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Abstract

Extending the results in Sargan (1976) and Tanaka (1984), we derive the asymptotic expansions, of the Edgeworth and Nagar type, of the MM and QML estimators of the 1^{st} order autocorrelation and the MA parameter for the $MA(1)$ model. It turns out that the asymptotic properties of the estimators depend on whether the mean of the process is known or estimated. A comparison of the Nagar expansions, either in terms of bias or MSE, reveals that there is not uniform superiority of neither of the estimators, when the mean of the process is estimated. This is also confirmed by simulations. In the zero-mean case, and on theoretical grounds, the $QMLEs$ are superior to the MM ones in both bias and MSE terms. The results presented here are important for deciding on the estimation method we choose, as well as for bias reduction and increasing the efficiency of the estimators.

Keywords: Edgeworth expansion, moving average process, method of moments, Quasi Maximum Likelihood, autocorrelation, asymptotic properties.

JEL: C10, C22

1 Introduction

Techniques for approximating probability distributions like the Edgeworth expansion have a long history in econometrics.[1] However, there are relatively few papers concerning the limiting distribution of estimators of the Moving Average (MA) parameters and their properties. Tanaka (1984) develops a technique for the first order Edgeworth expansion of the normal $MLEs$ for autoregressive moving-average ($ARMA$) models and presents the first order expansion of the MLE for the $MA(1)$ model with and without mean.[2] Developing a Nagar type expansion, Bao and Ullah (2007) present the second order bias and Mean Square Error (MSE) of the Quasi MLE ($QMLE$) for the $MA(1)$ but without mean and they do not develop a valid Edgeworth expansion.

In this paper we develop the second order Edgeworth expansions of two estimators of θ, the MA parameter, and ρ, the 1^{st} order autocorrelation, of the following $MA(1)$ model with mean, $MA(1|\mu)$ say,

$$y_t = \mu + u_t + \theta u_{t-1}, \qquad t = ..., -1, 0, 1, ..., \quad |\theta| < 1, \quad u_t \overset{iid}{\backsim} (0, \sigma^2),$$

where θ is the true parameter value. The asymptotic distribution of the estimators of θ and ρ depends on whether the mean is estimated, or it is known and not estimated. In the latter case, we set $\mu = 0$ without loss of generality, and we are using $MA(1)$ to denote the model.

The first estimator is the popular Quasi Maximum Likelihood Estimator ($QMLE$). Its expansion is based on techniques developed in Mitrofanova (1967) (see also Linton 1997 and Corradi and Iglesias 2008) and applied in Tanaka (1984).[3] We denote

[1]Nagar (1959), Sargan (1974), Phillips (1977), Tanaka (1984), Sargan and Satchell (1986), Kakizawa (1999) and Ogasawara (2006) to quote only a few papers. Rothenberg (1986) gives a review on the asymptotic techniques employed in econometrics. For a book treatment of Edgeworth expansions see e.g. Hall (1992), Barndorff-Nielsen and Cox (1989), and Taniguchi and Kakizawa (2000).

[2]From now on we will refer to the up to $n^{-\frac{1}{2}}$ order expansion as first order one and for the up to n^{-1} order as second order expansion, where n is the sample size.

[3]For an alternative methodology based on a Whittle type estimator see Taniguchi (1987),

the $QMLEs$ as $\widetilde{\theta}$, for the $MA(1|\mu)$ model, and $\widetilde{\theta}_0$ when we consider the $MA(1)$ one. Employing $\widetilde{\theta}$ and $\widetilde{\theta}_0$ we can evaluate the $QMLEs$ of ρ and ρ_0, denoted by $\widetilde{\rho} = \frac{\widetilde{\theta}}{1+\widetilde{\theta}^2}$ and $\widetilde{\rho}_0 = \frac{\widetilde{\theta}_0}{1+\widetilde{\theta}_0^2}$, respectively (for the expansion of $\widetilde{\rho}_0$, only, see Ali 1984).

On the other hand, one could equate the sample 1^{st} order autocorrelation, say $\widehat{\rho}$, or $\widehat{\rho}_0$ when there is no mean, with the theoretical one, $\frac{\theta}{1+\theta^2}$, and solve for the unknown parameter. We call these the MM estimators of θ and θ_0, and denote them by $\widehat{\theta}$ and $\widehat{\theta}_0$, respectively, although strictly speaking they are $z - type$ estimators. Notice that $\widetilde{\rho}$ is the Indirect estimator of ρ, when the true model is an $AR(1)$ and the auxiliary is an $MA(1)$, where the parameter θ is estimated by MM, or by ML in the Constraint Indirect estimation setup (see Calzolari, Fiorentini and Sentana 2004). On the other hand, $\widehat{\theta}$ is an Indirect estimator of θ when the true model is an $MA(1)$ and the auxiliary is an $AR(1)$ one (see Gourieroux, Monfort and Renault 1993).

Utilizing an extension of the result in Sargan (1976), presented in Section 2, we develop the second order Edgeworth expansions of $\widehat{\rho}$, $\widehat{\rho}_0$, $\widetilde{\theta}$, and $\widehat{\theta}_0$ in Section 3, whereas Section 4 presents the expansions of the $QMLEs$. Employing these expansions, we derive second order Nagar type expansions of all estimators. Notice that this is the first time that second order Edgeworth and moment expansions of $\widehat{\theta}$, $\widehat{\theta}_0$, $\widetilde{\theta}$, and $\widetilde{\rho}$ appear in the literature. In section 5, the expansions are employed to compare all estimators in terms of bias and MSE. These comparisons are complemented by a simulation exercise. Section 6 concludes. All proofs, rather lengthy and tedious, are collected in Appendices at the end.

2 Edgeworth Expansion

In general, let $\widehat{\varphi}$ be an estimator of φ and

$$\overline{\varphi} = \sqrt{n}\left(\widehat{\varphi} - \varphi\right) = f\left(A_0, A_1, A_2, ..., A_l\right)$$

Lieberman et al (2003), and Andrews and Lieberman (2005).

where f is a function of the statistics A_i, $i = 0, 1, ..., l$, with the following assumptions:

Assumption 1 All the derivatives of f of order 4 and less are continuous, bounded in a neighborhood of $(0, ..., 0)$, such that $f^i = \frac{\partial f}{\partial A_i} \neq 0$ for some $i = 0, 1, ..., l$, and that there are functions h^{ij} and h^{ijk} independent of n such that $f^{ij} = \frac{\partial^2 f}{\partial A_i \partial A_j} = \frac{1}{\sqrt{n}} h^{ij}$, and $f^{ijk} = \frac{\partial^3 f}{\partial A_i \partial A_j \partial A_k} = \frac{1}{n} h^{ijk}$, where all derivatives are evaluated at $(0, ..., 0)$.

The $A_i^{\prime} s$ are functions of the data standardized in such a way so that their cumulants $c_i = cum(A_i)$, $c_{ij} = cum(A_i, A_j)$, etc. obey the following assumption:

Assumption 2

$$\begin{aligned} c_i &= n^{-\frac{1}{2}} c_i^{(1)} + n^{-1} c_i^{(2)} + o\left(n^{-1}\right), \quad c_{ij} = c_{ij}^{(1)} + n^{-\frac{1}{2}} c_{ij}^{(2)} + n^{-1} c_{ij}^{(3)} + o\left(n^{-1}\right), \\ c_{ijk} &= n^{-\frac{1}{2}} c_{ijk}^{(1)} + n^{-1} c_{ijk}^{(2)} + o\left(n^{-1}\right), \quad c_{ijkl} = n^{-1} c_{ijkl}^{(1)} + o\left(n^{-1}\right), \quad \textit{and} \\ c_{ijklm} &= O\left(n^{-\frac{3}{2}}\right), \end{aligned}$$

where $c_i^{(r)}$, $c_{ij}^{(r)}$, $c_{ijk}^{(r)}$ and $c_{ijkl}^{(r)}$ are independent of n, for $r = 1, 2, 3$.

Assumption 3 (Cramer's condition) If the characteristic function of $A = (A_0, A_1, ..., A_l)^{\prime}$ is $\Psi(z) = \int \exp\left(iz^{\prime} A\right) dF(A)$, then $\int_{\|z\| > Kn^{\alpha}} |\Psi(z)| \, dz = O\left(n^{\varepsilon - \frac{3}{2}}\right)$ for all $K > 0$, $0 < \alpha < \frac{1}{2}$ and some $\varepsilon < 0$, and where F is the distribution function of A.

These are standard assumptions in the relevant literature (see Chambers 1967, Sargan 1976, and Bhattacharya and Ghosh 1978). Under these assumptions we have the following Theorem.

Theorem 1 *Under Assumptions 1, 2 and 3, the second order Edgeworth expansion of $\overline{\varphi}$ is given by*

$$P(\overline{\varphi} \leq m) = \Phi\left(\frac{m}{\omega}\right) - \phi\left(\frac{m}{\omega}\right) \begin{bmatrix} \psi_0 + \psi_1\left(\frac{m}{\omega}\right) + \psi_2\left(\frac{m}{\omega}\right)^2 \\ +\psi_3\left(\frac{m}{\omega}\right)^3 + \psi_4\left(\frac{m}{\omega}\right)^4 + \psi_5\left(\frac{m}{\omega}\right)^5 \end{bmatrix} + o\left(n^{-1}\right), \quad (1)$$

where m is any real number, $\phi(.)$ and $\Phi(.)$ are the standard normal density and distribution functions, and $\psi_0, ..., \psi_5$, and ω are given in Appendix A.

Sargan (1976) assumes that $c_i^{(1)} = c_i^{(2)} = c_{ij}^{(2)} = c_{ij}^{(3)} = c_{ijk}^{(2)} = 0$. In this respect, Theorem 1 is a necessary generalization needed in the expansions of all estimators considered in this paper. Next, we have the following Lemma, which is very useful for the evaluation of the cumulants of $\overline{\varphi}$.

Lemma 1 *Under Assumptions 1, 2 and 3, the second order approximate cumulants of* $\overline{\varphi}$ *are*

$$\begin{aligned} k_1^{\widehat{\varphi}} &= \frac{a_4^{(1)} + 2a_{11}^{(1)}}{2\sqrt{n}} + \frac{a_4^{(2)} + 2a_{11}^{(2)}}{2n}, \\ k_2^{\widehat{\varphi}} &= \omega^2 + \frac{\omega^{(2)}}{\sqrt{n}} + \frac{a_9^{(1)} + 2\left(a_5^{(1)} + a_7^{(1)} + \omega^{(3)} + 2a_{12}^{(1)}\right)}{2n}, \\ k_3^{\widehat{\varphi}} &= \frac{a_1^{(1)} + 3a_3^{(1)}}{\sqrt{n}} + \frac{a_1^{(2)} + 6a_3^{(2)}}{n}, \ k_4^{\widehat{\varphi}} = \frac{a_2^{(1)} + 4a_6^{(1)} + 12\left(a_8^{(1)} + a_{10}^{(1)}\right) + \frac{9}{4}\left(\omega^{(2)}\right)^2}{n}, \end{aligned}$$

where the so-called Edgeworth coefficients, $a_j^{(i)}$, *for* $i = 1, 2$ *and* $j = 1, ..., 12$, $\omega^{(i)}$, *for* $i = 2, 3$, *and* ω *are given in the proof of Theorem 1 in Appendix A. Furthermore,*

$$E\left(\overline{\varphi}^2\right) = k_2^{\widehat{\varphi}} + \frac{\left(2a_{11}^{(1)} + a_4^{(1)}\right)^2}{4n}.$$

The proof of Lemma 1 is also given in Appendix A. We can now proceed in finding the expansions of the MM estimators of ρ and θ. The expansions of the MM and QML estimators of μ are not presented for space considerations.

3 The Expansions of the MM Estimators

The following analysis is based on Kakizawa (1999). Given observations $y = (y_0, ..., y_n)'$, the MM estimators of ρ and μ are given by:

$$\widehat{\rho} = \frac{\sum_{t=1}^n \left(y_t - \frac{1}{n}\sum_{t=1}^n y_t\right)\left(y_{t-1} - \frac{1}{n}\sum_{t=1}^n y_{t-1}\right)}{\sum_{t=1}^n \left(y_{t-1} - \frac{1}{n}\sum_{t=1}^n y_{t-1}\right)^2} \quad \text{and} \quad \widehat{\mu} = \frac{1}{n}\sum_{t=1}^n y_{t-1}.$$

Hence

$$\sqrt{n}\left(\widehat{\rho} - \rho\right) = \frac{\left(1+\theta^4\right)A_1 + \left(1+\theta^2\right)A_2 - \left(1-\theta+\theta^2\right)\frac{1}{\sqrt{n}}\left(A_0\right)^2}{\frac{1}{\sqrt{n}}\left(1+\theta^2\right)^2 A_3 + \frac{1}{\sqrt{n}}2\theta\left(1+\theta^2\right)A_1 - \frac{1}{n}\left(1+\theta^2\right)\left(A_0\right)^2 + \left(1+\theta^2\right)^2\sigma^2}, \tag{2}$$

where

$$
\begin{aligned}
A_0 &= \frac{1}{\sqrt{n}} \sum_{t=1}^{n} (y_{t-1} - \mu), \quad A_1 = \frac{\sum_{t=2}^{n} u_{t-1} u_{t-2}}{\sqrt{n}}, \\
A_2 &= \frac{1}{\sqrt{n}} \left[\begin{array}{c} \left[(y_1 - \mu)(y_0 - \mu) - \theta\sigma^2\right] + \theta \sum_{t=2}^{n} u_t u_{t-2} + u_n u_{n-1} - u_1 u_0 \\ -\theta \frac{\theta^2 \left(u_0^2 - \sigma^2\right) - \theta^2 \left(u_{n-1}^2 - \sigma^2\right) + \left[(y_0 - \mu)^2 - \left(1+\theta^2\right)\sigma^2\right]}{\left(1+\theta^2\right)} \\ + \frac{1}{n} \left[(y_0 - \mu) - (y_n - \mu)\right] \left[(1+\theta) \sum_{t=2}^{n} u_{t-1} + \theta u_0 - \theta u_{n-1} + (y_0 - \mu)\right] \end{array} \right], \\
A_3 &= \frac{\sum_{t=2}^{n} \left(u_{t-1}^2 - \sigma^2\right) + \frac{\theta^2 \left(u_0^2 - \sigma^2\right) - \theta^2 \left(u_{n-1}^2 - \sigma^2\right) + \left[(y_0 - \mu)^2 - \left(1+\theta^2\right)\sigma^2\right]}{\left(1+\theta^2\right)}}{\sqrt{n}}.
\end{aligned}
$$

It is now obvious that $\sqrt{n}\left(\widehat{\rho} - \rho\right)$ is a function of $A_i\prime s$ $i = 0, ..., 3$, $f\left(A_0, A_1, A_2, A_3\right)$ say, with $f(0,0,0,0) = 0$. From Appendix B1, where the cumulants of the $A_i\prime s$ are presented, it is easily seen that Assumption 2 is satisfied and if $E\left(u_0^{10}\right)$ is finite we can apply Theorem 1. Notice that most of the second order cumulants of the $A_i\prime s$ include terms of $O\left(n^{-1}\right)$. Hence, the generalization of Sargan (1976) presented in section 2 is a necessary one. Let us now turn our attention to $\widehat{\rho}$.

3.1 The Expansion of the MM 1^{st} Order Autocorrelation

Lemma 2 *Under the Assumptions that $u_t^{\prime}s$ are identically and independently distributed, $E\left(u_0^{10}\right) < \infty$, $\left(u_0, u_0^2\right)$ satisfy the Cramer's condition and $\theta \in (-1, 1)$, the second order asymptotic expansion of $P\left(\sqrt{n}\left(\widehat{\rho} - \rho\right) < m\right)$ is given by:*

$$
G(m) = \Phi\left(\frac{m}{\omega}\right) - \phi\left(\frac{m}{\omega}\right)\left(\psi_0 + \psi_1 \frac{m}{\omega} + \psi_2 \left(\frac{m}{\omega}\right)^2 + \psi_3 \left(\frac{m}{\omega}\right)^3 + \psi_5 \left(\frac{m}{\omega}\right)^5\right), \quad (3)
$$

where the polynomial coefficients ψ_i, $i = 0, ..., 5$ are as in Theorem 1 and the Edgeworth coefficients are given in Appendix B2.

To evaluate the approximate bias, MSE and cumulants, needed in the sequel, we employ Lemma 1. Letting κ_3 and κ_4 to denote the 3^{rd} and 4^{th} order cumulants of u_0, respectively, the cumulants of $\sqrt{n}\left(\widehat{\rho} - \rho\right)$ are:

$$
k_1^{\widehat{\rho}} = -\frac{1}{\sqrt{n}}\left(\theta^4 + 2\theta^3 - 2\theta^2 + 2\theta + 1\right)\frac{\theta^2 + \theta + 1}{\left(\theta^2 + 1\right)^3} + o\left(n^{-1}\right)
$$

$$k_2^{\widehat{\rho}} = \omega_{\widehat{\rho}}^2 - \frac{1}{n}\left(\xi_{2,1}^{\widehat{\rho}} + \xi_{2,2}^{\widehat{\rho}}\right) + o\left(n^{-1}\right)$$

where $\omega_{\widehat{\rho}}^2 = \frac{\theta^2+4\theta^4+\theta^6+\theta^8+1}{\left(1+\theta^2\right)^4}$, i.e. the asymptotic variance, $\xi_{2,1}^{\widehat{\rho}} = 2\frac{\left(-4\theta-\theta^2+6\theta^3-12\theta^5+6\theta^7-\theta^8-4\theta^9+\theta^{10}+1\right)(\theta+1)^2}{\left(\theta^2+1\right)^6}$, and $\xi_{2,2}^{\widehat{\rho}} = 4\frac{\theta\left(1+\theta^4\right)^2}{\left(1+\theta^2\right)^5}\kappa_3^2 + \frac{\theta^2+4\theta^4-\theta^6+\theta^8+1}{\left(\theta^2+1\right)^4}\kappa_4$,

$$k_3^{\widehat{\rho}} = -\frac{6}{\sqrt{n}}\theta\left(\theta^4+1\right)\frac{6\theta^4+\theta^8+1}{\left(\theta^2+1\right)^7} + \frac{1}{\sqrt{n}}\frac{\left(1+\theta^4\right)^3+\theta^3\left(1+\theta^2\right)^3}{\left(1+\theta^2\right)^6}\kappa_3^2 + o\left(n^{-1}\right),$$

$$k_4^{\widehat{\rho}} = \frac{1}{n}\left(\xi_{4,1}^{\widehat{\rho}} + \xi_{4,2}^{\widehat{\rho}} + \xi_{4,3}^{\widehat{\rho}}\right),$$

where $\xi_{4,1}^{\widehat{\rho}} = 6\frac{-1+10\theta^2\left(1+\theta^{16}\right)-30\theta^4\left(1+\theta^{12}\right)+106\theta^6\left(1+\theta^8\right)-129\theta^8\left(1+\theta^4\right)+216\theta^{10}-\theta^{20}}{\left(\theta^2+1\right)^{10}}$,

$\xi_{4,2}^{\widehat{\rho}} = 12\theta\left(\theta^4+1\right)\frac{\theta\left(1-\theta^4\right)^2\left(1+\theta^2\right)-10\theta^4\left(1+\theta^4\right)+4\theta^6-2\left(1-\theta^2\right)\left(1-\theta^{10}\right)}{\left(\theta^2+1\right)^9}\kappa_3^2$,

and $\xi_{4,3}^{\widehat{\rho}} = \frac{5\theta^4+4\theta^6+12\theta^8+4\theta^{10}+5\theta^{12}+\theta^{16}+1}{\left(\theta^2+1\right)^8}\kappa_4^2$. Furthermore, the second order approximate MSE ($AMSE$) is

$$E\left[\sqrt{n}\left(\widehat{\rho}-\rho\right)\right]^2 = k_2^{\widehat{\rho}} + \frac{1}{n}\left(\theta^4+2\theta^3-2\theta^2+2\theta+1\right)^2\frac{\left(\theta^2+\theta+1\right)^2}{\left(\theta^2+1\right)^6}. \tag{4}$$

It is worth noticing first, that the sign of the asymmetry of the distribution of the errors (κ_3) does not affect the $AMSE$, i.e. positively and negatively skewed error distributions of the same magnitude have the same effect on the $AMSE$. Second, the $AMSE$ is a decreasing function of κ_4, for any value of θ in the admissible region. It seems that higher probability of extreme values of the errors increases the accuracy of the estimator. This is not true for the asymmetry parameter κ_3. For positive (negative) values of θ, the $AMSE$ of $\widehat{\rho}$ is a decreasing (increasing) function of κ_3^2. Further, for $\theta = 0$ and under elliptical error distributions, the presented moments are known in the literature (see e.g. Kan and Wang 2010) Let us now proceed to the expansion of the MM 1^{st} order autocorrelation when the mean is 0.

