Allocating Capital to Time: Introducing Credit Migration for Measuring Time-Related Risks

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ALLOCATING CAPITAL TO TIME: INTRODUCING CREDIT MIGRATION FOR MEASURING TIME-RELATED RISKS

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Abstract. Assessing time-related risks in long-tailed insurance is challenging. Regulatory capital allocation rules may underestimate credit deterioration risk by not requiring insurers to hold solvency capital early, while actuarial practices often allocate capital sooner than mandated. We propose a framework to quantify these time-associated risks and evaluate capital allocation strategies based on time to ultimate, aiming to manage long-tail business effectively. By modeling the impact of exogenous credit migration risk, we evaluate six strategies, including costs associated with potential company bankruptcy until long-term claims are settled. Using a numerical example of a future heavy-tailed insurance risk, we estimate a Markov chain credit migration model with insurance market data and analyze liability values from various capital management strategies. Our findings show that early capital raising is costly, even with penalties for avoided credit risk, unless the company's initial credit rating is poor. In such cases, purchasing protection through a credit derivative may be more efficient, if available.

1. Introduction

The appropriate management of capital is an important challenge in banking and insurance, see e.g. Wilson (2015); Klaassen and Van Eeghen (2009). In the insurance industry (particularly in Europe), a modern management of a company typically assigns capital to particular business lines (or even individual major risks) with the goal of optimizing the risk/return profile of the portfolio, see e.g. Dacorogna (2018). In this case, the pricing of the underlying policies needs to take into account the available capital 'budget', with the philosophy that the risk margin included in the premiums together with that capital must suffice for dealing with the underlying risk up to a defined threshold. The discount factor used to compute the risk margin for pricing is then the target profit the company wishes to achieve.

In general, it encompasses the cost of capital, which includes the credit-rating-related spread, along with an extra spread linked to the company's strategy. When it comes to risks that only unfold over longer time horizons, one can interpret that the required capital for those risks are held throughout the entire period, which is an extremely cautious and costly strategy. In contrast, regulators require solvency capital only for all the payments/claims that will be due within the next calendar year, implying that

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it overlooks long-term risks. This discrepancy can be costly from both perspectives, whether due to the extra expense of holding that capital or from underestimating the risk, as will be discussed. The regulator typically works with the assumption of a fixed (and known-in-advance) cost-of-capital rate of 6% (recently, this value was reduced to 4.75% by the European regulators). This may not be realistic. Indeed, following strictly the regulatory suggestions with the one-year time horizon in mind can have considerable drawbacks, as raising capital in later years during the management of the underlying policies may have a severe credit rating risk involved (as mentioned above, the credit spread is included in the discount factor for pricing the risk margin). Concretely, the credit rating of the insurance company at the future time when the capital needs to be raised from a regulatory perspective may be worse than today, making capital much more costly, so that there is a financial risk involved in that other extreme approach as well. This is why actuaries often consider the regulatory approach to be too dangerous, because it implicitly assumes that no capital is allocated to this risk until one year before it may materialize even though the risk is already on the book at time $t = 0$. Undoubtedly, from a management perspective, neither method proves satisfactory for developing long-tail business. The regulatory approach tends to be too inexpensive and carries significant risk, whereas the actuarial method makes long-tail business overly costly.

Beyond regulatory concerns, this study serves a broader twofold purpose: first, to develop a framework for quantifying risks tied to time, and second, to evaluate strategies for capital allocation as a function of the time to ultimate, aiming to effectively manage long-tail business without impeding its growth. We identify the time-related risk to the evolution of the credit rating of the insurance company holding this risk. Therefore, our method for quantifying it includes using a realistic dynamic model for the credit migration of an insurance company, as well as evaluating the impacts of progressively raising (or allocating) portions of the ultimately required capital beforehand. The final objective is to strike a balance that makes long-tail business appealing while safeguarding the company against risks arising from the time it takes to develop to ultimate. This paper addresses a gap in actuarial methodology related to the treatment of 'risk-of-time' and proposes a solution for this issue. Considering the pressing requirement for long-term investment in climate change, we view this objective as highly significant, timely, and relevant. It may also encourage additional welcomed research into the risk associated with time. While the focus of this paper is on insurance, the methods used here can also be applied for evaluating other financial long-term liabilities held by banks, financial institutions and pension funds. The latter often face even greater risks tied to time. However, to address the challenges presented by these extended time horizons, the methodology would require adjustments.

In order to keep the approach transparent and to peel out the characteristics of the credit migration risk that we want to study and understand here, we will take a number of simplifying assumptions (many of which can be relaxed in a rather straight-forward way, but at the expense of an overlay of different factors in the final results that may not be easy to disentangle, and an increased complexity of calculations). For example, we will assume the riskless interest rate to be zero (no discounting), and we will consider the assets of the company to be invested in a riskless way, so that we can focus on the liability side.

Our methodology relies on economic valuation of liabilities (see for instance chapter 15 in Koch-Medina and Munari (2020)). It is heavily used by insurers and reinsurers for valuing their deals besides the fact that it is now required by the European regulations, being Solvency II or the Swiss Solvency Test. For more general approaches to actuarial valuation of insurance liabilities in a multi-period setting (to which the considerations of this paper may be generalized in future work), see for instance Dhaene et al. (2017); Delong et al. (2019a,b); Barigou and Dhaene (2019); Chen et al. (2021) as well as Barigou et al. (2022), cf. also Bauer and Zanjani (2021) and Guo et al. (2023) for a dynamic view. For a recent survey on capital allocation techniques we refer to Guo et al. (2021) and a recent general account on market-consistent pricing can be found in Koch-Medina and Munari (2020). For a nice comparison of actuarial and financial valuation schemes in practice see Vedani et al. (2017), and a general overview of asset-liability management for long-tailed insurance risk can for instance be found in Albrecher et al. (2018).

Ever since the Swiss and European Union regulatory bodies introduced their risk-based solvency directives (Federal Office of Private Insurance (2006); European Union (2009)) there has been an on-going debate about the suitability and interpretation of the required solvency capital and the implied required risk margin in the technical provisions (see for instance Eling et al. (2008); Eisele and Artzner (2011); Floreani (2011); as well as Pelkiewicz et al. (2020) and Korn and Stahl (2024) for recent rejoinders after the first years of industry experience). Further recent contributions on how to appropriately price insurance risk in a market-consistent way are Salahnejhad Ghalehjooghi and Pelsser (2023), Engsner et al. (2023) and Artzner et al. (2024). Effects of limited liability on the cost of capital were for instance addressed in Filipović et al. (2015) and Albrecher et al. (2022). For a discussion on possible dependencies of the cost of capital on the risk in traded financial assets, see Schmutz et al. (2023).

In Section 2, we provide a detailed description of the multi-period valuation model adopted in this paper. The approach used for formalizing the liability valuation is inspired by Engler and Lindskog (2023), see also Engsner et al. (2020). We focus our considerations on facing an insurance risk X at a future time point $t = n$, with no other insurance risks in between. In Section **??**, we first recap the consequences of deterministic cost-of-capital rates (i.e., the cost of capital applicable later already being known at the beginning) on the valuation of liabilities, which represents to some extent the current regulatory practice. Note that while the European Union decided in December 2023 to reduce the regulatory cost-of-capital rate for Solvency II to 4.75%, we will still work in this paper with the previously demanded rate of 6%, as that number is only a minimal requirement (which in part was reduced to boost growth of the European economy and to free additional capital for green investments, cf. Jones

(2023)) and big insurance companies will most likely keep the 6% as their target return on equity in any case, so that it remains the natural benchmark for our study.

After defining the used credit migration model in Section **??**, we show in Section **??** how in general a stochastic nature of the future cost-of-capital affects the calculations of liability values. In particular, we also include the concrete costs of bankruptcy into the multi-period liability valuation (see e.g. Belkin et al. (1998) for background), and develop the respective framework in Section 3. In Section 4, we describe and discuss the six capital management strategies that we compare in this paper, and work out the concrete ingredients needed for their respective implementations. The focus of Section 5 is then on the practical application of these strategies within a simple framework, specifically dealing with a Pareto-distributed insurance risk over a five-year period without encountering any other insurance risks in the interim. For that purpose, we first determine the parameters of a Markov chain model for the credit migration of an insurance company from S&P data over the last 40 years and subsequently evaluate the corresponding credit spreads as a function of credit state. We then determine the numerical results of initial liability values of the considered long-term insurance risk for each of the strategies and discuss and interpret the results. Indeed, we see in that example that the regulatory practice of ignoring credit risk along the way considerably underestimates the actual value of liabilities, but the actuarial practice of raising the required capital already at the beginning is very expensive even when the avoided credit risk is penalized appropriately. Under the assumptions of the paper, buying a call option (when available) to hedge the credit risk may be an interesting alternative, yet buying protection through a credit derivative directly seems to be the most efficient strategy, if such a product can be purchased. Finally, we draw conclusions and outline potential future research in Section 6.

2. A multi-period valuation model

Let us consider a financial institution (here, concretely, an insurance company) whose value can be expressed in terms of the economic value of its assets, A, and of its liabilities, L. Let us assume that the time evolution of these two variables is expressed in terms of two discrete-time stochastic processes:

$$\mathbf{A} = (A_t,\, t = 0, 1, 2, \ldots) \text{ and } \mathbf{L} = (L_t,\, t = 0, 1, 2, \ldots)$$

defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where time t is measured in years. Let us denote by $\mathcal{F}_t$ the σ-algebra generated by all the information available at time t (including the realized sequence of $\mathbf{A}$ and $\mathbf{L}$ up to time t). In the sequel, all random variables indexed by t are $\mathcal{F}_t$-measurable, i.e. their realization is known at time t (in mathematical terms, $L_t = \mathbb{E}(L_t|\mathcal{F}_t)$ for the example of the liability process; although this notation may look convoluted, it will turn out to be convenient in the sequel, as we will, in certain cases, look at different times for L_t and the information set $\mathcal{F}_t$).

The value of the institution is then described by the stochastic process

$$\mathbf{Z} = (Z_t,\ t = 0, 1, 2, \ldots) \quad \text{defined as} \quad Z_t = A_t - L_t.$$

We denote by $\Delta Z_{t+1} = -(Z_{t+1} - Z_t), t = 0, 1, 2, \ldots,$ the negative change in value of the company over a time horizon of 1 year (from year t to year $t+1$), which is the time span relevant for solvency regulations.

