ..., h_{Kt})',\ \mathbf{h}_t = \left(h_t', ..., h_{t-r+1}'\right)' \\ d_t &= \left(\pi_1(t)\,\omega_t(1), ..., \pi_K(t)\,\omega_t(K)\right)',\ \text{and } \mathbf{d}_t = \left(d_t', 0_{K\times 1}', ..., 0_{K\times 1}'\right)_{rK\times 1}'. \end{aligned}$$

Therefore, in view of the latter equality and the nonnegativity of the model coefficients, an equivalent condition to (6.40) *for second-order periodic stationarity (and periodic ergodicity)* *is*

$$\rho\left(\prod_{v=0}^{S-1}\Lambda_{S-v}\right)<1.$$

ii) Instead of the vectorial Markov equation (6.38)*, an equivalent representation inspired by Francq et al* (2021) *could be used to get equivalent periodic stationarity conditions.*

6.3.3 A Markov-switching periodic positive conditional mean model

Let F_λ be a cumulative distribution function (cdf) with nonnegative support and mean $\lambda=\int_0^{+\infty}xdF_\lambda(x)>0$. An nonnegative-valued process $(Y_t,t\in\mathbb{Z})$ is said to be a *periodic positive conditional mean* (PPCM) of orders p and q, and period $S\geq 1$, if its conditional distributions are given by

$$Y_t|\mathcal{F}_{t-1}\sim F_{t,\lambda_t},\ t\in\mathbb{Z}, \tag{6.45a}$$

where $\mathcal{F}_{t-1}$ is the σ-algebra generated by $\{Y_{t-u},u\geq 1\}$ and the conditional mean λ_t is given by

$$\lambda_t=\omega_t+\sum_{i=1}^{q}\alpha_{it}Y_{t-i}+\sum_{j=1}^{p}\beta_{jt}\lambda_{t-j},\ t\in\mathbb{Z}. \tag{6.45b}$$

The parameters $\omega_t>0$, $\alpha_{ti}\geq 0$ $(i=1,...,q)$ and $\beta_{tj}\geq 0$ $(j=1,...,p)$ are S-periodic in t in the sense $\omega_t=\omega_{t+S}$, $\alpha_{it}=\alpha_{i,t+S}$ and $\beta_{jt}=\beta_{j,t+S}$ for every $t\in\mathbb{Z}$. Assume that $F_{t,\lambda_t}:=F_{\lambda_t}$ satisfies the following property (cf. Aknouche and Francq, 2021)

$$\lambda\leq\lambda^*\quad\Rightarrow\quad F_\lambda^-(u)\leq F_{\lambda^*}^-(u),\ \forall u\in(0,1), \tag{6.46}$$

where F_λ^- is the quantile function associated with F_λ. The class of distributions satisfying (6.46) is quite large and encompasses, for instance, the one-parameter exponential family.

Model (6.45) is a time-periodic extension of the *positive linear conditional mean* model in Aknouche and Francq (2021, 2023). It encompasses, in particular, the periodic autoregressive conditional duration model (PACD, Aknouche et *al*, 2022a) and the periodic INGARCH (integer-valued GARCH) model (see e.g. Aknouche et *al*, 2018; Almohaimeed, 2024) with various distributions in the class (6.46).

We now consider a Markov-Switching extension of the PPCM model (6.45). Let $(X_t, t \in \mathbb{N})$ be a periodically stationary and periodically regular Markov chain defined as in the above subsections and let $(X_t, t \in \mathbb{Z})$ its $\mathbb{Z}$-extended copy. A random sequence $(Y_t, t \in \mathbb{Z})$ is said to be a Markov-Switching periodic positive conditional mean (henceforth MS-PPCM(p,q)) if it satisfies

$$\begin{aligned} Y_t|\mathcal{F}_{t-1}^{Y,X} &\sim F_{t,X_t,\lambda_t} && (6.47a)\\ \lambda_t &:= \lambda_{X_t,t} = \omega_t(X_t) + \sum_{i=1}^{q} \alpha_{it}(X_t) Y_{t-i} + \sum_{j=1}^{p} \beta_{jt}(X_t) \lambda_{t-j}, && (6.47b) \end{aligned}$$