3.1.1 The Zero-mean Expansion

In case that μ is zero, or known and subtracted from the data, we have that

$$\widehat{\rho}_0 = \frac{\sum_{t=1}^n y_t y_{t-1}}{\sum_{t=1}^n y_{t-1}^2}.$$

Hence

$$\sqrt{n}\left(\widehat{\rho_0}-\rho\right)=\frac{\left(1+\theta^4\right)A_1+\left(1+\theta^2\right)A_2}{\left(1+\theta^2\right)^2\frac{1}{\sqrt{n}}A_3+2\theta\left(1+\theta^2\right)\frac{1}{\sqrt{n}}A_1+\left(1+\theta^2\right)^2\sigma^2},$$

where the A_i's are now given by

$$\begin{aligned}
A_1 &= \frac{\sum_{t=2}^{n}u_{t-1}u_{t-2}}{\sqrt{n}},\quad A_2=\frac{1}{\sqrt{n}}\left[\begin{array}{c}\left(y_1y_0-\theta\sigma^2\right)+\theta\sum_{t=2}^{n}u_tu_{t-2}+u_nu_{n-1}-u_1u_0\\ -\theta\frac{-\theta^2\left(u_{n-1}^2-\sigma^2\right)+\theta^2\left(u_0^2-\sigma^2\right)+\left(y_0^2-\left(1+\theta^2\right)\sigma^2\right)}{\left(1+\theta^2\right)}\end{array}\right],\\
A_3 &= \frac{1}{\sqrt{n}}\left[\sum_{t=2}^{n}\left(u_{t-1}^2-\sigma^2\right)+\frac{-\theta^2\left(u_{n-1}^2-\sigma^2\right)+\theta^2\left(u_0^2-\sigma^2\right)+\left(y_0^2-\left(1+\theta^2\right)\sigma^2\right)}{\left(1+\theta^2\right)}\right].
\end{aligned}$$

Notice that A_1 and A_3 are the same as in the non-zero mean case. However, the term $\frac{1}{n}\left[(y_0-\mu)-(y_n-\mu)\right]\left[(1+\theta)\sum_{t=2}^{n}u_{t-1}+\theta u_0-\theta u_{n-1}+(y_0-\mu)\right]$ is not included in A_2. Furthermore, $\sqrt{n}\left(\widehat{\rho_0}-\rho\right)$ has the same functional form with respect to A_1, A_2 and A_3. Consequently, the derivatives are the same, but now all sums determining the Edgeworth coefficients run from $i=1$ up to 3.

Hence, the asymptotic variance of $\sqrt{n}\left(\widehat{\rho_0}-\rho\right)$ is the same as the asymptotic variance of $\sqrt{n}\left(\widehat{\rho}-\rho\right)$, i.e. $\omega_{\widehat{\rho_0}}^2=\omega_{\widehat{\rho}}^2=\frac{1+\theta^2+4\theta^4+\theta^6+\theta^8}{\left(1+\theta^2\right)^4}$. Further, all Edgeworth coefficients are the same as in the non-zero mean case (see Appendix B2) apart from $a_4^{(1)}$, $a_5^{(1)}$, $a_7^{(1)}$, and $a_9^{(1)}$, which are also presented in Appendix B2.

We can now evaluate the bias, the MSE and the cumulants of $\sqrt{n}\left(\widehat{\rho_0}-\rho\right)$. The 1^{st} order cumulant is

$$k_1^{\widehat{\rho_0}}=k_1^{\widehat{\rho}}-\frac{1}{\sqrt{n}}\left(\theta-\theta^2-1\right)\frac{\left(\theta+1\right)^2}{\left(\theta^2+1\right)^2}=-\frac{2}{\sqrt{n}}\theta\frac{\theta^4+1}{\left(\theta^2+1\right)^3}+o\left(n^{-1}\right). \tag{5}$$

Comparing the absolute values of the two approximate biases (see Figure 1) it is clear that for $\theta\in(-1,-0.2)$ the absolute bias of $\widehat{\rho}$, multiplied by $\sqrt{n}$, is less than the one of $\widehat{\rho_0}$. The opposite is true for $\theta\in(-0.2,1)$.

The $AMSE$ is

$$\begin{aligned}
E\left[\sqrt{n}\left(\widehat{\rho_0}-\rho\right)\right]^2 &= E\left[\sqrt{n}\left(\widehat{\rho}-\rho\right)\right]^2\\
&\quad+\frac{1}{n}\left(1+8\theta-7\theta^2+6\theta^3+8\theta^4+6\theta^5-7\theta^6+8\theta^7+\theta^8\right)\frac{\left(\theta+1\right)^2}{\left(\theta^2+1\right)^5}.
\end{aligned}$$

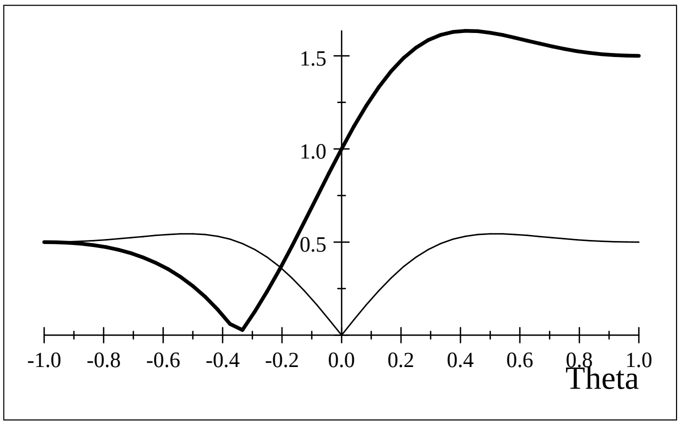

Figure 1: $\left|E\left[n\left(\widehat{\rho}-\rho\right)\right]\right|$ (thick line) and $\left|E\left[n\left(\widehat{\rho_0}-\rho_0\right)\right]\right|$.

Obviously, the sign of the difference between the zero and the non-zero mean case $AMSEs$ depends on the sign of the 8^{th} degree polynomial. As now the limit of the polynomial is -32, for $\theta \to -1$, and 24, for $\theta \to 1$, it follows that that there are intervals of θ, within $(-1,1)$, such that the $AMSE$ of $\widehat{\rho_0}$ is lower than the one of $\widehat{\rho}$ and vice versa, for any number of observations, n. However, notice that the asymmetry and kurtosis parameters, κ_3 and κ_4, have the same effect on the $AMSE$, for any values of θ in the admissible region. Of course, the two $AMSEs$ are equal to the common asymptotic variance $\omega^2_{\widehat{\rho}}$, as $n \to \infty$.

Applying again Lemma 1, we get that the second order cumulant of $\sqrt{n}\left(\widehat{\rho_0}-\rho\right)$ is given by:

$$k_2^{\widehat{\rho_0}} = k_2^{\widehat{\rho}} - \frac{4\theta}{n}\frac{\left[\left(1-\theta\right)\left(1-\theta^5\right)+2\theta^3\right]\left(\theta+1\right)^2}{\left(\theta^2+1\right)^5} + o\left(n^{-1}\right).$$

As now the Edgeworth coefficients involved in the evaluation of the 3^{rd} and 4^{th} order cumulants are the same in the non-zero and the zero mean case (see Lemma 1), i.e. $k_3^{\widehat{\rho_0}} = k_3^{\widehat{\rho}}$ and $k_4^{\widehat{\rho_0}} = k_4^{\widehat{\rho}}$, we can conclude that the non-normality of the estimators of ρ is not affected by the estimation or not of the mean μ, up to $o\left(n^{-1}\right)$.

3.2 The Expansion of the MM MA Coefficient

For $|\widehat{\rho}| < 0.5$ the solution for $\widehat{\theta}$ is:

$$\widehat{\theta} = \frac{1-\sqrt{1-4\widehat{\rho}^2}}{2\widehat{\rho}} \quad and \quad \widehat{\theta}-\theta = \frac{1-\sqrt{1-4\widehat{\rho}^2}}{2\widehat{\rho}} - \frac{1-\sqrt{1-4\rho^2}}{2\rho} = f\left(\widehat{\rho}\right). \quad (6)$$

Hence, given the cumulants of $\sqrt{n}\left(\widehat{\rho}-\rho\right)$ presented in Section 3.1, we can apply Theorem 1. The Edgeworth coefficients of $\sqrt{n}\left(\widehat{\theta}-\theta\right)$ are given in Appendix B4. Applying Lemma 1, we can prove the following Proposition:

Proposition 1 *Under the Assumptions of Lemma 2 we have that the* 1^{st} *order cumulant and the MSE of* $\sqrt{n}\left(\widehat{\theta}-\theta\right)$ *are*

$$k_1^{\widehat{\theta}} = \frac{1}{\sqrt{n}} \frac{2\theta^2+6\theta^3-2\theta^4+3\theta^5+2\theta^6-\theta^8-\theta^9-1}{\left(1-\theta^2\right)^3} + o\left(n^{-1}\right)$$

and

$$E\left[\sqrt{n}\left(\widehat{\theta}-\theta\right)\right]^2 = \frac{1+\theta^2+4\theta^4+\theta^6+\theta^8}{\left(1-\theta^2\right)^2} + \frac{1}{n}\left(\xi_3^{\widehat{\theta}}+\xi_4^{\widehat{\theta}}\right) + o\left(n^{-1}\right),$$

$$where\ \xi_3^{\widehat{\theta}} = \frac{\begin{array}{c} 1-8\theta-36\theta^2-56\theta^3-93\theta^4-150\theta^5+2\theta^6-192\theta^7+747\theta^8-72\theta^9+3019\theta^{10} \\ +192\theta^{11}+4765\theta^{12}+418\theta^{13}+5421\theta^{14}+352\theta^{15}+2539\theta^{16}+24\theta^{17}+460\theta^{18} \\ -216\theta^{19}-933\theta^{20}-210\theta^{21}-442\theta^{22}-96\theta^{23}-141\theta^{24}-8\theta^{25}+21\theta^{26}+16\theta^{27} \\ +29\theta^{28}+6\theta^{29}+3\theta^{30} \end{array}}{\left(1-\theta^4\right)^6},$$

and $\xi_4^{\widehat{\theta}} = -2\theta\frac{-1+2\theta-5\theta^2+5\theta^3-6\theta^4+2\theta^5-2\theta^6+\theta^7+\theta^8-2\theta^9+\theta^{10}}{\left(1-\theta^2\right)^2(1-\theta)^2}\kappa_3^2 - \frac{1+\theta^2+4\theta^4-\theta^6+\theta^8}{\left(1-\theta^2\right)^2}\kappa_4.$

Notice first, that the approximate bias of $\widehat{\theta}$ is not affected by the non-normality of the errors, and second that the effect of κ_4 on the $AMSE$ of $\widehat{\theta}$ is the same as the effect on the $AMSE$ of $\widehat{\rho}$, i.e. the $AMSE$ is a decreasing function of κ_4 for all $\theta \in (-1,1)$. However, for positive (negative) values of θ the $AMSE$ of $\widehat{\theta}$ is an increasing (decreasing) function of κ_3^2. This is exactly opposite from the effect that κ_3^2 has on the $AMSE$ of $\widehat{\rho}$. Let us now proceed to the expansion of the MM MA coefficient when the mean is 0.

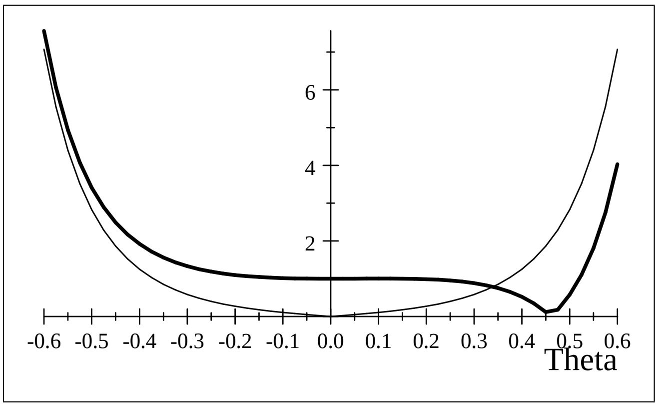

Figure 2: $\left|E\left[n\left(\widehat{\theta}-\theta\right)\right]\right|$ (thick line) and $\left|E\left[n\left(\widehat{\theta_0}-\theta\right)\right]\right|$

3.2.1 The Zero-Mean Expansion

For the zero mean case, all Edgeworth coefficients are the same as in the non-zero mean one, apart from $\omega^{(3)}$, $a_{11}^{(1)}$ and $a_{12}^{(1)}$, which are given in Appendix A4. Consequently, applying Lemma 1 and keeping terms up to order $O\left(n^{-1}\right)$, the approximate bias of $\sqrt{n}\left(\widehat{\theta_0}-\theta\right)$ is

$$k_1^{\widehat{\theta_0}} = k_1^{\widehat{\theta}} + \frac{1}{\sqrt{n}}\left(1+\theta\right)\frac{1-\theta+\theta^2}{1-\theta} = \frac{1}{\sqrt{n}}\theta\frac{1+5\theta^2+2\theta^4+\theta^6-\theta^8}{\left(1-\theta^2\right)^3}. \tag{7}$$

Plotting, again, the absolute values of the two approximate biases (multiplied by $\sqrt{n}$), i.e. $\left|E\left(n\left(\widehat{\theta}-\theta\right)\right)\right|$ and $\left|E\left(n\left(\widehat{\theta_0}-\theta\right)\right)\right|$, we observe that for values of θ higher than about 0.3 the approximate bias of $\widehat{\theta}$ is less than the one of $\widehat{\theta_0}$ (see Figure 2).

In terms of $AMSE$ we have that, keeping the relevant terms,

$$E\left[\sqrt{n}\left(\widehat{\theta_0}-\theta\right)\right]^2 = E\left[\sqrt{n}\left(\widehat{\theta}-\theta\right)\right]^2 + \frac{1}{n}\lambda$$

where $\lambda = \dfrac{\begin{array}{c}8\theta+7\theta^2+56\theta^3+65\theta^4+150\theta^5+204\theta^6+192\theta^7+297\theta^8+72\theta^9\\ +51\theta^{10}-192\theta^{11}-481\theta^{12}-418\theta^{13}-656\theta^{14}-352\theta^{15}-199\theta^{16}\\ -24\theta^{17}+285\theta^{18}+216\theta^{19}+327\theta^{20}+210\theta^{21}+132\theta^{22}+96\theta^{23}\\ -\theta^{24}+8\theta^{25}-23\theta^{26}-16\theta^{27}-7\theta^{28}-6\theta^{29}-3\end{array}}{\left(1-\theta^4\right)^6}$, indicating that, first, the non-normality of the errors affects the $AMSE$ of $\widehat{\theta}$ and $\widehat{\theta_0}$ in the same way and second, the asymptotic variance of $\widehat{\theta}$ and $\widehat{\theta}_0$ is the same. However, the sign of λ depends on the sign of the numerator, a polynomial of 29^{th} degree. As the limit of this polynomial changes sign as $\theta \to \pm 0.6$, we can conclude that there are values of θ, in its admissible interval, such that the $AMSE$ of $\widehat{\theta}$ is less than the one of $\widehat{\theta}_0$. Let us now turn our attention to the expansions of the QML estimators of θ, μ and ρ.

4 The Expansions of the QML Estimators

In this section we extend the analysis in Tanaka (1984) by dropping normality and including terms of second order in the approximation of the $QMLE$ of the $MA\,(1|\mu)$ parameters, θ and μ, say $\widetilde{\theta}$ and $\widetilde{\mu}$.[4] These are the solutions to the following equations:

$$\begin{aligned} \frac{\partial \ell\left(\widetilde{\theta}\right)}{\partial \theta} &= 0 \Rightarrow \frac{1}{\sigma^2}\sum_{t=1}^{n} u_t \left(\theta \frac{\partial u_{t-1}}{\partial \theta} + u_{t-1}\right)\Bigg|_{\theta=\widetilde{\theta}} = 0 \\ &\quad and \\ \frac{\partial \ell\left(\widetilde{\mu}\right)}{\partial \mu} &= 0 \Rightarrow \frac{1}{\sigma^2}\sum_{t=1}^{n} u_t \frac{\partial u_t}{\partial \mu}\Bigg|_{\mu=\widetilde{\mu}} = 0 \end{aligned} \tag{8}$$

where

$$\ell\left(\theta,\mu\right) = -\frac{n\log(2\pi\sigma^2)}{2} - \frac{\sum_{t=1}^{n} u_t^2}{2\sigma^2} \qquad and \qquad u_t = y_t - \mu - \theta u_{t-1}.$$

In Appendix C1 we express $\sqrt{n}\left(\widetilde{\theta}-\theta\right)$ and $\sqrt{n}\left(\widetilde{\mu}-\mu\right)$ as functions of the first, second and third order derivatives of $\ell\left(\theta,\mu\right)$ standardized appropriately and evaluated at the true parameter values. We also present their expectations. In Appendix C2 we evaluate the needed cumulants of these derivatives, so that Theorem 1 can be applied. Let us now turn our attention to the expansion of $\widetilde{\theta}$.

[4]For various approximations of the MLE see Davidson (1981).

4.1 The Expansion of the QML MA Coefficient Estimator

Lemma 3 *Under the Assumptions of Lemma 2, the second order Edgeworth expansion of* $P\left(\sqrt{n}\left(\widetilde{\theta}-\theta\right)<m\right)$ *is given by:*

$$P(m)=\Phi\left(\frac{m}{\omega}\right)-\phi\left(\frac{m}{\omega}\right)\left(\psi_0+\psi_1\frac{m}{\omega}+\psi_2\left(\frac{m}{\omega}\right)^2+\psi_3\left(\frac{m}{\omega}\right)^3+\psi_5\left(\frac{m}{\omega}\right)^5\right),$$

where the coefficients ψ_i, $i=0,...,5$ *are as in Theorem 1 and the Edgeworth coefficients are given in Appendix C3.*

Applying Lemma 1 we get the first four approximate cumulants, up to $o\left(n^{-1}\right)$, of $\sqrt{n}\left(\widetilde{\theta}-\theta\right)$ as

$$k_1^{\widetilde{\theta}}=\frac{2\theta-1}{\sqrt{n}},$$

$$k_2^{\widetilde{\theta}}=\omega_{\widetilde{\theta}}^2+\frac{1}{n}\left(\theta+6\right)\left(2-\theta\right)+\frac{1}{n}\xi_2^{\widetilde{\theta}},$$

where $\omega_{\widetilde{\theta}}^2=1-\theta^2$, and $\xi_2^{\widetilde{\theta}}=2\frac{\theta^2-\theta-1}{\theta^2-\theta+1}\frac{\theta\left(1-\theta^2\right)^2}{1+\theta^3}\kappa_3^2-\left(1-\theta^2\right)\kappa_4$.

$$k_3^{\widetilde{\theta}}=\frac{1}{\sqrt{n}}\frac{\left(1-\theta^2\right)^3}{1+\theta^3}\kappa_3^2 \quad \text{and} \quad k_4^{\widetilde{\theta}}=\frac{1}{n}6\left(1-\theta^2\right)\left(\theta^2+3\right)+\frac{1}{n}\xi_4^{\widetilde{\theta}},$$

where $\xi_4^{\widetilde{\theta}}=12\theta\frac{\theta-\theta^2-2}{1-\theta+\theta^2}\frac{\left(1-\theta^2\right)^3}{1+\theta^3}\kappa_3^2+\frac{\left(1-\theta^2\right)^3}{1+\theta^2}\kappa_4^2$.

It is worth noticing that the 3^{rd} approximate cumulant of $\widetilde{\theta}$ is positive even if the errors $u_t s$ are negatively skewed, whereas is symmetrically distributed for symmetric error distribution. Furthermore, $k_4^{\widetilde{\theta}}$ is an increasing function of κ_4^2. Consequently, for either platykurtic or leptokurtic error distribution, the distribution of $\widetilde{\theta}$ becomes platykurtic.

The second order approximate $AMSE$ of $\widetilde{\theta}$ is given by

$$E\left[\sqrt{n}\left(\widetilde{\theta}-\theta\right)\right]^2=\omega_{\widetilde{\theta}}^2+\frac{1}{n}\left[-8\theta+3\theta^2+13+2\frac{\theta^2-\theta-1}{\theta^2-\theta+1}\frac{\theta\left(1-\theta^2\right)^2}{1+\theta^3}\kappa_3^2-\left(1-\theta^2\right)\kappa_4\right]. \tag{9}$$

Notice that the $AMSE$ is a decreasing function of κ_4. This property of $\widetilde{\theta}$ is shared with $\widehat{\rho}$ and $\widehat{\theta}$, as well (see sections 3.1 and 3.3). Let us now proceed to the expansion of the QML MA coefficient when the mean is 0.

4.1.1 The Zero-Mean Expansion

Now for the case that $\mu = 0$, or known and subtracted from the data, we can repeat the procedure of section 4.1, appropriately modified (see Appendix C3). Notice that the derivatives with respect to g_1, w_{11} and q_{111}, and the cumulants of these variables remain the same. Further, as in the expansion of $\widehat{\rho}_0$, all Edgeworth coefficients are the same as in the non-zero mean case apart from $a_4^{(1)}$, $a_5^{(1)}$, $a_7^{(1)}$, and $a_9^{(1)}$, which are presented in Appendix C3.

In terms of cumulants, from Lemma 1, we have that the first order approximate cumulant, up to $o\left(n^{-1}\right)$, of $\sqrt{n}\left(\widetilde{\theta}_0 - \theta\right)$ is

$$k_1^{\widetilde{\theta}_0} = \frac{1}{\sqrt{n}}\theta, \tag{10}$$

which is the same result as in Tanaka (1984), where the 1^{st} order expansion is presented, and Bao and Ullah (2007). Comparing with the non-zero mean case, it is obvious that estimating the mean increases the absolute approximate bias of the QML estimator of θ for $\theta \in (-1, 0.3)$, whereas for $\theta \in (0.34, 1)$ the approximate bias of $\widetilde{\theta}$ is less than that of $\widetilde{\theta}_0$.

Further, the up to $o\left(n^{-1}\right)$ 2^{nd} order cumulant of $\sqrt{n}\left(\widetilde{\theta}_0 - \theta\right)$ is

$$k_2^{\widetilde{\theta}_0} = k_2^{\widetilde{\theta}} - \frac{4}{n}\left(1 - \theta\right),$$

whereas the 3^{rd} and 4^{th} order approximate cumulants are the same as the ones of $\sqrt{n}\left(\widetilde{\theta} - \theta\right)$. This can be explained by the fact that these approximate cumulants do not depend on any of the Edgeworth coefficients that change in the zero mean case.

Finally, the second order $AMSE$ of $\sqrt{n}\left(\widetilde{\theta}_0 - \theta\right)$ is

$$E\left[\sqrt{n}\left(\widetilde{\theta}_0 - \theta\right)\right]^2 = E\left[\sqrt{n}\left(\widetilde{\theta} - \theta\right)\right]^2 - \frac{3\theta^2 - 8\theta + 5}{n}.$$

Comparing the above $AMSE$ with the $AMSE$ of $\widetilde{\theta}$ we can conclude that the $AMSE$ of the estimator of θ when we estimate the mean is higher than the one when the mean is zero and not estimated, for all $\theta \in (-1, 1)$. Let us now derive the expansion of the ρ $QMLE$.