In risk-based solvency, both assets and liabilities must be valued economically. This means that asset valuation should be *mark-to-market* (market-consistent) as long as there is a market liquid enough for it; if not they should be valued *mark-to-model*, while liabilities should be mark-to-model, as there is no market for insurance liabilities. The model for valuing economically insurance liabilities is based on the law of one price (that states that any cash flow has only one market price) and on the concept of a replicating portfolio: the economic value L_t at time t of the insurance liabilities is then the present value RP_t of the replicating portfolio (at that time t) plus a risk margin RM_t to account for the cost-of-capital needed for the non-hedgeable part (the insurance risk beyond the expectation):

$$L_t = \mathbb{E}[L_t|\mathcal{F}_t] = \mathrm{RP}_t + \mathrm{RM}_t,\ t = 0, 1, 2, \ldots. \tag{1}$$

Note that here and later in the paper, the expectation operator without an index refers to the one with respect to the physical measure $\mathbb{P}$, i.e. $\mathbb{E}[X] := \mathbb{E}_{\mathbb{P}}[X]$ for any random variable X. At this point, we would like to clarify a commonly misunderstood term: 'replicating portfolio'. It is important to note that financial markets do not offer a true replicating portfolio for insurance liabilities. However, insurance liabilities are associated with premiums, which are typically determined based on expected payout patterns. These premiums form the reserves, which are then invested in financial markets to mimic the projected payouts. Through this investment, insurers effectively create a 'replicating portfolio' that hedges against expected liabilities. In insurance practice, this is referred to as the replicating portfolio. However, the insurance risk extends beyond this portfolio and is covered by the insurer's risk-based capital. This capital is used to calculate the risk margin added to the replicating portfolio.

Our focus here is on determining the necessary capital over time to support this risk. Then, L_t can be interpreted as the cash amount that would have to be transferred along with those liabilities to a third party, so that she would be willing to take over these liabilities (their run-off and potential associated capital costs). If the insurer wants to earn L_t as the aggregate premium for these liabilities, one can also interpret RP_t as the expected aggregate claim amount (*best estimate* of the liabilities) and RM_t as the safety loading on top of this expectation. Clearly, all these quantities are evaluated in the physical probability space. This economic valuation is central to insurance deals and is the way insurers have solved the problem of valuing illiquid assets. Similar ideas have been developed in financial mathematics, see, for instance Froot and Stein (1998), but other ideas have already been developed by actuaries as early as in the 1940's by de Finetti (1940) who advocated to use the portfolio to evaluate the risk or later by introducing a deformation to the physical probability Wang (1996); Denneberg and Kaplan (1988) for pricing the risk.

Coming back to the rationale to determine the capital, it can be understood as follows: At the beginning of each year, the regulator will require the company to hold sufficient capital

$$\mathrm{SCR}_t := \rho(\Delta Z_{t+1}|\mathcal{F}_t), \tag{2}$$

to safeguard against adverse developments (losses) of both the asset and liability side during that year, that is to ensure a sufficient amount of safety to meet the obligations towards the policyholders (see e.g. Federal Office of Private Insurance (2006)). Here $\rho(\cdot)$ denotes an appropriate risk measure. Solvency II defines this Solvency Capital Requirements (SCR) as:

$$\rho(\Delta Z_{t+1}|\mathcal{F}_t) = \mathrm{VaR}_\alpha(\Delta Z_{t+1}|\mathcal{F}_t) = \inf\{x \in \mathbb{R} \mid \mathbb{P}(\Delta Z_{t+1} > x|\mathcal{F}_t) \le 1-\alpha\}, \tag{3}$$

where $\alpha = 99.5\%$. Alternatively, the Swiss Solvency Test defines the SCR through

$$\rho(\Delta Z_{t+1}|\mathcal{F}_t) = \mathrm{ES}_\alpha(\Delta Z_{t+1}|\mathcal{F}_t) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_\beta(\Delta Z_{t+1}|\mathcal{F}_t)\, d\beta \tag{4}$$

with $\alpha = 99\%$. If the random variable ΔZ_{t+1} is continuous, the latter expression is equivalent to $\mathrm{ES}_\alpha(\Delta Z_{t+1}|\mathcal{F}_t) = \mathbb{E}(\Delta Z_{t+1}|\Delta Z_{t+1} \ge \mathrm{VaR}_\alpha(\Delta Z_{t+1}), \mathcal{F}_t)$. In the numerical illustrations later, we will only use the Value-at-Risk, but the considerations can easily be generalized to the expected shortfall.

The analysis of this paper will heavily rely on (conditional) translation invariance and homogeneity of ρ, i.e.

$$\rho(aX + b|\mathcal{F}_t) = a\,\rho(X|\mathcal{F}_t) + b, \quad a > 0, b \in \mathbb{R},$$

which is fulfilled for both the VaR and the ES (see e.g. Pflug and Roemisch (2007)).

This required capital SCR_t is to be compared to the available capital $c_a(t) = A_t - L_t$ at time t to obtain the solvency ratio

$$\frac{c_a(t)}{\mathrm{SCR}_t}.$$

For the company to be considered as solvent by the regulators, the solvency ratio must obviously be bigger than 1 at time t.

Holding the capital amount SCR_t (raising/allocating it, or blocking it from other investment opportunities on the asset side for shareholders or investors of the insurance company) is costly, and that cost is quantified by a spread η_t above the risk-free interest rate, compensating for the involved insurance risk. The risk margin RM_t can be interpreted as the amount dedicated to cover these capital costs (it also implicitly contains information on the *amount of capital allocated* to this risk at time t). It is one purpose of the present paper to contribute to the determination and clarification of this quantity for a multi-period risk, and to examine how its value is affected by the early allocation of capital, beyond what is mandated by the regulator.

In order to simplify the analysis and focus on the essentials for the present purpose, we assume in this paper that once the portfolio is set up at time $t = 0$, there is no new business underwritten until the run-off of the portfolio is completed. We also assume that the risk-free rate is 0 and thus neglect discounting. In addition, we want to focus on the liability side only (i.e. we ignore the asset risk here; another interpretation of that assumption is that $A_0 = A_1 = \cdots = A_n$, where n is the time horizon of our considerations). The choice of this time horizon implies $L_n = 0$, and we will now be interested in determining L_t for $t < n$ recursively, and particularly L_0, which as mentioned above can be interpreted as the required premium amount for the policyholders for being offered the insurance protection underwritten at time $t = 0$, neglecting costs. Correspondingly, L_t can be seen as the part of the premium that at time t should still be reserved for future run-off of the underlying insurance risk.

Let X_t be the random variable representing the claim payment to be paid by the insurance company in year t ($t = 1, \ldots, n$). The required solvency capital (2) at time t in this situation then amounts to

$$\mathrm{SCR}_t = \rho(\Delta Z_{t+1}|\mathcal{F}_t) = \rho(X_{t+1} + L_{t+1} - L_t|\mathcal{F}_t) = \rho(X_{t+1} + L_{t+1}|\mathcal{F}_t) - L_t, \tag{5}$$

$t = 0, 1, \ldots, n-1$, using translation-invariance of the risk measure ρ. This amount is made available by the capital provider for an (excess) expected return η_t on the provided capital between time t and $t+1$. The actual return at time $t+1$ is $\mathrm{SCR}_t + (L_t - L_{t+1}) - X_{t+1}$, so that

$$\mathbb{E}(\mathrm{SCR}_t + L_t - L_{t+1} - X_{t+1}|\mathcal{F}_t) = (1 + \eta_t)\,\mathrm{SCR}_t,$$

or equivalently

$$L_t = \eta_t\,\mathrm{SCR}_t + \mathbb{E}(L_{t+1} + X_{t+1}|\mathcal{F}_t). \tag{6}$$

By virtue of (5) this leads to

$$L_t = \frac{1}{1+\eta_t}\,\mathbb{E}(X_{t+1} + L_{t+1}|\mathcal{F}_t) + \frac{\eta_t}{1+\eta_t}\,\rho(X_{t+1} + L_{t+1}|\mathcal{F}_t), \tag{7}$$

$t = 0, \ldots, n-1$. Equation (7) in fact corresponds to (Engler and Lindskog, 2023, Eq.8), where the focus is subsequently put on asymptotic considerations.

In practice, at time $t = 0$, it will not always be easy or even feasible to specify the distributions of all claim payments (and further cash-flows) X_t of future times $t = 1, \ldots, n$ (conditional on the information $\mathcal{F}_{t-1}$ at the previous time point $t-1$). However, if we assume this to be the case, and if further the cost-of-capital rates η_t, $t = 0, \ldots, n-1$, are known at the beginning as well, then Equation (7) can be used recursively (starting with $L_n = 0$) to determine the economic value of the liability L_0 at time $t = 0$. According to the above derivations, L_0 then represents the necessary amount to handle the run-off of the respective risks (including the financing of the necessary solvency capital along the way).

A special case is the one-period model with a risk $X_1 = X$ at time $t = 1$. In this situation, with $L_1 = 0$, (7) simplifies to

$$L_0 = \frac{\mathbb{E}(X)}{1+\eta_0} + \frac{\eta_0}{1+\eta_0}\,\rho(X). \tag{8}$$

Indeed, if we interpret this in the setting of Albrecher et al. (2022), then $L_0 = \mathbb{E}(X) + \mathrm{RM}_0$ is the premium that the insurance company should ask for X from the policyholders at time $t = 0$ (neglecting expenses). Using the fact that $\rho(X - \mathbb{E}(X) - \mathrm{RM}_0|\mathcal{F}_0) = \mathrm{SCR}_0$ or equivalently $\rho(X) = \mathbb{E}(X) + \mathrm{RM}_0 + \mathrm{SCR}_0$, Equation (8) then reduces to

$$\mathrm{RM}_0 = \eta_0 \cdot \mathrm{SCR}_0,$$

which is the natural and well-known interpretation (or even definition) of the cost of carrying the capital, since the 'safety loading' RM_0 in the premium L_0 in expectation remains as a return at time $t = 1$ for the investor providing the amount SCR_0.

3. Capital management for an insurance risk in the further future

Let us now consider in more detail how to value and manage an insurance liability X that is only due in the further future, say at time $n > 1$, i.e. $X_n = X$ (and it is certain that it will not be faced earlier, like e.g. insuring a building that by contract will only be built by the end of Year $n - 1$). Also, the distribution of X is already known at $t = 0$, so that $\mathbb{E}(X|\mathcal{F}_{n-1}) = \mathbb{E}(X|\mathcal{F}_0) := \mathbb{E}(X)$ and $\mathbb{E}(\rho|\mathcal{F}_{n-1}) = \rho(X|\mathcal{F}_0) := \rho(X)$. For simplicity, we treat it as a stand-alone event, so there are no additional claims from previous years to consider, and no new information emerges about it until the year it occurs[1], cf. Figure 1.