where $F_\lambda := F_{t,j,\lambda}$ satisfies (6.46) and $\mathcal{F}_{t-1}^{Y,X}$ denotes the σ-algebra generated by $\{Y_{t-u}, X_{t-u+1}, u \geq 1\}$. The coefficients $\omega_t(X_t) > 0$, $\alpha_{it}(X_t) \geq 0$ and $\beta_{jt}(X_t) \geq 0$ are S-periodic in t. Equation (6.47b) can be rewritten as

$$\lambda_{nS+v} = \omega_v(X_{nS+v}) + \sum_{i=1}^{q} \alpha_{iv}(X_{nS+v}) Y_{nS+v-i} + \sum_{j=1}^{p} \beta_{jv}(X_{nS+v}) \lambda_{nS+v-j},\ n \in \mathbb{Z},\ 0 \leq v \leq S-1,$$

where, for instance, $\omega_v(j)$ $(0 \leq v \leq S-1,\ 1 \leq j \leq K)$ represents the value of the intensity intercept at channel v and regime j. As in the above Markov-Switching models, the past recent values of $\lambda_{X_t,t}$ depend on the past values of the regime variable X_t so the likelihood of the model depends on the whole path history of X_t. When $K = 1$, the MS-PPCM model (6.47) is simply the PPCM(p,q) model (6.45), and when the chain is ipd_S, model (6.47) reduces to the model considered by Almohaimeed (2023).

As in Aknouche and Francq (2022), we assume that X_t contains all the information of the σ-algebra $\mathcal{I}_t = \sigma\{(U_u, \Delta_u), u \leq t\}$, as stated by the following assumption.

A0 $P(X_t = j|X_{t-1} = i, A) = P_{ij}(t-1)$ for every event $A \in \mathcal{I}_{t-1}$.

We now give periodic stationarity and periodic ergodicity conditions for the MS-PPCM model (6.47). Let the $K \times K$ matrix $M_t^{(l)}$ be given as in (6.41a),

$$M_t^{(l)}(i,j) = P_{ji}^{(l)}(t-l)\left(\alpha_{lt}(i) + \beta_{lt}(i)\right),\ 1 \leq i,j \leq K,\ l = 1, ..., r = \max(p,q) \qquad (6.48a)$$

$(t \in \mathbb{Z})$ and define the companion block-matrix Ω_t by

$$\Omega_t = \begin{pmatrix} M_t^{(1)} & M_{t-1}^{(2)} & \cdots & M_{t-r+2}^{(r-1)} & M_{t-r+1}^{(r)} \\ I_K & 0_{K\times K} & \cdots & 0_{K\times K} & 0_{K\times K} \\ 0_{K\times K} & I_K & \cdots & 0_{K\times K} & 0_{K\times K} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{K\times K} & 0_{K\times K} & \cdots & I_K & 0_{K\times K} \end{pmatrix}. \tag{6.48b}$$

The following result gives a necessary and sufficient periodic ergodicity condition for the MS-PPCM model (6.47).

Proposition 6.5 *Let* $F_\lambda = F_{t,j,\lambda}$ $(\lambda > 0)$ *be a family of cdf's satisfying* (4.46). *There exists a periodically stationary and periodically ergodic sequence* $(Y_t, t \in \mathbb{Z})$ *whose conditional distribution is given by* $(4.47a)$ *where* λ_t *satisfies* $(4.47b)$ *if*

$$\rho\left(\prod_{v=0}^{S-1} \Omega_{S-v}\right) < 1. \tag{6.49}$$

Conversely, if there exists a sequence $(Y_t, t \in \mathbb{Z})$ *satisfying* $(6.47a)$ *such that* $E(Y_t) < \infty$ *(for all* t*), and the periodically irreducible and periodically stationary Markov chain* $(X_t, t \in \mathbb{Z})$ *satisfies* ***A0****, then* (6.49) *holds true.*

Proof The proof mainly follows the lines of the proof of Theorem 3.1 in Aknouche and Francq (2022) with a slight adaptation to time-varying settings.