4.2 The Expansion of the 1^{st} order Autocorrelation QMLE

Let us define the $QMLE$ of ρ as

$$\widetilde{\rho} = \frac{\widetilde{\theta}}{1+\widetilde{\theta}^2}.$$

In Appendix C4 we present the Edgeworth coefficients of the second order approximation of the distribution of $\sqrt{n}\left(\widetilde{\rho}-\rho\right)$. To find the approximate bias and $AMSE$ of $\sqrt{n}\left(\widetilde{\rho}-\rho\right)$, up to $o\left(n^{-1}\right)$, we can apply Lemma 1 and get

$$k_1^{\widetilde{\rho}} = -\frac{(1-\theta)\left(1+2\theta+3\theta^2\right)}{\sqrt{n}}\frac{\left(1-\theta^2\right)}{\left(1+\theta^2\right)^3}$$

and

$$E\left[\sqrt{n}\left(\widetilde{\rho}-\rho\right)\right]^2 = \frac{\left(1-\theta^2\right)^3}{\left(1+\theta^2\right)^4} + \frac{1}{n}\left(\xi_1^{\widetilde{\rho}}+\xi_2^{\widetilde{\rho}}\right), \tag{11}$$

where $\xi_1^{\widetilde{\rho}} = \frac{\left(10\theta+62\theta^2-4\theta^3-65\theta^4-14\theta^5+24\theta^6+7\right)\left(1-\theta^2\right)^2}{\left(1+\theta^2\right)^6}$ and $\xi_2^{\widetilde{\rho}} = 4\theta\frac{\theta-\theta^2-\theta^3+\theta^4-2}{(\theta+1)\left(-\theta+\theta^2+1\right)^2}\frac{\left(1-\theta^2\right)^4}{\left(1+\theta^2\right)^5}\kappa_3^2 - \frac{\left(1-\theta^2\right)^3}{\left(1+\theta^2\right)^4}\kappa_4$. We next concentrate on the expansion in the zero-mean case.

4.2.1 The Zero-Mean Case

For the zero mean case, all Edgeworth coefficients are the same as in the non-zero mean one, apart from $\omega^{(3)}$, $a_{11}^{(1)}$ and $a_{12}^{(1)}$ (see Appendix C4). Consequently, applying Lemma 1 and keeping terms up to order $O\left(n^{-1}\right)$, we can find the approximate bias of $\sqrt{n}\left(\widetilde{\rho}_0-\rho\right)$ as

$$k_1^{\widetilde{\rho}_0} = k_1^{\widetilde{\rho}} + \frac{1}{\sqrt{n}}\frac{(1-\theta)^2(1+\theta)}{\left(1+\theta^2\right)^2}. \tag{12}$$

It is obvious that the absolute values of the approximate bias of $\widetilde{\rho}_0$ is less than the one of $\widetilde{\rho}$.

In terms of $AMSE$ we have that, keeping relevant terms,

$$\begin{aligned} E\left[\sqrt{n}\left(\widetilde{\rho}_0-\rho\right)\right]^2 &= E\left[\sqrt{n}\left(\widetilde{\rho}-\rho\right)\right]^2 \\ &\quad -\frac{1}{n}\left(14\theta+87\theta^2-12\theta^3-70\theta^4-26\theta^5+31\theta^6+14\right)\frac{\left(1-\theta^2\right)^2}{\left(1+\theta^2\right)^6}. \end{aligned}$$

This is different from the non-zero mean case. However, notice that the asymmetry and kurtosis parameters, κ_3 and κ_4, have the same effect on the $AMSE$, for any values of θ in the admissible region. In fact, the $AMSE$ of $\widetilde{\rho}_0$ is always lower than the one of $\widetilde{\rho}$ for all $\theta \in (-1, 1)$. Of course, for higher values of n the two $AMSEs$ collapse to the common asymptotic variance. Let us now proceed with the comparisons between all estimators.

5 Comparing the Estimators

To compare all estimators in terms of bias and MSE we run a simulation exercise. We draw a random sample of $n \in \{50, 200\}$ observations from a non-central Student-t distribution with non-centrality parameter $\eta \in \{-1, 1\}$ and $\nu \in \{11, 20\}$ degrees of freedom. Notice that for these values of η and ν we have that $\kappa_3 \in \{\pm 0.400, \pm 0.17\}$ and $\kappa_4 \in \{1.250, 0.42\}$. For each random sample, we generate the $MA(1|\mu)$ process y_t for $\theta \in \{-0.9, -0.8, ..., 0.9\}$, $\mu = 5.0$ and $\sigma^2 = 1.0$. We evaluate $\widehat{\rho}$ and if the estimate is in the $(-0.5, 0.5)$ interval we estimate all estimators, otherwise we throw away the sample and draw another one. This will introduce some bias in the estimation of the biases and the $MSEs$ of the estimators, for which the closer θ is at the boundary of the admissible space the fiercer it will be. Furthermore, this will probably affect more the estimation of bias and MSE of the MM estimator of θ, as the maximization of the quasi likelihood is not restricted in any way. For each retained sample we evaluate the MM ($\widehat{\rho}$, $\widehat{\theta}$, and $\widehat{\mu}$), the QML ($\widetilde{\theta}$, $\widetilde{\rho}$ and $\widetilde{\mu}$) and the feasibly bias corrected estimators, i.e. when the estimated value of θ is employed for bias correction, employing the approximate bias formulae of the previous sections (see Iglesias and Phillips 2008, as well). We set the number of replications to 20000.

Only the results for $n \in \{50, 200\}$, $\eta = 1$ and $\nu \in \{11, 20\}$ are presented, as first, the results with $\eta = -1$ and $\nu \in \{11, 20\}$ are almost identical to the reported ones, and second for space considerations.

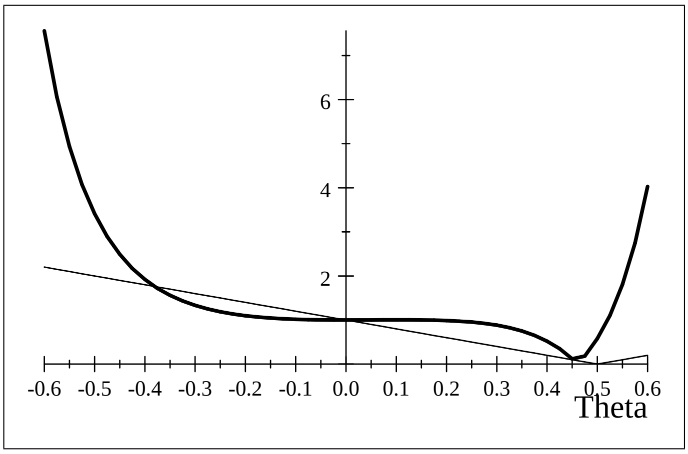

Figure 3: $\left|E\left[n\left(\widehat{\theta}-\theta\right)\right]\right|$ (thick line) and $\left|E\left[n\left(\widetilde{\theta}-\theta\right)\right]\right|$

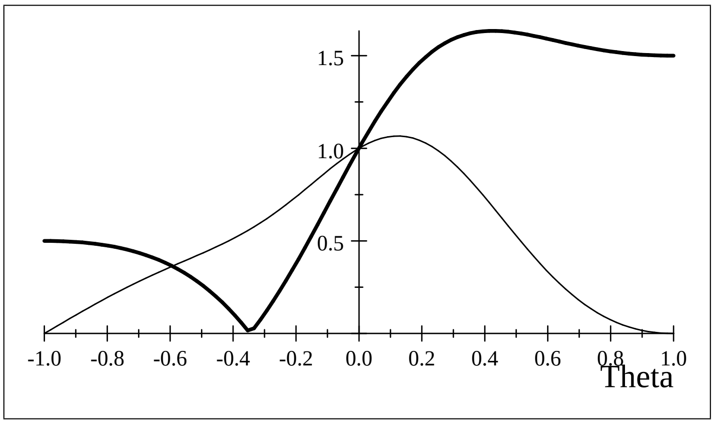

Figure 4: $\left|E\left[n\left(\widehat{\rho}-\rho\right)\right]\right|$ (thick line) and $\left|E\left[n\left(\widetilde{\rho}-\rho\right)\right]\right|$

5.1 Bias of the Estimators

On $o\left(n^{-1}\right)$ approximations grounds, it is apparent that, when μ is estimated, there are areas of the admissible region of θ that the MM estimators of either θ or ρ are less (approximately) biased than the $QMLEs$ (see Figure 3 and Figure 4). For example, for $-.3 \leq \theta \leq 0$, both $\widehat{\theta}$ and $\widehat{\rho}$ are less biased than $\widetilde{\theta}$ and $\widetilde{\rho}$, respectively. However, the opposite is true for $\theta \geq 0$.

In terms of the simulation results, the same is more-or-less true for the estimated values of the biases of $\widehat{\theta}$ and $\widetilde{\theta}$ (compare the 3^{rd} with the 6^{th} column of Table 1, for non-central Student-t with $\nu = 20$, and the same ones in Table 2, for $\nu = 11$). However, there are important differences between the two estimators. Regarding the MM estimator, the approximate biases are far away from the estimated ones for values of θ near the ends of the admissible parameter region. In fact, for θ lower than -0.4 (for $n = 50$) and -0.5 (for $n = 200$), the approximate bias continuously underestimates the estimated one. The opposite is true for θ higher than 0.5 for both samples. For $\theta = -0.9$ or $\theta = 0.9$, the under and over estimation is massive, respectively. On the other hand, regarding the $QMLE$, the estimated bias of $\widetilde{\theta}$ is higher than the approximate one for $\theta < 0.4$, when $n = 50$, and for $\theta < -0.4$, when $n = 200$. In terms of the bias corrected estimators, it is apparent that when the approximate biases are close to the estimated ones, the corrected estimators are, by all terms, unbiased. Furthermore, it seems that the decrease in the degrees of freedom affects the estimated bias of $\widehat{\theta}$ more than that of $\widetilde{\theta}$. This is an indication that the assumption $E\left(u_0^{10}\right)$ is more important for the MM estimator of θ than for the $QMLE$.

For the estimators of ρ (see Tables 3 and 4), the estimated biases of the feasibly corrected estimators of both estimators $\widehat{\rho}$ and $\widetilde{\rho}$ are less, in absolute value, from the equivalent ones of the estimated biases. Furthermore, the estimated biases of the feasibly corrected $\widehat{\rho}$ are less, in absolute values, than the ones of the feasibly corrected $\widetilde{\rho}$ when $\theta \in [-0.3, 0.0]$ for $n = 50$, and $\theta \in [-0.4, 0.0]$ for $n = 200$, which partly confirms Figure 4. It seems that near the ends of the admissible region of θ the approximate bias of $\widetilde{\rho}$ is more accurate as compared with the one of $\widehat{\rho}$, i.e. it is closer to the estimated bias. Finally, the decrease in the degrees of freedom of the distribution of the errors affects the bias results, of both estimators, only marginally.

However, for the zero-mean case notice that the $QMLEs$ of either θ or ρ are less (approximately) biased than the MM ones, for all $\theta \in (-1, 1)$. To see this, compare (7) with (10), and (5) with (12), respectively.

Hence, in terms of bias and when μ is estimated, for negative values of θ, but

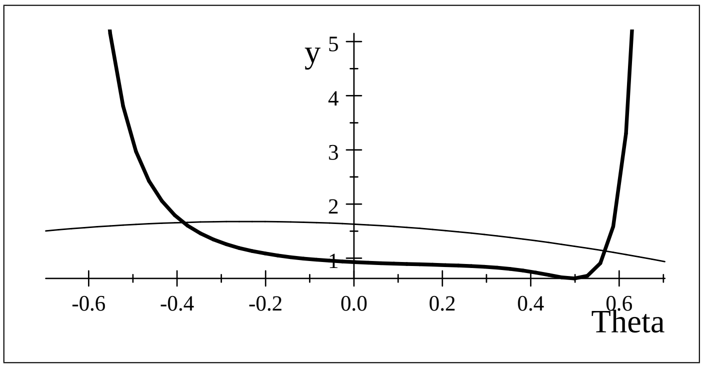

Figure 5: MSE of $\sqrt{20}\left(\widehat{\theta}-\theta\right)$ (thick) and $\sqrt{20}\left(\widetilde{\theta}-\theta\right)$, for $\kappa_3 = 0.17$ and $\kappa_4 = 0.42$.

close to 0, the approximations of $\widehat{\theta}$ and $\widehat{\rho}$ work better than those of $\widetilde{\theta}$ and $\widetilde{\rho}$, whereas for $\theta > 0$ or θ close to -1 the $QMLEs$ approximations are better.

5.2 MSE of Estimators

In terms of second order $AMSEs$, we plot the ones of the two estimators of θ in Figure 5 and the corresponding ones of the estimators of ρ in Figure 6. Notice that in both graphs we set $n = 20$ and in both cases μ is estimated. It is apparent that there is not uniform superiority of neither the $QMLEs$ nor the MM ones, over the whole range of the admissible values of θ. In fact, it seems that for $\theta \in (-0.3, 0.3)$, and for the above sample size, the MSE of the MM estimators are smaller than the ones of the $QMLEs$.

These findings can be explained by the following facts: i) the asymptotic variance of $\widetilde{\theta}$, $AV\left(\widetilde{\theta}\right)$, is less than or equal to $AV\left(\widehat{\theta}\right)$, a well known result, and the same is true for $AV\left(\widetilde{\rho}\right)$ and $AV\left(\widehat{\rho}\right)$. In fact, only for $\theta = 0$ $AV\left(\widetilde{\theta}\right) = AV\left(\widehat{\theta}\right)$ and $AV\left(\widetilde{\rho}\right) = AV\left(\widehat{\rho}\right)$, and we have strict inequality for all other values of θ. ii) For the $\frac{1}{n}$ terms, which do not include κ_3 or κ_4, and for $\theta \in (-0.5, 0.6)$ the term of $E\left[\sqrt{n}\left(\widehat{\theta}-\theta\right)\right]^2$ is lower than the one of $E\left[\sqrt{n}\left(\widetilde{\theta}-\theta\right)\right]^2$, for any sample size. The same is true for the equivalent terms of the estimators of ρ for $\theta \in (-0.8, 0.5)$. iii) For $\theta \in (-1, 0)$, $E\left[\sqrt{n}\left(\widehat{\theta}-\theta\right)\right]^2$ is a decreasing function of κ_3^2, whereas $E\left[\sqrt{n}\left(\widetilde{\theta}-\theta\right)\right]^2$,

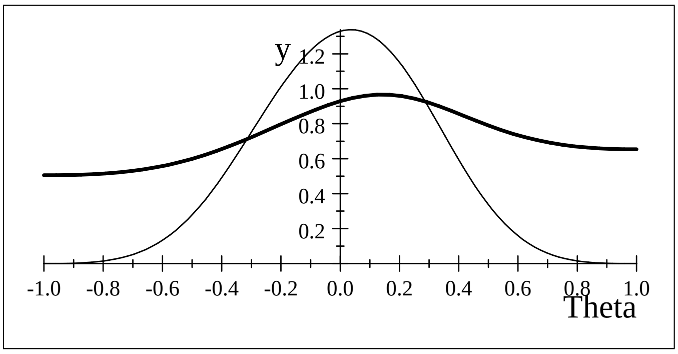

Figure 6: MSE of $\sqrt{20}\,(\widehat{\rho}-\rho)$ (thick) and $\sqrt{20}\,(\widetilde{\rho}-\rho)$, for $\kappa_3 = 0.17$ and $\kappa_4 = 0.42$.

$E\left[\sqrt{n}\,(\widehat{\rho}-\rho)\right]^2$ and $E\left[\sqrt{n}\,(\widetilde{\rho}-\rho)\right]^2$ are increasing functions of κ_3^2. The opposite is true for $\theta \in (0,1)$. iv) All $MSEs$ are decreasing functions of κ_4, for $\theta \in (-1,1)$. However, $E\left[\sqrt{n}\left(\widehat{\theta}-\theta\right)\right]^2$ and $E\left[\sqrt{n}\,(\widehat{\rho}-\rho)\right]^2$ are decreasing at a higher rate.

In terms of the simulations, it is immediately obvious that the $AMSEs$ are close to the estimated ones for the MM estimator of θ (see Tables 5 and 6) in the middle range of values of θ, and are massively higher than the estimated ones at the two ends of the admissible range. On the other hand, the estimated $MSEs$ of $\widetilde{\theta}$ are almost always underestimated by the approximate ones over the whole interval of θ. The underestimation is worse for values of θ less than -0.6 and higher than 0.6. For $n = 50$, the estimated MSE of $\widehat{\theta}$ is less than the one of $\widetilde{\theta}$ for $\theta \in (-0.1, 0.1)$, partially confirming Figure 5. The estimated $MSEs$ of the bias corrected $\widetilde{\theta}$ are less than the ones of $\widehat{\theta}$ for all values of θ apart for $\theta = 0$, and this is true for both sample sizes. By decreasing the degrees of freedom of the error distribution, the estimated $MSEs$ are lower for $\widehat{\theta}$ and higher for $\widetilde{\theta}$ (compare the 3^{rd} and 6^{th} columns of Table 5 with the respective ones of Table 6). This is in agreement with the approximate results for $\widehat{\theta}$ but not for $\widetilde{\theta}$. Finally, apart from the central part of the admissible range of θ, the MSE of the corrected $\widetilde{\theta}$ is almost always less than the one of $\widehat{\theta}$.

The estimated $MSEs$ of $\widehat{\rho}$ are close to the $AMSE$ ones (closer for $n = 200$ than for $n = 50$) and they are more so for $\theta \in (-0.6, 0.6)$ (see Table 7 and Table 8). The same is true for the $MSEs$ of $\widetilde{\rho}$. Comparing the $MSEs$ of $\widehat{\rho}$ with those of $\widetilde{\rho}$, for

$\nu = 20$ and for both sample sizes, it is apparent that the estimated $MSEs$ of $\widehat{\rho}$ are less than those of $\widetilde{\rho}$, for $\theta \in (-0.1, 0.1)$ partially confirming Figure 6. The same is true for the $MSEs$ of the two estimators, for 11 degrees of freedom. The biased corrected $\widetilde{\rho}$ has, more or less, a smaller MSE than the corrected $\widehat{\rho}$ and for both samples.

Hence, to conclude this section, we can say that in terms of MSE and for small sample size, the QML method is more efficient for the estimation of θ and ρ only for the interval $(-1.0, -0.6) \cup (0.0, 1.0)$.

6 Conclusions

This paper, by extending the results in Sargan (1976) and Tanaka (1984), derives the asymptotic expansions of the MM and QML estimators of the 1^{st} order autocorrelation and the MA parameter for the $MA(1)$ model. First, the second order Edgeworth and Nagar-type expansions of the MM estimators are derived in a more general setup of Sargan (1976) and second, the first order expansions in Tanaka (1984) are extended to include terms of second order for the QML ones. It is worth noticing that the second order approximate bias of all estimators is not affected by the non-normality of the errors. A comparison of the expansions, either in terms of approximate bias or $AMSE$, reveals that there is not uniform superiority of neither of the estimators of θ and ρ, something which is also confirmed by the simulation results. Furthermore, it seems that the approximations work well for the middle rage of the admissible values of θ, whereas when θ takes values near the two ends, -1 and $+1$, the approximation are very poor with the MM approximations being affected more the $QMLE$ ones. Finally, the approximate bias and $AMSE$ of the estimators depend on whether the mean of the process is known or estimated. In the zero-mean case, and on approximate grounds, the $QMLEs$ of θ and ρ are superior the MM ones in both approximate bias and $AMSE$ terms.

The results can be utilized to provide finer approximations of the distributions of the estimators, as compared to the asymptotically normal ones. In fact, the bias results were employed to correct the up to $O(n^{-1})$ bias of the estimators. It turned out that the feasibly corrected $\widetilde{\rho}$ is, almost always, less biased than $\widehat{\rho}$, for the whole interval of θ, without considerable alteration of its MSE. This indicates that the presented expansion works well for as small sample size as 50. On the other end, the approximation of $\widehat{\theta}$ works well only for values of θ close to 0, with even as much as 200 observations. The presented approximations of $\widetilde{\theta}$ and $\widetilde{\rho}$ are somewhere in the middle, i.e. work well for a large interval of values of θ. Furthermore, in the Indirect Inference literature, our results constitute an application of the general results in Arvanitis and Demos (2009).

The analysis presented here can be extended to any $ARMA(p, q|\mu)$ model. However, the algebra involved is becoming extremely tedious even for small values of p and q. Furthermore, one could consider the stochastic process $y_t = \mu + u_t + \theta_s u_{t-s}$, where $s = 1, 2, ...$ For specific values of s, this class of models could capture seasonal effects, e.g. for quarterly data $s = 4$, for monthly data $s = 12$, etc. (see e.g. Ghysels and Osborn 2001). In this case, the cumulants, at least up to 2^{nd} order, of the various statistics employed in sections 3 and 4 will become functions of s, complicating further the evaluations of the Edgeworth coefficients and the moments of the estimators.

Another interesting issue could be the expansion of the estimators as the parameter θ reaches the boundary of the admissible region, i.e. when $\theta \to \pm 1$ (in this respect see Andrews 1999, and Iglesias and Linton 2007). Furthermore, along the lines of Durbin (1959) and Gourieroux et al. (1993), the properties of the MM estimators can be improved by considering the expansions not only of the first order autocorrelation but higher order ones. Finally, one could, utilising the presented expansions, consider adjusted Box-Pierce tests along the lines of Kan and Wang (2010), or develop asymptotic expansions of the error variance estimators, as well, and consider expansions of various tests, e.g. Wald etc. We leave these issues for future research.