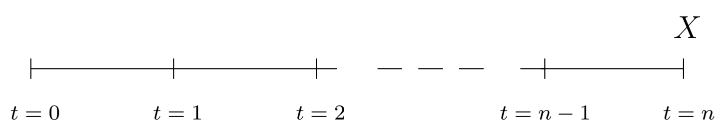

Figure 1. Facing a risk X at some future time n (no information in between).

Beyond the regulatory requirements, the broader challenge here is to determine the appropriate amount of capital to allocate at each step until the ultimate point. In this section, we propose to explore six strategies, starting with the one required by regulation. We then move on to various strategies that differ in their level of conservativeness in allocation, and conclude with hedging strategies. Note that our focus here is not on the issue of capital allocation within a portfolio (refer to Tasche 2007, for example), but rather on a distinct problem: the allocation of capital over time.

3.1. **Deterministic cost-of-capital rate.** Let us start with the situation that all one-year cost-of-capital rates η_t up to time $t = n - 1$ are known at $t = 0$ (and therefore

[1] An example of this would be an insurance written for the launching of a satellite. The contract is signed before the construction starts and the satellite is typically launched several years later.

deterministic). This is the situation for the solvency requirement where the cost-of-capital is fixed at 6%. From (7) we first get

$$L_{n-1} = \frac{1}{1+\eta_{n-1}}\,\mathbb{E}(X) + \frac{\eta_{n-1}}{1+\eta_{n-1}}\,\rho(X). \tag{9}$$

The quantity L_{n-1} is then (already) deterministic at time $t = n-2$ (there is no randomness left), and we get, by applying (7) with $X_{n-1} = 0$ a.s. at $t = n-2$,

$$\begin{aligned} L_{n-2} &= \frac{1}{1+\eta_{n-2}}\,\mathbb{E}(L_{n-1}|\mathcal{F}_{n-2}) + \frac{\eta_{n-2}}{1+\eta_{n-2}}\,\rho(L_{n-1}|\mathcal{F}_{n-2}) \\ &= \frac{L_{n-1}}{1+\eta_{n-2}} + \frac{\eta_{n-2}}{1+\eta_{n-2}}\,L_{n-1} = L_{n-1}, \end{aligned}$$

assuming that $\rho(c) = c$ for any constant c (which is for instance the case when ρ is the Value-at-Risk or the Expected Shortfall). Subsequently, one then gets in the same manner $L_0 = L_1 = \cdots = L_{n-1}$. In other words, if the risk-free interest rate is 0, at time $t = 0$ one needs to charge as premium (and reserve) only the amount

$$L_0 = L_{n-1} = \rho(X) - \frac{1}{1+\eta_{n-1}}\,(\rho(X) - \mathbb{E}(X)) \tag{10}$$

needed for raising the additional required solvency capital $\mathrm{SCR}_{n-1} = \rho(X - L_{n-1}) = \rho(X) - L_{n-1}$ at time $t = n-1$, satisfying the one-year time horizon view of the regulator.

The above could be interpreted as the regulator's view on how, at time $t = 0$, to deal with the risk X that will lead to possible claim payments between times $t = n-1$ and $t = n$. In this view, it would not be required to hold capital against this risk before $t = n-1$, one only would have to set aside the capital cost $L_0 - \mathbb{E}(X)$ at $t = 0$ (needed later at $t = n-1$). However, we reiterate that this approach tacitly assumes that the value of the cost-of-capital rate η_{n-1} is known at time $t = 0$ already. Indeed, in Solvency II and in the Swiss Solvency Test a regulatory cost-of-capital rate η_r is assumed and set at 6%, so under that assumption one can use $\eta_0 = \ldots = \eta_{n-1} = 0.06$ in multi-period calculations and reserving. For a detailed discussion and possible economic justifications of this value of 6% in one time period, see e.g. Albrecher et al. (2022).

However, in practice, the insurer would fix the required cost of capital for pricing purposes every year based on the target profit of the company, which is related to its credit rating. Thus, on one hand, an actuary may be concerned about the fact that this value of 6% can perhaps not be realized in the market, in particular when considering the future uncertainty about the credit rating of the insurance company and, on the other hand, risk management could be concerned to hold a risk on the books without capital associated to it. It may therefore be more prudent to already allocate (parts of) the capital that will be needed at time $t = n-1$ at earlier time points, when the rating (and resulting rate) is possibly more favorable. This will clearly be in a trade-off with the fact that holding capital for longer than the required year (starting at $t = n-1$ in the above example) comes at an additional cost, and it is in this direction where we want to provide some theoretical and practical input in this paper.

Clearly, η_0 is known at time $t = 0$, but all the future cost-of-capital rates $\eta_1, \ldots \eta_{n-1}$ are unknown and it makes sense to model them as random variables, which we will do in the sequel.

3.2. **A Markov chain model for credit migration.** The randomness of the cost-of-capital rate stems from the dynamics (randomness) of the credit rating of the company at that time, and for future reference we define the model that we use for the credit migration in this paper. Let the ($\mathcal{F}_t$-measurable) random variable κ_t denote the credit rating of the company at time t, which can be one of the eight S&P rating states

$$\text{AAA, AA, A, BBB, BB, B, CCC/C, D}$$

respectively (we summarize all $\pm$ notches within their letter rating). Here AAA refers to state 1 and D ('default') is state 8. In this paper we will use a homogeneous Markov chain transition model to assign probabilities for being in state k at time t. Let $\boldsymbol{P} = (p_{ij})_{1\le i\le 8, 1\le j\le 8}$ denote the transition probability matrix of this Markov chain, where p_{ij} is the probability to move within one year from state i to state j. Then $\mathbb{P}(\kappa_m = j|\kappa_0 = i) := p_{ij}^{(m)}$ is the m-step transition probability from state i to state j in that Markov chain (recall that $\boldsymbol{P}^{(m)} = \boldsymbol{P}^m$, i.e. $p_{ij}^{(m)}$ can be read off the mth power of the matrix $\boldsymbol{P}$).

Let $\eta_t^{(k)}(s)$ denote the s-period cost-of-capital rate at time t for capital that will be held by the company (with credit rating k at time t) during the time period $[t, t+s]$. In this paper we make the further assumption that $\eta_t^{(k)}(s)$ does not depend on the time t, so that solely the rating status k determines the value of $\eta_t^{(k)}(s) \equiv \eta^{(k)}(s)$. In other words, we assume that for the same maturity and current rating status, the cost-of-capital rates are constant over time. Then the uncertainty of the company concerning applicable rates for raising capital in the future only comes from the uncertainty about its rating status at future time points, governed by the Markov chain model above.

3.3. **Random cost-of-capital rate.** Consider now the situation where in every single year we want to fulfill the regulatory requirement of future obligations and raise the necessary solvency capital according to the then available one-year cost of capital (according to our respective credit rating) until the end. The resulting L_0 then represents the value of future liabilities in the situation when (in each year up to expiry) investors demand a spread for one-year time borrowing and renewing of capital to the next needed amount, depending on the then applicable credit rating of the company.

To simplify notation, define for any $i = 0, \ldots, n-1$

$$\delta(\kappa_i) := \frac{1}{1 + \eta^{(\kappa_i)}(1)} < 1 \quad \text{and} \quad K(X) := \rho(X) - \mathbb{E}(X).$$

Equation (9) at $t = n-1$ then reads

$$L_{n-1}(\kappa_{n-1}) = \mathbb{E}(X) + (1 - \delta(\kappa_{n-1})) \cdot K(X) = \rho(X) - \delta(\kappa_{n-1}) \cdot K(X), \tag{11}$$

where $\delta(\kappa_{n-1})$ is an $\mathcal{F}_{n-1}$-measurable random variable, i.e. it is only known at time $t = n-1$ (when the credit rating at that point in time is known). Under the assumption

$$X_1 = \ldots = X_{n-1} = 0 \quad \text{a.s.,} \tag{12}$$

one can rewrite (7) for any earlier times $t = 0, \ldots, n-2$ as

$$L_t(\kappa_t) = \delta(\kappa_t) \cdot \mathbb{E}(L_{t+1}|\mathcal{F}_t) + (1 - \delta(\kappa_t)) \cdot \rho(L_{t+1}|\mathcal{F}_t). \tag{13}$$

Note that according to this representation the value of the future liabilities at time t can be interpreted as a convex combination of its expected value at time $t+1$ and its risk measure at time $t+1$, with the weights being given by the current one-year 'discount rate' due to the cost of capital. A little calculation, using the translation-invariance and the homogeneity of the risk measure ρ, now shows that (13) applied at $t = n-2$ gives

$$L_{n-2}(\kappa_{n-2}) = \rho(X) + f_{n-2}(\kappa_{n-2}) \cdot K(X), \tag{14}$$

where

$$f_{n-2}(\kappa_{n-2}) := \delta(\kappa_{n-2}) \cdot \mathbb{E}(-\delta(\kappa_{n-1})|\mathcal{F}_{n-2}) + (1 - \delta(\kappa_{n-2})) \cdot \rho(-\delta(\kappa_{n-1})|\mathcal{F}_{n-2}) \tag{15}$$

is (solely) a function of the credit rating κ_{n-2} of the insurance company at time $t = n-2$. Moreover, this function is again simply the convex combination of expected value and risk measure of the future cost-of-capital discount rate, with the weights composed by its current value, just one time unit earlier. Iterative application of the same principle then gives for any $t = 0, \ldots, n-2$

$$L_t(\kappa_t) = \rho(X) + f_t(\kappa_t) \cdot K(X), \tag{16}$$

where

$$f_t(\kappa_t) = \delta(\kappa_t) \cdot \mathbb{E}(f_{t+1}(\kappa_{t+1})|\mathcal{F}_t) + (1 - \delta(\kappa_t)) \cdot \rho(f_{t+1}(\kappa_{t+1})|\mathcal{F}_t) \tag{17}$$

is determined recursively, and, in view of (15), starting with $f_{n-1}(\kappa_{n-1}) = -\delta(\kappa_{n-1})$. Note that all these factors $f_t(\kappa_t)$ are negative, so the liability values $L_t(\kappa_t)$ are all smaller than $\rho(X)$. The final goal is the value of the future liabilities as a function of the credit rating κ_0 of the company at time $t = 0$, and we get

$$L_0(\kappa_0) = \rho(X) + f_0(\kappa_0) \cdot K(X). \tag{18}$$

When comparing the latter expression to (10), one sees that $-1/(1+\eta_{n-1})$ got replaced by $f_0(\kappa_0)$, which depends on the initial credit rating and the Markov chain dynamics of the credit rating model. However, we have not yet properly included the event of bankruptcy into this approach, which we will do next.

