# A Novel Dominance Principle based Approach to the Solution of Two Persons General Sum Games with n by m moves

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7 October 2024

# A Novel Dominance Principle based Approach to the Solution of Two Persons General Sum Games with $n$ by $m$ Moves

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**Submitted for publication**

**Abstract**

In a previous paper [1] the application of the dominance principle was proposed to find the non-cooperative solution of the two by two general sum game with mixed strategies; in this way it was possible to choose the equilibrium point among the classical solutions avoiding the ambiguity due to their non-interchangeability, moreover the non-cooperative equilibrium point was determined by a new geometric approach based on the dominance principle. Starting from that result it is here below proposed the extension of the method to two persons general sum games with $n$ by $m$ moves. The algebraic two multi-linear forms of the expected payoffs of the two players are studied. From these expressions of the expected payoffs the derivatives are obtained and they are used to express the probabilities distribution on the moves after the two definitions as Nash and prudential strategies [1].

The application of the dominance principle allows to choose the equilibrium point between the two solutions avoiding the ambiguity due to their non-interchangeability and a conjecture about the uniqueness of the solution is proposed in order to solve the problem of the existence and uniqueness of the non-cooperative solution of a two persons $n$ by $m$ game. The uniqueness of the non-cooperative solution could be used as a starting point to find out the cooperative solution of the game too. Some games from the sound literature are discussed in order to show the effectiveness of the presented procedure.

***Keywords**: Dominance principle; General sum game; two persons n by m moves game.*

# 1 Introduction

The main references for the development of the present paper are my previous paper [1], the master paper by Nash [2] and the texts of Luce and Raiffa [6], Owen [7], Straffin [8], Maschler et al. [12].

It is proposed to use the dominance principle as the only tool to find the non-cooperative solution of a two persons game on the basis that a rational player should never play a dominated move [8, 12]. Straffin [8] argues that there is a conflict between the dominance principle and the Pareto-optimality, but it has to be noted that the dominance principle is cogent for individual rationality whereas the Pareto-optimality is cogent for the group rationality. The individual rationality is here considered suitable to find the non-cooperative equilibrium strategies of a two persons game, thus the dominance principle is applied to find the solution.

This paper is devoted to the study of the non-cooperative solution of a two persons $n$ by $m$ moves game with no dominated pure strategies, therefore it is not considered the trivial case that can be solved by the elimination of all the dominated moves.

On the other hand, the maximin [16] value of any particular player is unaffected by the elimination of his dominated moves, whether those strategies are weakly or strictly dominated; moreover the iterated elimination of weakly dominated moves does not lead to the creation of new equilibria (Maschler et al. [12]).

As it is well known the mixed strategies method to find the solution of a game is suitable only for repeatable games. A mixed strategy for a player is defined as the probability distribution on the set of his pure strategies [7]; the expected payoff from a mixed strategy is defined as the corresponding probability-weighted average of the payoffs from its constituent pure strategies [10]. The search of the non-cooperative solution with the mixed strategies could bring to find more than one equilibrium point, but these equilibrium points represent the acceptable non-cooperative solution of the game only if they are equivalent and interchangeable [2, 6].

In Part 1 of the paper it is proposed to look for the non-cooperative solution of a two persons 3 by 3 game by applying the dominance principle on the mixed strategies and the relationship is studied among the two classical mixed strategies, prudential and Nash strategy [1], and the expected payoff.

In Part 3 seven numerical examples are discussed to show the application of the dominance principle and the so found solutions are compared and discussed with respect to the literature solutions.

Part 4 is an extension of the application of the dominance principle to the two persons game with $n$ by $m$ moves.

In Part 5, in order to show its powerful meaning, the proposed method is applied to two examples by showing that it is possible to find the equilibrium point of a game also when the algebraic method fails.

The conclusion summarizes the main features of the proposed method recognizing it as a powerful tool to find the non-cooperative solution of a two person general sum game larger than two by two moves.

# 2 Non-cooperative solution of the normal form of two persons 3 by 3 game

## 2.1 Theory

As it is well known the normal form of the two persons 3-by-3 game is the following one:

**Table 1**

| | | Moves of player $B$ | | |
|---|---|---|---|---|
| | | $y_1$ | $y_2$ | $y_3$ |
| | $x_1$ | $a_{11}, b_{11}$ | $a_{12}, b_{12}$ | $a_{13}, b_{13}$ |
| Moves of player $A$ | $x_2$ | $a_{21}, b_{21}$ | $a_{22}, b_{22}$ | $a_{23}, b_{23}$ |
| | $x_3$ | $a_{31}, b_{31}$ | $a_{32}, b_{32}$ | $a_{33}, b_{33}$ |

$$(x) = (x_1, x_2, x_3) = (x_1, x_2, 1 - x_1 - x_2) \tag{1}$$

and

$$(y) = (y_1, y_2, y_3) = (y_1, y_2, 1 - y_1 - y_2) \tag{2}$$

are the vectors of the probability distribution on the moves respectively for player $A$ and $B$, with the constraints

$$0 \leq x_i \leq 1 \tag{3}$$

and

$$0 \leq y_j \leq 1 \tag{4}$$

Associated to each possible outcome of the game is a collection of numerical payoffs, one to each player.

The expected payoff for each player is then given by:

$$\begin{aligned} z_A = {} & (a_{11} + a_{33} - a_{13} - a_{31})x_1y_1 + (a_{12} + a_{33} - a_{13} - a_{32})x_1y_2 + (a_{21} + a_{33} - a_{23} - a_{31})x_2y_1 + \\ & + (a_{22} + a_{33} - a_{23} - a_{32})x_2y_2 + (a_{13} - a_{33})x_1 + (a_{23} - a_{33})x_2 + (a_{31} - a_{33})y_1 + (a_{32} - a_{33})y_2 + a_{33} = \\ & = A_1x_1y_1 + A_2x_1y_2 + A_3x_2y_1 + A_4x_2y_2 + A_5x_1 + A_6x_2 + A_7y_1 + A_8y_2 + A_0 \end{aligned} \tag{5}$$

$$\begin{aligned} z_B = {} & (b_{11} + b_{33} - b_{13} - b_{31})x_1y_1 + (b_{12} + b_{33} - b_{13} - b_{32})x_1y_2 + (b_{21} + b_{33} - b_{23} - b_{31})x_2y_1 + \\ & + (b_{22} + b_{33} - b_{23} - b_{32})x_2y_2 + (b_{13} - b_{33})x_1 + (b_{23} - b_{33})x_2 + (b_{31} - b_{33})y_1 + (b_{32} - b_{33})y_2 + b_{33} = \\ & = B_1x_1y_1 + B_2x_1y_2 + B_3x_2y_1 + B_4x_2y_2 + B_5x_1 + B_6x_2 + B_7y_1 + B_8y_2 + B_0 \end{aligned} \tag{6}$$

These formulas will be used throughout the paper from here on.

In the case of three by three moves in order to find maximum and minimum points [14] it is not sufficient to calculate the first derivatives of the expected payoffs and to study the Hessian whose ranking is greater than three, but some more complicated methods should be used.