Assume that $(Y_t, t \in \mathbb{Z})$ is a strictly periodically stationary process satisfying (6.47) with $E(Y_t) < \infty$ and $E(\lambda_t) < \infty$ for all $t \in \mathbb{Z}$. Under **A0**, similarly to (6.42) and (6.43), we have

$$E(\lambda_t | X_t = j) = E(Y_t | X_t = j)$$

and

$$\pi_j(t) E(\lambda_t | X_t = j) = \sum_{i=1}^{K} P_{ij}^{(l)}(t-l) \pi_i(t-l) E(\lambda_{t-l} | X_{t-l} = i), \text{ for all } 1 \le l \le r. \tag{6.50}$$

As for $(6.44a)$, equality (6.50) entails

$$\pi_j(t) E(\lambda_t | X_t = j) =$$
$$\pi_j(t)\omega_t(j) + \sum_{l=1}^{r}\sum_{k=1}^{K} P_{kj}^{(l)}(t-l)(\alpha_{lt}(j) + \beta_{lt}(j))\pi_k(t-l) E(\lambda_{t-l} | X_{t-l} = k). \tag{6.51a}$$

Taking

$$\begin{aligned} v_{jt} &= \pi_j(t) E(\lambda_t | X_t = j), \ v_t = (h_{1t}, ..., h_{Kt})', \ \mathbf{v}_t = \left(h_t', ..., h_{t-r+1}'\right)' \\ d_t &= (\pi_1(t)\omega_t(1), ..., \pi_K(t)\omega_t(K))', \text{ and } \mathbf{d}_t = \left(d_t', 0_{K\times 1}', ..., 0_{K\times 1}'\right)_{rK\times 1}', \end{aligned}$$

equality $(6.51a)$ can be embedded in a block-matrix form

$$\mathbf{v}_t = \mathbf{d}_t + \Omega_t \mathbf{v}_{t-1}. \tag{6.51b}$$

Iterating $(6.51b)$ S times we obtain

$$\mathbf{v}_t = \sum_{j=0}^{S-1} \prod_{i=0}^{j-1} \Omega_{t-i} \mathbf{d}_{t-j} + \left(\prod_{v=0}^{S-1} \Omega_{t-v} \right) \mathbf{v}_{t-S}. \tag{6.51c}$$

The S-periodicity of the PPCM model (6.47) therefore implies that $\mathbf{v}_t = \mathbf{v}_{t-S}$ so $(6.51c)$ writes as a periodic fixed-point (monodromy) identity

$$\mathbf{v}_t = H_t + G_t \mathbf{v}_t,$$

that is

$$\mathbf{v}_t = H_{t,K} + G_t^K \mathbf{v}_t, \tag{6.51d}$$

where

$$G_t = \prod_{v=0}^{S-1} \Omega_{t-v}, \ H_t = \sum_{j=0}^{S-1} \prod_{i=0}^{j-1} \Omega_{t-i} \mathbf{d}_{t-j}, \text{ and } H_{t,K} = H_t + G_t H_t + \cdots + G_t^{K-1} H_t > 0.$$

By the positivity of the coefficients of $H_{t,K}$ and G_t^K in $(6.51d)$, and since

$$\rho(G_t^K) = (\rho(G_t))^K,$$

Lemma A.1 in Aknouche and Francq (2022) and Corollary 8.1.29 of Horn and Johnson (2013) show that condition (6.49) holds. This complete the necessity part.