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Appendix A

Proof of Theorem 1

As the validity of Theorem 1 is dealt in Sargan (1976) or Bhattacharya and Ghosh (1978) we proceed with the coefficient derivation. Let us denote by $cf_{\overline{\varphi}}(s)$ the characteristic function of $\overline{\varphi}$. The Taylor series expansion of $\overline{\varphi}$ is:

$$\overline{\varphi} = \sum_{i=0}^{l} f^i A_i + \frac{1}{2}\sum_{i,j=0}^{l} f^{ij} A_i A_j + \frac{1}{6}\sum_{i,j,k=0}^{l} f^{ijk} A_i A_j A_k + o_p\left(n^{-1}\right)$$

where $f^i = \frac{\partial f}{\partial A_i}$, $f^{ij} = \frac{\partial^2 f}{\partial A_i \partial A_j}$, and $f^{ijk} = \frac{\partial^3 f}{\partial A_i \partial A_j \partial A_k}$, all evaluated at 0.

Adapting the summation convention, i.e. $f^{ij} A_i A_j = \sum_{i,j=0}^{l} f^{ij} A_i A_j$, the characteristic function of $\overline{\varphi}$ is:

$$cf_{\overline{\varphi}}(s) = \int \begin{bmatrix} \exp\left(isf^i A_i\right)\exp\left(\frac{is}{2} f^{ij} A_i A_j\right) \\ \exp\left(\frac{is}{6} f^{ij} A_i A_j A_k\right) \end{bmatrix} dF(A) + o\left(n^{-1}\right).$$

where $A = (A_0, ..., A_l)^{/}$.

Now expanding $\exp\left(\frac{is}{2} f^{ij} A_i A_j\right)$ and $\exp\left(\frac{is}{6} f^{ijk} A_i A_j A_k\right)$ around $(0, ..., 0)^{/}$ the characteristic function of $\overline{\varphi}$ becomes:

$$cf_{\overline{\varphi}}(s) = \int \left[\exp\left(isf^i A_i\right)\left(1 + \frac{is}{2\sqrt{n}} h^{ij} A_i A_j + \frac{is}{6n} h^{ijk} A_i A_j A_k - \frac{s^2}{8n}\left(h^{ij} A_i A_j\right)^2\right)\right] dF(A) + o\left(n^{-1}\right),$$

where $h^{ij} = \sqrt{n} f^{ij}$ and $h^{ijk} = n f^{ijk}$.

Setting $s\left(f^1, ..., f^l\right)^{/} = z$ and noticing that

$$\begin{aligned} \frac{\partial cf_A(z)}{\partial z_i} &= \int iA_i \exp\left(iz^{/} A\right) dF(A), \quad \frac{\partial^2 cf_A(z)}{\partial z_i \partial z_j} = -\int A_i A_j \exp\left(iz^{/} A\right) dF(A), \\ \frac{\partial^3 cf_A(z)}{\partial z_i \partial z_j \partial z_k} &= -\int iA_i A_j A_k \exp\left(iz^{/} A\right) dF(A) \quad \textit{and} \\ \frac{\partial^4 cf_A(z)}{\partial z_i \partial z_j \partial z_k \partial z_m} &= \int A_i A_j A_k A_m \exp\left(iz^{/} A\right) dF(A), \end{aligned}$$

we get

$$\begin{aligned} cf_{\overline{\varphi}}(s) &= cf_A(z) - \frac{is}{2\sqrt{n}} h^{ij} \frac{\partial^2 cf_A(z)}{\partial z_i \partial z_j} - \frac{s}{6n} h^{ijk} \frac{\partial^3 cf_A(z)}{\partial z_i \partial z_j \partial z_k} \qquad \text{(app-1)} \\ &\quad - \frac{s^2}{8n} h^{ij} h^{km} \frac{\partial^4 cf_A(z)}{\partial z_i \partial z_j \partial z_k \partial z_m} + o\left(n^{-1}\right) \end{aligned}$$

By definition, the characteristic function of A is:

$$cf_A(z) = \exp\left(ic_i z_i - \frac{1}{2}c_{ij}z_i z_j - \frac{i}{6}c_{ijk}z_i z_j z_k + \frac{1}{24}c_{ijkm}z_i z_j z_k z_m\right) + o\left(n^{-1}\right)$$

and expanding $\exp(ic_i z_i)$, $\exp\left(-\frac{i}{6}c_{ijk}z_i z_j z_k\right)$, $\exp\left(\frac{1}{24}c_{ijkm}z_i z_j z_k z_m\right)$ up to $o(n^{-1})$ we get

$$cf_A(z) = \exp\left(-\frac{1}{2}c_{ij}z_i z_j\right)\begin{pmatrix} 1 + ic_i z_i - \frac{1}{2}(c_i z_i)^2 + \frac{1}{24}c_{ijkm}z_i z_j z_k z_m \\ -\frac{i}{6}c_{ijk}z_i z_j z_k + \frac{1}{6}(c_{ijk}z_i z_j z_k)(c_i z_i) \\ -\frac{1}{72}(c_{ijk}z_i z_j z_k)^2 \end{pmatrix} + o\left(n^{-1}\right).$$

Employing the above formula we can find the, up to 4^{th} order, derivatives of the characteristic function. Substituting into (app-1) and setting for $z_i = sf^i$ we get:

$$\begin{aligned} cf_{\overline{\varphi}}(s) &= \exp\left(-\frac{s^2}{2}c_{ij}f^i f^j\right) \\ &\times \begin{pmatrix} 1 - \frac{is^3}{6}c_{ijk}f^i f^j f^k + \frac{s^4}{24}c_{ijkm}f^i f^j f^k f^m - \frac{s^6}{72}\left(c_{ijk}f^i f^j f^k\right)^2 \\ +\frac{s^4}{6}c_{ijk}f^i f^j f^k\left(c_i f^i\right) - \frac{s^2}{2}\left(c_i f^i\right)^2 + isc_j f^j \\ -\frac{is}{2}h^{pq}\begin{pmatrix} \frac{s^2}{\sqrt{n}}\left(c_{qj}f^j\right)\left(c_{pj}f^j\right) + \frac{i}{2}\frac{s^3}{\sqrt{n}}\left(c_{qj}f^j\right)\left(c_{pjk}f^j f^k\right) - \frac{1}{\sqrt{n}}c_{pq} \\ +\frac{i}{2}\frac{s^3}{\sqrt{n}}\left(c_{pj}f^j\right)\left(c_{qjk}f^j f^k\right) - \frac{i}{6}\frac{s^5}{\sqrt{n}}\left(c_{qj}f^j\right)\left(c_{pj}f^j\right)\left(c_{ijk}f^i f^j f^k\right) \\ -i\frac{s}{\sqrt{n}}\left(c_{pqk}f^k\right) + \frac{i}{6}\frac{s^3}{\sqrt{n}}c_{pq}\left(c_{ijk}f^i f^j f^k\right) + \frac{s^3}{\sqrt{n}}i\left(c_{qk}f^k\right)\left(c_{pj}f^j\right)\left(c_i f^i\right) \\ -\frac{s}{\sqrt{n}}ic_p\left(c_{iq}f^i\right) - \frac{s}{\sqrt{n}}ic_{pq}\left(c_i f^i\right) - \frac{s}{\sqrt{n}}ic_q\left(c_{pj}f^j\right) \end{pmatrix} \\ -\frac{s}{6}h^{pqr}\left(-\frac{s^3}{n}\left(c_{rj}f^j\right)\left(c_{qj}f^j\right)\left(c_{pj}f^j\right) + \frac{s}{n}\left[c_{qr}\left(c_{pj}f^j\right) + c_{pr}\left(c_{qj}f^j\right) + c_{pq}\left(c_{rj}f^j\right)\right]\right) \\ -\frac{s^2}{8}h^{pq}h^{rs}\begin{pmatrix} \frac{s^4}{n}\left(c_{sj}f^j\right)\left(c_{rj}f^j\right)\left(c_{qj}f^j\right)\left(c_{pj}f^j\right) \\ -\frac{s^2}{n}c_{sj}f^j\left[c_{qr}\left(c_{pj}f^j\right) + c_{pr}\left(c_{qj}f^j\right) + c_{pq}\left(c_{rj}f^j\right)\right] \\ -\frac{s^2}{n}\left[c_{rs}\left(c_{qj}f^j\right)\left(c_{pj}f^j\right) + c_{qs}\left(c_{rj}f^j\right)\left(c_{pj}f^j\right) + c_{ps}\left(c_{rj}f^j\right)\left(c_{qj}f^j\right)\right] \\ +\frac{1}{n}\left(c_{qr}c_{ps} + c_{pr}c_{qs} + c_{pq}c_{rs}\right) \end{pmatrix} \end{pmatrix} \end{aligned}$$

with a remainder of $o(n^{-1})$.

However as $c_{ij} = c_{ij}^{(1)} + n^{-1/2}c_{ij}^{(2)} + n^{-1}c_{ij}^{(3)} + o(n^{-1})$ there are terms of $O\left(n^{-1/2}\right)$ and $O(n^{-1})$, in the exponential. Consequently, we have that

$$\exp\left(-\frac{s^2}{2}c_{ij}f^i f^j\right) = \exp\left(-\frac{s^2 c_{ij}^{(1)}f^i f^j}{2}\right)\left(1 - \frac{s^2 c_{ij}^{(2)}f^i f^j}{2\sqrt{n}} - \frac{s^2 c_{ij}^{(3)}f^i f^j}{2n} + \frac{s^4\left(c_{ij}^{(2)}f^i f^j\right)^2}{8n}\right) + o\left(n^{-1}\right)$$

and it follows that, with the same order of remainder,

$$cf_{\overline{\varphi}}(s) = \exp\left(-\frac{s^2}{2}\omega^2\right) \times \left(\begin{array}{c} 1+\frac{is}{2\sqrt{n}}a_4^{(1)}+\frac{is}{2n}a_4^{(2)}-\frac{s^2}{2\sqrt{n}}\omega^{(2)}-\frac{s^2}{2n}\omega^{(3)}-\frac{s^2}{2n}h^{pq}\gamma_{pq}^{(1)} \\ -\frac{s^2}{6n}h^{pqr}\left(c_{qr}^{(1)}\gamma_p^{(1)}+c_{pr}^{(1)}\gamma_q^{(1)}+c_{pq}^{(1)}\gamma_r^{(1)}\right) \\ -\frac{s^2}{8n}h^{pq}h^{rs}\left(c_{qr}^{(1)}c_{ps}^{(1)}+c_{pr}^{(1)}c_{qs}^{(1)}+c_{pq}^{(1)}c_{rs}^{(1)}\right) \\ -\frac{is^3}{6\sqrt{n}}a_1^{(1)}-\frac{is^3}{6n}a_1^{(2)}-i\frac{s^3}{2\sqrt{n}}h^{pq}\gamma_q^{(1)}\gamma_p^{(1)}-i\frac{s^3}{2n}h^{pq}\gamma_q^{(2)}\gamma_p^{(1)}-i\frac{s^3}{2n}h^{pq}\gamma_q^{(1)}\gamma_p^{(2)} \\ -\frac{is^3}{4n}\omega^{(2)}a_4^{(1)}+\frac{s^4}{24n}a_2^{(1)}+\frac{s^4}{2n}h^{pq}\gamma_q^{(1)}\beta_p^{(1)}+\frac{s^4}{12n}a_4^{(1)}a_1^{(1)}+\frac{s^4}{6n}h^{pqr}\gamma_r^{(1)}\gamma_q^{(1)}\gamma_p^{(1)} \\ +\frac{s^4}{8n}\left(\omega^{(2)}\right)^2-\frac{s^6}{72n}\left(a_1^{(1)}\right)^2-\frac{s^6}{12n}a_1^{(1)}h^{pq}\gamma_q^{(1)}\gamma_p^{(1)}-\frac{s^6}{8n}h^{pq}h^{rs}\gamma_s^{(1)}\gamma_r^{(1)}\gamma_q^{(1)}\gamma_p^{(1)} \\ +\frac{is^5}{12n}\omega^{(2)}a_1^{(1)}+\frac{is^5}{4n}\omega^{(2)}h^{pq}\gamma_q^{(1)}\gamma_p^{(1)} \\ -\frac{s^2}{n}h^{pq}c_p^{(1)}\gamma_q^{(1)}+\frac{s^4}{8n}h^{pq}h^{rs}\left[\begin{array}{c} \gamma_s^{(1)}\left(c_{qr}^{(1)}\gamma_p^{(1)}+c_{pr}^{(1)}\gamma_q^{(1)}+c_{pq}^{(1)}\gamma_r^{(1)}\right) \\ +c_{rs}^{(1)}\gamma_q^{(1)}\gamma_p^{(1)}+c_{qs}^{(1)}\gamma_r^{(1)}\gamma_p^{(1)}+c_{ps}^{(1)}\gamma_r^{(1)}\gamma_q^{(1)} \end{array}\right] \\ +\frac{s^4}{6n}a_1^{(1)}a_{11}^{(1)}-\frac{s^2}{2n}\left(a_{11}^{(1)}\right)^2+\frac{is}{\sqrt{n}}a_{11}^{(1)}+\frac{is}{n}a_{11}^{(2)}-i\frac{s^3}{2n}\omega^{(2)}a_{11}^{(1)} \\ -\frac{s^2}{2n}a_4^{(1)}a_{11}^{(1)}+\frac{s^4}{2n}h^{pq}\gamma_q^{(1)}\gamma_p^{(1)}a_{11}^{(1)} \end{array}\right)$$

where

$$\begin{aligned} \omega^2 &= c_{ij}^{(1)}f^i f^j, \quad \omega^{(2)} = c_{ij}^{(2)}f^i f^j, \quad \omega^{(3)} = c_{ij}^{(3)}f^i f^j, \\ a_1^{(1)} &= c_{ijk}^{(1)}f^i f^j f^k, \quad a_1^{(2)} = c_{ijk}^{(2)}f^i f^j f^k, \quad a_2^{(1)} = c_{ijkm}^{(1)}f^i f^j f^k f^m, \\ \beta_p^{(1)} &= c_{pjk}^{(1)}f^j f^k, \quad \gamma_p^{(1)} = c_{pj}^{(1)}f^j, \quad \gamma_{pq}^{(1)} = c_{pqk}^{(1)}f^k, \quad \gamma_{pq}^{(2)} = c_{pqk}^{(2)}f^k, \\ a_4^{(1)} &= h^{pq}c_{pq}^{(1)}, \quad a_4^{(2)} = h^{pq}c_{pq}^{(2)}, \quad a_{11}^{(1)} = c_i^{(1)}f^i \quad \textit{and} \quad a_{11}^{(2)} = c_i^{(2)}f^i. \end{aligned}$$

As now $h^{pq}\gamma_q^{(2)}\gamma_p^{(1)} = h^{pq}\gamma_q^{(1)}\gamma_p^{(2)}$, $h^{pq}h^{rs}c_{qr}^{(1)}c_{ps}^{(1)} = h^{pq}h^{rs}c_{pr}^{(1)}c_{qs}^{(1)}$, $h^{pq}h^{rs}c_{pq}^{(1)}c_{rs}^{(1)} = \left(h^{pq}c_{pq}^{(1)}\right)^2$, $h^{pq}h^{rs}\gamma_s^{(1)}c_{pq}^{(1)}\gamma_r^{(1)} = h^{pq}h^{rs}c_{rs}^{(1)}\gamma_q^{(1)}\gamma_p^{(1)} = \left(h^{rs}c_{rs}^{(1)}\right)\left(h^{pq}\gamma_q^{(1)}\gamma_p^{(1)}\right)$, $h^{pq}h^{rs}\gamma_s^{(1)}c_{qr}^{(1)}\gamma_p^{(1)} = h^{pq}h^{rs}\gamma_s^{(1)}c_{pr}^{(1)}\gamma_q^{(1)} = h^{pq}h^{rs}c_{qs}^{(1)}\gamma_r^{(1)}\gamma_p^{(1)} = h^{pq}h^{rs}c_{ps}^{(1)}\gamma_r^{(1)}\gamma_q^{(1)}$, $h^{pq}h^{rs}\gamma_s^{(1)}\gamma_r^{(1)}\gamma_q^{(1)}\gamma_p^{(1)} = \left(h^{pq}\gamma_q^{(1)}\gamma_p^{(1)}\right)^2$, it follows that

$$cf_{\overline{\varphi}}(s) = \exp\left(-\frac{s^2}{2}\omega^2\right)\left(\begin{array}{c} 1+\left(\frac{1}{2\sqrt{n}}a_4^{(1)}+\frac{1}{2n}a_4^{(2)}+\frac{1}{\sqrt{n}}a_{11}^{(1)}+\frac{1}{n}a_{11}^{(2)}\right)is \\ -s^2\left(\begin{array}{c} \frac{1}{2n}a_5^{(1)}+\frac{1}{2n}a_7^{(1)}+\frac{1}{2\sqrt{n}}\omega^{(2)}+\frac{1}{2n}\omega^{(3)}+\frac{1}{4n}a_9^{(1)} \\ +\frac{1}{8n}\left(a_4^{(1)}\right)^2+\frac{1}{n}a_{12}^{(1)}+\frac{1}{2n}\left(a_{11}^{(1)}\right)^2+\frac{1}{2n}a_4^{(1)}a_{11}^{(1)} \end{array}\right) \\ -is^3\left[\frac{1}{6\sqrt{n}}a_1^{(1)}+\frac{1}{6n}a_1^{(2)}+\frac{1}{2\sqrt{n}}a_3^{(1)}+\frac{1}{n}a_3^{(2)}+\frac{\omega^{(2)}}{4n}\left(a_4^{(1)}+2a_{11}^{(1)}\right)\right] \\ +s^4\left(\begin{array}{c} \frac{1}{24n}a_2^{(1)}+\frac{1}{2n}a_{10}^{(1)}+\frac{1}{12n}a_4^{(1)}a_1^{(1)}+\frac{1}{6n}a_6^{(1)}+\frac{1}{2n}a_8^{(1)} \\ +\frac{1}{4n}a_3^{(1)}a_4^{(1)}+\frac{1}{8n}\left(\omega^{(2)}\right)^2+\frac{1}{6n}a_1^{(1)}a_{11}^{(1)}+\frac{1}{2n}a_3^{(1)}a_{11}^{(1)} \end{array}\right) \\ +is^5\left(\frac{1}{4n}\omega^{(2)}a_3^{(1)}+\frac{1}{12n}\omega^{(2)}a_1^{(1)}\right) \\ -s^6\left(\frac{1}{72n}\left(a_1^{(1)}\right)^2+\frac{1}{12n}a_1^{(1)}a_3^{(1)}+\frac{1}{8n}\left(a_3^{(1)}\right)^2\right) \end{array}\right)$$

where

$$\begin{aligned} a_3^{(1)} &= \gamma_p^{(1)}h^{pq}\gamma_q^{(1)}, \quad a_3^{(2)} = \gamma_p^{(1)}h^{pq}\gamma_q^{(2)}, \quad a_5^{(1)} = h^{pq}\gamma_{pq}^{(1)}, \quad a_6^{(1)} = h^{pqr}\gamma_p^{(1)}\gamma_r^{(1)}\gamma_q^{(1)}, \\ a_7^{(1)} &= h^{pqr}c_{pq}^{(1)}\gamma_r^{(1)}, \quad a_8^{(1)} = h^{pq}h^{rs}\gamma_s^{(1)}c_{pr}^{(1)}\gamma_q^{(1)}, \quad a_9^{(1)} = h^{pq}h^{rs}c_{qr}^{(1)}c_{ps}^{(1)}, \\ a_{10}^{(1)} &= \beta_p^{(1)}h^{pq}\gamma_q^{(1)}, \quad and \quad a_{12}^{(1)} = h^{pq}c_p^{(1)}\gamma_q^{(1)}. \end{aligned}$$

Inverting the characteristic function of $\overline{\varphi}$ term by term, we deduce the corresponding asymptotic expansion of the density, say $g(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp(-isx)\, cf_{\overline{\theta}}(s)\, ds$, and the probability function $G(m) = \Pr\left[\sqrt{n}\left(\widehat{\varphi}-\varphi\right) \leq m\right]$ as $n \to \infty$.