As mentioned in my previous paper [1], in literature there are two ways to calculate the probability distribution for each player: a prudential strategy [8] and a Nash strategy [2]. These two different strategies can be determined by calculating the first derivatives of the expected payoffs and equating them to zero. First of all the Nash strategies are determined.

$$\partial z_A / \partial x_1 = A_1y_1 + A_2y_2 + A_5 = 0 \tag{7}$$

$$\partial z_A / \partial x_2 = A_3y_1 + A_4y_2 + A_6 = 0 \tag{8}$$

implies

$$y_1 = (A_2A_6 - A_4A_5)/(A_1A_4 - A_2A_3) = y_{N1} \tag{9}$$

$$y_2 = (A_5A_3 - A_6A_1)/(A_1A_4 - A_2A_3) = y_{N2} \tag{10}$$

and

$$\partial z_B/\partial y_1 = B_1 x_1 + B_3 x_2 + B_7 = 0 \tag{11}$$

$$\partial z_B/\partial y_2 = B_2 x_1 + B_4 x_2 + B_8 = 0 \tag{12}$$

implies

$$x_1 = (B_8 B_3 – B_7 B_4)/(B_1 B_4 – B_2 B_3) = x_{N1} \tag{13}$$

$$x_2 = (B_2 B_7 – B_1 B_8)/(B_1 B_4 – B_2 B_3) = x_{N2} \tag{14}$$

that is the probability distribution for player $A$ and $B$ after Nash.

The meaning of the Nash strategy is that if player $B$ chooses $(y_N)$ then there is no variation of $z_A$ irrespective of the choice of player $A$; if player $A$ chooses $(x_N)$ there is no variation of $z_B$ irrespective of the choice of player $B$.

The prudential strategies are following.

$$\partial z_A/\partial y_1 = A_1 x_1 + A_3 x_2 + A_7 = 0 \tag{15}$$

$$\partial z_A/\partial y_2 = A_2 x_1 + A_4 x_2 + A_8 = 0 \tag{16}$$

implies

$$x_1 = (A_3 A_8 – A_4 A_7)/(A_1 A_4 – A_2 A_3) = x_{p1} \tag{17}$$

$$x_2 = (A_2 A_7 – A_1 A_8)/(A_1 A_4 – A_2 A_3) = x_{p2} \tag{18}$$

implies

$$\partial z_B/\partial x_1 = B_1 y_1 + B_2 y_2 + B_5 = 0 \tag{19}$$

$$\partial z_B/\partial x_2 = B_2 y_1 + B_4 y_2 + B_8 = 0 \tag{20}$$

implies

$$y_1 = (B_2 B_6 – B_4 B_5)/(B_1 B_4 – B_2 B_3) = y_{p1} \tag{21}$$

$$y_2 = (B_3 B_5 – B_1 B_6)/(B_1 B_4 – B_2 B_3) = y_{p2} \tag{22}$$

that is the prudential probability distribution for player $A$ and $B$.

The meaning of the prudential strategy is that if player $B$ chooses $(y_p)$ then there is no variation of $z_B$ irrespective of the choice of player $A$; if player $A$ chooses $(x_p)$ there is no variation of $z_A$ irrespective of the choice of player $B$.

By substituting in the formulas of the expected payoffs of each player respectively the prudential strategies and the Nash's strategies it can easily be seen that the expected payoffs are equal in the two cases, thus the two couples of strategies $(x_p, y_p)$ and $(x_N, y_N)$ are equivalent.

Moreover combining the prudential strategies with the Nash's strategies it is found that:

$$\begin{aligned} z_A(x_p, y_N) = z_A(x_p, y_p) = z_A(x_N, y_N) = \\ = (A_2 A_6 A_7 – A_1 A_6 A_8 + A_3 A_5 A_8 – A_4 A_5 A_7)/(A_1 A_4 – A_2 A_3) + A_0 = z_A^* \end{aligned} \tag{23}$$

but

$$z_B(x_p, y_N) \neq z_B(x_p, y_p) = z_B(x_N, y_N) = z_B^* \tag{24}$$

and

$$\begin{aligned} z_B(x_N, y_p) = z_B(x_p, y_p) = z_B(x_N, y_N) = \\ = (B_2 B_6 B_7 – B_1 B_6 B_8 + B_3 B_5 B_8 – B_4 B_5 B_7)/(B_1 B_4 – B_2 B_3) + B_0 = z_B^* \end{aligned} \tag{25}$$

but

$$z_A(x_N, y_p) \neq z_A(x_p, y_p) = z_A(x_N, y_N) = z_A^* \tag{26}$$

this means that the two couples of strategies are generally not interchangeable.

It can be concluded that the couple of strategies does not represent a solution of the game because they are equivalent, but not interchangeable, as it is stated by Nash [2]. The non-cooperative solution does not take into account the possibility of an agreement between the two players, thus it is possible that the players choose non homogeneous strategies because they are equivalent, but this is not optimal because they are not interchangeable.

The outcome of the possible choices of the two players is depicted in the following Table 2.

**Table 2**

| | Strategies | | Expected payoffs | |
|---|---|---|---|---|
| | $x$ | $y$ | $z_A$ | $z_B$ |
| Nash | $x_{N1}, x_{N2}$ | $y_{N1}, y_{N2}$ | $z_A^*$ | $z_B^*$ |
| Prudential | $x_{p1}, x_{p2}$ | $y_{p1}, y_{p2}$ | $z_A^*$ | $z_B^*$ |
| Nash/Prud. | $x_{N1}, x_{N2}$ | $y_{p1}, y_{p2}$ | $\alpha$ | $z_B^*$ |
| Prud./Nash | $x_{p1}, x_{p2}$ | $y_{N1}, y_{N2}$ | $z_A^*$ | $\beta$ |

where
$\alpha = z_A(x_N, y_p)$
$\beta = z_B(x_p, y_N)$

It comes out that in order to choose the optimal strategy the player $A$ should look whether the value of $z_A(x_N, y_p)$ is greater or lower than $z_A^*$: if it is greater, the strategy $(x_N)$ is dominant irrespective of the choice of player $B$, if it is lower, the strategy $(x_p)$ becomes dominant irrespective of the choice of player $B$.

The player $B$ should look whether the value of $z_B(x_p, y_N)$ is greater or lower than $z_B^*$: if it is greater, the strategy $(y_N)$ is dominant irrespective of the choice of player $A$, if it is lower, the strategy $(y_p)$ becomes dominant irrespective of the choice of player $A$.

The discussion of the first derivatives of the expected payoffs gives a rationale of the two different ways to calculate the probability distribution on the strategies:

- the prudential strategy

$$\partial z_A/\partial y = 0 \text{ implies } (x) = (x_p) \tag{27}$$
$$\partial z_B/\partial x = 0 \text{ implies } (y) = (y_p) \tag{28}$$

  guarantees that each player receives a payoff irrespective of the choice of the other player: i.e. $(x_p)$ is the best replay of player $A$ whatever it is the strategy of $B$ and $(y_p)$ is the best replay of player $B$ whatever it is the strategy of $A$; this explains why:

$$z_A(x_p, y_N) = z_A(x_p, y_p) = z_A(x_p, y) = z_A^* \tag{29}$$
$$z_B(x_N, y_p) = z_B(x_p, y_p) = z_B(x, y_p) = z_B^* \tag{30}$$

- the Nash's strategy

$$\partial z_B/\partial y = 0 \text{ implies } (x) = (x_N) \tag{31}$$
$$\partial z_A/\partial x = 0 \text{ implies } (y) = (y_N) \tag{32}$$

guarantees that each player receives a payoff irrespective of his own choice: i.e. $(x_N)$ makes indifferent the replay of player $B$ and $(y_N)$ makes indifferent the replay of player $A$; this explains why:

$$z_A(x_p, y_N) = z_A(x_N, y_N) \tag{33}$$
$$z_B(x_N, y_p) = z_B(x_N, y_N) \tag{34}$$

### 2.2 Remarks about the solution of two persons 3 by 3 games

It is worth to note that the proposed procedure to determine the equilibrium strategies of a game does not depend upon the value of the payoffs of the bi-matrix, nevertheless the resulting equilibrium strategies depend totally upon those values.