Now, let $(U_t, t \in \mathbb{Z})$ denote an ipd_S sequence of uniformly distributed variables in $[0,1]$ and independent of $(\lambda_t, t \in \mathbb{Z})$. For every $t \in \mathbb{Z}$ set

$$\lambda_t^{(k)} = \begin{cases} \omega_t(X_t) + \sum_{i=1}^{q} \alpha_{it}(X_t) Y_{t-i}^{(k-i)} + \sum_{j=1}^{p} \beta_{jt}(X_t) \lambda_{t-j}^{(k-j)} & \text{if } k \geq 1 \\ 0 & \text{if } k \leq 0 \end{cases} \tag{6.52}$$

and

$$Y_t^{(k)} = \begin{cases} F^{-}_{t,X_t,\lambda_t^{(k)}}(U_t) & \text{if } k \geq 1 \\ 0 & \text{if } k \leq 0. \end{cases} \tag{6.53}$$

When $k \geq 2$, (6.52) and (6.53) together yield

$$\lambda_t^{(k)} = \psi_{kt}(U_{t-1}, \ldots, U_{t-k+1};\, X_t, ..., X_{t-k+1}),$$

where $\psi_{kt} : [0,1]^{k-1} \times \{1, ..., K\}^k \to [0, \infty)$ is a measurable S-periodic function in t, in the sense $\psi_{k,t} = \psi_{k,t+S}$ for all $t, k \in \mathbb{Z}$. Hence, $\left(\lambda_t^{(k)}, t \in \mathbb{Z}\right)$ and $\left(Y_t^{(k)}, t \in \mathbb{Z}\right)$ are periodically stationary and periodically ergodic for every $k \in \mathbb{Z}$.

Let $\mathcal{F}_{t-1}^{k,X}$ and $\mathcal{F}_{t-1}^{*}$ be the σ-fields generated by $\left\{Y_{t-i}^{(k-i)}, X_{t-i+1}, i > 0\right\}$ and $\{U_u, X_{u+1}, u < t\}$, respectively. Since $F^{-}_{t,j,\lambda}(U)$ has the cdf $F_{t,j,\lambda}$ when U is uniformly distributed in $[0, 1]$, we have

$$E\left(Y_t^{(k)} \mid \mathcal{F}_{t-1}^{k,X}\right) = E\left(Y_t^{(k)} \mid \mathcal{F}_{t-1}^{*}\right) = \lambda_t^{(k)}$$
$$P\left(Y_t^{(k)} \leq y \mid \mathcal{F}_{t-1}^{k,X}\right) = P\left(F^{-}_{t,X_t,\lambda_t^{(k)}}(U_t) \leq y \mid \mathcal{F}_{t-1}^{*}\right) = F_{t,X_t,\lambda_t^{(k)}}(y).$$

The existence of a process satisfying $(6.47a)$ with $\mathcal{F}_{t-1}^{*}$ replacing $\mathcal{F}_{t-1}^{Y,X}$ follows if we show

$$\lim_{k\to\infty} \lambda_t^{(k)} = \lambda_t \quad \text{a.s.} \tag{6.54}$$

so that taking the limit on both sides of (6.52) and (6.53) as $k \to \infty$, this gives

$$Y_t = \lim_{k\to\infty} Y_t^{(k)} = F^{-}_{t,X_t,\lambda_t}(U_t) \quad \text{a.s.}$$

Since λ_t is $\mathcal{F}_{t-1}^{Y,X}$-measurable, we have

$$Y_t | \mathcal{F}_{t-1}^{*} \stackrel{d}{=} Y_t | \mathcal{F}_{t-1}^{X,Y}.$$

To show (6.54) under (6.49), let us prove that for all positive integer k,

$$0 \leq \lambda_t^{(k-1)} \leq \lambda_t^{(k)} \quad \text{a.s.} \tag{6.55}$$

and

$$E\left(Y_t^{(k)} - Y_t^{(k-1)}\right) = E\left(\lambda_t^{(k)} - \lambda_t^{(k-1)}\right) \in [0, \infty). \tag{6.56}$$