3.4. **The inclusion of bankrupty costs.** In this paper we focus on a stand-alone risk X at a future time $t = n$, and it is natural to assume that X is small compared to the overall assets and liabilities of the insurance company (outside our considered portfolio), and correspondingly that the credit rating of the company at each point in time is determined exogeneously. Under that assumption, bankruptcy (then caused by other business lines or more generally other events in the company) could occur at any point in time, and we would like to explicitly include the bankruptcy costs in the considerations of this paper. If bankruptcy occurs at time t, a potential buyer of the bankrupt company would provide the solvency capital to run-off the portfolio in the future, but ask (immediately at that time t) for the available reserves L_{t-1} plus the capital costs for the remaining needed amount $\rho(X)-L_{t-1}$ (under the then unfavorable conditions, which is the 'penalty' of the bankruptcy scenario), see e.g. Bohn and Hall (1998). If that indeed happens, the trajectory stops there (i.e. $X_{t+1}=\ldots=X_n=0$), as the portfolio is liquidated. If bankruptcy occurs, we assume for simplicity that one can still raise capital at conditions as if one were in state CCC/C (i.e., state 7). Recall that $\eta^{(7)}(n-t)$ denotes the cost-of-capital rate at time t for the remaining $n-t$ time units. According to the above assumptions, bankruptcy at time t leads to overall liquidation costs

$$L_{t-1}+\eta_t^{(7)}(n-t)\cdot\rho(X-L_{t-1}|\mathcal{F}_t)=L_{t-1}(1-\eta_t^{(7)}(n-t))+\eta_t^{(7)}(n-t)\cdot\rho(X)$$

at that time, which we can include in our model as costs X_t at time t. For each $t=0,\ldots,n-1$, we then get from (7) and the abbreviation $Y_{t+1}:=X_{t+1}+L_{t+1}$ that under the inclusion of bankruptcy costs we have

(19)
$$Y_{t+1}|\kappa_t=\begin{cases} L_t(\kappa_t)(1-\eta^{(7)}(n-t-1))+\eta^{(7)}(n-t-1)\cdot\rho(X) & \text{with prob. } p_{\kappa_t,D}\\ L_{t+1}(j) \text{ with prob. } p_{\kappa_t,j} \text{ for } j=1,\ldots,7. \end{cases}$$

4. Six strategies under a Markov chain credit migration model

We now discuss six different strategies how to cope with that risk X appearing in time period n (cf. Figure 1). In order to distinguish the liability values resulting from each strategy, we will define $L_t^{(i)}$ as the liability value at time t under Strategy i.

4.1. **Strategy 1: The coarse regulatory approach.**

To provide a benchmark, let us start with the classical (but somewhat inaccurate[2] that is why we call it "coarse") regulatory approach to assume the cost-of-capital rate in the future is known at time $t=0$ and equal to $\eta_{n-1}=\eta_r=0.06$. In that case, from (10) we get

$$L_0^{(1)}=\mathbb{E}(X)+\frac{\eta_r}{1+\eta_r}(\rho(X)-\mathbb{E}(X))=\mathbb{E}(X)+0.0566\cdot K(X). \tag{20}$$

As mentioned before, according to modern risk management the actuaries would use a value of the cost-of-capital that is higher than 6%, taking into account the credit spread and the profit target. Moreover, even in cases where regulators do not mandate capital reserves for this risk at time $t=0$, it could be risky from a risk management perspective to maintain such a risk on the books without assigned capital. This situation may encourage underwriters to engage extensively in long-tail business, as these risks do not require additional capital returns in the following year. However, in the long run, these risks will require capital, potentially more than initially anticipated. Thus, there

[2]because not taking into account the credit risk of the company. Once the risk is recorded on the books, even if it is expected to materialize only in the future, the company bears it and is required to determine a cost of capital at time $t=n$, which may differ from the current one.

is a need to explore other ways of allocating capital to long-tail business. Like in the case of equalization reserves, we see here the opposition between the short and long term perspective as discussed in Dacorogna et al. (2013).

4.2. **Strategy 2: The consequent regulator's rationale.** If the company follows the regulator's approach, then it will indeed not allocate solvency capital for dealing with X before time $t = n-1$ and at that point will raise the needed solvency capital SCR_{n-1} for the last year. However, in the presence of credit rating risk, there is a need for extra capital to protect against adverse developments of the credit rating and correspondingly reserves need to be sufficient to afford the potentially higher costs of raising SCR_{n-1} at $t = n-1$ (here 'sufficiency' is again w.r.t. the risk measure ρ imposed by the regulator. However, we could choose any capital. The formalism is valid for any risk measure that presents transitional invariance and homogeneity). In addition, we also take into account the possibility of bankruptcy according to the approach outlined in Section 3.4 that is why we call it the "consequent" approach.

Since $L_n = 0$ and $X_n = X$, we get from Equation (7)

$$\begin{aligned} L^{(2)}_{n-1}(\kappa_{n-1}) &= \delta(\kappa_{n-1})\,\mathbb{E}(X) + (1-\delta(\kappa_{n-1}))\,\rho(X) \\ &= \rho(X) - \delta(\kappa_{n-1})\,(\rho(X) - \mathbb{E}(X)), \end{aligned} \tag{21}$$

the liability value for each credit state κ_{n-1} at time $t = n-1$. Note that at that time, potential future bankruptcy does not any more influence the liability value. For earlier time periods, it becomes more complex now. Concretely, for any time $t = 0, \ldots, n-2$, using (19),

$$\begin{aligned} L^{(2)}_{t}(\kappa_t) &= \delta(\kappa_t)\,\mathbb{E}(X_{t+1} + L^{(2)}_{t+1}|\kappa_t) + (1-\delta(\kappa_t))\,\rho(X_{t+1} + L^{(2)}_{t+1}|\kappa_t) \\ &= \delta(\kappa_t)\,\mathbb{E}(Y_{t+1}|\kappa_t) + (1-\delta(\kappa_t))\,\rho(Y_{t+1}|\kappa_t), \end{aligned} \tag{22}$$

cf. (19). Under the rather natural assumption

$$L^{(2)}_{t+1}(1) < L^{(2)}_{t+1}(2) < \ldots < L^{(2)}_{t+1}(7) < \eta^{(7)}(n-t-1)\cdot\rho(X) + L^{(2)}_{t}(\kappa_t)(1-\eta^{(7)}_{t}(n-t-1)) \tag{23}$$

for all κ_t (which will be empirically checked in the calculations later on), the quantity $\rho(Y_{t+1}|\kappa_t)$ can easily be determined for each credit state κ_t. Starting with $L^{(2)}_{n-1}(\kappa_{n-1})$ from (21), the implicit equation (22) then leads to a simple way of calculating $L^{(2)}_{n-2}(\kappa_{n-2})$ for each credit state κ_{n-2}. This recursive procedure can be applied for all smaller values of t as well, eventually arriving at $L^{(2)}_{0}(\kappa_0)$ for any initial credit state κ_0. More details on how to compute these quantities concretely will be given along the numerical illustrations in Section 5.

Note that another way to think about (21) is that

$$L^{(2)}_{n-1}(\kappa_{n-1}) = \mathbb{E}(X) + \eta^{(\kappa_{n-1})}(1)\cdot\big(\rho(X) - L^{(2)}_{n-1}(\kappa_{n-1})\big),$$

the liability value at time $t = n-1$, is the expected loss at $t = n$ plus the cost of raising the remaining required capital at $t = n-1$. In turn, the difference $L^{(2)}_{n-1}(\kappa_{n-1}) - \mathbb{E}(X)$

is the expected return for the investor on the provided capital $\rho(X) - L^{(2)}_{n-1}(\kappa_{n-1})$. In contrast, if the company goes bankrupt at $t = n-2$, then according to (19) the cost of bankruptcy, seen from time $t = n-2$ with present credit rating κ_{n-2}, is

$$Y_{n-1}(\kappa_{n-2}) = L^{(2)}_{n-2}(\kappa_{n-2}) + \eta^{(7)}(1) \cdot \big(\rho(X) - L^{(2)}_{n-2}(\kappa_{n-2})\big),$$

so all the reserves $L^{(2)}_{n-2}(\kappa_{n-2})$ plus the (expensive) capital cost for raising the remaining needed capital $\rho(X) - L^{(2)}_{n-2}(\kappa_{n-2})$ for the last time period are due (no 'updating' of the liability value is possible at the time of bankruptcy). One observes that replacing the expected claim cost $\mathbb{E}(X)$ by the available reserves is an additional 'penalty' of bankruptcy, that the buyer of the bankrupt company receives as a bonus for being willing to buy it (this principle of *"Vae Victis"*, or cost of financial distress, is indeed observed in insurance practice).

4.3. **Strategy 3: The conservative actuary.** In order to avoid the uncertainty about the credit state of the company at time $t = n-1$ completely, a conservative actuary may instead prefer to 'purchase' (allocate) the capital needed at time $t = n-1$ already at time 0, and hold it throughout the entire time period $[0, n]$. The conservative actuary would argue that since the risk is on the book, the capital to support it should also be on it.[3] In that case, one can log in the current credit rating κ_0 for the entire period. Whatever the economic value of the resulting liability value $L^{(3)}_0$ at time $t = 0$ is, this value is reserved at the beginning (typically this amount will be asked as the premium), and at time $n-1$ this amount needs to be complemented by the required solvency capital

$$\mathrm{SCR}_{n-1} = \rho(X) - L^{(3)}_0,$$

which under this strategy is already raised at the beginning. The cost for this at time $t = 0$ is

$$\eta^{(\kappa_0)}(n) \cdot (\rho(X) - L^{(3)}_0),$$

which leads to

$$L^{(3)}_0(\kappa_0) = \mathbb{E}(X) + \frac{\eta^{(\kappa_0)}(n)}{1+\eta^{(\kappa_0)}(n)}(\rho(X) - \mathbb{E}(X)). \tag{24}$$

This will typically be considerably larger than (20), see the numerical illustrations in Section 5. It is evident that the longer the time until ultimate, the more penalizing this strategy becomes, as it ties up capital for the entire period.