Moreover, as already said in my previous paper [1], there is a possible flaw in the proposed procedure. The prudential strategy is calculated for each player on the basis of its own matrix of the payoffs, but the expected payoff for each player is based also on the knowledge of the prudential strategy of the other player. Something similar happens for the Nash's way because the strategy of each player is based on the matrix of the payoffs of the other player, so the expected payoff of a player is depending upon the matrix of the payoffs of the other. In both cases there is a possible flaw of the method because also if a player should be able to state precisely his payoffs matrix corresponding to each of his own pure strategies, he could not be able to state precisely the payoffs matrix of the competitor. This flaw is overcome by the theorem that every finite $n$-person game with perfect information has an equilibrium $n$-tuple of strategies [7]. Nevertheless the theorem gives a demonstration of the existence of a solution, but it does not give the way to find it.

The proposed procedure could not work both in the case three by three moves and in the case of different number of moves between the two players: it depends whether the algebraic requirements for the existence of a solution of the system of equations are satisfied or not. If the requirements are not satisfied another procedure should be adopted: this situation will be presented in the following numerical examples.

## 3 Numerical solutions of a 3 x 3 game in normal form

### 3.1 Example 1

As a first example a general sum game published and solved by Owen [7] is shown in Table 3.

**Table 3**

| | | Moves of player $B$ | | |
|---|---|---|---|---|
| | | $y_1$ | $y_2$ | $y_3$ |
| | $x_1$ | $2,2$ | $1,1$ | $0,0$ |
| Moves of player $A$ | $x_2$ | $1,1$ | $0,0$ | $2,2$ |
| | $x_3$ | $0,0$ | $2,2$ | $1,1$ |

Formulas 17, 18, 21, 22 and 9, 10, 13, 14 give the vectors of the probability distribution on the moves respectively for player $A$ and $B$ and the expected payoffs are

$$z_A = 3x_1y_1 + 0x_1y_2 + 0x_2y_1 - 3x_2y_2 - x_1 + x_2 - y_1 - y_2 + 1$$

$$z_B = 3x_1y_1 + 0x_1y_2 + 0x_2y_1 - 3x_2y_2 - x_1 + x_2 - y_1 - y_2 + 1$$

The strategies are following:

- first way (prudential strategy)

$$(x_p) = (1/3, 1/3, 1/3) \qquad \text{for player A}$$
$$(y_p) = (1/3, 1/3, 1/3) \qquad \text{for player B}$$

  with $z_A(x_p, y_p) = 1$ and $z_B(x_p, y_p) = 1$

- second way (Nash's strategy)

$$(x_N) = (1/3, 1/3, 1/3) \qquad \text{for player A}$$
$$(y_N) = (1/3, 1/3, 1/3) \qquad \text{for player B}$$

  with $z_A(x_N, y_N) = 1$ and $z_B(x_N, y_N) = 1$

For $A$ it comes out that
$z_A(x_N, y_p) = 1 = z_A(x_p, y_N) = z_A(x_p, y_p) = z_A(x_N, y_N) = 1$
thus $(x_N)$ and $(x_p)$ are totally equivalent strategies for $A$.
For $B$ it comes out that
$z_B(x_p, y_N) = 1 = z_B(x_N, y_p) = z_B(x_p, y_p) = z_B(x_N, y_N) = 1$
thus $(y_N)$ and $(y_p)$ are totally equivalent strategies for $B$.

Moreover for $A$ it comes out that
$z_A(x, y_N) = 1 = z_A(x, y_p)$
whatever $(x)$ strategy is adopted by player $A$.

Analogously for $B$ it comes out that
$z_B(x_N, y) = 1 = z_B(x_p, y)$
whatever $(y)$ strategy is adopted by player $B$.

Nevertheless the non-cooperative solution should be based on an independent choice of the strategies by each player thus the solution (whatever $(x)$, whatever $(y)$) is not suitable because the expected payoff becomes undefined and it cannot be calculated by each of the players and the only one solution is one of the above determined solution.

Thus the two solutions of the game are equivalent and interchangeable and due to the symmetry of the game the two players have the same probability distribution on the moves and the same expected payoff.

This solution is different from that proposed by Owen [7].

### 3.2 Example 2

As a second example a zero sum game published and solved by Owen [7] is shown in Table 4.

**Table 4**

| | | Moves of player $B$ | | |
|---|---|---|---|---|
| | | $y_1$ | $y_2$ | $y_3$ |
| | $x_1$ | $0, 0$ | $1, -1$ | $-2, 2$ |
| Moves of player $A$ | $x_2$ | $-1, 1$ | $0, 0$ | $3, -3$ |
| | $x_3$ | $2, -2$ | $-3, 3$ | $0, 0$ |

Formulas 17, 18, 21, 22 and 9, 10, 13, 14 give the vectors of the probability distribution on the moves respectively for player $A$ and $B$ and the expected payoffs are
$z_A = 0x_1y_1 + 6x_1y_2 - 6x_2y_1 + 0x_2y_2 - 2x_1 + 3x_2 + 2y_1 - 3y_2 + 0$

$z_B = 0x_1y_1 - 6x_1y_2 + 6x_2y_1 + 0x_2y_2 + 2x_1 - 3x_2 - 2y_1 + 3y_2 + 0$

The strategies are following:

- first way (prudential strategy)

$$(x_p) = (1/2, 1/3, 1/6) \qquad \text{for player A}$$
$$(y_p) = (1/2, 1/3, 1/6) \qquad \text{for player B}$$

  with $z_A(x_p, y_p) = 0$ and $z_B(x_p, y_p) = 0$

- second way (Nash's strategy)

$$(x_N) = (1/2, 1/3, 1/6) \qquad \text{for player A}$$
$$(y_N) = (1/2, 1/3, 1/6) \qquad \text{for player B}$$

  with $z_A(x_N, y_N) = 0$ and $z_B(x_N, y_N) = 0$

For $A$ it comes out that
$z_A(x_N, y_p) = 0 = z_A(x_p, y_N) = z_A(x_p, y_p) = z_A(x_N, y_N) = 0$
thus $(x_N)$ and $(x_p)$ are totally equivalent strategies for $A$.

For $B$ it comes out that $z_B(x_p, y_N) = 0 = z_B(x_N, y_p) = z_B(x_p, y_p) = z_B(x_N, y_N) = 0$
thus $(y_N)$ and $(y_p)$ are totally equivalent strategies for $B$.

Thus the two solutions of the game are equivalent and interchangeable, moreover due to the symmetry of the game the strategies of the two players are equal, $(x_N) = (y_N)$ and $(x_p) = (y_p)$, and the expected payoffs are opposite, in this case both are zero.

This result is in agreement with the finding of a unique equilibrium pair by Owen [7].

### 3.3 Example 3

As a third example a general sum game published as exercise by Owen [7] is shown in Table 5.