Now the probability function $G(m)$ is given as $G(m) = \Pr\left[\sqrt{n}\left(\widehat{\varphi}-\varphi\right) \leq m\right] = \int_{-\infty}^{m} g(x)dx$. Employing again the connection between the derivatives of the standard normal and the Hermite polynomials we get:

$$\begin{aligned} G(m) &= \Phi\left(\tfrac{m}{\omega}\right) - \left(\frac{1}{2\sqrt{n}}a_4^{(1)}+\frac{1}{2n}a_4^{(2)}+\frac{1}{\sqrt{n}}a_{11}^{(1)}+\frac{1}{n}a_{11}^{(2)}\right)\frac{1}{\omega}\phi\left(\tfrac{m}{\omega}\right) \\ &- \left(\begin{array}{c} \frac{1}{2n}a_5^{(1)}+\frac{1}{2n}a_7^{(1)}+\frac{1}{2\sqrt{n}}\omega^{(2)}+\frac{1}{2n}\omega^{(3)}+\frac{1}{4n}a_9^{(1)} \\ +\frac{1}{8n}\left(a_4^{(1)}\right)^2+\frac{1}{n}a_{12}^{(1)}+\frac{1}{2n}\left(a_{11}^{(1)}\right)^2+\frac{1}{2n}a_4^{(1)}a_{11}^{(1)} \end{array}\right)\frac{1}{\omega^2}H_1\left(\tfrac{m}{\omega}\right)\phi\left(\tfrac{m}{\omega}\right) \\ &- \left(\begin{array}{c} \frac{1}{6\sqrt{n}}a_1^{(1)}+\frac{1}{6n}a_1^{(2)}+\frac{1}{2\sqrt{n}}a_3^{(1)}+\frac{1}{n}a_3^{(2)} \\ +\frac{1}{4n}\omega^{(2)}a_4^{(1)}+\frac{1}{2n}\omega^{(2)}a_{11}^{(1)} \end{array}\right)\frac{1}{\omega^3}H_2\left(\tfrac{m}{\omega}\right)\phi\left(\tfrac{m}{\omega}\right) \\ &- \left(\begin{array}{c} \frac{1}{24n}a_2^{(1)}+\frac{1}{2n}a_{10}^{(1)}+\frac{1}{12n}a_4^{(1)}a_1^{(1)}+\frac{1}{6n}a_6^{(1)}+\frac{1}{2n}a_8^{(1)}+\frac{1}{4n}a_3^{(1)}a_4^{(1)} \\ +\frac{1}{8n}\left(\omega^{(2)}\right)^2+\frac{1}{6n}a_1^{(1)}a_{11}^{(1)}+\frac{1}{2n}a_3^{(1)}a_{11}^{(1)} \end{array}\right)\frac{1}{\omega^4}H_3\left(\tfrac{m}{\omega}\right)\phi\left(\tfrac{m}{\omega}\right) \\ &- \left(\frac{1}{4n}\omega^{(2)}a_3^{(1)}+\frac{1}{12n}\omega^{(2)}a_1^{(1)}\right)\frac{1}{\omega^5}H_4\left(\tfrac{m}{\omega}\right)\phi\left(\tfrac{m}{\omega}\right) \\ &- \left(\frac{1}{72n}\left(a_1^{(1)}\right)^2+\frac{1}{12n}a_1^{(1)}a_3^{(1)}+\frac{1}{8n}\left(a_3^{(1)}\right)^2\right)\frac{1}{\omega^6}H_5\left(\tfrac{m}{\omega}\right)\phi\left(\tfrac{m}{\omega}\right)dx + o\left(n^{-1}\right), \end{aligned}$$

where $H_i\left(\frac{m}{\omega}\right)$ the i^{th} order Hermite polynomial. Substituting the values of these polynomials we get the Edgeworth approximation of the distribution function of $\sqrt{n}\left(\widehat{\varphi}-\varphi\right)$, written compactly, as:

$$G(m)=\Phi\left(\frac{m}{\omega}\right)-\phi\left(\frac{m}{\omega}\right)\left[\psi_0+\psi_1\left(\frac{m}{\omega}\right)+\psi_2\left(\frac{m}{\omega}\right)^2+\psi_3\left(\frac{m}{\omega}\right)^3+\psi_4\left(\frac{m}{\omega}\right)^4+\psi_5\left(\frac{m}{\omega}\right)^5\right],$$

where

$$\psi_0=\tfrac{1}{\sqrt{n}}\psi_0^{(1)}+\tfrac{1}{n}\psi_0^{(2)},\quad \psi_1=\tfrac{1}{\sqrt{n}}\psi_1^{(1)}+\tfrac{1}{n}\psi_1^{(2)},\quad \psi_2=\tfrac{1}{\sqrt{n}}\psi_2^{(1)}+\tfrac{1}{n}\psi_2^{(2)},$$

$$\psi_3=\tfrac{1}{n}\psi_3^{(2)},\quad \psi_5=\tfrac{1}{72n}\frac{\left(a_1^{(1)}+3a_3^{(1)}\right)^2}{\omega^6},\quad \psi_4=\tfrac{1}{12n}\frac{\omega^{(2)}\left(a_1^{(1)}+3a_3^{(1)}\right)}{\omega^5},$$

$$\psi_0^{(1)}=\tfrac{1}{6\omega}\left\{3\left(a_4^{(1)}+2a_{11}^{(1)}\right)-\frac{\left(a_1^{(1)}+3a_3^{(1)}\right)}{\omega^2}\right\},$$

$$\psi_0^{(2)}=\tfrac{1}{6\omega}\left\{3\left[a_4^{(2)}+2a_{11}^{(2)}\right]-\frac{a_1^{(2)}+6a_3^{(2)}+\frac{3}{2}\omega^{(2)}a_4^{(1)}+3\omega^{(2)}a_{11}^{(1)}}{\omega^2}+\frac{3}{2}\frac{\omega^{(2)}\left(a_1^{(1)}+3a_3^{(1)}\right)}{\omega^4}\right\},$$

$$\psi_1^{(1)}=\frac{\omega^{(2)}}{2\omega^2},\quad \psi_1^{(2)}=\frac{1}{24\omega^2}\left\{\begin{array}{c}3\left[4\left(a_5^{(1)}+a_7^{(1)}+\omega^{(3)}+2a_{12}^{(1)}\right)+2a_9^{(1)}+\left(2a_{11}^{(1)}+a_4^{(1)}\right)^2\right]\\ -3\varsigma+5\frac{\left(a_1^{(1)}+3a_3^{(1)}\right)^2}{\omega^4}\end{array}\right\},$$

$$\psi_2^{(1)}=\frac{\left(a_1^{(1)}+3a_3^{(1)}\right)}{6\omega^3},\quad \psi_2^{(2)}=\frac{1}{6\omega^3}\left[\begin{array}{c}a_1^{(2)}+6a_3^{(2)}+\frac{3}{2}\omega^{(2)}a_4^{(1)}+3\omega^{(2)}a_{11}^{(1)}\\ -3\frac{\omega^{(2)}\left(a_1^{(1)}+3a_3^{(1)}\right)}{\omega^2}\end{array}\right],$$

$$\psi_3^{(2)}=-\frac{1}{72\omega^2}\left[10\frac{\left(a_1^{(1)}+3a_3^{(1)}\right)^2}{\omega^4}-3\varsigma\right],\text{ and }\varsigma=\frac{a_2^{(1)}+2\left(a_4^{(1)}+2a_{11}^{(1)}\right)\left(a_1^{(1)}+3a_3^{(1)}\right)+4a_6^{(1)}+12\left(a_8^{(1)}+a_{10}^{(1)}\right)+3\left(\omega^{(2)}\right)^2}{\omega^2}.$$

In Sargan (1976) we have that $\omega^{(2)}=\omega^{(3)}=\gamma_{pq}^{(2)}=0$, $a_i^{(2)}=0$ for $i=1,2,3,4$ and $a_{11}^{(1)}=a_{11}^{(2)}=a_{12}^{(1)}=0$. Under these assumptions our coefficients become identical to the ones in Sargan (1988) (the corrected version of the 1976 paper).

Proof of Lemma 1

To easy the notation, let $w=\frac{m}{\omega}$. Then we would like to find $d_0^{(1)}$, $d_1^{(1)}$, $d_2^{(1)}$, and $d_0^{(2)}$, $d_1^{(2)}$, $d_2^{(2)}$, and $d_3^{(2)}$ such that

$$\begin{aligned}&\Phi(w)-\phi(w)\left[\begin{array}{c}\left(\psi_0^{(1)}+\psi_1^{(1)}w+\psi_2^{(1)}w^2\right)y\\ +\left(\psi_0^{(2)}+\psi_1^{(2)}w+\psi_2^{(2)}w^2+\psi_3w^3+\psi_4w^4+\psi_5w^5\right)y^2\end{array}\right]\\ =\;&\Phi\left[w+\left(d_0^{(1)}+d_1^{(1)}w+w^2d_2^{(1)}\right)y+\left(d_0^{(2)}+d_1^{(2)}w+w^2d_2^{(2)}+d_3^{(2)}w^3\right)y^2\right]+o\left(n^{-1}\right)\end{aligned}$$

where $y=\frac{1}{\sqrt{n}}$. Employing a Taylor series expansion of the right-hand side around $y=0$ and equating terms of the same order of y we get:

$$\begin{aligned} d_0^{(1)} &= -\psi_0^{(1)}, \quad d_0^{(2)}=-\psi_0^{(2)}, \quad d_1^{(1)}=-\psi_1^{(1)}, \quad d_1^{(2)}=-\psi_1^{(2)}+\frac{1}{2}\left(\psi_0^{(1)}\right)^2 \\ d_2^{(1)} &= -\psi_2^{(1)}, \quad d_2^{(2)}=-\psi_2^{(2)}+\psi_0^{(1)}\psi_1^{(1)}, \\ d_3^{(2)} &= -\psi_3+\frac{1}{2}\left(\psi_1^{(1)}\right)^2+\psi_0^{(1)}\psi_2^{(1)}, \end{aligned}$$

and

$$\psi_4=d_2^{(1)}d_1^{(1)}, \quad \textit{and} \quad \psi_5=\frac{1}{2}\left(d_2^{(1)}\right)^2$$

which are always true.

As $\Phi\left[w+\left(d_0^{(1)}+d_1^{(1)}w+w^2d_2^{(1)}\right)\frac{1}{\sqrt{n}}+\left(d_0^{(2)}+d_1^{(2)}w+w^2d_2^{(2)}+d_3^{(2)}w^3\right)\frac{1}{n}\right]+o\left(n^{-1}\right)$ one can find a standard normal variate, say z, such that $z=w+\left(d_0^{(1)}+d_1^{(1)}w+w^2d_2^{(1)}\right)\frac{1}{\sqrt{n}}+\left(d_0^{(2)}+d_1^{(2)}w+w^2d_2^{(2)}+d_3^{(2)}w^3\right)\frac{1}{n}+o\left(n^{-1}\right)$.

Let $w=a+bz+cz^2+ez^3+o\left(n^{-1}\right)$ where the coefficients a, b, c, and e are to be determined. Then substituting out z, by employing the above formula, letting $a=a^{(0)}+\frac{1}{\sqrt{n}}a^{(1)}+\frac{1}{n}a^{(2)}$ and the same for b, c, and e, and equating coefficients we get a, b, c, and e as functions of the $d_i^{(j)}s$. Hence
$w=-\frac{1}{\sqrt{n}}d_0^{(1)}+\frac{1}{n}\left(d_0^{(1)}d_1^{(1)}-d_0^{(2)}\right)+\left(1-\frac{1}{\sqrt{n}}d_1^{(1)}+\frac{1}{n}\left(2d_2^{(1)}d_0^{(1)}+\left(d_1^{(1)}\right)^2-d_1^{(2)}\right)\right)z$ $+\left(-\frac{1}{\sqrt{n}}d_2^{(1)}+\frac{1}{n}\left(3d_1^{(1)}d_2^{(1)}-d_2^{(2)}\right)\right)z^2+\frac{1}{n}\left(2\left(d_2^{(1)}\right)^2-d_3^{(2)}\right)z^3+o\left(n^{-1}\right)$ or in terms of the $\psi_j^{(i)}s$

$$\begin{aligned} w &= \frac{1}{\sqrt{n}}\psi_0^{(1)}+\frac{1}{n}\left(\psi_0^{(1)}\psi_1^{(1)}+\psi_0^{(2)}\right) \\ &+\left(1+\frac{1}{\sqrt{n}}\psi_1^{(1)}+\frac{1}{n}\left(2\psi_2^{(1)}\psi_0^{(1)}+\left(\psi_1^{(1)}\right)^2+\psi_1^{(2)}-\frac{1}{2}\left(\psi_0^{(1)}\right)^2\right)\right)z \\ &+\left(\frac{1}{\sqrt{n}}\psi_2^{(1)}+\frac{1}{n}\left(3\psi_1^{(1)}\psi_2^{(1)}+\psi_2^{(2)}-\psi_0^{(1)}\psi_1^{(1)}\right)\right)z^2 \\ &+\frac{1}{n}\left(2\left(\psi_2^{(1)}\right)^2+\psi_3^{(2)}-\frac{1}{2}\left(\psi_1^{(1)}\right)^2-\psi_0^{(1)}\psi_2^{(1)}\right)z^3+o\left(n^{-1}\right). \end{aligned}$$

Hence, employing the connection between the $\psi_j^{(i)}s$ and the Edgeworth coefficients, $a_l^{(k)}$, setting $w=\overline{\varphi}$ we get the results of Lemma 1.

Appendix B1 cumulants needed for $\widehat{\rho}$

A_0, A_1, A_2, and A_3 can be expressed as $A_0 = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left(u_{t-1}+\theta u_{t-2}\right)$, $A_1 = \frac{\sum_{t=2}^{n}u_{t-1}u_{t-2}}{\sqrt{n}}$

$$A_2 = \frac{1}{\sqrt{n}}\left[\begin{array}{c}\theta u_1 u_{-1} + \theta^2 u_0 u_{-1} + \theta\sum_{t=2}^{n}u_t u_{t-2} + u_n u_{n-1} - \theta\frac{-\theta^2\left(u_{n-1}^2-\sigma^2\right)+2\theta u_0 u_{-1}+\theta^2\left(u_{-1}^2-\sigma^2\right)}{\left(1+\theta^2\right)} \\ +\frac{1}{n}\left[\left(u_0+\theta u_{-1}\right)-u_n-\theta u_{n-1}\right]\left[\left(1+\theta\right)\sum_{t=1}^{n}u_{t-1}-\theta u_{n-1}+\theta u_{-1}\right]\end{array}\right]$$

and $A_3 = \frac{\sum_{t=1}^{n}\left(u_{t-1}^2-\sigma^2\right)+\frac{-\theta^2\left(u_{n-1}^2-\sigma^2\right)+2\theta u_0 u_{-1}+\theta^2\left(u_{-1}^2-\sigma^2\right)}{\left(1+\theta^2\right)}}{\sqrt{n}}$.

It is obvious that

$$E\left(A_0\right) = E\left(A_1\right) = E\left(A_3\right) = 0, \quad E\left(A_2\right) = o\left(n^{-1}\right),$$

and consequently

$$c_0 = c_1 = c_3 = 0, \quad and \quad c_2 = o\left(n^{-1}\right).$$

Hence

$$c_i^{(1)} = c_i^{(2)} = 0 \quad for \quad i = 0,1,2,3$$

In terms of second moments, notice that $E\left(A_0^2\right) = \left(1+\theta\right)^2\sigma^2 - \frac{2}{n}\theta\sigma^2 + o\left(n^{-1}\right)$, $E\left(A_1^2\right) = \sigma^4 - \frac{\sigma^4}{n} + o\left(n^{-1}\right)$ $E\left(A_2^2\right) = \theta^2\sigma^4 + \frac{1}{n}\left[2\frac{\theta^6}{\left(\theta^2+1\right)^2}\kappa_4 + \sigma^4\left(\theta^4+1\right)\right] + o\left(n^{-1}\right)$ and $E\left(A_3^2\right) = \sigma^4\kappa_4 + 2\sigma^4 - 2\frac{1}{n}\frac{\theta^2}{\left(1+\theta^2\right)^2}\kappa_4 + o\left(n^{-1}\right)$ where $\kappa_4 = \frac{E\left(u_0^4\right)-3\sigma^4}{\sigma^4}$. Furthermore, $E\left(A_0A_1\right) = 0$, $E\left(A_0A_2\right) = \frac{1}{n}\theta^3\frac{1-\theta}{\theta^2+1}\kappa_3\sigma^3 + o\left(n^{-1}\right)$, $E\left(A_0A_3\right) = \left(1+\theta\right)\kappa_3\sigma^3 - \frac{1}{n}\theta\frac{\theta+1}{\theta^2+1}\kappa_3\sigma^3 + o\left(n^{-1}\right)$, $E\left(A_1A_2\right) = o\left(n^{-1}\right)$, $E\left(A_1A_3\right) = 0$ and $E\left(A_2A_3\right) = \frac{1}{n}\frac{\theta^3\left(1-\theta^2\right)}{\left(1+\theta^2\right)^2}\sigma^4\kappa_4 + o\left(n^{-1}\right)$ where $\kappa_3 = \frac{E\left(u_0^3\right)}{\sigma^3}$.

Hence

$$\begin{aligned}
c_{00} &= \left(1+\theta\right)^2\sigma^2 - \frac{2}{n}\theta\sigma^2, \quad c_{01} = 0, \quad c_{02} = \frac{1}{n}\theta^3\frac{1-\theta}{\theta^2+1}\kappa_3\sigma^3 + o\left(n^{-1}\right) \\
c_{03} &= \left(1+\theta\right)\kappa_3\sigma^3 - \frac{1}{n}\theta\frac{\theta+1}{\theta^2+1}\kappa_3\sigma^3 + o\left(n^{-1}\right), \quad c_{11} = \sigma^4 - \frac{\sigma^4}{n}, \quad c_{12} = 0 + o\left(n^{-1}\right), \quad c_{13} = 0, \\
c_{22} &= \theta^2\sigma^4 + \frac{1}{n}\left[2\frac{\theta^6}{\left(\theta^2+1\right)^2}\kappa_4 + \left(\theta^4+1\right)\right]\sigma^4 + o\left(n^{-1}\right), \quad c_{23} = \frac{1}{n}\frac{\theta^3\left(1-\theta^2\right)}{\left(1+\theta^2\right)^2}\sigma^4\kappa_4 + o\left(n^{-1}\right) \\
c_{33} &= \sigma^4\left(\kappa_4+2\right) - 2\frac{1}{n}\frac{\theta^2}{\left(1+\theta^2\right)^2}\sigma^4\kappa_4 + o\left(n^{-1}\right).
\end{aligned}$$

For the cubes, $E\left(A_0^3\right) = \frac{1}{\sqrt{n}}\left(1+\theta\right)^3\sigma^3\kappa_3 + o\left(n^{-1}\right)$, $E\left(A_1^3\right) = \frac{1}{\sqrt{n}}\sigma^6\kappa_3^2 + o\left(n^{-1}\right)$, $E\left(A_2^3\right) = \frac{1}{\sqrt{n}}\theta^3\sigma^6\kappa_3^2 + o\left(n^{-1}\right)$, $E\left(A_0^2A_1\right) = \frac{2}{\sqrt{n}}\left(1+\theta\right)^2\sigma^4 + o\left(n^{-1}\right)$, $E\left(A_0^2A_2\right) =$

$\frac{1}{\sqrt{n}}2(1+\theta)^2\theta\sigma^4+o(n^{-1})$, $E(A_1^2A_2)=2\frac{1}{\sqrt{n}}\theta\sigma^6+o(n^{-1})$, $E(A_1^2A_3)=\frac{1}{\sqrt{n}}2\sigma^2V(u_0^2)+o(n^{-1})$, $E(A_1A_2^2)=o(n^{-1})$, $E(A_2^2A_3)=\frac{2}{\sqrt{n}}\theta^2\sigma^2V(u_0^2)+o(n^{-1})$ and $E(A_1A_2A_3)=o(n^{-1})\,m$

Now, as $E(A_j)=0$ for all js, we have, up to $o(n^{-1})$,

$$\begin{aligned} c_{000} &= \frac{1}{\sqrt{n}}(1+\theta)^3\sigma^3\kappa_3,\ c_{001}=\frac{2}{\sqrt{n}}(1+\theta)^2\sigma^4,\ c_{002}=\frac{1}{\sqrt{n}}2(1+\theta)^2\theta\sigma^4,\ c_{111}=\frac{1}{\sqrt{n}}\sigma^6\kappa_3^2, \\ c_{112} &= \frac{2\theta\sigma^6}{\sqrt{n}},\ c_{113}=\frac{2\sigma^2}{\sqrt{n}}(\kappa_4+2),\ c_{122}=c_{123}=0,\ c_{222}=\frac{\theta^3\sigma^6}{\sqrt{n}}\kappa_3^2,\ c_{223}=\frac{2\theta^2\sigma^6}{\sqrt{n}}(\kappa_4+2). \end{aligned}$$

With some tedious algebra we get:
$E(A_1^4)=3\sigma^8+\frac{\sigma^8(\kappa_4^2+12\kappa_4+12)}{n}+o(n^{-1})$, $E(A_1A_2^3)=o(n^{-1})$, $E(A_1^3A_2)=\frac{1}{n}6\theta\sigma^8\kappa_3^2+o(n^{-1})$, $E(A_2^4)=3\theta^4\sigma^8+\frac{1}{n}\sigma^8\left[\theta^4\kappa_4^2+12\theta^4\kappa_4\frac{2\theta^2+2\theta^4+1}{(\theta^2+1)^2}+6\theta^2\left(3\theta^2+\theta^4+1\right)\right]$, $E(A_2^2A_1^2)=\theta^2\sigma^8+\frac{1}{n}\sigma^8\left[\frac{13\theta^2+24\theta^4+21\theta^6+\theta^8+1}{(\theta^2+1)^2}+2\theta^2\kappa_3^2+2\frac{(4\theta^2+3\theta^4+2)\theta^2}{(\theta^2+1)^2}\kappa_4\right]+o(n^{-1})$, and $E(A_0^4)=3(1+\theta)^4\sigma^4+\frac{1}{n}\left((1+\theta)^4E(u_0^4)-3(1+\theta)^4\sigma^4+12(1+\theta)^2\sigma^4-12(1+\theta)^3\sigma^4\right)+o(n^{-1})$.

Please see Technical Appendix (TA) for detailed proof (www.aueb.gr/users/demos/WorkingPapers/MA-TA.pdf).

Consequently, and due to zero mean we get

$$\begin{aligned} c_{0000} &= \frac{(1+\theta)^4\sigma^4}{n}\kappa_4,\quad c_{1111}=\frac{\sigma^8}{n}\left(\kappa_4^2+12\kappa_4+18\right),\quad c_{2222}=\frac{\sigma^8}{n}\left(\theta^4\kappa_4^2+12\theta^4\kappa_4+18\theta^4\right) \\ c_{1122} &= \frac{\sigma^8}{n}\left(4\theta^2\kappa_4+2\theta^2\kappa_3^2+12\theta^2\right),\quad c_{1112}=\frac{6\theta\sigma^8}{n}\kappa_3^2,\quad c_{1222}=0 \end{aligned}$$

with an error of order $o(n^{-1})$.