4.4. **Strategy 4: The prudent actuary.** As Strategy 3 will typically be too costly, it may be interesting to target for a compromise: raise (allocate) every year already some part of the eventually needed capital until the time $t = n-1$ when it is finally required by the regulator. In this way, the credit migration risk is improved in a cheaper way than under Strategy 3, while still maintaining an initial allocation of capital. To differentiate between strategy 3 and strategy 4, we describe the actuary in strategy

[3] Another benefit of this approach, from a risk management standpoint, is that it removes the temptation for underwriters to engage in long-term business without considering future implications.

4 as "prudent" because he raises some capital, though he is less conservative than the actuary in strategy 3. However, mathematically this leads to a considerably more complex procedure than for the previous strategies: SCR_{n-1} now not only depends on κ_{n-1}, but also on all other earlier credit states $\kappa_0, .., \kappa_{n-2}$, since the capital costs at $t = n-1$ depend on all the previously visited credit states. For our proposed recursive backward approach, this means that instead of evaluating L_{n-1} once for each of the 7 possible values of κ_{n-1} as before, we now need to do it for all 7^{n-1} possible sample paths of the Markov chain up to time point $n-1$.

Concretely, assume that at the times $t = 0, \dots, n-2$ one already raises capital C_t, respectively, so that at $t = n-1$ one only needs to raise the remaining amount $\mathrm{SCR}_{n-1} - \sum_{i=0}^{n-2} C_i$. Let us assume that all values C_t are already fixed at time $t = 0$ (and may depend on the initial credit state κ_0).

In that case, at time $t = n-1$ we have:

$$\begin{aligned} L_{n-1}^{(4)}(\kappa_0, .., \kappa_{n-1}) = \mathbb{E}(X) &+ \sum_{j=0}^{n-2} (s_1^{(\kappa_j)} + \eta_r) C_j(\kappa_0) \\ &+ \eta^{(\kappa_{n-1})}(1) \left(\rho(X) - \sum_{i=0}^{n-2} C_i(\kappa_0) - L_{n-1}^{(4)}(\kappa_0, .., \kappa_{n-1}) \right), \end{aligned}$$

where $s_j^{(k)}$ denotes the annual spread for remaining maturity j with current credit rating k (cf. Section 5 for details). This translates into

$$L_{n-1}^{(4)}(\kappa_0, .., \kappa_{n-1}) = \frac{\mathbb{E}(X) + \sum_{j=0}^{n-2} (s_1^{(\kappa_j)} + \eta_r) C_j + \eta^{(\kappa_{n-1})}(1) \left(\rho(X) - \sum_{i=0}^{n-2} C_i \right)}{1 + \eta^{(\kappa_{n-1})}(1)}. \tag{25}$$

For any $t = 0, \dots, n-2$, interpreting the capital costs $\sum_{j=0}^{t} (s_{n-t}^{(\kappa_j)} + \eta_r) C_j$ due at time $t+1$ as a 'claim cost' X_{t+1}, we can adapt (19) to the present situation to obtain for $Y_{t+1} = X_{t+1} + L_{t+1}^{(4)}$:

$$Y_{t+1} | \mathcal{F}_t = \begin{cases} L_t^{(4)}(\kappa_0, \dots, \kappa_t) + \eta^{(7)}(n-t-1) \cdot (\rho(X) - \sum_{j=0}^{t} C_j - L_t^{(4)}(\kappa_0, \dots, \kappa_t)) \\ \quad + \sum_{l=1}^{n-t} \sum_{j=0}^{t} (s_l^{(\kappa_j)} + \eta_r) C_j \quad \text{with prob. } p_{\kappa_t, D} \\ \sum_{j=0}^{t} (s_{n-t}^{(\kappa_j)} + \eta_r) C_j + L_{t+1}^{(4)}(\kappa_0, \dots, \kappa_t, j) \quad \text{with prob. } p_{\kappa_t, j} \text{ for } j = 1, \dots, 7. \end{cases} \tag{26}$$

That is, in case of bankruptcy at time $t+1$, the buying company will demand the liability value $L_t^{(4)}$, the remaining (present and future) capital costs of the already raised capital $\sum_{j=0}^{t} C_j$ as well as the capital cost for the remaining required capital amount at $t = n$, but raised immediately, and charged at the rate of a credit state CCC/C. With (5) adapted to

$$\mathrm{SCR}_t = \rho(Y_{t+1} | \mathcal{F}_t) - L_t^{(4)}(\kappa_0, \dots, \kappa_t) - \sum_{j=0}^{t-1} C_j, \tag{27}$$

we get from (6)

$$L_t^{(4)}(\kappa_0,\dots,\kappa_t) = \mathbb{E}(Y_{t+1}|\mathcal{F}_t) + \eta^{(\kappa_t)}(1)\left(\rho(Y_{t+1}|\mathcal{F}_t) - L_t^{(4)}(\kappa_0,\dots,\kappa_t) - \sum_{j=0}^{t-1} C_j\right)$$

or equivalently,

$$L_t^{(4)}(\kappa_0,\dots,\kappa_t) = \mathbb{E}(Y_{t+1}|\mathcal{F}_t) + (1-\delta(\kappa_t))\left(\rho(Y_{t+1}|\mathcal{F}_t) - \sum_{j=0}^{t-1} C_j - \mathbb{E}(Y_{t+1}|\mathcal{F}_t)\right), \tag{28}$$

for $t = 0,\dots,n-2$ (with the convention $\sum_{j=0}^{-1} C_j = 0$). For the calculation of $\rho(Y_{t+1}|\mathcal{F}_t)$ in (28), we again pose an intuitive monotonicity assumption, that we then verify in the numerical illustrations in Section 5, which here is

$$\begin{aligned} L_{t+1}^{(4)}(\kappa_0,\dots,\kappa_t,1) < \dots < L_{t+1}^{(4)}(\kappa_0,\dots,\kappa_t,7) &< \eta^{(7)}(n-t-1)\cdot(\rho(X) - \sum_{j=0}^{t} C_j) \\ &+ L_t^{(4)}(\kappa_0,\dots,\kappa_t)(1-\eta^{(7)}(n-t-1)) + \sum_{l=1}^{n-t-1}\sum_{j=0}^{t}(s_l^{(\kappa_j)} + \eta_r)C_j. \end{aligned} \tag{29}$$

Note that $L_t^{(4)}(\kappa_0,\dots,\kappa_t)$ also appears on the right-hand side of (28) in the expression $\mathbb{E}(Y_{t+1}|\mathcal{F}_t)$ and (potentially) in $\rho(Y_{t+1}|\mathcal{F}_t)$ (depending on the credit state κ_t), so that for each combination $(\kappa_0,\dots,\kappa_t)$ this equation needs to be considered separately. Since $L_{t+1}^{(4)}(\kappa_0,\dots,\kappa_{t+1})$ also appears on the right-hand side, (28) can now be used recursively, starting with $L_{n-1}^{(4)}$ from (25). From the latter equation it also becomes clear, however, that every path $(\kappa_0,..,\kappa_{n-1})$ leads to another value of $L_{n-1}^{(4)}(\kappa_0,..,\kappa_{n-1})$, and the latter is the starting point of the recursion (28), so that its value intrinsically affects the earlier liability values. Consequently, in order to determine $L_0^{(4)}(\kappa_0)$ we have to calculate the respective future liability values along each possible sample path starting in κ_0, see Section 5 for details.

To simplify the analysis, it is somewhat natural to assume that we allocate the same capital level $C_0 = \dots = C_{n-2} = C(\kappa_0)$ for all $t = 0,\dots,n-2$ (at time $n-1$ we then raise the actually needed remaining amount), and its value is determined at the beginning according to the credit state κ_0. An intuitive choice for raising a constant amount is

$$C(\kappa_0) = (\rho(X) - L_0^{(2)}(\kappa_0))/n, \tag{30}$$

where $L_0^{(2)}(\kappa_0)$ is the initial liability value under Strategy 2 (the rationale being that at each time point up to $t = n-2$ an equal fraction of the (initially and very roughly estimated) eventually needed SCR under Strategy 2, namely without early capital, is allocated.

However, any other fixed choice

$$C_0(\kappa_0) = c_0,\dots,C_{n-2}(\kappa_0) = c_{n-2} \tag{31}$$

of not necessarily equal values $c_0, \ldots, c_2$ can be handled in the analysis as well. One may even wonder whether it is feasible to determine the capital amounts C_t, $t = 0, \ldots, n-2$ adaptively and in an optimal fashion along the credit rating path, i.e., using the full information $\mathcal{F}_t$: $C_t = C_t(\kappa_0, .., \kappa_t)$ (the strategy of an *optimizing actuary*). A priori, this is also feasible, but with the recursive nature of the proposed method, one will have to optimize over $7^4+7^3+7^2+7^4+7 = 2800$ variables, which we consider a computational overkill for the purpose at hand. Moreover, the solution would be path-dependent, thus not general enough to be easily implemented in practice.

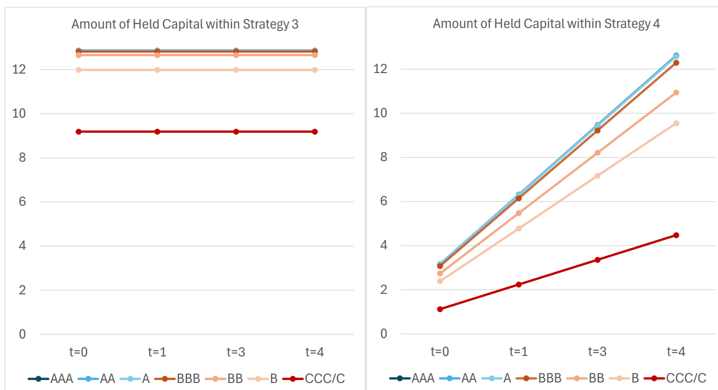

FIGURE 2. Capital held at each time t as a function of initial credit rating: Strategy 3 (left) and Strategy 4 (right).