Formulas 17, 18, 21, 22 and 9, 10, 13, 14 give the vectors of the probability distribution on the moves respectively for player $A$ and $B$ and the expected payoffs are

**Table 5**

| | | Moves of player $B$ | | |
|---|---|---|---|---|
| | | $y_1$ | $y_2$ | $y_3$ |
| Moves of player $A$ | $x_1$ | $2, 1$ | $0, 0$ | $1, 2$ |
| | $x_2$ | $1, 2$ | $2, 1$ | $0, 0$ |
| | $x_3$ | $0, 0$ | $1, 2$ | $2, 1$ |

$$z_A = 3x_1y_1 + 0x_1y_2 + 3x_2y_1 + 3x_2y_2 - x_1 - 2x_2 - 2y_1 - 2y_2 + 2$$

$$z_B = 0x_1y_1 - 3x_1y_2 + 3x_2y_1 + 0x_2y_2 + x_1 - x_2 - y_1 + y_2 + 1$$

The strategies are following:

- first way (prudential strategy)

$$(x_p) = (0, 2/3, 1/3) \qquad \text{for player A}$$
$$(y_p) = (1/3, 1/3, 1/3) \qquad \text{for player B}$$

  with $z_A(x_p, y_p) = z_A(x_p, y_N) = 2/3$ and $z_B(x_p, y_p) = z_B(x_p, y_N) = 1$

- second way (Nash's strategy)

$$(x_N) = (1/3, 1/3, 1/3) \qquad \text{for player A}$$
$$(y_N) = (1/3, 1/3, 1/3) \qquad \text{for player B}$$

  with $z_A(x_N, y_N) = z_A(x_N, y_p) = 2/3$ and $z_B(x_N, y_N) = z_B(x_N, y_p) = 1$

For $A$ it comes out that
$z_A(x_N, y_p) = 2/3 = z_A(x_p, y_N) = z_A(x_p, y_p) = z_A(x_N, y_N) = 2/3$
thus $(x_N)$ and $(x_p)$ are totally equivalent strategies for $A$.

For $B$ it comes out that $z_B(x_p, y_N) = 1 = z_B(x_N, y_p) = z_B(x_p, y_p) = z_B(x_N, y_N) = 1$
thus $(y_N)$ and $(y_p)$ are totally equivalent strategies for $B$.

Thus the two solutions of the game are equivalent and interchangeable. Owen [7] finds only the Nash solution.

## 3.4 Example 4

As a forth example a general sum game, symmetric bimatrix game, published as exercise by Owen [7] is shown in Table 6.

**Table 6**

| | | Moves of player $B$ | | |
|---|---|---|---|---|
| | | $y_1$ | $y_2$ | $y_3$ |
| Moves of player $A$ | $x_1$ | $1, 1$ | $2, 2$ | $3, 2$ |
| | $x_2$ | $2, 2$ | $1, 1$ | $4, 3$ |
| | $x_3$ | $2, 3$ | $3, 4$ | $1, 1$ |

Formulas 17, 18, 21, 22 and 9, 10, 13, 14 give the vectors of the probability distribution on the moves respectively for player $A$ and $B$ and the expected payoffs are

$$z_A = -3x_1y_1 - 3x_1y_2 - 3x_2y_1 - 5x_2y_2 + 2x_1 + 3x_2 + y_1 + 2y_2 + 1$$

$$z_B = -3x_1y_1 - 3x_1y_2 - 3x_2y_1 - 5x_2y_2 + x_1 + 2x_2 + 2y_1 + 3y_2 + 1$$

First of all the Nash strategies are here below determined.

$$\begin{aligned} \partial z_A/\partial x_1 &= -3y_1 - 3y_2 + 2 = 0 \\ \partial z_A/\partial x_2 &= -3y_1 - 5y_2 + 2 = 0 \end{aligned}$$

implies

$$\begin{aligned} y_1 &= 1/6 = y_{N1} \\ y_2 &= 1/2 = y_{N2} \end{aligned}$$

and

$$\begin{aligned} \partial z_B/\partial y_1 &= -3x_1 - 3x_2 + 2 = 0 \\ \partial z_B/\partial y_2 &= -3x_1 - 5x_2 + 3 = 0 \end{aligned}$$

implies

$$\begin{aligned} x_1 &= 1/6 = x_{N1} \\ x_2 &= 1/2 = x_{N2} \end{aligned}$$

The Nash strategies are following:

$$\begin{aligned} (x_N) &= (1/6, 1/2, 1/3) && \text{for player A} \\ (y_N) &= (1/6, 1/2, 1/3) && \text{for player B} \end{aligned}$$

with $z_A(x_N, y_N) = 13/6$ and $z_B(x_N, y_N) = 13/6$

The prudential strategies are here below determined.

$$\begin{aligned} \partial z_A/\partial y_1 &= -3x_1 - 3x_2 + 1 = 0 \\ \partial z_A/\partial y_2 &= -3x_1 - 5x_2 + 2 = 0 \end{aligned}$$

implies

$$x_1 = -1/6 = x_{p1}$$
$$x_2 = 1/2 = x_{p2}$$

and

$$\partial z_B/\partial x_1 = -3y_1 - 3y_2 + 1 = 0$$
$$\partial z_B/\partial x_2 = -3y_1 - 5y_2 + 2 = 0$$

implies

$$y_1 = -1/6 = y_{p1}$$
$$y_2 = 1/2 = y_{p2}$$

For the prudential probability distribution the result is not acceptable nor for player $A$ neither for player $B$.

Moreover for $A$ it comes out that $z_A(x, y_N) = 13/16$ and it is independent from the strategy $(x)$ of player $A$.

Analogously for $B$ it comes out that $z_B(x_N, y) = 13/16$ and it is independent from the strategy $(y)$ of player $B$.

Nevertheless for $A$ it comes out that $z_A(x_N, y)$ is depending upon the $(y)$ strategy adopted by player $B$ and for $B$ it comes out that $z_B(x, y_N)$ is depending upon the $(x)$ strategy adopted by player $A$.

Thus the only solution is the Nash's strategy and the prudential strategy is not existing and due to the symmetry of the game the two players have the same probability distribution on the moves and the same expected payoff.

### 3.5 Example 5

As a fifth example a zero sum game, symmetric bimatrix game, published by Gambarelli [14] is shown in Table 7.

**Table 7**

| | | Moves of player $B$ | | |
|---|---|---|---|---|
| | | $y_1$ | $y_2$ | $y_3$ |
| | $x_1$ | $2, -2$ | $1, -1$ | $0, 0$ |
| Moves of player $A$ | $x_2$ | $1, -1$ | $2, -2$ | $1, -1$ |
| | $x_3$ | $0, 0$ | $1, -1$ | $2, -2$ |

First of all the Nash strategies are determined. Formulas 11 and 12 gives the system of equations to be solved to find the vector of the probability distribution on the moves for player $A$

$$\partial z_B/\partial y_1 = -4x_1 - 2x_2 + 2 = 0$$
$$\partial z_B/\partial y_2 = -2x_1 - 2x_2 + 1 = 0$$

The solution of this system of equations is:

$$x_1 = 1/2 = x_{N1}$$
$$x_2 = 0 = x_{N2}$$
$$x_3 = 1/2 = x_{N3}$$

Formulas 7 and 8 gives the system of equations to be solved to find the vector of the probability distribution on the moves for player $B$

$$\partial z_A/\partial x_1 = 4y_1 + 0y_2 - 2 = 0$$
$$\partial z_A/\partial x_2 = 2y_1 + 2y_2 - 1 = 0$$

implies

$$y_1 = 1/2 = y_{N1}$$
$$y_2 = 0 = y_{N2}$$
$$y_3 = 1/2 = y_{N3}$$

The Nash strategies are following:

$(x_N) = (1/2, 0, 1/2)$ for player A
$(y_N) = (1/2, 0, 1/2)$ for player B

with $z_A(x_N, y_N) = 1$ and $z_B(x_N, y_N) = -1$

The prudential strategies are here below determined. Formulas 15 and 16 gives the system of equations to be solved to find the vector of the probability distribution on the moves for player $A$

$$\partial z_A/\partial y_1 = 4x_1 - 2 = 0$$
$$\partial z_A/\partial y_2 = 2x_1 + 2x_2 - 1 = 0$$