Appendix B2 Expansion of $\widehat{\rho}$

As the validity of the approximation is established in Kakizawa (1999), let us concentrate on deriving the Edgeworth coefficients. As $\sqrt{n}(\widehat{\rho}-\rho)=f(A_0,A_1,A_2,A_3)$ by (2), the first derivatives evaluated at 0 are $f^0=0$, $f^1=\frac{(1+\theta^4)}{(1+\theta^2)^2\sigma^2}$, $f^2=\frac{1}{(1+\theta^2)\sigma^2}$, and $f^3=0$. The non-zero second order derivatives, evaluated at 0, are $f^{00}=\frac{1}{\sqrt{n}}\frac{-2(1-\theta+\theta^2)}{(1+\theta^2)^2\sigma^2}$, $f^{11}=-\frac{1}{\sqrt{n}}4\theta\frac{(1+\theta^4)}{(1+\theta^2)^3\sigma^4}$, $f^{12}=-\frac{1}{\sqrt{n}}\frac{2\theta}{(1+\theta^2)^2\sigma^4}$, $f^{13}=-\frac{1}{\sqrt{n}}\frac{(1+\theta^4)}{(1+\theta^2)^2\sigma^4}$, $f^{23}=-\frac{1}{\sqrt{n}}\frac{1}{(1+\theta^2)\sigma^4}$. Consequently, $h^{ij}=\sqrt{n}f^{ij}$, e.g. $h^{00}=\frac{-2(1-\theta+\theta^2)}{(1+\theta^2)^2\sigma^2}$, etc. Finally, the non-zero third order derivatives, evaluated at 0, are $f^{001} = \frac{1}{n}2\frac{\left(2\theta-2\theta^2+2\theta^3+\theta^4+1\right)}{\left(\theta^2+1\right)^3\sigma^4}$, $f^{002} = \frac{1}{n}\frac{2}{\left(1+\theta^2\right)^2\sigma^4}$, $f^{003} = \frac{1}{n}2\frac{\left(1-\theta+\theta^2\right)}{\left(1+\theta^2\right)^2\sigma^4}$, $f^{111} = \frac{1}{n}24\frac{\theta^2\left(1+\theta^4\right)}{\left(1+\theta^2\right)^4\sigma^6}$, $f^{112} = \frac{1}{n}\frac{8\theta^2}{\left(1+\theta^2\right)^3\sigma^6}$, $f^{113} = \frac{1}{n}\frac{8\theta\left(1+\theta^4\right)}{\left(1+\theta^2\right)^3\sigma^6}$, $f^{123} = 4\frac{1}{n}\frac{\theta}{\left(1+\theta^2\right)^2\sigma^6}$, $f^{133} = 2\frac{1}{n}\frac{\left(1+\theta^4\right)}{\left(1+\theta^2\right)^2\sigma^6}$, $f^{233} = \frac{1}{n}\frac{2}{\left(1+\theta^2\right)\sigma^6}$, whereas $h^{ijk} = nf^{ijk}$, e.g. $h^{001} = 2\frac{\left(2\theta-2\theta^2+2\theta^3+\theta^4+1\right)}{\left(\theta^2+1\right)^3\sigma^4}$ etc.

Now from Theorem 1 (Appendix A) we have $\omega^2 = \frac{\theta^2+4\theta^4+\theta^6+\theta^8+1}{\left(1+\theta^2\right)^4}$, the asymptotic variance of $\sqrt{T}\left(\widehat{\rho}-\rho\right)$. Further $\omega^{(2)} = 0$ and $\omega^{(3)} = \frac{2\theta^2\left(\theta^4+1\right)}{\left(1+\theta^2\right)^4} + 2\frac{\theta^6}{\left(\theta^2+1\right)^4}\kappa_4$. Next $a_1^{(1)} = \frac{6\theta\left(1+\theta^4\right)^2}{\left(1+\theta^2\right)^5} + \frac{\left(1+\theta^4\right)^3+\theta^3\left(1+\theta^2\right)^3}{\left(1+\theta^2\right)^6}\kappa_3^2$, $a_2^{(1)} = \kappa_4^2\frac{5\theta^4+4\theta^6+12\theta^8+4\theta^{10}+5\theta^{12}+\theta^{16}+1}{\left(\theta^2+1\right)^8} + 12\kappa_4\frac{\left(\theta^2+4\theta^4+\theta^6+\theta^8+1\right)^2}{\left(\theta^2+1\right)^8} + 12\theta\left(\theta+\theta^3+2\theta^4+2\right)\frac{\left(\theta^4+1\right)^2}{\left(\theta^2+1\right)^7}\kappa_3^2 + 18\frac{4\theta^2+13\theta^4+16\theta^6+28\theta^8+16\theta^{10}+13\theta^{12}+4\theta^{14}+\theta^{16}+1}{\left(\theta^2+1\right)^8}$, $a_3^{(1)} = -4\frac{\left(\theta^2+4\theta^4+\theta^6+\theta^8+1\right)\left(\theta^4+1\right)\theta}{\left(\theta^2+1\right)^7}$, $a_4^{(1)} = -2\left(2\theta-2\theta^2+2\theta^3+\theta^4+1\right)\frac{\theta+\theta^2+1}{\left(\theta^2+1\right)^3}$, $a_5^{(1)} = -4\left(-\theta+\theta^2+1\right)\frac{5\theta+9\theta^2+6\theta^3+9\theta^4+5\theta^5+3\theta^6+3}{\left(\theta^2+1\right)^4} - 4\frac{\theta\left(1+\theta^4\right)^2}{\left(1+\theta^2\right)^5}\kappa_3^2$. Next

$a_6^{(1)} = 24\theta^2\left(\theta^2+4\theta^4+\theta^6+\theta^8+1\right)\frac{\left(\theta^4+1\right)^2}{\left(\theta^2+1\right)^{10}}$,

$a_7^{(1)} = 2\frac{4\theta+23\theta^2+6\theta^3+29\theta^4+14\theta^5+66\theta^6+14\theta^7+29\theta^8+6\theta^9+23\theta^{10}+4\theta^{11}+3\theta^{12}+3}{\left(\theta^2+1\right)^6} + 2\frac{\theta^2+4\theta^4+\theta^6+\theta^8+1}{\left(\theta^2+1\right)^4}\kappa_4$,

$a_8^{(1)} = 2\left(\theta^2+4\theta^4+\theta^6+\theta^8+1\right)\frac{11\theta^2+9\theta^4+30\theta^6+9\theta^8+11\theta^{10}+\theta^{12}+1}{\left(\theta^2+1\right)^{10}} + \frac{\left(\theta^2+4\theta^4+\theta^6+\theta^8+1\right)^2}{\left(\theta^2+1\right)^8}\kappa_4$,

$a_9^{(1)} = 8\frac{\theta+5\theta^2+3\theta^3+8\theta^4+4\theta^5+16\theta^6+4\theta^7+8\theta^8+3\theta^9+5\theta^{10}+\theta^{11}+\theta^{12}+1}{\left(\theta^2+1\right)^6} + 2\kappa_4\frac{\left(1+\theta^4\right)^2+\theta^2\left(1+\theta^2\right)^2}{\left(1+\theta^2\right)^4}$, Finally, $a_{10}^{(1)} = -2\frac{\left(\theta^2+4\theta^4+\theta^6+\theta^8+1\right)^2}{\left(\theta^2+1\right)^8}\kappa_4 - 2\theta\left(\theta^4+1\right)\frac{\theta^2+\theta^3+8\theta^4+3\theta^5+2\theta^6+3\theta^7+8\theta^8+\theta^9+\theta^{10}+2\theta^{12}+2}{\left(\theta^2+1\right)^9}\kappa_3^2 - 4\frac{7\theta^2+11\theta^4+29\theta^6+24\theta^8+29\theta^{10}+11\theta^{12}+7\theta^{14}+\theta^{16}+1}{\left(\theta^2+1\right)^8}$ and $a_1^{(2)} = a_3^{(2)} = a_4^{(2)} = a_{11}^{(1)} = a_{11}^{(2)} = a_{12}^{(1)} = 0$.

For the **zero-mean** case, all Edgeworth coefficients are the same as in the non-zero mean one, apart from $a_4^{(1)}$, $a_5^{(1)}$, $a_7^{(1)}$, and $a_9^{(1)}$, which now stand as: $a_4^{(1)} = -4\theta\frac{\theta^4+1}{\left(\theta^2+1\right)^3}$, $a_5^{(1)} = -8\frac{\left(1-\theta+\theta^2\right)\left(\theta+\theta^2+1\right)}{\left(1+\theta^2\right)^2} - 4\frac{\theta\left(1+\theta^4\right)^2}{\left(1+\theta^2\right)^5}\kappa_3^2 - 4\frac{\left(1+\theta^4\right)^2+\theta^2\left(1+\theta^2\right)^2}{\left(1+\theta^2\right)^4}\kappa_4$, $a_7^{(1)} = 4\frac{9\theta^2+9\theta^4+26\theta^6+9\theta^8+9\theta^{10}+\theta^{12}+1}{\left(\theta^2+1\right)^6} + 2\frac{\theta^2+4\theta^4+\theta^6+\theta^8+1}{\left(\theta^2+1\right)^4}\kappa_4$, and $a_9^{(1)} = 4\frac{7\theta^2+9\theta^4+22\theta^6+9\theta^8+7\theta^{10}+\theta^{12}+1}{\left(\theta^2+1\right)^6} + 2\kappa_4\frac{\left(1+\theta^4\right)^2+\theta^2\left(1+\theta^2\right)^2}{\left(1+\theta^2\right)^4}$.

Appendix B3 Expansion of $\widehat{\theta}$

For $|\widehat{\rho}| < 0.5$ the solution for $\widehat{\theta}$ is given in equation (6). Hence $f(\rho) = 0$, $\frac{\partial f(\rho)}{\partial \widehat{\rho}} = \frac{(1+\theta^2)^2}{(1-\theta^2)} > 0$, $\frac{\partial^2 f(\rho)}{\partial \widehat{\rho}^2} = \frac{2\theta(3-\theta^2)(1+\theta^2)^3}{(1-\theta^2)^3}$, and $\frac{\partial^3 f(\rho)}{\partial \widehat{\rho}^3} = 6\theta^4 \frac{11\theta^2-5\theta^4+\theta^6+1}{(1-\theta^2)^5}$. It follows that for $\overline{\theta} = \sqrt{n}\left(\widehat{\theta} - \theta\right)$ we have

$$\overline{\theta} = \frac{\partial f(\rho)}{\partial \widehat{\rho}} \sqrt{n}(\widehat{\rho} - \rho) + \frac{1}{2\sqrt{n}} \frac{\partial^2 f(\rho)}{\partial \widehat{\rho}^2} \left[\sqrt{n}(\widehat{\rho} - \rho)\right]^2 + \frac{1}{6n} \frac{\partial^3 f(\rho)}{\partial \widehat{\rho}^3} \left[\sqrt{n}(\widehat{\rho} - \rho)\right]^3 + o\left(n^{-1}\right)$$

where the cumulants of $\sqrt{n}(\widehat{\rho} - \rho)$, $k_1^{\widehat{\rho}}$, $k_2^{\widehat{\rho}}$, $k_3^{\widehat{\rho}}$ and $k_4^{\widehat{\rho}}$, are presented in section 3.1.

Hence Theorem 1 can be applied with $f^1 = \frac{(1+\theta^2)^2}{(1-\theta^2)}$, $h^{11} = \frac{2\theta(3-\theta^2)(1+\theta^2)^3}{(1-\theta^2)^3}$, $h^{111} = 6\theta^4 \frac{11\theta^2-5\theta^4+\theta^6+1}{(1-\theta^2)^5}$, and $c_1^{(1)} = -\frac{(\theta+\theta^2+1)(2\theta-2\theta^2+2\theta^3+\theta^4+1)}{(\theta^2+1)^3}$, $c_{11}^{(1)} = \frac{\theta^2+4\theta^4+\theta^6+\theta^8+1}{(1+\theta^2)^4}$, $c_{11}^{(3)} = -\frac{2(-4\theta-\theta^2+6\theta^3-12\theta^5+6\theta^7-\theta^8-4\theta^9+\theta^{10}+1)(\theta+1)^2}{(\theta^2+1)^6} - \frac{4\theta(1+\theta^4)^2}{(1+\theta^2)^5}\kappa_3^2 - \frac{\theta^2+4\theta^4-\theta^6+\theta^8+1}{(\theta^2+1)^4}\kappa_4$, $c_{111}^{(1)} = \sqrt{n}k_3^{\widehat{\rho}}$, $c_{1111}^{(1)} = nk_4^{\widehat{\rho}}$, and $c_1^{(2)} = c_2^{(2)} = c_{111}^{(2)} = 0$.

It follows that the Edgeworth coefficients are: $\omega^2 = \frac{\theta^2+4\theta^4+\theta^6+\theta^8+1}{(1-\theta^2)^2}$,
$\omega^{(3)} = -\frac{2(-4\theta-\theta^2+6\theta^3-12\theta^5+6\theta^7-\theta^8-4\theta^9+\theta^{10}+1)(\theta+1)^2}{(1-\theta^4)^2} - \frac{4\theta(1+\theta^4)^2}{(1+\theta^2)(1-\theta^2)^2}\kappa_3^2 - \frac{\theta^2+4\theta^4-\theta^6+\theta^8+1}{(1-\theta^2)^2}\kappa_4$,
$a_1^{(1)} = -6\theta\left(\theta^4+1\right)\frac{6\theta^4+\theta^8+1}{(\theta^2+1)(1-\theta^2)^3} + \frac{(1+\theta^4)^3+\theta^3(1+\theta^2)^3}{(1-\theta^2)^3}\kappa_3^2$,

$$a_2^{(1)} = -6\frac{\begin{pmatrix} 1-10\theta^2+30\theta^4-106\theta^6+129\theta^8-216\theta^{10} \\ +129\theta^{12}-106\theta^{14}+30\theta^{16}-10\theta^{18}+\theta^{20} \end{pmatrix}}{(\theta^2+1)^2(1-\theta^2)^4}$$

$$-12\theta\left(\theta^4+1\right)\frac{\begin{pmatrix} -\theta-2\theta^2-\theta^3+10\theta^4+2\theta^5-4\theta^6+2\theta^7 \\ +10\theta^8-\theta^9-2\theta^{10}-\theta^{11}+2\theta^{12}+2 \end{pmatrix}}{(\theta^2+1)(1-\theta^2)^4}\kappa_3^2 + \frac{5\theta^4+4\theta^6+12\theta^8+4\theta^{10}+5\theta^{12}+\theta^{16}+1}{(1-\theta^2)^4}\kappa_4^2,$$

$a_3^{(1)} = 2\theta\left(3-\theta^2\right)\frac{(\theta^2+4\theta^4+\theta^6+\theta^8+1)^2}{(1+\theta^2)(1-\theta^2)^5}$, $a_4^{(1)} = 2\theta\left(3-\theta^2\right)\frac{\theta^2+4\theta^4+\theta^6+\theta^8+1}{(1+\theta^2)(1-\theta^2)^3}$,
$a_5^{(1)} = -12\theta^2\left(\theta^4+1\right)\left(3-\theta^2\right)\frac{6\theta^4+\theta^8+1}{(\theta^2+1)^2(1-\theta^2)^4} + 2\theta\left(3-\theta^2\right)\frac{(1+\theta^4)^3+\theta^3(1+\theta^2)^3}{(1+\theta^2)(1-\theta^2)^4}\kappa_3^2$, $a_6^{(1)} = 6\theta^4\frac{(11\theta^2-5\theta^4+\theta^6+1)(\theta^2+4\theta^4+\theta^6+\theta^8+1)^3}{(1+\theta^2)^6(1-\theta^2)^8}$, $a_7^{(1)} = 6\theta^4\frac{11\theta^2-5\theta^4+\theta^6+1}{(1-\theta^2)^6}\frac{(\theta^2+4\theta^4+\theta^6+\theta^8+1)^2}{(1+\theta^2)^6}$, $a_8^{(1)} = 4\theta^2\left(3-\theta^2\right)^2\frac{(\theta^2+4\theta^4+\theta^6+\theta^8+1)^3}{(1+\theta^2)^2(1-\theta^2)^8}$, $a_9^{(1)} = \frac{4\theta^2(3-\theta^2)^2}{(1-\theta^2)^6}\frac{(\theta^2+4\theta^4+\theta^6+\theta^8+1)^2}{(1+\theta^2)^2}$,
$a_{10}^{(1)} = -12\theta^2\left(\theta^4+1\right)\left(3-\theta^2\right)\frac{6\theta^4+\theta^8+1}{(1-\theta^2)^6}\frac{\theta^2+4\theta^4+\theta^6+\theta^8+1}{(1+\theta^2)^2}$

$+ 2\theta\left(3-\theta^2\right)\left(1+\theta^2\right)\frac{\left(1+\theta^4\right)^3+\theta^3\left(1+\theta^2\right)^3}{\left(1-\theta^2\right)^6}\frac{\theta^2+4\theta^4+\theta^6+\theta^8+1}{\left(1+\theta^2\right)^2}\kappa_3^2,$
$a_{11}^{(1)} = -\frac{\left(\theta+\theta^2+1\right)\left(2\theta-2\theta^2+2\theta^3+\theta^4+1\right)}{\left(1-\theta^4\right)},$
$a_{12}^{(1)} = -\left(\theta+\theta^2+1\right)\left(2\theta-2\theta^2+2\theta^3+\theta^4+1\right)\frac{2\theta\left(3-\theta^2\right)}{\left(1-\theta^2\right)^4}\frac{\theta^2+4\theta^4+\theta^6+\theta^8+1}{\left(1+\theta^2\right)^2}$, and $\omega^{(2)} = a_1^{(2)} = a_3^{(2)} = a_4^{(2)} = a_{11}^{(2)} = 0.$

For the **zero mean** case, all Edgeworth coefficients which are different from the coefficients given above are: $\omega^{(3)} = -2\frac{\left(1-\theta^2\right)^4}{\left(\theta^2+1\right)^2} - 4\frac{\theta\left(1+\theta^4\right)^2}{\left(1+\theta^2\right)\left(1-\theta^2\right)^2}\kappa_3^2 - \frac{\theta^2+4\theta^4-\theta^6+\theta^8+1}{\left(1-\theta^2\right)^2}\kappa_4,$ $a_{11}^{(1)} = -2\theta\frac{\theta^4+1}{\left(\theta^2+1\right)\left(1-\theta^2\right)}$, and $a_{12}^{(1)} = -4\theta^2\left(3-\theta^2\right)\frac{\theta^4+1}{\left(1-\theta^2\right)^4}\frac{\theta^2+4\theta^4+\theta^6+\theta^8+1}{\left(1+\theta^2\right)^2}.$

Appendix C1 Expansion of QMLEs

Consider the first order conditions in equation (8). Now let $\overline{\varphi} = \left(\overline{\theta}_1, \overline{\theta}_2\right)^{/} = \left(\sqrt{n}\left(\widetilde{\theta}-\theta\right), \sqrt{n}\left(\widetilde{\mu}-\mu\right)\right)^{/}$. The Taylor expansion of $\frac{1}{\sqrt{n}}\frac{\partial \ell(\widetilde{\varphi})}{\partial \varphi}$, where $\widetilde{\varphi} = \left(\widetilde{\theta}_1, \widetilde{\theta}_2\right)^{/} = \left(\widetilde{\theta}, \widetilde{\mu}\right)^{/}$ around the true value $\varphi = (\theta_1, \theta_2)^{/} = (\theta, \mu)^{/}$ can be written as:

$$\begin{aligned} 0 &= \frac{1}{\sqrt{n}}\frac{\partial \ell}{\partial \theta_j} + \sum_{i=1}^{2}\left(M_{ji} + \frac{w_{ji}}{\sqrt{n}}\right)\overline{\theta}_i + \frac{1}{2\sqrt{n}}\sum_{k,i=1}^{2}\left(M_{jik} + \frac{q_{jik}}{\sqrt{n}}\right)\overline{\theta}_i\overline{\theta}_k \\ &\quad + \frac{1}{6n}\sum_{l,k,i=1}^{2} M_{jikl}\overline{\theta}_l\overline{\theta}_i\overline{\theta}_k + O_p\left(n^{-\frac{3}{2}}\right), \\ &\equiv g_j\left(\overline{\varphi}, v\right) + O_p\left(n^{-\frac{3}{2}}\right), \quad j = 1, 2, \end{aligned}$$

where $j = 1, 2$, $A_{ij} = \frac{1}{n}E\left(\frac{\partial^2 \ell(\varphi)}{\partial \theta_j \partial \theta_i}\right)$, $K_{jik} = \frac{1}{n}E\left(\frac{\partial^3 \ell(\varphi)}{\partial \theta_j \partial \theta_i \partial \theta_k}\right)$, $M_{jikl} = \frac{1}{n}E\left(\frac{\partial^4 \ell(\varphi)}{\partial \theta_j \partial \theta_i \partial \theta_k \partial \theta_l}\right)$, $w_{ij} = \frac{1}{\sqrt{n}}\left(\frac{\partial^2 \ell(\varphi)}{\partial \theta_j \partial \theta_i} - nA_{ij}\right)$, $q_{ijk} = \frac{1}{\sqrt{n}}\left(\frac{\partial^3 \ell(\varphi)}{\partial \theta_j \partial \theta_i \partial \theta_k} - nK_{jik}\right)$, for $i, j, k = 1, 2$ and all derivatives are evaluated at the true values. Let us define a vector A containing the non-zero elements of $\frac{1}{\sqrt{n}}\frac{\partial \ell}{\partial \theta_i}$, w_{ij}, q_{ijk}, for $i, j, k = 1, 2$. As however $w_{22} = q_{122} = q_{222} = 0$ (see below) we define A as $A = (A_1, A_2, A_3, A_4, A_5, A_6)^{/} = \left(\frac{1}{\sqrt{n}}\frac{\partial \ell}{\partial \theta_1}, \frac{1}{\sqrt{n}}\frac{\partial \ell}{\partial \theta_2}, w_{11}, w_{12}, q_{111}, q_{112}\right)^{/}$. Solving for $\overline{\theta}_j$, and $j = 1, 2$, as continuously differentiable functions of A, gives:

$$\begin{aligned} \overline{\theta}_j\left(A\right) &= \sum_{a=1}^{6}\frac{\partial \overline{\theta}_j\left(0\right)}{\partial A_a}A_a + \frac{1}{2}\sum_{a,b=1}^{6}\frac{\partial^2 \overline{\theta}_j\left(0\right)}{\partial A_a \partial A_b}A_a A_b + \frac{1}{6}\sum_{a,b,c=1}^{6}\frac{\partial^3 \overline{\theta}_j\left(0\right)}{\partial A_a \partial A_b \partial A_c}A_a A_b A_c + O_p\left(n^{-\frac{3}{2}}\right) \\ &\equiv \sum_{a=1}^{6} f_j^a A_a + \frac{1}{2\sqrt{n}}\sum_{a,b=1}^{6} h_j^{ab} A_a A_b + \frac{1}{6n}\sum_{a,b,c=1}^{6} h_j^{abc} A_a A_b A_c + O_p\left(n^{-\frac{3}{2}}\right) \end{aligned}$$

where $f_j^a = \frac{\partial \bar{\theta}_j(0)}{\partial A_a}$, $h_j^{ab} = \sqrt{n}\frac{\partial^2 \bar{\theta}_j(0)}{\partial A_a \partial A_b}$ and $h_j^{abc} = n\frac{\partial^3 \bar{\theta}_j(0)}{\partial A_a \partial A_b \partial A_c}$ (employing the notation of Theorem 1).