In Figure 2, we illustrate the differentiation in capital allocation over time for strategies 3 and 4, corresponding to each initial credit rating. The left graph of Figure 2 shows that capital is allocated at the beginning and sustained throughout to the ultimate. This approach represents the strategy of a conservative actuary. In the right graph of Figure 2, the strategy of a prudent actuary is depicted, where capital is incrementally built up linearly over time, culminating in the full amount in the final year

4.5. **Strategy 5: Buying a call option.** Another alternative we consider here is to buy a (capped) call option on the excess of the eventual loss over the expected value of risk X (the excess needed is bounded from above by $\rho(X) - \mathbb{E}(X)$, so it is in fact a bull spread in financial terms), see for instance Dacorogna et al. (2013) for details on such a financial product in the context of equalization reserves. The price of this call option together with $\mathbb{E}(X)$ then constitutes the initial liability value

$$L_0^{(5)} = \mathbb{E}(X) + \mathbb{E}_{\mathbb{Q}}((\min(X, \rho(X)) - \mathbb{E}(X))_+). \tag{32}$$

Note that this strategy circumvents any credit migration risk, as the call option serves as a complete hedge up to the amount required by the regulator (although a fee for a protection against counterparty default risk may have to be added). Here we introduce classically the call option price as $\mathbb{E}_{\mathbb{Q}}$ (expectation under the risk neutral measure) since we assume that, contrary to insurance liabilities, there would be a market for these options. However, a crucial question in the implementation of such a strategy in practice will be the availability of an option seller as well as the specification of an appropriate price. Since a respective market is clearly not liquid, it will not be realistic to infer the pricing measure $\mathbb{Q}$ from a market, but its choice is important for any quantitative comparison of this solution to the above alternatives. In this paper, the simple pricing kernel

$$\mathbb{E}_{\mathbb{Q}}(\cdot) = (1+\theta^{(5)})\mathbb{E}_{\mathbb{P}}(\cdot) \tag{33}$$

for some $\theta^{(5)} > 0$, where $\mathbb{P}$ is the physical measure (other choices of $\mathbb{Q}$ are of course easily possible), see e.g. Hansen and Renault (2010). In particular, we will look for the limiting value of relative 'safety loading' $\theta^{(5)}$ in the price so that this solution is still attractive relative to its alternatives. Note also that due to our assumption of a zero risk-less interest rate in this paper, we do not have to consider discounting in formula (32), although the maturity of this option is n years. The longer maturity of this option will in fact lead to a smaller value of $\theta^{(5)}$, as the option seller can invest the paid price over the longer time, which might tradeoff with a fee for protection against counterparty default risk of the option seller, resulting in some final value of $\theta^{(5)}$. While a priori the payoff of this option is independent of the credit rating of the company, in the numerical illustrations in Section 5 we will still choose $\theta^{(5)}$ as a function of initial credit state, to reflect that option sellers might typically have a preference for going into a trade with a company with better credit rating (for instance because of the better reputation, the specification of the distribution of the risk X possibly being more trustworthy, general liquidity of the market for this rating, etc.).

One could also interpret (32) as a reinsurance cover with a bounded layer above a retention $\mathbb{E}(X)$, priced with an expected value principle with relative safety loading θ according to an expected value premium principle. Although reinsurance is typically written for one-year periods, reinsurance contracts with liabilities beyond one year do of course exist, although, in most cases, the underlying risk materializes in the first year (accident year), which is not the situation here. Also, the above coverage would correspond to an elimination of all the insurance risk by reinsurance, which a regulator will usually not accept (if X were the insurer's only risk, this contract would turn the insurer merely into a management firm). In any case, the market for such a product is clearly not liquid, and such a coverage is currently not available in the market at all, but it could emerge if there is a clear recognition of its necessity.

Finally, note that in case the call option is extremely expensive (for instance in view of the above reasons), one may think of reserving an additional amount c in the initial liability and buy a (cheaper) option with a suitably raised strike value, which would replace (32) by

$$\mathbb{E}(X) + c + \mathbb{E}_{\mathbb{Q}}((\min(X, \rho(X)) - \mathbb{E}(X) - c)_+).$$

However, for all reasonable magnitudes in numerical illustrations, it turns out that $c = 0$ is optimal.

4.6. **Strategy 6: Buying credit rating protection.** Yet another (and, as it turns out, attractive) alternative to manage the risk of a worsening of the credit state of the company is the following. One can lock in the liability $L_0^{(1)}$ of Strategy 1 (which the regulator requires), and – instead of any other measures – buy a credit derivative, which protects the company against any adverse future developments (i.e., against a bad credit state at time $n-1$ when the allocation of the capital is due[4]). In addition, one then also needs to add protection against the costs of bankruptcy which might occur at any point in time up to $t = n-1$. This results in a financial product whose payoff at time $n-1$ is contingent on the credit state of the company at that time and of the form

$$W(\kappa_0) = \begin{cases} s_1^{(j)} \cdot (\rho(X) - L_0^{(1)}) & \text{with prob. } \hat{p}_{\kappa_0,j}^{(n-1)} \text{ for } j = 1, \dots, 7. \\ \eta^{(7)}(n-t) \cdot \rho(X - L_0^{(1)}) & \text{with prob. } \hat{p}_{\kappa_0,D}^{(t)}. \end{cases} \tag{34}$$

Note that, if there is no bankruptcy, this payoff will exactly provide the additional cost (expressed through the credit spread) when at time $n-1$ the required capital $\rho(X) - L_0^{(1)}$ is more expensive than η_r (the rate for AAA, as $s_1^{(1)} = 0$), because one is in a worse credit state. The latter happens with the above $(n-1)$-step transition probabilities ($\hat{p}$ signifies that these are the transition probabilities conditional on the event that bankruptcy has not occurred up to $t = n-1$, which one can ensure by using the transition probability matrix $\boldsymbol{P}$ restricted to its first 7 lines and columns, e.g. $\hat{p}_{i,j}^{(2)} = \sum_{k=1}^{7} p_{i,k} p_{k,j}$). On the other hand, in case of bankruptcy at some time t, in alignment with the above assumptions for that case, the buyer of the bankrupt company will require $L_0^{(1)} + \eta^{(7)}(n-t) \cdot \rho(X - L_0^{(1)})$, so that the second summand should be the payoff in (34) for that case. Here, accordingly $\hat{p}_{\kappa_0,D}^{(t)} = \sum_{k=1}^{7} \hat{p}_{\kappa_0,j}^{(t-1)} p_{j,D}$ refers to the probability that bankruptcy occurs at time t. Clearly, this payoff (and therefore the price of this financial product) depends on the initial credit state κ_0.

This credit derivative fully hedges the future cash-flows in excess of $L_0^{(1)}$ that may be needed under worsening of the credit state or bankruptcy. The fact that the bankruptcy costs are being paid by the product only at the potentially later time $n-1$ does not matter under our assumption of a zero riskless interest rate. One can expect the existence of a market for this credit derivative (in particular for larger insurers), and in that case one may be able to determine a price from traded credit spreads. However, for simplicity of exposition, in this paper we choose to price this credit derivative using again $\mathbb{E}_{\mathbb{Q}}$ and the same pricing kernel as for the call option under Strategy 5, i.e.

$$L_0^{(6)} = L_0^{(1)} + \mathbb{E}_{\mathbb{Q}}(W(\kappa_0)) = L_0^{(1)} + (1 + \theta^{(6)})\, \mathbb{E}_{\mathbb{P}}(W(\kappa_0)), \tag{35}$$

where in comparison to Strategy 5 a lower value for $\theta^{(6)}$ may be realistic here, as the market for credit derivatives is quite liquid.

[4]The drawback of this technique is the lack of initial capital allocation. This creates the wrong incentives for underwriters, as they are not required to generate a return on risk capital in the first year.

5. Numerical Illustration

A major challenge with multi-step models is their numerical complexity, particularly when using Monte Carlo simulations. The necessity for nested simulations results in a dramatically increased number of simulations required, escalating exponentially to the n^{th} power in our scenario. However, we develop in this section an analytical approach that eliminates the need for Monte Carlo simulations. All findings are derived from the analytical solutions detailed in this section.

5.1. **Transition probabilities for the Markov chain migration model.** Let us start by estimating actual transition probabilities for the Markov chain model from market data. Denzler et al. (2006) have developed a model to link credit spread to default probability which fits market data very well by including fat-tailed jumps. We use their Formula (5) for the credit spread to infer the respective η-values:

$$s_{j,i} = (1+\bar{Y}_{j,i})\left[\frac{1}{[R+(1-R)(1-q_{j,i})]^{1/j}} - 1\right] \tag{36}$$

is the (annual) spread for maturity j (in years) at time i, R is the recovery rate, $\bar{Y}_{j,i}$ is the default free yield and $q_{j,i}$ is the default probability up to maturity j at time i. While among all financial institutions a recovery of 40% is considered typical (cf. Denzler et al. (2006)), for insurance companies a recovery rate of 60% ($R = 0.6$) seems more appropriate, and leads to resulting numbers that match empirical magnitudes. We want to concentrate on the cost of capital and not mix this with the effect of discounting, so we choose $\bar{Y}_{j,i} = 0$ for the risk-free rate in this paper. Recall that we assume that the spread depends on time i only through the credit rating k at that time point, so that we can reformulate this formula for our purposes in the following way: The annual spread for remaining maturity j with current credit rating k is given by

$$s_j^{(k)} = \frac{1}{[0.6+0.4(1-q_j^{(k)})]^{1/j}} - 1, \tag{37}$$

where $q_j^{(k)}$ is the probability to default within the next j years, given a current credit rating of k. To compute these values, we use our Markov chain assumption for the annual development of the rating. For the one-year transition matrix, we choose a recent study of the rating agency S&P (Nick W Kraemer, S&P (2021), Table 21) that extends over 40 years from 1981 to 2020. In that table, for each rated state there is also a probability given for the transition to a non-rated (NR) status. However, empirically, becoming non-rated is not a concern for insurance companies, so that for our considerations we remove that category and reweight the remaining transition rates accordingly, leading to the transition probability matrix given in Table 1. From this we simply derive $q_j^{(k)}$ (restricting to maturities of 5 years) as given in Table 2, and using (37) this leads to the credit spreads given in Table 3 (here and in the sequel, all depicted values are rounded to four digits after the comma, although the actual calculations are done with larger precision).