The solution of this system of equations is:

$$x_1 = 1/2 = x_{p1}$$
$$x_2 = 0 = x_{p2}$$
$$x_3 = 1/2 = x_{p3}$$

Formulas 19 and 20 gives the system of equations to be solved to find the vector of the probability distribution on the moves for player $B$

$$\partial z_B/\partial x_1 = -4y_1 - 2y_2 + 2 = 0$$
$$\partial z_B/\partial x_2 = -2y_1 - 2y_2 + 2 = 0$$

implies

$$y_1 = 1/2 = y_{p1}$$
$$y_2 = 0 = y_{p2}$$
$$y_3 = 1/2 = y_{p3}$$

The prudential strategies are following:

$$(x_p) = (1/2, 0, 1/2) \qquad \text{for player A}$$
$$(y_p) = (1/2, 0, 1/2) \qquad \text{for player B}$$

with $z_A(x_p, y_p) = 1$ and $z_B(x_p, y_p) = -1$

It can be concluded that:

$$z_A(x_p, y_p) = z_A(x_N, y_N) = z_A(x_p, y_N) = 1$$
$$z_B(x_p, y_p) = z_B(x_N, y_N) = z_B(x_N, y_p) = -1$$

moreover

$$z_A(x_N, y_p) = 1$$
$$z_B(x_p, y_N) = -1$$

Therefore the two solutions are equivalent and interchangeable, moreover due to the symmetry of the game the strategies of the two players are equal, $(x_N) = (y_N)$ and $(x_p) = (y_p)$, and the expected payoffs are opposite.

### 3.6 Example 6

As a sixth example a zero sum game published by Dixit and Skeath [10] to show the application of the mixed strategy concept is shown in Table 8.
The game is a simplified representation of a penalty kick in soccer; both players have just three pure strategies: the kicker, row player $A$, can kick to his left, center or right and the goalie, column player $B$, can move to left, center or right (left and right are referred for both to the kicker).

**Table 8**

| | | Moves of player $B$ | | |
|---|---|---|---|---|
| | | $y_1$ | $y_2$ | $y_3$ |
| | $x_1$ | $45, -45$ | $90, -90$ | $90, -90$ |
| Moves of player $A$ | $x_2$ | $85, -85$ | $0, 0$ | $85, -85$ |
| | $x_3$ | $95, -95$ | $95, -95$ | $60, -60$ |

First of all the Nash strategies are determined. Formulas 11 and 12 gives the system of equations to be solved to find the vector of the probability distribution on the moves for player $A$

$$\partial z_B/\partial y_1 = 80x_1 + 35x_2 - 35 = 0$$
$$\partial z_B/\partial y_2 = 35x_1 + 120x_2 - 35 = 0$$

The solution of this system of equations is:

$$x_1 = 119/335 = 0,355 = x_{N1}$$
$$x_2 = 63/335 = 0,188 = x_{N2}$$
$$x_3 = 153/335 = 0,457 = x_{N3}$$

Formulas 7 and 8 gives the system of equations to be solved to find the vector of the probability distribution on the moves for player $B$

$$\partial z_A/\partial x_1 = -80y_1 - 35y_2 + 30 = 0$$
$$\partial z_A/\partial x_2 = -35y_1 - 120y_2 + 25 = 0$$

implies

$$y_1 = 109/335 = 0,325 = y_{N1}$$
$$y_2 = 38/335 = 0,113 = y_{N2}$$
$$y_3 = 188/335 = 0,562 = y_{N3}$$

The Nash strategies are following:

$$(x_N) = (119/335, 63/335, 153/335) \qquad \text{for player A}$$
$$(y_N) = (109/335, 38/335, 188/335) \qquad \text{for player B}$$

with $z_A(x_N, y_N) = 75,4$ and $z_B(x_N, y_N) = -75,4$

The prudential strategies are here below determined. Formulas 15 and 16 gives the system of equations to be solved to find the vector of the probability distribution on the moves for player $A$

$$\partial z_B/\partial y_1 = -80x_1 - 35x_2 + 35 = 0$$
$$\partial z_B/\partial y_2 = -35x_1 - 120x_2 + 35 = 0$$

The solution of this system of equations is:

$$x_1 = 119/335 = 0,355 = x_{p1}$$
$$x_2 = 63/335 = 0,188 = x_{p2}$$
$$x_3 = 153/335 = 0,457 = x_{p3}$$

Formulas 19 and 20 gives the system of equations to be solved to find the vector of the probability distribution on the moves for player $B$

$$\partial z_A/\partial x_1 = 80y_1 + 35y_2 - 30 = 0$$
$$\partial z_A/\partial x_2 = 35y_1 + 120y_2 - 25 = 0$$

implies

$$y_1 = 109/335 = 0,325 = y_{p1}$$
$$y_2 = 38/335 = 0,113 = y_{p2}$$
$$y_3 = 188/335 = 0,562 = y_{p3}$$

The prudential strategies are following:

$$(x_p) = (119/335, 63/335, 153/335) \qquad \text{for player A}$$
$$(y_p) = (109/335, 38/335, 188/335) \qquad \text{for player B}$$

with $z_A(x_p, y_p) = 75,4$ and $z_B(x_p, y_p) = -75,4$

It can be concluded that:

$$z_A(x_p, y_p) = z_A(x_N, y_N) = z_A(x_p, y_N) = 75,4$$
$$z_B(x_p, y_p) = z_B(x_N, y_N) = z_B(x_N, y_p) = -75,4$$

moreover

$$z_A(x_N, y_p) = 75,4$$
$$z_B(x_p, y_N) = -75,4$$

Therefore the two solutions are equivalent and interchangeable because in the zero sum games the Nash and the prudential strategies are equal and the expected payoffs are opposite.

The same solution is given by Skeath [10].

### 3.7 Example 7

As a seventh example a tricky zero sum game published by Dixit and Skeath [10] Rock-Scissors-Paper is shown in Table 9. For both player the first move is Rock, the second one is Scissors and the third one is Paper; Paper wins against Rock, but it looses against Scissors and Scissors loses against Rock; if two players choose the same object, they tie.

**Table 9**

| | | Moves of player $B$ | | |
|---|---|---|---|---|
| | | $y_1$ | $y_2$ | $y_3$ |
| Moves of player $A$ | $x_1$ | $0,0$ | $10,-10$ | $-10,10$ |
| | $x_2$ | $-10,10$ | $0,0$ | $10,-10$ |
| | $x_3$ | $10,-10$ | $-10,10$ | $0,0$ |

First of all the Nash strategies are determined. Formulas 11 and 12 gives the system of equations to be solved to find the vector of the probability distribution on the moves for player $A$

$$\partial z_B/\partial y_1 = 0x_1 + 30x_2 - 10 = 0$$
$$\partial z_B/\partial y_2 = -30x_1 + 0x_2 + 10 = 0$$