Clearly, (6.55) and (6.56) hold true when $k \leq 0$. Assume (6.55) holds up to k. In view of (6.46) we have

$$\begin{aligned}\lambda_t^{(k)} &= \omega_t(X_t) + \sum_{i=1}^{q} \alpha_{it}(X_t) F^{-}_{t,X_{t-i},\lambda_{t-i}^{(k-i)}}(U_{t-i}) + \sum_{j=1}^{p} \beta_{jt}(X_t) \lambda_{t-j}^{(k-j)} \\ &\leq \omega_t(X_t) + \sum_{i=1}^{q} \alpha_{it}(X_t) F^{-}_{t,X_{t-i},\lambda_{t-i}^{(k+1-i)}}(U_{t-i}) + \sum_{j=1}^{p} \beta_{jt}(X_t) \lambda_{t-j}^{(k+1-j)} = \lambda_t^{(k+1)}.\end{aligned}$$

Hence (6.55) and (6.56) are satisfied by induction. From (6.46), (6.52), (6.53), (6.55) and (6.56) we have

$$E\left(\left|Y_t^{(k)} - Y_t^{(k-1)}\right| \mid X_t = j\right) = E\left(Y_t^{(k)} - Y_t^{(k-1)} | X_t = j\right) = E\left(\left|\lambda_t^{(k)} - \lambda_t^{(k-1)}\right| | X_t = j\right)$$

and

$$\begin{aligned}&\pi_j(t) E\left(\left|\lambda_t^{(k)} - \lambda_t^{(k-1)}\right| | X_t = j\right) \\ &= \sum_{l=1}^{r} \sum_{i=1}^{K} p_{ij}^{(l)}(t-l)\left(\alpha_{lt}(j) + \beta_{lt}(j)\right)\pi_i(t-l) E\left(\left|\lambda_{t-l}^{(k-l)} - \lambda_{t-l}^{(k-l-1)}\right| | X_{t-l} = i\right), \quad (6.57)\end{aligned}$$

for all $1 \leq i \leq K$. Thus, (6.57) can be embedded in

$$\mathbf{h}_t^{(k)} = \Omega_t \mathbf{h}_{t-1}^{(k-1)}, \quad (6.58)$$

where $\mathbf{h}_t^{(k)} = \left(h_t^{(k)\prime}, ..., h_t^{(k-r+1)\prime}\right)'$, $h_t^{(k)} = \left(h_{1t}^{(k)}, ..., h_{Kt}^{(k)}\right)'$, and $h_{jt}^{(k)} = E\left(\left|\lambda_t^{(k)} - \lambda_t^{(k-1)}\right| |X_t = j\right)$. Iterating (6.58) S times gives

$$\begin{aligned}\mathbf{h}_t^{(k)} &= (\Omega_t \cdots \Omega_{t-S+1}) \mathbf{h}_{t-S}^{(k-S)} \\ &= (\Omega_t \cdots \Omega_{t-S+1}) \mathbf{h}_t^{(k-S)}\end{aligned}$$

where the latter equality stems from the S-periodicity of the model. Under (6.49), we finally get

$$\mathbf{u}_t^{(k)} \to 0, \quad \text{as } k \to \infty$$

exponentially fast as $k \to \infty$ so $\left(\lambda_t^{(k)}\right)_k$ converges in L^1 and a.s. In addition, since

$$\lambda_t = \psi_t(U_{t-1}, U_{t-2}, \ldots; X_t, X_{t-1}, ...),$$

where $\psi_t : [0,1]^\infty \times \{1,..,K\}^\infty \to [0,\infty)$ is a measurable function, the sequence $(\lambda_t, t \in \mathbb{Z})$ is strictly periodically stationary and periodically ergodic and hence so is $(Y_t, t \in \mathbb{Z})$ (cf. Aknouche, 2008). $\square$

7 Conclusion

We presented a basic theory regarding countable periodically homogeneous Markov chains and the stochastic equations that involve them as inputs. Such processes constitute a rather general class of dynamic stochastic models and have occupied an important part of the literature of the last three decades. For these processes, we have studied two main classes of properties, namely:

- *Communication properties* such as v-irreducibility, v-recurrence, v-positive recurrence v-periodicity, v-ergodicity, etc.