Now the derivatives can be found by solving the following system of equations, for $j,k = 1,2$ and $a,b,c = 1,...,6$:

$0 = \sum_{k=1}^{2} M_{jk} f_k^a + \frac{\partial g_j(0,0)}{\partial A_a}$,

$0 = \sum_{k=1}^{2} \left(\frac{1}{\sqrt{n}} \sum_{l=1}^{2} M_{jkl} f_l^b + \frac{\partial^2 g_j(0,0)}{\partial A_b \partial \bar{\theta}_k} \right) f_k^a + \sum_{k=1}^{2} \frac{\partial^2 g_j(0,0)}{\partial A_a \partial \bar{\theta}_k} f_k^b + \frac{1}{\sqrt{n}} \sum_{k=1}^{2} M_{jk} h_k^{ab}$, and

$0 = \sum_{k=1}^{2} \left(\frac{1}{n} \sum_{p,l=1}^{2} M_{jlkp} f_b^l f_c^p + \frac{1}{n} \sum_{l=1}^{2} M_{jkl} h_l^{bc} + \sum_{l=1}^{2} \frac{\partial^3 g_j(0,0)}{\partial A_c \partial \bar{\theta}_l \partial \bar{\theta}_k} f_l^b + \sum_{p=1}^{2} \frac{\partial^3 g_j(0,0)}{\partial \bar{\theta}_p \partial A_b \partial \bar{\theta}_k} f_c^p \right) f_k^a +$ $\sum_{k=1}^{2} \left(\frac{1}{n} \sum_{l=1}^{2} M_{jkl} h_l^{ac} + \sum_{p=1}^{2} \frac{\partial^3 g_j(0,0)}{\partial \bar{\theta}_p \partial \bar{\theta}_k \partial A_a} f_c^p \right) f_k^b + \sum_{k=1}^{2} \left(\frac{1}{n} \sum_{p=1}^{2} M_{jkp} f_c^p + \frac{1}{\sqrt{n}} \frac{\partial^2 g_j(0,0)}{\partial A_c \partial \bar{\theta}_k} \right) h_k^{ab} +$ $\frac{1}{\sqrt{n}} \sum_{k=1}^{2} \frac{\partial^2 f_j(0,0)}{\partial \bar{\theta}_k \partial A_a} h_k^{bc} + \frac{1}{\sqrt{n}} \sum_{k=1}^{2} \frac{\partial^2 f_j(0,0)}{\partial A_b \partial \bar{\theta}_k} h_k^{ac} + \frac{1}{n} \sum_{k=1}^{2} M_{jk} h_k^{abc}$. Notice that the first two equations are as in Tanaka (1984). However, the third is completely new (Tanaka 1984 is developing a 1^{st} order expansion).

Hence, first consider $j = 1$ and observe that $\frac{\partial g_1(0,0)}{\partial A_1} = 1$, and $\frac{\partial g_1(0,0)}{\partial A_a} = 0$ for $a = 2,...,6$. It follows that $f_1^1 = 1 - \theta^2$, and $f_1^2 = f_1^3 = f_1^4 = f_1^5 = f_1^6 = 0$. Applying the same logic and by the notation of Theorem 1 we find that the non-zero second derivatives for $j = 1$ are: $h_1^{11} = -6\theta\left(1-\theta^2\right)$, $h_1^{13} = \left(1-\theta^2\right)^2$, $h_1^{22} = 2\sigma^2(1+\theta)\left(1-\theta^2\right)$, and $h_1^{24} = \sigma^2(1+\theta)^2\left(1-\theta^2\right)$, and also $h_1^{111} = \left(-12+72\theta^2\right)\left(1-\theta^2\right)$, $h_1^{113} = -18\theta\left(1-\theta^2\right)^2$, $h_1^{115} = \left(1-\theta^2\right)^3$, $h_1^{122} = 2\sigma^2(1-7\theta)(1+\theta)\left(1-\theta^2\right)$, $h_1^{124} = 2\sigma^2\left(2-3\theta-5\theta^2\right)(1+\theta)\left(1-\theta^2\right)$, $h_1^{126} = \sigma^2(1+\theta)^2\left(1-\theta^2\right)^2$, $h_1^{133} = 2\left(1-\theta^2\right)^3$, $h_1^{144} = 2\sigma^2(1+\theta)^2\left(1-\theta^2\right)^2$, $h_1^{234} = \sigma^2(1+\theta)^2\left(1-\theta^2\right)^2$, $h_1^{223} = 2\sigma^2(1+\theta)\left(1-\theta^2\right)^2$, whereas all the other derivatives are 0. For the expansion of $\widetilde{\theta}$ we do not need the derivatives for $j = 2$. These can be found in TA.

Appendix C2 cumulants needed for $\widetilde{\theta}$

Taking the derivatives of $\ell(\theta,\mu)$ w.r.t. θ and μ, at the true parameter values we have that $\frac{\partial u_t}{\partial \theta} = -u_{t-1} - \theta\frac{\partial u_{t-1}}{\partial \theta} = ... = -\sum_{i=0}^{\infty}(-\theta)^i u_{t-1-i}$, $\frac{\partial^2 u_t}{\partial \theta^2} = -2\frac{\partial u_{t-1}}{\partial \theta} - \theta\frac{\partial^2 u_{t-1}}{\partial \theta^2} =$

$2\sum_{i=0}^{\infty}(i+1)(-\theta)^i u_{t-2-i}$, $\frac{\partial^3 u_t}{\partial\theta^3} = -3\frac{\partial^2 u_{t-1}}{\partial\theta^2} - \theta\frac{\partial^3 u_{t-1}}{\partial\theta^3} = -6\sum_{i=0}^{\infty}\frac{(i+1)(i+2)}{2}(-\theta)^i u_{t-3-i}$, $\frac{\partial^4 u_t}{\partial\theta^4} = -4\frac{\partial^3 u_{t-1}}{\partial\theta^3} - \theta\frac{\partial^4 u_{t-1}}{\partial\theta^4} = ... = 4\sum_{i=0}^{\infty}(i+1)(i+2)(i+3)(-\theta)^i u_{t-4-i}$, $\frac{\partial u_t}{\partial\mu} = -1 - \theta\frac{\partial u_{t-1}}{\partial\mu} = -\sum_{i=0}^{\infty}(-\theta)^i = -\frac{1}{1+\theta}$, $\frac{\partial^2 u_t}{\partial\mu^2} = \frac{\partial^3 u_t}{\partial\mu^3} = \frac{\partial^4 u_t}{\partial\mu^4} = 0$, $\frac{\partial^2 u_t}{\partial\mu\partial\theta} = \frac{1}{(1+\theta)^2}$, $\frac{\partial^3 u_t}{\partial\mu\partial\theta^2} = -\frac{2}{(1+\theta)^3}$, and $\frac{\partial^3 u_t}{\partial\theta\partial\mu^2} = 0$. It follows that

$$\begin{aligned} M_{11} &= \frac{1}{n}E\left(\frac{\partial^2\ell(\varphi)}{\partial\theta^2}\right) = -\frac{1}{1-\theta^2}, \quad M_{111} = \frac{1}{n}E\left(\frac{\partial^3\ell(\varphi)}{\partial\theta^3}\right) = -6\frac{\theta}{\left(1-\theta^2\right)^2}, \\ M_{1111} &= \frac{1}{n}E\left(\frac{\partial^4\ell(\varphi)}{\partial\theta^4}\right) = -12\frac{1+3\theta^2}{\left(1-\theta^2\right)^3}, \end{aligned}$$

$$M_{22} = \frac{1}{n}E\left(\frac{\partial^2\ell(\varphi)}{\partial\mu^2}\right) = -\frac{1}{(1+\theta)^2\sigma^2}, \; M_{222} = \frac{1}{n}E\left(\frac{\partial^3\ell(\varphi)}{\partial\mu^3}\right) = M_{2222} = \frac{1}{n}E\left(\frac{\partial^4\ell(\varphi)}{\partial\mu^4}\right) = 0.$$

Furthermore,

$$M_{12} = E\left(\frac{\partial^2\ell}{\partial\mu\partial\theta}\right) = M_{112} = \frac{1}{n}E\left(\frac{\partial^3\ell(\varphi)}{\partial\theta^2\partial\mu}\right) = 0, \quad M_{122} = \frac{1}{n}E\left(\frac{\partial^3\ell(\varphi)}{\partial\theta\partial\mu^2}\right) = \frac{2}{(1+\theta)^3\sigma^2},$$

and

$$\begin{aligned} M_{1112} &= \frac{1}{n}E\left(\frac{\partial^4\ell(\varphi)}{\partial\theta^3\partial\mu}\right) = 0, \quad M_{1122} = \frac{1}{n}E\left(\frac{\partial^4\ell(\varphi)}{\partial\theta^2\partial\mu^2}\right) = -6\frac{1}{(1+\theta)^4\sigma^2}, \\ M_{1222} &= \frac{1}{n}E\left(\frac{\partial^4\ell(\varphi)}{\partial\theta\partial\mu^3}\right) = 0. \end{aligned}$$

For the cumulants of $v_i s$, the $A_i s$ in terms of Theorem 1, notice that in the maximization of the likelihood we have that for any admissible θ and μ we have that $u_t = y_t - \mu - \theta u_{t-1}$, with u_0 drawn from the stationary distribution. Hence we have that the derivatives of the $u_t\prime s$ with respect to the parameters θ and μ are: $\frac{\partial u_t}{\partial\theta} = -u_{t-1} - \theta\frac{\partial u_{t-1}}{\partial\theta} = ... = -\sum_{i=0}^{t-1}(-\theta)^i u_{t-1-i}$, for $t > 1$ and $\frac{\partial u_0}{\partial\theta} = 0$. $\frac{\partial^2 u_t}{\partial\theta^2} = -2\frac{\partial u_{t-1}}{\partial\theta} - \theta\frac{\partial^2 u_{t-1}}{\partial\theta^2} = 2\sum_{i=0}^{t-2}(i+1)(-\theta)^i u_{t-2-i}$, for $t > 2$ and $\frac{\partial^2 u_0}{\partial\theta^2} = \frac{\partial^2 u_1}{\partial\theta^2} = 0$. $\frac{\partial^3 u_t}{\partial\theta^3} = -3\frac{\partial^2 u_{t-1}}{\partial\theta^2} - \theta\frac{\partial^3 u_{t-1}}{\partial\theta^3} = -6\sum_{i=0}^{t-3}\frac{(i+1)(i+2)}{2}(-\theta)^i u_{t-3-i}$, for $t > 3$ and $\frac{\partial u_0}{\partial\theta} = \frac{\partial^2 u_1}{\partial\theta^2} = \frac{\partial^2 u_2}{\partial\theta^2} = 0$. $\frac{\partial u_t}{\partial\mu} = -1 - \theta\frac{\partial u_{t-1}}{\partial\mu} - \sum_{i=0}^{t-1}(-\theta)^i = -\frac{1-(-\theta)^t}{1+\theta}$, $\frac{\partial^2 u_t}{\partial\mu^2} = \frac{\partial^3 u_t}{\partial\mu^3} = \frac{\partial^4 u_t}{\partial\mu^4} = 0$, and $\frac{\partial^2 u_t}{\partial\mu\partial\theta} = -\frac{[t+(t+1)(-\theta)](-\theta)^{t-1}-1}{(1+\theta)^2}$, $\frac{\partial^3 u_t}{\partial\mu\partial\theta^2} = \frac{t(t-1)(-\theta)^{t-2}+2t(t+1)(-\theta)^{t-1}+(t+1)(t+2)(-\theta)^t-2}{(1+\theta)^3}$, and $\frac{\partial^3 u_t}{\partial\theta\partial\mu^2} = 0$.

Hence, adapting the notation of Theorem 1, and as all first order cumulants of the $A_i s$ are 0, we have that. $c_i^{(1)} = c_i^{(2)} = 0$ for $i = 1, .., 6$. The second order cumulants are: $c_{11}^{(1)} = \frac{1}{1-\theta^2}$, $c_{12}^{(1)} = c_{14}^{(1)} = c_{16}^{(1)} = 0$, $c_{13}^{(1)} = \frac{4\theta}{\left(1-\theta^2\right)^2}$, $c_{15}^{(1)} = 6\frac{3\theta^2+1}{\left(1-\theta^2\right)^3}$, $c_{22}^{(1)} = \frac{1}{\sigma^2}\frac{1}{(1+\theta)^2}$, $c_{23}^{(1)} = -\frac{1}{\sigma^4}\frac{E\left(u_0^3\right)}{(1+\theta)\left(1-\theta^2\right)}$, $c_{24}^{(1)} = -\frac{1}{\sigma^2}\frac{2}{(1+\theta)^3}$, $c_{25}^{(1)} = -\frac{6\theta}{\sigma(1+\theta)\left(1-\theta^2\right)^2}\kappa_3$, $c_{26}^{(1)} = 6\frac{1}{\sigma^2}\frac{1}{(1+\theta)^4}$, $c_{33}^{(1)} = 2\frac{7\theta^2+3}{\left(1-\theta^2\right)^3} + \kappa_4\frac{1}{\left(1-\theta^2\right)^2}$, $c_{44}^{(1)} = \frac{4}{\sigma^2(1+\theta)^4}$, $c_{ij}^{(2)} = 0$ for $i,j = 1,...,6$. From the $c_{ij}^{(3)}s$ we need only $c_{11}^{(3)}$, mainly due to fact that $f_1^2 = f_1^3 = f_1^4 = f_1^5 = f_1^6 = 0$. Hence, $c_{11}^{(3)} = -\frac{\theta^2}{\left(1-\theta^2\right)^2}$.

Out of all 3^{rd}order cumulants we only need c_{111}, c_{113}, c_{122} and c_{124}. Employing the notation of Theorem 1, we have: $c_{111}^{(1)} = -6\frac{\theta}{\left(1-\theta^2\right)^2} + \frac{\kappa_3^2}{1+\theta^3}$, $c_{113}^{(1)} = -\frac{19\theta^2+8}{\left(1-\theta^2\right)^3} + \frac{2\theta}{1+\theta^3}\left(\frac{1}{1-\theta^2} - \frac{\theta}{1+\theta^3}\right)\kappa_3^2 - \frac{2}{\left(1-\theta^2\right)^2}\kappa_4$, $c_{122}^{(1)} = \frac{2}{\sigma^2}\frac{1}{(1+\theta)^3}$, $c_{124}^{(1)} = -\frac{4}{\sigma^2}\frac{1}{(1+\theta)^4}$, and $c_{113}^{(2)} = c_{122}^{(2)} = c_{124}^{(2)} = 0$.

From all the 4^{th} order cumulants we only need c_{1111}, i.e. $c_{1111}^{(1)} = \kappa_4^2\frac{1}{1-\theta^4} + 12\frac{1}{\left(1-\theta^2\right)^2}\kappa_4 - 12\frac{\theta}{1+\theta^3}\kappa_3^2\left(\frac{2}{1-\theta^2} - \frac{\theta}{1+\theta^3}\right) + 6\frac{7\theta^2+3}{\left(1-\theta^2\right)^3}$.

Appendix C3 Expansion of $\widetilde{\theta}$

For the validity of the expansion we have that under the assumptions of Lemma 2, $A = (A_1, A_2, A_3, A_4, A_5, A_6)^{/}$ is a martingale satisfying all the assumptions of Götze and Hipp (1983, 1994) and Hall and Horowitz (1996) (see also Corradi and Iglesias 2008).

Now applying the results of Theorem 1 (see Appendix A) we get: $\omega^2 = \left(1-\theta^2\right)$, $\omega^{(3)} = -\theta^2$, $a_1^{(1)} = -6\theta\left(1-\theta^2\right) + \frac{\kappa_3^2\left(1-\theta^2\right)^3}{1+\theta^3}$, $a_2^{(1)} = \kappa_4^2\frac{\left(1-\theta^2\right)^3}{1+\theta^2} + 12\left(1-\theta^2\right)^2\kappa_4 - 12\frac{\theta\left(1-\theta^2\right)^4}{1+\theta^3}\kappa_3^2\left(\frac{2}{1-\theta^2} - \frac{\theta}{1+\theta^3}\right) + 6\left(7\theta^2+3\right)\left(1-\theta^2\right)$, $a_3^{(1)} = 2\theta\left(1-\theta^2\right)$, $a_4^{(1)} = 2\left(2\theta-1\right)$, $a_5^{(1)} = 2\left(4\theta-3\theta^2-10\right) - \frac{\kappa_3^2}{1+\theta^3}6\theta\left(1-\theta^2\right)^2 + 4\left(\frac{1}{1-\theta^2} - \frac{\theta}{1+\theta^3}\right)\frac{\theta\left(1-\theta^2\right)^3}{\left(1+\theta^3\right)}\kappa_3^2 - 4\left(1-\theta^2\right)\kappa_4$, $a_6^{(1)} = 6\left(1-\theta^4\right)$, $a_7^{(1)} = 4\left(-2\theta+\theta^2+6\right) + 2\left(1-\theta^2\right)\kappa_4$, $a_8^{(1)} = 2\left(1-\theta^2\right)\left(\theta^2+3\right) + \left(1-\theta^2\right)^2\kappa_4$, $a_9^{(1)} = 4\left(-2\theta+\theta^2+4\right) + 2\left(1-\theta^2\right)\kappa_4$, $a_{10}^{(1)} = -\left(1-\theta^2\right)\left(7\theta^2+8\right) - 2\frac{(1-\theta)\theta^2}{\left(\theta^2-\theta+1\right)}\frac{\left(1-\theta^2\right)^3}{1+\theta^3}\kappa_3^2 - 2\kappa_4\left(1-\theta^2\right)^2$, and $\omega^{(2)} = a_1^{(2)} = a_3^{(2)} = a_4^{(2)} = a_{11}^{(1)} = a_{11}^{(2)} = a_{12}^{(1)} = 0$.

Now from Lemma 1 we get that

$$k_1^{\widetilde{\theta}} = \frac{2\theta-1}{\sqrt{n}} + o\left(n^{-1}\right), \quad k_2^{\widetilde{\theta}} = \omega^2 + \frac{1}{n}\left(\theta+6\right)\left(2-\theta\right) + \frac{1}{n}\xi_2^{\widetilde{\theta}} + o\left(n^{-1}\right),$$

where $\omega^2 = 1-\theta^2$, and $\xi_2^{\widetilde{\theta}} = 2\frac{\theta^2-\theta-1}{\theta^2-\theta+1}\frac{\theta\left(1-\theta^2\right)^2}{1+\theta^3}\kappa_3^2 - \left(1-\theta^2\right)\kappa_4$.

Also

$$E\left[\sqrt{n}\left(\widetilde{\theta}-\theta\right)\right]^2 = (1-\theta^2)+\frac{1}{n}\left[-8\theta+3\theta^2+13+2\frac{\theta^2-\theta-1}{\theta^2-\theta+1}\frac{\theta\left(1-\theta^2\right)^2}{1+\theta^3}\kappa_3^2-\left(1-\theta^2\right)\kappa_4\right],$$

$$k_3^{\widetilde{\theta}} = \frac{1}{\sqrt{n}}\frac{\left(1-\theta^2\right)^3}{1+\theta^3}\kappa_3^2 + o\left(n^{-1}\right), \quad k_4^{\widetilde{\theta}} = \frac{1}{n}6\left(1-\theta^2\right)\left(\theta^2+3\right)+\frac{1}{n}\xi_4^{\widetilde{\theta}}+o\left(n^{-1}\right),$$

where $\xi_4^{\widetilde{\theta}} = 12\theta\frac{\theta-\theta^2-2}{1-\theta+\theta^2}\frac{\left(1-\theta^2\right)^3}{1+\theta^3}\kappa_3^2 + \frac{\left(1-\theta^2\right)^3}{1+\theta^2}\kappa_4^2$.

For $\mu=0$, we play the above procedure with the difference that now the vector A is $A=(A_1,A_2,A_3)^{/} = (g_1, w_{11}, q_{111})^{/}$. The coefficients which are different from the above ones are: $a_4^{(1)} = 2\theta$, $a_5^{(1)} = -2\left(\theta^2+8\right)+2\frac{-\theta+\theta^2-1}{-\theta+\theta^2+1}\theta\frac{\left(1-\theta^2\right)^2}{1+\theta^3}\kappa_3^2 - 4\left(1-\theta^2\right)\kappa_4$, $a_7^{(1)} = 2\left(\theta^2+9\right)+2\left(1-\theta^2\right)\kappa_4$, and $a_9^{(1)} = 12+2\left(1-\theta^2\right)\kappa_4$.

Hence

$$k_1^{\widetilde{\theta}_0} = \frac{1}{\sqrt{n}}\theta + o\left(n^{-1}\right),$$

$$E\left[\sqrt{n}\left(\widetilde{\theta}_0-\theta\right)\right]^2 = 1-\theta^2+\frac{1}{n}\left[8+2\frac{\theta^2-\theta-1}{\theta^2-\theta+1}\frac{\theta\left(1-\theta^2\right)^2}{1+\theta^3}\kappa_3^2-\left(1-\theta^2\right)\kappa_4\right],$$

$$k_2^{\widetilde{\theta}_0} = \omega^2+\frac{1}{n}\left(8-\theta^2\right)+\frac{1}{n}\xi_2^{\widetilde{\theta}}+o\left(n^{-1}\right),$$

where ω^2 and $\xi_2^{\widetilde{\theta}}$ given above. Finally, $k_3^{\widetilde{\theta}_0} = k_3^{\widetilde{\theta}}$ and $k_4^{\widetilde{\theta}_0} = k_4^{\widetilde{\theta}}$ as neither of these cumulants are functions of the Edgeworth coefficients which are different in the non-zero mean case, i.e. $a_4^{(1)}$, $a_5^{(1)}$, $a_7^{(1)}$ and $a_9^{(1)}$.