The solution of this system of equations is:

$$x_1 = 1/3 = x_{N1}$$
$$x_2 = 1/3 = x_{N2}$$
$$x_3 = 1/3 = x_{N3}$$

Formulas 7 and 8 gives the system of equations to be solved to find the vector of the probability distribution on the moves for player $B$

$$\partial z_A/\partial x_1 = 0y_1 + 30y_2 - 10 = 0$$
$$\partial z_A/\partial x_2 = -30y_1 + 0y_2 + 10 = 0$$

implies

$$y_1 = 1/3 = y_{N1}$$
$$y_2 = 1/3 = y_{N2}$$
$$y_3 = 1/3 = y_{N3}$$

The Nash strategies are following:

$$x_N = (1/3, 1/3, 1/3) \qquad \text{for player A}$$
$$y_N = (1/3, 1/3, 1/3) \qquad \text{for player B}$$

with $z_A(x_N, y_N) = 0$ and $z_B(x_N, y_N) = 0$

The prudential strategies are here below determined. Formulas 15 and 16 gives the system of equations to be solved to find the vector of the probability distribution on the moves for player $A$

$$\partial z_B/\partial y_1 = 0x_1 - 30x_2 + 10 = 0$$
$$\partial z_B/\partial y_2 = 30x_1 + 0x_2 - 10 = 0$$

The solution of this system of equations is:

$$x_1 = 1/3 = x_{p1}$$
$$x_2 = 1/3 = x_{p2}$$
$$x_3 = 1/3 = x_{p3}$$

Formulas 19 and 20 gives the system of equations to be solved to find the vector of the probability distribution on the moves for player $B$

$$\partial z_A/\partial x_1 = 0y_1 - 30y_2 + 10 = 0$$
$$\partial z_A/\partial x_2 = -30y_1 + 0y_2 + 10 = 0$$

implies

$$y_1 = 1/3 = y_{p1}$$
$$y_2 = 1/3 = y_{p2}$$
$$y_3 = 1/3 = y_{p3}$$

The prudential strategies are following:

$$x_p = (1/3, 1/3, 1/3) \qquad \text{for player A}$$
$$y_p = (1/3, 1/3, 1/3) \qquad \text{for player B}$$

with $z_A(x_p, y_p) = 0$ and $z_B(x_p, y_p) = 0$

It can be concluded that:

$$z_A(x_p, y_p) = z_A(x_N, y_N) = z_A(x_p, y_N) = 0$$
$$z_B(x_p, y_p) = z_B(x_N, y_N) = z_B(x_N, y_p) = 0$$

moreover

$$z_A(x_N, y_p) = 0$$
$$z_B(x_p, y_N) = 0$$

The equilibrium strategies are trivial, as it could be expected: the two solutions are equivalent and interchangeable because in the zero sum games the Nash and the prudential strategies are equal and the expected payoffs are opposite and in this case both are equal to zero.

# 4 Non-cooperative solution of the normal form of two persons $n$ by $m$ game

## 4.1 Theory

As it is well known the normal form of the two persons $n$-by-$m$ game is the following one:

**Table 10**

| | | Moves of player $B$ | | |
|---|---|---|---|---|
| | | $y_1$ | $...y_j...$ | $y_m$ |
| | $x_1$ | $a_{11}, b_{11}$ | $...a_{1j}, b_{1j}...$ | $a_{1m}, b_{1m}$ |
| | $...$ | $..., ...$ | $..., ...$ | $..., ...$ |
| Moves of player $A$ | $x_i$ | $a_{i1}, b_{i1}$ | $...a_{ij}, b_{ij}...$ | $a_{im}, b_{im}$ |
| | $...$ | $..., ...$ | $..., ...$ | $..., ...$ |
| | $x_n$ | $a_{n1}, b_{n1}$ | $...a_{nj}, b_{nj}...$ | $a_{nm}, b_{nm}$ |

being

$$\sum_{i=1}^{n} x_i = 1 \tag{35}$$

and

$$\sum_{j=1}^{m} y_j = 1 \tag{36}$$

with the constraints

$$0 \leq x_i \leq 1 \tag{37}$$

and

$$0 \leq y_j \leq 1 \tag{38}$$

the raw vectors of the probability distribution on the moves respectively for player $A$ and $B$ are following:

$$x = (x_1, x_i, x_n) = (x_1, x_i, 1 - \sum_{i=1}^{n-1} x_i) \tag{39}$$

and

$$y = (y_1, y_j, y_m) = (y_1, y_j, 1 - \sum_{j=1}^{m-1} y_j) \tag{40}$$

Associated to each possible outcome of the game is a collection of numerical payoffs, one to each player.

The expected payoff for each player is then given by:

$$z_A = (x)(H)_A(y)^T \tag{41}$$

$$z_B = (x)(H)_B(y)^T \tag{42}$$

where $(x)$ is the vector probability distribution of player $A$, $(H)$ is the matrix of the payoff of $A$ and $B$, and $(y)^T$ is the transposed of vector $(y)$. These formulas will be used throughout the paper from here on.

As mentioned in my previous paper [1], in literature there are two ways to calculate the probability distribution for each player: a prudential strategy [8] and a Nash strategy [2]. These two different strategies can be determined by calculating the first derivatives of the expected payoffs and equating them to zero. First of all the Nash strategies are determined.

$$\partial z_A/\partial x_i = \sum_{k=1}^{m-1}(a_{ik} - a_{im} - a_{nk} + a_{nm})y_k + a_{im} - a_{nm} = 0 \tag{43}$$

these partial derivatives equated to zero are $n-1$ equations in $m-1$ $y_j$ unknowns and

$$\partial z_B/\partial y_j = \sum_{k=1}^{n-1}(b_{kj} - b_{nj} - b_{km} + b_{nm})x_k + b_{nj} - b_{nm} = 0 \tag{44}$$

these are $m-1$ equations in $n-1$ $x_i$ unknowns.

The solution of the two systems gives the probability distribution for player $B$ and $A$ after Nash.

The meaning of the Nash strategy is that if player $B$ chooses $(y_N)$ there is no variation of $z_A$ irrespective of the choice of player $A$; if player $A$ chooses $(x_N)$ there is no variation of $z_B$ irrespective of the choice of player $B$.

The prudential strategies are following.

$$\partial z_B/\partial x_i = \sum_{k=1}^{m-1}(b_{ik} - b_{im} - b_{nk} + b_{nm})y_k + b_{im} - b_{nm} = 0 \tag{45}$$

these partial derivatives equated to zero are $n-1$ equations in $m-1$ $y_j$ unknowns and

$$\partial z_A/\partial y_j = \sum_{k=1}^{n-1}(a_{kj} - a_{nj} - a_{km} + a_{nm})x_k + a_{nj} - a_{nm} = 0 \tag{46}$$

these are $m-1$ equations in $n-1$ $x_i$ unknowns.

The solution of the two systems give the prudential probability distribution for player $B$ and $A$.

The meaning of the prudential strategy is that if player $B$ chooses $(y_p)$ there is no variation of $z_B$ irrespective of the choice of player $A$; if player $A$ chooses $(x_p)$ there is no variation of $z_A$ irrespective of the choice of player $B$.

By substituting in the formulas of the expected payoffs of each player respectively the prudential strategies and the Nash's strategies it can easily be seen that the expected payoffs are equal in the two cases, thus the two couples of strategies $(x_p, y_p)$ and $(x_N, y_N)$ are equivalent.

Moreover combining the prudential strategies with the Nash's strategies it is found that:

$$z_A(x_p, y_p) = z_A(x_N, y_N) = z_A(x_p, y_N) = z_A^* \tag{47}$$

and

$$z_B(x_p, y_p) = z_B(x_N, y_N) = z_B(x_N, y_p) = z_B^* \tag{48}$$

but in general

$$z_B(x_p, y_N) \neq z_B(x_p, y_p) = z_B(x_N, y_N) = z_B^* \tag{49}$$

and

$$z_A(x_N, y_p) \neq z_A(x_p, y_p) = z_A(x_N, y_N) = z_A^* \tag{50}$$

this means that in general the two couples of strategies are not interchangeable.