TABLE 1. S&P Global Corporate Average Transition Rates (1981-2020) for 1 year in percentage

From/to	**AAA**	**AA**	**A**	**BBB**	**BB**	**B**	**CCC/C**	**D**
AAA	89.85	9.35	0.55	0.05	0.11	0.03	0.05	0
AA	0.50	90.77	8.08	0.49	0.05	0.06	0.02	0.02
A	0.03	1.67	92.61	5.23	0.27	0.12	0.02	0.05
BBB	0	0.10	3.45	91.93	3.78	0.46	0.11	0.17
BB	0.01	0.03	0.12	5.03	85.99	7.51	0.61	0.70
B	0	0.02	0.08	0.17	5.18	85.08	5.66	3.81
CCC/C	0	0	0.12	0.20	0.65	14.72	50.90	33.41

TABLE 2. Default probabilities $q_j^{(k)}$ within the next j years by current credit rating class k

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
AAA	0	0.0002	0.0005	0.0010	0.0015
AA	0.0002	0.0005	0.0010	0.0016	0.0022
A	0.0005	0.0012	0.0021	0.0032	0.0047
BBB	0.0017	0.0041	0.0071	0.0109	0.0152
BB	0.0070	0.0179	0.0324	0.0497	0.0691
B	0.0381	0.0897	0.1440	0.1969	0.2457
CCC/C	0.3341	0.5098	0.6069	0.6645	0.7017

TABLE 3. Credit spreads $s_j^{(k)}$ for maturity j and credit rating class k (in basis points)

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
AAA	0	0.4307	0.7528	1.0106	1.2301
AA	0.8325	1.0893	1.3263	1.5557	1.7849
A	2.0914	2.4645	2.8607	3.2812	3.7266
BBB	6.8074	8.1601	9.5122	10.8829	12.2699
BB	27.9231	36.047	43.6120	50.36	56.2201
B	154.73	184.521	200.077	207.149	209.116
CCC/C	1542.65	1207.93	971.341	803.063	681.051

The over-all cost of capital $\eta^{(k)}(j)$ for a maturity of j years is then given by

$$\eta^{(k)}(j) = \sum_{i=1}^{j} (s_i^{(k)} + \eta_r). \tag{38}$$

That is, in each year i the regulatory cost of capital for one year, η_r, is augmented by the annual credit spread $s_i^{(k)}$ for the remaining maturity i at that point. Note that since the arrangement is settled j years before maturity, the current credit rating state

k is secured throughout. This approach leads to the numbers given in Table 4.

TABLE 4. Cost of capital $\eta^{(k)}(j)$ as a function of maturity j and current credit rating state k

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
AAA	0.06	0.1200	0.1801	0.2402	0.3003
AA	0.0601	0.1202	0.1803	0.2405	0.30079
A	0.0602	0.1205	0.1807	0.2411	0.3014
BBB	0.0607	0.1215	0.1824	0.2435	0.3048
BB	0.0628	0.1264	0.1908	0.2558	0.3214
B	0.0755	0.1539	0.2339	0.3146	0.3956
CCC/C	0.2143	0.3951	0.5522	0.6925	0.8206

5.2. **A Pareto risk with a time horizon of 5 years and VaR.** To assess the different strategies, it is necessary to choose a distribution for the risk X and a time horizon. These will be used to determine the value of the liability and to calculate and compare the values of the six strategies. Let us now assume that the risk X is Pareto distributed with distribution function $F_X(x) = 1-(x/x_0)^{-\alpha_P}$, $x > x_0$, where we choose the parameters to be $x_0 = 1$ and $\alpha_P = 1.8$, which is fat tailed with a non-converging second moment. We furthermore choose $n = 5$ years and the risk measure ρ to be the Value-at-Risk at safety level $\alpha = 99.5\%$ (cf. (3)), leading to $\mathbb{E}(X) = x_0\alpha_P/(\alpha_P-1) = 2.25$ and $\rho_\alpha(X) = x_0(1-\alpha)^{-1/\alpha_P} = 18.9824$ (note that any other choice of a distribution for X would lead to the same analysis, all that is needed here is the specification of $\mathbb{E}(X)$ and $\rho(X)$). Under Strategy 1 (where a constant $\eta_r = \eta_4 = 0.06$ is assumed and applied), (20) yields

$$L_0^{(1)} = \mathbb{E}(X) + 0.0566\,K(X) = 3.19705.$$

This represents the liability value required by the regulators, but neglecting the risk of credit migration as we have seen earlier. Accordingly, this value underestimates the appropriate liability value (an effect that is further exacerbated by the recent replacement of the 6% by 4.75%, which here would lead to the even lower value $L_0^{(1)} = 2.9705$) and does not allocate any capital at time $t = 0$, for a risk that is on the books.

For Strategy 2, one first calculates $L_4^{(2)}(\kappa_4)$ according to (21). Subsequently, we use (22) for $t = 3$ to calculate $L_3^{(2)}(\kappa_3)$ for all credit ratings κ_3 at time $t = 3$. Note that

$$Y_4|\kappa_3 = \begin{cases} L_4^{(2)}(j) & \text{with prob. } p_{\kappa_3,j} \text{ for } j = 1,\dots,7 \\ \eta^{(7)}(1)\cdot\rho(X) + (1-\eta^{(7)}(1))L_3^{(2)}(\kappa_3) & \text{with prob. } p_{\kappa_3,8}, \end{cases}$$

so that

$$\mathbb{E}(Y_4|\mathcal{F}_3) = \mathbb{E}(Y_4|\kappa_3) = \sum_{j=1}^{7} p_{\kappa_3,j}L_4^{(2)}(j) + p_{\kappa_3,8}(\eta^{(7)}(1)\cdot\rho(X) + (1-\eta^{(7)}(1))L_3^{(2)}(\kappa_3))$$

and

$$\rho(Y_4|\mathcal{F}_3) = \rho(Y_4|\kappa_3) = \inf\left\{x : \mathbb{P}\left(Y_4 > x|\kappa_3\right) \le 0.005\right\}.$$

Under the monotonicity assumption (23), for the latter one needs to look for the smallest value j^* such that $\sum_{l=1}^{j^*} p_{k,l} \ge 0.995$ and then $\rho(Y_4|\kappa_3 = k) = L_4^{(2)}(j^*)$ for $j^* < 8$ and $\eta^{(7)}(1) \cdot (\rho(X) - L_3^{(2)}(k)) + L_3^{(2)}(k)$ if $j^* = 8$. For the transition probabilities of Table 1 this yields $j^* = \{3, 4, 4, 6, 8, 8, 8\}$ as a function of $k = 1, \ldots, 7$. This approach leads to linear equations for the values $L_3^{(2)}(\kappa_3)$. One can then proceed analogously to obtain the values $L_2^{(2)}(\kappa_2)$.

The iterative and analogous application of the above procedure eventually leads to $L_0^{(2)}(\kappa_0)$. One indeed can verify that Assumption (23) is fulfilled for all $t = 0, \ldots, 3$. Table 5 depicts the resulting values for our concrete numerical example, in particular the final value $L_0^{(2)}$ as a function of the initial credit rating. Indeed, for all κ_0 the resulting value $L_0^{(2)}(\kappa_0)$ is higher than the one for Strategy 1. The difference is much smaller for the best rating from 0.6% to 2% for A-ratings, while it jumps at more than 10% for a BBB rating and prohibitive percentages for the non-investment grade ratings (inferior to BBB). For the good ratings, the difference might appear too small. However, in this case, credit risk is accounted for in terms of expected changes in the credit spread, without including it in the risk-adjusted capital, which is evidently a minimal requirement. These outcomes are expected, since the consideration of credit risk leads to larger capital costs in each scenario. It is interesting to see by how much the value $L_0^{(1)}$ under Strategy 1 underestimates the 'true' value of $L_0^{(2)}$ depending on the credit rating of the company. The monotonicity of $L_t^{(2)}$ as a function of the credit rating at any time t and also as a function of time to maturity matches the intuition and quantifies it for the concrete example.

Table 5. Liability values for Strategy 2

	$L_4^{(2)}(\kappa_4)$	$L_3^{(2)}(\kappa_3)$	$L_2^{(2)}(\kappa_2)$	$L_1^{(2)}(\kappa_1)$	$L_0^{(2)}(\kappa_0)$
AAA	3.1971	3.1985	3.2013	3.2065	3.2150
AA	3.1984	3.2003	3.2047	3.2148	3.2344
A	3.2002	3.2034	3.2113	3.2289	3.2623
BBB	3.2073	3.2289	3.2926	3.4211	3.6276
BB	3.2386	3.4961	3.9506	4.5663	5.3078
B	3.4242	3.8997	4.7184	5.8035	7.0463
CCC/C	5.2025	6.9108	9.3333	11.5922	13.3924

Next, let us turn to Strategy 3, which can be interpreted as actuarial practice of a conservative actuary, as well as cautionary risk management practice. Equation (24) together with Table 4 leads to the values of $L_0^{(3)}(\kappa_0)$ given in Table 6. It is not surprising that holding the capital for the entire period of 5 years is typically more costly than only anticipating the credit rating risk via regulatory measures as in Strategy 2, when the credit rating is rather good. However, we observe that if the initial credit rating is rather low (B or CCC/C), then raising the capital immediately is in fact cheaper than dealing with the possible bankruptcy risk along the way.

TABLE 6. Result for Strategy 3

	$L_0^{(3)}(\kappa_0)$
AAA	6.1147
AA	6.1178
A	6.1256
BBB	6.1583
BB	6.3199
B	6.9926
CCC/C	9.7918

We now turn to Strategy 4. To get a feeling for the effects of allocating capital earlier, we first consider the choice (30). The respective values are depicted in the left column of Table 7. Next, one has to evaluate $L_4^{(4)}$ using (25) for each of the possible 7^4 possible Markov chain sample paths up to this point. Then one calculates $L_3^{(4)}$ using (28) for each of the 7^3 possible combinations $\kappa_0, .., \kappa_3$, using the previously obtained $L_4^{(4)}$-values. This is quite a complex task, as for each of the values κ_3 one needs to separately solve a linear equation, for all possible values of $\kappa_0, .., \kappa_2$. One can continue along these lines and finally arrive at the result $L_0^{(4)}(\kappa_0)$ (which is the last column in Table 7). For illustration, in the additional columns in Table 7 we depict liability values $L_t^{(4)}$ for the specific credit migration paths $\kappa_0 = \cdots = \kappa_t$ (which are named κ_t* in the table), but of course all $7^5 = 16807$ paths enter the calculation of $L_0^{(4)}(\kappa_0)$. Note that we numerically verified that assumption (29) indeed holds for all t and credit state combinations, and therefore the j^*-calculations for Strategy 2 apply here as well. We observe that the benefit of allocating all the capital upfront now drops to the lowest rating of CCC/C, and otherwise could be considered too expensive in comparison with this strategy. Moreover, the latter offers an interesting alternative to the most conservative one (Strategy 3), while still adhering to the risk management principle of allocating capital for each risk present in the books.