- *Asymptotic stability properties* such as periodic ergodic theorems, v-regularity, periodic Harris-ergodicity, periodic geometric ergodicity, etc.

It is worth mentioning the following two points:

i) Communication properties for a periodically homogeneous Markov chain are relative to a given channel. Thus, a property may hold along one channel and not along another. This is especially reflected on solidarity properties such as v-irreducibility, v-recurrence, v-positive recurrence, state-v-periodicity, and v-ergodicity.

ii) On the contrary, periodic stability, when it occurs for a time-invariant state space E, is satisfied along all channels. Indeed, we have seen that the periodic ergodic theorem, the periodic regularity and the periodic Harris ergodicity are each satisfied for all channels under the same conditions.

Finally, Markov chains on uncountable state spaces with time-periodic transition kernels are worth exploring.

8 Appendix

A. Discrete renewal theorem

Theorem A.1 (e.g. Karlin and Taylor, 1975)

i) *Let* $(u_k)_{k\in\mathbb{N}}$, $(a_k)_{k\in\mathbb{N}}$, *and* $(b_k)_{k\in\mathbb{N}}$ *be sequences of non-negative real numbers satisfying:* $\sum_{k=0}^{\infty} a_k = 1,\ 0 < \sum_{k=0}^{\infty} ka_k < \infty,\ \sum_{k=0}^{\infty} b_k < \infty,$ *and* $gcd\{k \in \mathbb{N} : a_k > 0\} = 1$. *If the renewal equation*

$$u_n = \sum_{k=0}^{n} a_{n-k}u_k + b_n,\ \ n \in \mathbb{N}$$

or equivalently

$$u_n = \sum_{k=0}^{n} a_k u_{n-k} + b_n,\ \ n \in \mathbb{N}$$

has a bounded solution $(u_k)_{k\in\mathbb{N}}$, *then*

$$\lim_{n\to\infty} u_n = \frac{\sum_{k=0}^{\infty} b_k}{\sum_{k=0}^{\infty} ka_k}.$$

If $\sum_{k=0}^{\infty} ka_k = \infty$ *and* $\sum_{k=0}^{\infty} b_k < \infty$, *then* $\lim_{n\to\infty} u_n = 0$.

ii) *Let* $(y_k)_{k\in\mathbb{N}}$, $(a_k)_{k\in\mathbb{N}}$, *and* $(x_k)_{k\in\mathbb{N}}$ *be sequences of non-negative real numbers satisfying:* $\sum_{k=0}^{\infty} a_k = 1,\ \lim_{n\to\infty} x_n < \infty,$ *and*

$$y_n = \sum_{k=0}^{n} a_{n-k}x_k,\ n \in \mathbb{N}.$$

Then

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n.$$

□

B. Periodic stationarity and periodic ergodicity

Let $(Y_t,\ t \in \mathbb{Z})$ be a random sequence defined on a probability space $(\Omega, \mathcal{F}, P)$ and valued in the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-algebra of $\mathbb{R}$. Let also PY^{-1} be the corresponding push-forward probability measure defined on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ as $PY^{-1}(A) = P\left(Y^{-1}(A)\right)$ for all $A \in \mathcal{B}\left(\mathbb{R}^{\mathbb{Z}}\right)$, where $\mathcal{B}\left(\mathbb{R}^{\mathbb{Z}}\right)$ stands for the Borel σ-algebra of

$$\mathbb{R}^{\mathbb{Z}} := \{(..., x_{-1}, x_0, x_1, ...) : x_i \in \mathbb{R}, i \in \mathbb{Z}\}$$

and

$$Y^{-1}(A) = \{\omega \in \Omega : Y(\omega) := (..., Y_{-1}(\omega), Y_0(\omega), Y_1(\omega), ...) \in A\}.$$

For all $v \in \{0, ..., S-1\}$, define $T_v : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ to be the right-shift transformation given for each $\mathbf{x}_v = (..., x_{v-S}, x_v, x_{v+S}, ...) \in \mathbb{R}^{\mathbb{Z}}$ by

$$T_v \mathbf{x}_v = (..., x_{v-S+1}, x_{v+1}, x_{v+S+1}, ...) := \mathbf{x}_{v+1}.$$