It can be concluded that the couple of strategies does not represent a solution of the game because they are equivalent, but not interchangeable, as it is stated by Nash [2]. The non-cooperative solution does not take into account the possibility of an agreement between the two players, thus it is possible that the players choose different strategies because they are equivalent, but this is not optimal because they are not interchangeable.

The outcome of the possible choices of the two players is depicted in the following Table 11.

**Table 11**

| | Strategies | | Expected payoffs | |
|---|---|---|---|---|
| | $x$ | $y$ | $z_A$ | $z_B$ |
| Nash | $x_N$ | $y_N$ | $z_A^*$ | $z_B^*$ |
| Prudential | $x_p$ | $y_p$ | $z_A^*$ | $z_B^*$ |
| Nash/Prud. | $x_N$ | $y_p$ | $\alpha$ | $z_B^*$ |
| Prud./Nash | $x_p$ | $y_N$ | $z_A^*$ | $\beta$ |

where
$\alpha = z_A(x_N, y_p)$
$\beta = z_B(x_p, y_N)$

It comes out that in order to choose the optimal strategy the player $A$ should look whether the value of $z_A(x_N, y_p)$ is greater or lower than $z_A^*$: if it is greater, the strategy $(x_N)$ is dominant irrespective of the choice of player $B$, if it is lower, the strategy $(x_p)$ becomes dominant irrespective of the choice of player $B$.

The player $B$ should look whether the value of $z_B(x_p, y_N)$ is greater or lower than $z_B^*$: if it is greater, the strategy $(y_N)$ is dominant irrespective of the choice of player $A$, if it is lower, the strategy $(y_p)$ becomes dominant irrespective of the choice of player $A$.

The discussion of the first derivatives of the expected payoffs gives a rationale of the two different ways to calculate the probability distribution on the moves:

- the prudential strategy

$$\partial z_A/\partial y = 0 \ \text{ implies } \ x = x_p \tag{51}$$

$$\partial z_B/\partial x = 0 \ \text{ implies } \ y = y_p \tag{52}$$

  guarantees that each player receives a payoff irrespective of the choice of the other player; this explains why:

$$z_A(x_p, y_N) = z_A(x_p, y_p) = z_A(x_p, y) = z_A^* \tag{53}$$

$$z_B(x_N, y_p) = z_B(x_p, y_p) = z_B(x, y_p) = z_B^* \tag{54}$$

- the Nash's strategy

$$\partial z_B/\partial y = 0 \text{ implies } x = x_N \quad (55)$$
$$\partial z_A/\partial x = 0 \text{ implies } y = y_N \quad (56)$$

guarantees that each player receives a payoff irrespective of his own choice; this explains why:

$$z_A(x_p, y_N) = z_A(x_N, y_N) \quad (57)$$
$$z_B(x_N, y_p) = z_B(x_N, y_N) \quad (58)$$

### 4.2 Remarks about the solution of two persons $n$ by $m$ games

The remarks in section 2.2 are totally applicable to the solution of two persons and $n$ by $m$ moves games: it is known that a solution of the game exists [7], but there are a lot of different ways to find out that solution. The proposed procedure is very simple for finding the solution also if in some cases it fails and some other ways should be used such as the search of the Nash equilibria. The application of the geometric approach, proposed [1] to find the non-cooperative solution of the two by two general sum game with mixed strategies, is not recommended in the case of $n$ by $m$ moves games, because it becomes too much troublesome in the $n$ by $m$ dimensions space.

## 5 Numerical solutions of some games larger than 3 by 3 moves

### 5.1 Example 1

As a first example a general sum game published as exercise by Maschler [12] is shown in Table 12.

**Table 12**

| | | Moves of player $B$ | | | |
|---|---|---|---|---|---|
| | | $y_1$ | $y_2$ | $y_3$ | $y_4$ |
| Moves of player $A$ | $x_1$ | $3,7$ | $0,13$ | $4,5$ | $5,3$ |
| | $x_2$ | $5,3$ | $4,5$ | $4,5$ | $3,7$ |
| | $x_3$ | $4,5$ | $3,7$ | $4,5$ | $5,3$ |
| | $x_4$ | $4,5$ | $4,5$ | $4,5$ | $4,5$ |

First of all the Nash strategies are determined. Formula 44 gives the system of equations to be solved to find the vector of the probability distribution on the moves for player $A$

$$\partial z_B/\partial y_1 = 2x_1 - 2x_2 + 2x_3 = 0$$
$$\partial z_B/\partial y_2 = 4x_1 - 4x_2 + 2x_3 = 0$$
$$\partial z_B/\partial y_3 = 10x_1 - 2x_2 + 4x_3 = 0$$

This is a homogeneous system and the trivial solution is

$$\begin{aligned} x_1 &= 0 = x_{N1} \\ x_2 &= 0 = x_{N2} \\ x_3 &= 0 = x_{N3} \\ x_4 &= 1 = x_{N4} \end{aligned}$$

Formula 43 gives the system of equations to be solved to find the vector of the probability distribution on the moves for player $B$

$$\begin{aligned} \partial z_A/\partial x_1 &= -2y_1 - 5y_2 - y_3 + 1 = 0 \\ \partial z_A/\partial x_2 &= 2y_1 + y_2 + y_3 - 1 = 0 \\ \partial z_A/\partial x_3 &= -y_1 - 2y_2 - y_3 + 1 = 0 \end{aligned}$$

implies

$$\begin{aligned} y_1 &= 0 = y_{N1} \\ y_2 &= 0 = y_{N2} \\ y_3 &= 1 = y_{N3} \\ y_4 &= 0 = y_{N4} \end{aligned}$$

The Nash strategies are following:

$$\begin{aligned} (x_N) &= (0,0,0,1) && \text{for player A} \\ (y_N) &= (0,0,1,0) && \text{for player B} \end{aligned}$$

with $z_A(x_N, y_N) = 4$ and $z_B(x_N, y_N) = 5$

The prudential strategies are here below determined. Formula 46 gives the system of equations to be solved to find the vector of the probability distribution on the moves for player $A$

$$\begin{aligned} \partial z_A/\partial y_1 &= -2x_1 + 2x_2 - x_3 = 0 \\ \partial z_A/\partial y_2 &= -5x_1 + x_2 - 2x_3 = 0 \\ \partial z_A/\partial y_3 &= -x_1 + x_2 - x_3 = 0 \end{aligned}$$

This is a homogeneous system and the trivial solution is the prudential strategy for player $A$:

$$\begin{aligned} x_1 &= 0 = x_{p1} \\ x_2 &= 0 = x_{p2} \\ x_3 &= 0 = x_{p3} \\ x_4 &= 1 = x_{p4} \end{aligned}$$

Formula 45 gives the system of equations to be solved to find the vector of the probability distribution on the moves for player $B$

$$\begin{aligned} \partial z_B/\partial x_1 &= 4y_1 + 10y_2 + 2y_3 - 2 = 0 \\ \partial z_B/\partial x_2 &= -4y_1 - 2y_2 - 2y_3 + 2 = 0 \\ \partial z_B/\partial x_3 &= 2y_1 + 4y_2 + 2y_3 - 2 = 0 \end{aligned}$$

implies

$$\begin{aligned} y_1 &= 0 = y_{p1} \\ y_2 &= 0 = y_{p2} \\ y_3 &= 1 = y_{p3} \\ y_4 &= 0 = y_{p4} \end{aligned}$$

The prudential strategies are following:

$$\begin{aligned} (x_p) &= (0,0,0,1) && \text{for player A} \\ (y_p) &= (0,0,1,0) && \text{for player B} \end{aligned}$$

with $z_A(x_p, y_p) = 4$ and $z_B(x_p, y_p) = 5$

It can be concluded that:

$$\begin{aligned} z_A(x_p, y_p) = z_A(x_N, y_N) = z_A(x_p, y_N) = 4 \\ z_B(x_p, y_p) = z_B(x_N, y_N) = z_B(x_N, y_p) = 5 \end{aligned}$$

moreover

$$\begin{aligned} z_A(x_N, y_p) = 4 \\ z_B(x_p, y_N) = 5 \end{aligned}$$

Therefore the two solutions are equal, equivalent and interchangeable, therefore there is only one solution:

$$\begin{aligned} (x) &= (0,0,0,1) && \text{for player A} \\ (y) &= (0,0,1,0) && \text{for player B} \end{aligned}$$

with $z_A(x, y) = 4$ and $z_B(x, y) = 5$.