TABLE 7. Allocated capital and liability values for Strategy 4 under (30)

	$C(\kappa_0)$	$L_4^{(4)}(\kappa_4*)$	$L_3^{(4)}(\kappa_3*)$	$L_2^{(4)}(\kappa_2*)$	$L_1^{(4)}(\kappa_1*)$	$L_0^{(4)}(\kappa_0)$
AAA	3.1536	3.1971	3.4193	3.6314	3.8347	4.0342
AA	3.1500	3.1984	3.4207	3.6334	3.8386	4.0399
A	3.1446	3.2002	3.4233	3.6381	3.8476	4.0576
BBB	3.0726	3.2072	3.4336	3.6689	3.940	4.2773
BB	2.7352	3.2386	3.5872	4.0761	4.7305	5.5598
B	2.3872	3.4242	3.9597	4.7551	5.8496	7.1952
CCC/C	1.1180	5.2025	7.3630	9.9331	12.1862	13.9259

TABLE 8. Allocated capital and liability values for Strategy 4 under (31)

	$C(\kappa_0)$	$L_0^{(4)}(\kappa_0)$
AAA	c_1	$3.2150 + 0.2586\ c_1$
AA	c_2	$3.2344 + 0.2557\ c_2$
A	c_3	$3.2623 + 0.2529\ c_3$
BBB	c_4	$3.6276 + 0.2114\ c_4$
BB	c_5	$5.3078 + 0.0921\ c_5$
B	c_6	$7.0463 + 0.0624\ c_6$
CCC/C	c_7	$13.3924 + 0.4772\ c_7$

Table 8 presents the values of $L_0^{(4)}(\kappa_0)$ for an arbitrary choice (31). Notably, when $c_0 = \cdots = c_3 = 0$, we recover the values of Strategy 2, as expected. Specifically, the initial liability values are minimal when no capital is allocated before time $t = n-1 = 4$. From this numerical implementation and under the expected value perspective, it appears that the penalization for credit risk does not justify the extra expenses of raising capital earlier than regulatory requirements.

However, two considerations arise: First, prudent risk management suggests that even if not mandated by regulators, it is unwise to avoid allocating capital to a risk already on the books. Second, our approach does not link the company's credit rating to its capital allocation strategy for long-tail business. One could argue that a company's credit rating should improve with more conservative strategies. Without a mechanism to reflect the potentially improved rating for each strategy (e.g., via a model for improved credit transition rates), we can still evaluate liability values under the fixed transition matrix. We might assume that adopting a more conservative strategy would enhance the initial credit rating by some notches, especially if such prudent capital management is applied across all company portfolios. Conversely, adopting less conservative strategies could increase credit risk and degrade the rating. Thus, Strategy 4 (or a variant) can still offer advantages over Strategy 2.

For ratings below BBB, Strategy 4 is equivalent to Strategy 2; for example, the liability value with initial credit state BB under Strategy 2 is 5.3065. If Strategy 4 improves the credit state by one notch from BB to BBB, according to Table 7 the new liability value is $L_0^{(4)}(4) = 4.2773$. If we can specify how earlier allocated capital amounts affect credit rating improvements, we can use Table 8 to refine the analysis and optimize the situation.

Next, let us consider Strategy 5 (buying a call option, cf. (32)). As mentioned earlier, that call option may typically not be available in the market, so specifying values of θ involves some arbitrariness. For the purpose of this illustration, we choose values that become higher for worse initial credit rating in the following way. Starting with a base value of $\theta^{(5)}(AAA) = 0.5$, we use the ratio of 1-year spreads between consecutive credit states to proportionally increase the values θ accordingly (and as an approximation of the spread ratio between AA and AAA we halve the one between state A and AA),

TABLE 9. Liability values for Strategy 5 (using (39)) and Strategy 6 with $\theta^{(6)} = 0.5$

	$L_0^{(5)}(\kappa_0)$	$L_0^{(6)}(\kappa_0)$	$\theta_*^{(6)}(\kappa_0)$
AAA	3.0521	3.2108	0.9040
AA	3.1206	3.2186	1.4637
A	3.4564	3.2427	1.0428
BBB	3.6550	3.3475	3.2112
BB	3.8815	3.8739	3.6749
B	4.2663	5.7787	1.2364
CCC/C	5.4504	12.5692	0.6317

leading to

$$\theta^{(5)}(\kappa_0) = \{0.5, 0.6280, 1.2560, 1.6275, 2.0510, 2.7707, 4.9850\}. \tag{39}$$

Table 9 presents the resulting liability values. We observe that Strategy 5 is quite attractive if such a call option is available. By varying θ, the liability value remains below $L_0^{(2)}(1)$ (the liability value of Strategy 2 with an initial credit rating of AAA) as long as $\theta < 0.804$. However, since the market for this call option is illiquid, it's unclear whether such a magnitude of θ is realistic compared to other strategies; if not, this reinsurance-type solution may not be more advantageous than previous strategies when purely evaluating liability value. The second column in Table 9 shows the liability values under Strategy 6, according to equations (34) and (35), with $\theta^{(6)} = 0.5$. Under this choice, the result is more attractive than Strategy 2 for any initial credit rating. We also calculate the critical value $\theta_*^{(6)}(\kappa_0)$ that leads to $L_0^{(6)}(\kappa_0) = L_0^{(2)}(\kappa_0)$, depicted in the third column of Table 9. Compared to Strategy 2, the credit derivative is particularly interesting for initial credit states between AA and B, as Strategy 6 yields a lower liability value even if the safety loading in the credit derivative price exceeds 100% (and actual values may be lower in the often liquid credit market). However, while this strategy covers credit migration risk, it leaves unresolved the risk management issue of allocating capital to every risk on the books.

Figure 3 presents a graphical comparison of liability values for each strategy across initial credit states. Recall that Strategy 1 completely disregards credit migration risk, while Strategy 2 acknowledges this risk but does not involve early capital raising. Strategy 3 proactively raises all required capital at the outset, whereas Strategy 4 incrementally raises portions of the necessary capital at each time point, as shown in Figure 3. Strategies 5 and 6 explore alternative approaches: purchasing a call option (Strategy 5) and buying a credit derivative to hedge against credit risk (Strategy 6). However, these last two strategies depend on the existence of a market for these financial derivatives, and their pricing is relatively speculative, making comparison with the other strategies less straightforward. Nevertheless, under our hypothesis, they could be viable alternatives worth exploring.

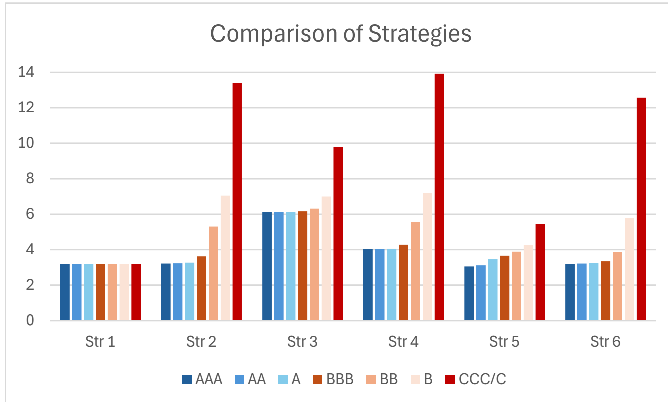

FIGURE 3. Resulting liability values for each strategy and each initial credit rating.

6. Conclusion and Outlook

Effectively handling time in risk estimation and management remains one of the fundamental challenges for actuarial mathematics and risk management, despite not being widely addressed in the literature. This paper advances the issue by developing a framework for evaluating capital allocation over time, focusing on using a liability valuation methodology to assess various capital allocation strategies. The approach is illustrated with a simple long-tail liability problem, highlighting the challenges involved. We find that the regulator-recommended approach overlooks both the credit risk on capital cost and the fundamental principles of prudent risk management, which advocate allocating capital for every risk on the books. Our study addresses credit risk migration, demonstrating that lower initial ratings significantly affect valuation—an aspect overlooked by current regulatory requirements. Different strategies for incorporating time in liability valuation and risk are examined. Allocating full capital at the start of a contract makes the liability excessively costly, exceeding regulatory requirements by over 90% in all cases. Therefore, we explore other strategies, and gradually building up capital emerges as a compromise, reducing the liability cost while requiring some capital at the onset of the contract. Additionally, we consider transferring the entire liability risk to another institution and hedging the credit migration risk. While risk transfer appears cost-effective, practical difficulties exist due to regulatory constraints and market reluctance. Hedging credit migration risk (Strategy 6) emerges as a viable alternative to Strategy 2 under our assumptions. However, it does not address the risk management issue of not allocating risk capital at the start of the contract and may encourage overly liberal underwriting of long-term risk, piling up liabilities that may be hard to manage—a problem observed in insurance practice, such as in professional liability in the U.S. in the early 2000s.

A potential area for future research is formalizing how prudent capital strategies could improve credit ratings, extending the current model. This would involve moving from an exogenous to an endogenous framework, where uniform application of a capital strategy across portfolios influences the company's overall credit rating. Technically, this could be implemented by assuming capital strategy choices impact the credit migration model's transition matrix, penalizing long-term risk underwriting without sufficient early capital allocation. While this extension would significantly increase model complexity and be sensitive to assumed transition probability changes, it could provide valuable insights into capital strategy effectiveness. Ultimately, this could help design optimal strategies balancing prudent risk management and regulatory requirements without penalizing insurers for holding long-tail business.

Other interesting extensions of the present work will be to relax the assumptions on the Markov chain to allow for time-inhomogeneous features that include market dynamics, to adapt and generalize the considered liability structure towards accommodating more complex claims run-off patterns of different business lines, to introduce stochastic discounting of the risk-free reference asset, to merge the present considerations with a general stochastic model for the asset side of the company as well as to study the effects of credit migration on a fully dynamic multi-period portfolio balancing premium risk and reserve risk over long time horizons.

In any case, we hope that the methodology introduced in this paper paves the way for extensive research opportunities aimed at better understanding how to integrate the time component in risk measurements. This is crucial in an era increasingly characterized by our growing awareness of the long-term impacts of human actions.

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