Denote by $\mathbb{T}_v^S$ the S-step (monodromy) right-shift transformation given by

$$\mathbb{T}_v^S = T_v \circ T_{v+1} \circ ... \circ T_{v+S-1},$$

so that

$$\mathbb{T}_v^S \mathbf{x}_v = (..., x_v, x_{v+S}, x_{v+2S}, ...).$$

The sequence $(Y_t,\ t \in \mathbb{Z})$ is said to be *strictly S-periodically stationary* if for all $v \in \{0, ..., S-1\}$, $\mathbb{T}_v^S$ preserves the probability measure $PY^{-1}(.)$ in the sense

$$PY^{-1}\left(\mathbb{T}_v^{-S}(A)\right) = PY^{-1}(A) \text{ for all } A \in \mathcal{B}\left(\mathbb{R}^{\mathbb{Z}}\right),$$

where $\mathbb{T}_v^{-S}(A) = \left\{\mathbf{x}_v \in \mathbb{R}^{\mathbb{Z}} : \mathbb{T}_v^S \mathbf{x}_v \in A\right\}$.

Thus the infinite-dimensional distributions of a strictly S-periodically stationary sequence are invariant by a translation multiple of the period. An equivalent but simpler definition of strict periodic stationarity can be given in terms of the finite-dimensional distributions. A sequence $(Y_t,\ t \in \mathbb{Z})$ is strictly S-periodically stationary if and only if

$$(Y_v, Y_{S+v}, ...Y_{nS+v}) \stackrel{d}{=} (Y_{v+h}, Y_{2S+v+h}, ...Y_{nS+v+h})$$

for all $v \in \{0, ..., S-1\}$, $n \in \mathbb{N}$ and $h \in \mathbb{Z}$. The latter equality can be replaced by

$$(Y_{t_1}, Y_{t_2}, ...Y_{t_n}) \stackrel{d}{=} (Y_{t_1+h}, Y_{t_2+h}, ...Y_{t_n+h})$$

for all $n \in \mathbb{N}^*$ and $t_1, ..., t_n, h \in \mathbb{Z}$.

A Borel set $C_v \in \mathcal{B}\left(\mathbb{R}^{\mathbb{Z}}\right)$ of the form

$$C_v = \{\mathbf{x}_v = (..., x_{v-S}, x_v, x_{v+S}, ...) : x_{v+kS} \in \mathbb{R}, k \in \mathbb{Z}\}$$

is said to be S-shift invariant along the channel v if

$$\mathbb{T}_v^{-S}(C_v) = C_v.$$

A strictly periodically stationary sequence $(Y_t,\ t \in \mathbb{Z})$ is said to be *S-periodically ergodic* if

$$P(\{..., Y_{v-S}, Y_v, Y_{v+S}, ...\} \in C_v) = 0 \text{ or } 1,$$

for each $v \in \{0, ..., S-1\}$ and each Borel set C_v, S-shift-invariant along the channel v (cf. Boyles and Gardner, 1983; Aknouche, 2008; Aknouche et *al*, 2020).

The simplest *strictly S-periodically stationary and S-periodically ergodic* process is an ipd_S sequence.

Let $f_t : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ be a measurable time-periodic sequence of functions (i.e. $f_t = f_{t+S}$ for all $t \in \mathbb{Z}$). *If $(Y_t,\ t \in \mathbb{Z})$ is strictly S-periodically stationary and S-periodically ergodic and if $(Z_t, t \in \mathbb{Z})$ is defined as*

$$Z_t = f_t(..., Y_{t-1}, Y_t, Y_{t+1}, ...), \quad t \in \mathbb{Z},$$

then $\{Z_t, t \in \mathbb{Z}\}$ is also strictly S-periodically stationary and S-periodically ergodic (Aknouche, 2008).

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