## 5.2 Example 2

As a second example a general sum game published by Dixit and Skeath [10] to show the application of the rationalizability concept is shown in Table 13. This concept by Skeath is based on the

**Table 13**

| | | Moves of player $B$ | | | |
|---|---|---|---|---|---|
| | | $y_1$ | $y_2$ | $y_3$ | $y_4$ |
| Moves of player $A$ | $x_1$ | $0, 7$ | $2, 5$ | $7, 0$ | $0, 1$ |
| | $x_2$ | $5, 2$ | $3, 3$ | $5, 2$ | $0, 1$ |
| | $x_3$ | $7, 0$ | $2, 5$ | $0, 7$ | $0, 1$ |
| | $x_4$ | $0, 0$ | $0, -2$ | $0, 0$ | $10, -1$ |

identification of strategies that are never a best response and it is deemed that this property is stronger than the simple dominance principle.

It can be seen that there are no dominances. First of all the Nash strategies are determined. Formula 44 gives the system of equations to be solved to find the vector of the probability distribution on the moves for player $A$

$$\begin{aligned}
\partial z_B/\partial y_1 &= 0x_1 + 0x_2 + 5x_3 = -1 \\
\partial z_B/\partial y_2 &= 5x_1 + 0x_2 - 2x_3 = -1 \\
\partial z_B/\partial y_3 &= 5x_1 + 3x_2 + 5x_3 = 1
\end{aligned}$$

The solution of the system of equations is

$$\begin{aligned}
x_1 &= -7/25 < 0 \\
x_2 &= 1/5 \\
x_3 &= -1/5 < 0 \\
x_4 &= 32/25 > 1
\end{aligned}$$

The solution is not acceptable because the probabilities should be non negative and lower than one. Formula 43 gives the system of equations to be solved to find the vector of the probability distribution on the moves for player $B$

$$\begin{aligned}
\partial z_A/\partial x_1 &= 10y_1 + 12y_2 + 17y_3 - 10 = 0 \\
\partial z_A/\partial x_2 &= 15y_1 + 13y_2 + 15y_3 - 10 = 0 \\
\partial z_A/\partial x_3 &= 17y_1 + 12y_2 + 10y_3 - 10 = 0
\end{aligned}$$

implies

$$\begin{aligned}
y_1 &= -10/9 < 0 \\
y_2 &= 10/3 > 1 \\
y_3 &= -10/9 < 0 \\
y_4 &= 19/9 > 1
\end{aligned}$$

The solution is not acceptable because the probabilities should be non negative and lower than one.
There is no Nash strategy and the equilibrium strategy should be found among the pure strategies.

The prudential strategies are here below determined. Formula 46 gives the system of equations to be solved to find the vector of the probability distribution on the moves for player $A$

$$\begin{aligned}
\partial z_A/\partial y_1 &= 10x_1 + 15x_2 + 17x_3 - 10 = 0\\
\partial z_A/\partial y_2 &= 12x_1 + 13x_2 + 12x_3 - 10 = 0\\
\partial z_A/\partial y_3 &= 17x_1 + 15x_2 + 10x_3 - 10 = 0
\end{aligned}$$

The solution of the system of equations is

$$\begin{aligned}
x_1 &= 20/9 > 1\\
x_2 &= -20/9 < 0\\
x_3 &= 20/9 > 1\\
x_4 &= -11/9 < 0
\end{aligned}$$

The system has negative solutions, therefore the solutions are not acceptable.

Formula 45 gives the system of equations to be solved to find the vector of the probability distribution on the moves for player $B$

$$\begin{aligned}
\partial z_B/\partial x_1 &= 5y_1 + 5y_2 - 2y_3 + 2 = 0\\
\partial z_B/\partial x_2 &= 0y_1 + 0y_2 + 3y_3 + 2 = 0\\
\partial z_B/\partial x_3 &= -2y_1 + 5y_2 + 5y_3 + 2 = 0
\end{aligned}$$

implies

$$\begin{aligned}
y_1 &= 4/9\\
y_2 &= -2/3 < 0\\
y_3 &= 4/9\\
y_4 &= 11/9 > 1
\end{aligned}$$

The system has negative solutions, therefore the solutions are not acceptable. There are no mixed strategies solutions either Nash or prudential and the equilibrium strategy should be found among the pure strategies.
Looking at the bimatrix of the game it can be seen that there is a Nash equilibrium:

$$\begin{aligned}
(x) &= (0, 1, 0, 0) && \text{for player A}\\
(y) &= (0, 1, 0, 0) && \text{for player B}
\end{aligned}$$

Therefore in this case the mixed strategies solution does not exist, nevertheless the solution of the game can still be found and it is the above found Nash equilibrium with:
$z_A(x, y) = 3$ and $z_B(x, y) = 3$. This is the same solution identified by Skeath [10].

Nevertheless the rationalization concept provides a possible way of solving games that do not have a Nash equilibrium [10].

# 6 Conclusions

The proposed non-cooperative solution of two persons $n$ by $m$ games is based on the application of the dominance principle, therefore the paper is dealing only with games with no dominances on the pure strategies and the dominance principle is applied to find the solution on the mixed strategies too.

The main conclusions holding independently from the specific values of the payoff matrix are following:

A) The value of the expected payoff corresponding to the prudential distribution for a player is not only independent either from the prudential or the Nash's distribution of the other player, but it is independent from every distribution of the other player; moreover when a player chooses the Nash's distribution the expected payoff of the other player is not depending upon his own strategy distribution;

B) Generally speaking the couples of prudential and Nash's strategies are not interchangeable, but by applying the dominance principle it is possible to choose the right equilibrium strategies avoiding the bad consequences due to the non-interchangeability of the strategies;

C) It is worth noting that in the case of zero sum game the prudential and the Nash strategy are coincident and they are the unique mixed strategies solution of the game; as it can easily be understood the zero sum game is a special case of the general sum games;

D) On the basis of the dominance principle the dominant mixed strategy is given by the point that has the greatest expected payoff: on the basis of point B) the so found equilibrium pair is candidate to be a perfect equilibrium pair [7];

E) A conjecture of the proposed way of solution is that the so found solution is unique (Nash [4]). In this case the so found equilibrium pair of the non-cooperative solution gives the perfect equilibrium pair of the game and the corresponding expected payoff could be the starting point for finding the cooperative solution of the game too [7].

**Interest Conflicts**
The author declares that there is no conflict of interest concerning the publishing of this paper.

**Funding Statement**
No funding was received for conducting this study.

**Acknowledgments**
The author wishes to thank Prof. Rosalba Ferrari, University of Bergamo Department of Engineering and Applied Sciences, for her invaluable help during the editorial preparation of the paper.

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