Appendix C4 Expansion of $\widetilde{\rho}$

With the definition of $\widetilde{\rho}$ let us call $m\left(\widetilde{\theta}\right) = \frac{\widetilde{\theta}}{1+\widetilde{\theta}^2}-\rho$, where ρ is the true value of the parameter. Then we have that

$$\sqrt{n}\left(\widetilde{\rho}-\rho\right) = \frac{\partial m(\theta)}{\partial\widetilde{\theta}}\sqrt{n}\left(\widetilde{\theta}-\theta\right)+\frac{1}{2\sqrt{n}}\frac{\partial^2 m(\theta)}{\partial\widetilde{\theta}^2}\left[\sqrt{n}\left(\widetilde{\theta}-\theta\right)\right]^2+\frac{1}{6n}\frac{\partial^3 m(\theta)}{\partial\widetilde{\theta}^3}\left[\sqrt{n}\left(\widetilde{\theta}-\theta\right)\right]^3,$$

with a $o\left(n^{-1}\right)$ error. Consequently, we can apply Theorem 1 with $f^1 = \frac{\left(1-\theta^2\right)}{\left(1+\theta^2\right)^2}$, $h^{11} = -2\frac{\theta\left(3-\theta^2\right)}{\left(1+\theta^2\right)^3}$, $h^{111} = -6\frac{\left(1-\theta^2\right)^2-4\theta^2}{\left(1+\theta^2\right)^4}$, and $c_1^{(1)} = 2\theta-1$, $c_{11}^{(1)} = 1-\theta^2$, $c_{11}^{(3)} =$

$(\theta+6)(2-\theta)+\xi_2^{\widetilde{\theta}}$, $c_{111}^{(1)}=\frac{\left(1-\theta^2\right)^3}{1+\theta^3}\kappa_3^2$, $c_{1111}^{(1)}=6\left(1-\theta^2\right)\left(\theta^2+3\right)+12\theta\frac{\theta-\theta^2-2}{1-\theta+\theta^2}\frac{\left(1-\theta^2\right)^3}{1+\theta^3}\kappa_3^2+\frac{\left(1-\theta^2\right)^3}{1+\theta^2}\kappa_4^2$, and all other cumulants being zero. Hence applying the formulae of Appendix A we get that the non-zero Edgeworth coefficients are: $\omega^2=\frac{\left(1-\theta^2\right)^3}{\left(1+\theta^2\right)^4}$, $\omega^{(3)}=\left[(\theta+6)(2-\theta)+2\frac{\theta^2-\theta-1}{\theta^2-\theta+1}\frac{\theta\left(1-\theta^2\right)^2}{1+\theta^3}\kappa_3^2-\left(1-\theta^2\right)\kappa_4\right]\frac{\left(1-\theta^2\right)^2}{\left(1+\theta^2\right)^4}$, $a_1^{(1)}=\frac{\left(1-\theta^2\right)^6}{\left(1+\theta^3\right)\left(1+\theta^2\right)^6}\kappa_3^2$, $a_2^{(1)}=\left(6\left(1-\theta^2\right)\left(\theta^2+3\right)+12\theta\frac{\theta-\theta^2-2}{1-\theta+\theta^2}\frac{\left(1-\theta^2\right)^3}{1+\theta^3}\kappa_3^2+\frac{\left(1-\theta^2\right)^3}{1+\theta^2}\kappa_4^2\right)\frac{\left(1-\theta^2\right)^4}{\left(1+\theta^2\right)^8}$, $a_3^{(1)}=-2\frac{\theta\left(3-\theta^2\right)\left(1-\theta^2\right)^4}{\left(1+\theta^2\right)^7}$, $a_4^{(1)}=-2\frac{\theta\left(3-\theta^2\right)}{\left(1+\theta^2\right)^3}\left(1-\theta^2\right)$, $a_5^{(1)}=-2\frac{\theta\left(3-\theta^2\right)}{1+\theta^3}\frac{\left(1-\theta^2\right)^4}{\left(1+\theta^2\right)^5}\kappa_3^2$, $a_6^{(1)}=-6\frac{\left(1-\theta^2\right)^2-4\theta^2}{\left(1+\theta^2\right)^4}\frac{\left(1-\theta^2\right)^6}{\left(1+\theta^2\right)^6}$, $a_7^{(1)}=-6\frac{\left(1-\theta^2\right)^2-4\theta^2}{\left(1+\theta^2\right)^4}\frac{\left(1-\theta^2\right)^3}{\left(1+\theta^2\right)^2}$, $a_8^{(1)}=4\frac{\theta^2\left(3-\theta^2\right)^2}{\left(1+\theta^2\right)^6}\frac{\left(1-\theta^2\right)^4}{\left(1+\theta^2\right)^4}\left(1-\theta^2\right)$, $a_9^{(1)}=4\frac{\theta^2\left(3-\theta^2\right)^2}{\left(1+\theta^2\right)^6}\left(1-\theta^2\right)^2$, $a_{10}^{(1)}=-2\frac{\theta\left(3-\theta^2\right)}{1+\theta^3}\frac{\left(1-\theta^2\right)^7}{\left(1+\theta^2\right)^9}\kappa_3^2$, $a_{11}^{(1)}=(2\theta-1)\frac{\left(1-\theta^2\right)}{\left(1+\theta^2\right)^2}$, and $a_{12}^{(1)}=-2\frac{\theta\left(3-\theta^2\right)}{\left(1+\theta^2\right)^3}(2\theta-1)\frac{\left(1-\theta^2\right)^2}{\left(1+\theta^2\right)^2}$.

Hence By Lemma 1

$$k_1^{\widetilde{\rho}}=-\frac{(1-\theta)\left(1+2\theta+3\theta^2\right)}{\sqrt{n}}\frac{\left(1-\theta^2\right)}{\left(1+\theta^2\right)^3},$$

and

$$\begin{aligned}E\left[\sqrt{n}\left(\widetilde{\rho}-\rho\right)\right]^2 &= \frac{\left(1-\theta^2\right)^3}{\left(1+\theta^2\right)^4}+\frac{1}{4n}\frac{\left(34\theta+239\theta^2-4\theta^3-245\theta^4-38\theta^5+69\theta^6+25\right)\left(1-\theta^2\right)^2}{\left(\theta^2+1\right)^6}\\ &\quad+\frac{1}{n}\left[-\theta(1-\theta)\frac{\left(1-\theta^2\right)^3\left(2-\theta+\theta^2+\theta^3-\theta^4\right)}{\left(1+\theta^2\right)^5\left(\theta^2-\theta+1\right)^2}\kappa_3^2-\frac{\left(1-\theta^2\right)^3}{\left(1+\theta^2\right)^4}\kappa_4\right].\end{aligned}$$

For the **zero-mean** case, notice that the Edgeworth coefficients that are different from the ones given above are: $\omega^{(3)}=\left(8-\theta^2+2\frac{\theta^2-\theta-1}{\theta^2-\theta+1}\frac{\theta\left(1-\theta^2\right)^2}{1+\theta^3}\kappa_3^2-\left(1-\theta^2\right)\kappa_4\right)\frac{\left(1-\theta^2\right)^2}{\left(1+\theta^2\right)^4}$, $a_{11}^{(1)}=\theta\frac{\left(1-\theta^2\right)}{\left(1+\theta^2\right)^2}$, and $a_{12}^{(1)}=-2\frac{\theta^2\left(3-\theta^2\right)}{\left(1+\theta^2\right)^3}\frac{\left(1-\theta^2\right)^2}{\left(1+\theta^2\right)^2}$. It follows that

$$k_1^{\widetilde{\rho_0}}=-\frac{2}{\sqrt{n}}\frac{\theta\left(1-\theta^2\right)^2}{\left(1+\theta^2\right)^3},$$

and

$$E\left[\sqrt{n}\left(\widetilde{\rho}_0-\rho\right)\right]^2=\frac{\left(1-\theta^2\right)^3}{\left(1+\theta^2\right)^4}+\frac{1}{n}\left(\begin{array}{c}2\left(32\theta^2-29\theta^4+6\theta^6+1\right)\frac{\left(1-\theta^2\right)^2}{\left(\theta^2+1\right)^6}\\ -\theta(1-\theta)\frac{\left(1-\theta^2\right)^3\left(2-\theta+\theta^2+\theta^3-\theta^4\right)}{\left(1+\theta^2\right)^5\left(\theta^2-\theta+1\right)^2}\kappa_3^2-\frac{\left(1-\theta^2\right)^3}{\left(1+\theta^2\right)^4}\kappa_4\end{array}\right).$$

	THETA MM			THETA QML		
$\mathbf{n=50}$, $u_t \overset{iid}{\sim}$ $non-central\ Student-t$ with $\mathbf{20}$ df, $and\ non-centrality = 1$						
Theta	**Appr. Bias**	**Est. Bias**	**Bias Feas.**	**Appr. Bias**	**Est. Bias**	**Bias Feas.**
-0.9	-119.9542	2.6921	69.8340	-0.3960	-0.3219	-0.0355
-0.8	-12.4383	1.8054	157.3671	-0.3677	-0.2655	-0.0002
-0.7	-3.0324	1.0435	188.8647	-0.3394	-0.1749	0.0686
-0.6	-1.0691	0.4293	30.4886	-0.3111	-0.1310	0.0917
-0.5	-0.4825	0.0414	228.7782	-0.2828	-0.1381	0.0646
-0.4	-0.2715	-0.1237	5.1714	-0.2546	-0.1601	0.0231
-0.3	-0.1884	-0.1315	0.3206	-0.2263	-0.1631	0.0002
-0.2	-0.1553	-0.1129	0.0167	-0.1980	-0.1481	-0.0051
-0.1	-0.1437	-0.0998	0.0070	-0.1697	-0.1260	-0.0035
0	-0.1414	-0.0958	0.0055	-0.1414	-0.1037	-0.0016
0.1	-0.1419	-0.0944	0.0047	-0.1131	-0.0817	-0.0001
0.2	-0.1397	-0.0902	-0.0011	-0.0849	-0.0608	0.0004
0.3	-0.1249	-0.0781	-0.1714	-0.0566	-0.0434	-0.0025
0.4	-0.0737	-0.0995	-1.3525	-0.0283	-0.0454	-0.0245
0.5	0.0818	-0.2445	-53.2987	0.0000	-0.0691	-0.0677
0.6	0.5699	-0.5946	-49.9888	0.0283	-0.0959	-0.1140
0.7	2.3447	-1.1736	-475.2946	0.0566	-0.1032	-0.1411
0.8	11.3308	-1.9416	-99.9750	0.0849	-0.1070	-0.1648
0.9	117.4888	-2.8222	-179.6870	0.1131	-0.1636	-0.2403
$\mathbf{n=200}$, $u_t \overset{iid}{\sim}$ $non-central\ Student-t$ with $\mathbf{20}$ df, $and\ non-centrality = 1$						
-0.9	-59.9771	3.0855	146.5577	-0.1980	-0.0446	0.1538
-0.8	-6.2191	1.8673	312.9910	-0.1838	-0.0733	0.1113
-0.7	-1.5162	0.8553	1457.2564	-0.1697	-0.0694	0.1010
-0.6	-0.5345	0.1583	23.1202	-0.1556	-0.0807	0.0757
-0.5	-0.2412	-0.1320	13.1220	-0.1414	-0.1141	0.0284
-0.4	-0.1357	-0.1449	0.1865	-0.1273	-0.1280	0.0006
-0.3	-0.0942	-0.1009	0.0095	-0.1131	-0.1183	-0.0040
-0.2	-0.0776	-0.0802	0.0017	-0.0990	-0.1041	-0.0041
-0.1	-0.0718	-0.0731	0.0002	-0.0849	-0.0891	-0.0033
0	-0.0707	-0.0716	-0.0005	-0.0707	-0.0735	-0.0020
0.1	-0.0709	-0.0716	-0.0010	-0.0566	-0.0581	-0.0009
0.2	-0.0698	-0.0701	-0.0021	-0.0424	-0.0435	-0.0006
0.3	-0.0624	-0.0611	-0.0083	-0.0283	-0.0300	-0.0015
0.4	-0.0368	-0.0365	-0.0953	-0.0141	-0.0209	-0.0065
0.5	0.0409	-0.0505	-8.2296	0.0000	-0.0315	-0.0312
0.6	0.2849	-0.3287	-61.1723	0.0141	-0.0623	-0.0758
0.7	1.1723	-0.9893	-603.7911	0.0283	-0.0833	-0.1108
0.8	5.6654	-1.9529	-1046.4400	0.0424	-0.0903	-0.1318
0.9	58.7444	-3.1666	-304.7226	0.0566	-0.1710	-0.2259

Table 1: Biases of the MA Coefficient Estimators

	THETA MM			THETA QML		
$\mathbf{n = 50}$, $u_t \overset{iid}{\sim} non-central\ Student-t$ with **11** df, $and\ non-centrality = 1$						
Theta	**Appr. Bias**	**Est. Bias**	**Bias Feas.**	**Appr. Bias**	**Est. Bias**	**Bias Feas.**
-0.9	-119.9542	2.3323	508.2411	-0.3960	-0.9820	-0.5467
-0.8	-12.4383	1.6906	3440.5556	-0.3677	-0.9269	-0.5221
-0.7	-3.0324	1.11025	194.5709	-0.3394	-0.6700	-0.3038
-0.6	-1.0691	0.61611	87.0193	-0.3111	-0.4516	-0.1224
-0.5	-0.4825	0.2364	915.3231	-0.2828	-0.3228	-0.0270
-0.4	-0.2715	-0.0088	46.2209	-0.2546	-0.2782	-0.0125
-0.3	-0.1884	-0.1279	2.2324	-0.2263	-0.2601	-0.0234
-0.2	-0.1553	-0.1583	10.1938	-0.1980	-0.2402	-0.0326
-0.1	-0.1437	-0.1565	14.6839	-0.1697	-0.2102	-0.0320
0	-0.1414	-0.1491	-0.0003	-0.1414	-0.1741	-0.0257
0.1	-0.1419	-0.1451	-0.0948	-0.1131	-0.1399	-0.0212
0.2	-0.1397	-0.14757	-1.5566	-0.0849	-0.1129	-0.0235
0.3	-0.1249	-0.1737	-8.4322	-0.0566	-0.1005	-0.0399
0.4	-0.0737	-0.2663	-15.5777	-0.0283	-0.1058	-0.0733
0.5	0.0818	-0.4777	-13.7759	0.0000	-0.1177	-0.1130
0.6	0.5699	-0.8195	-193.7762	0.0283	-0.1170	-0.1406
0.7	2.3447	-1.2831	-94.4679	0.0566	-0.0833	-0.1365
0.8	11.3308	-1.8498	-96.4860	0.0849	-0.0234	-0.1074
0.9	117.4888	-2.4850	-1230.0230	0.1131	0.0168	-0.0970
$\mathbf{n = 200}$, $u_t \overset{iid}{\sim} non-central\ Student-t$ with **11** df, $and\ non-centrality = 1$						
-0.9	-59.9771	3.0777	2082.6805	-0.1980	-0.0479	0.1506
-0.8	-6.2191	1.8795	126.6579	-0.1838	-0.0718	0.1128
-0.7	-1.5162	0.8624	67.4715	-0.1697	-0.0666	0.1038
-0.6	-0.5345	0.1755	146.7298	-0.1556	-0.0757	0.0806
-0.5	-0.2412	-0.1280	16.4453	-0.1414	-0.1133	0.0292
-0.4	-0.1357	-0.1445	0.9975	-0.1273	-0.1251	0.0034
-0.3	-0.0942	-0.1004	0.0101	-0.1131	-0.1163	-0.0020
-0.2	-0.0776	-0.0793	0.0027	-0.0990	-0.1022	-0.0022
-0.1	-0.0718	-0.0720	0.0014	-0.0849	-0.0873	-0.0016
0	-0.0707	-0.0702	0.0010	-0.0707	-0.0719	-0.0004
0.1	-0.0709	-0.0700	0.0006	-0.0566	-0.0566	0.0006
0.2	-0.0698	-0.0681	-0.0001	-0.0424	-0.0421	0.0008
0.3	-0.0624	-0.0580	-0.0043	-0.0283	-0.0286	0.0000
0.4	-0.0368	-0.0297	-104.2074	-0.0141	-0.0199	-0.0055
0.5	0.0409	-0.0571	-2.3114	0.0000	-0.0342	-0.0339
0.6	0.2849	-0.3269	-26.6365	0.0141	-0.0653	-0.0788
0.7	1.1723	-0.9924	-43.5185	0.0283	-0.0813	-0.1088
0.8	5.6654	-1.9865	-52.7870	0.0424	-0.0926	-0.1341
0.9	58.7444	-3.1849	-111.6753	0.0566	-0.1725	-0.2274

Table 2: Biases of the MA Coefficient Estimators

	RHO MM			RHO QML		
$\mathbf{n=50}$, $u_t \overset{iid}{\sim} non-central\ Student-t$ with **20** df, $and\ non-centrality = 1$						
Theta	**Appr. Bias**	**Est. Bias**	**Bias Feas.**	**Appr. Bias**	**Est. Bias**	**Bias Feas.**
-0.9	0.0699	0.5784	0.5426	-0.0140	0.0290	0.0357
-0.8	0.0672	0.5235	0.4886	-0.0274	0.0135	0.0301
-0.7	0.0617	0.4277	0.3951	-0.0397	0.0103	0.0365
-0.6	0.0522	0.2981	0.2697	-0.0507	0.0111	0.0458
-0.5	0.0373	0.1621	0.1413	-0.0611	-0.0018	0.0406
-0.4	0.0153	0.0565	0.0486	-0.0725	-0.0282	0.0224
-0.3	-0.0150	0.0017	0.0133	-0.0866	-0.0527	0.0079
-0.2	-0.0534	-0.0328	0.0041	-0.1043	-0.0713	0.0012
-0.1	-0.0972	-0.0628	0.0032	-0.1241	-0.0868	-0.0016
0	-0.1414	-0.0928	0.0026	-0.1414	-0.1001	-0.0041
0.1	-0.1801	-0.1194	0.0021	-0.1504	-0.1075	-0.0059
0.2	-0.2085	-0.1396	0.0017	-0.1468	-0.1068	-0.0072
0.3	-0.2250	-0.1550	-0.0015	-0.1301	-0.0987	-0.0093
0.4	-0.2309	-0.1890	-0.0300	-0.1041	-0.0939	-0.0206
0.5	-0.2297	-0.2625	-0.1023	-0.0747	-0.0914	-0.0368
0.6	-0.2250	-0.3687	-0.2091	-0.0472	-0.0808	-0.0444
0.7	-0.2197	-0.4822	-0.3233	-0.0253	-0.0595	-0.0387
0.8	-0.2154	-0.5719	-0.4136	-0.0104	-0.0402	-0.0305
0.9	-0.2129	-0.6231	-0.4651	-0.0024	-0.0321	-0.0285
$\mathbf{n=200}$, $u_t \overset{iid}{\sim} non-central\ Student-t$ with **20** df, $and\ non-centrality = 1$						
-0.9	0.0350	0.5627	0.5341	-0.0070	0.0119	0.0187
-0.8	0.0336	0.4860	0.4581	-0.0137	0.0068	0.0201
-0.7	0.0308	0.3535	0.3271	-0.0198	0.0078	0.0272
-0.6	0.0261	0.1948	0.1715	-0.0253	0.0041	0.0291
-0.5	0.0187	0.0700	0.0527	-0.0305	-0.0144	0.0158
-0.4	0.0077	0.0131	0.0061	-0.0362	-0.0332	0.0027
-0.3	-0.0075	-0.0079	-0.0001	-0.0433	-0.0439	-0.0009
-0.2	-0.0267	-0.0271	-0.0008	-0.0521	-0.0540	-0.0023
-0.1	-0.0486	-0.0487	-0.0012	-0.0620	-0.0641	-0.0030
0	-0.0707	-0.0705	-0.0016	-0.0707	-0.0723	-0.0031
0.1	-0.0901	-0.0896	-0.0017	-0.0752	-0.0764	-0.0028
0.2	-0.1043	-0.1037	-0.0017	-0.0734	-0.0747	-0.0028
0.3	-0.1125	-0.1122	-0.0017	-0.0650	-0.0673	-0.0032
0.4	-0.1154	-0.1191	-0.0052	-0.0521	-0.0576	-0.0057
0.5	-0.1148	-0.1537	-0.0399	-0.0373	-0.0537	-0.0158
0.6	-0.1125	-0.2564	-0.1436	-0.0236	-0.0517	-0.0271
0.7	-0.1098	-0.3964	-0.2847	-0.0127	-0.0410	-0.0273
0.8	-0.1077	-0.5122	-0.4012	-0.0052	-0.0260	-0.0200
0.9	-0.1065	-0.5876	-0.4770	-0.0012	-0.0194	-0.0176

Table 3: Biases of First Order Autocorrelation Estimators

	RHO MM			RHO QML		
$\mathbf{n=50}$, $u_t \overset{iid}{\sim}$ *non − central Student − t* with **11** *df, and non − centrality* = 1						
Theta	**Appr. Bias**	**Est. Bias**	**Bias Feas.**	**Appr. Bias**	**Est. Bias**	**Bias Feas.**
-0.9	0.0699	0.5935	0.5529	-0.0140	0.0998	0.0956
-0.8	0.0672	0.5553	0.5163	-0.0274	0.0647	0.0743
-0.7	0.0617	0.4851	0.4493	-0.0397	0.0404	0.0670
-0.6	0.0522	0.3848	0.3547	-0.0507	0.0281	0.0705
-0.5	0.0373	0.2634	0.2431	-0.0611	0.0152	0.0716
-0.4	0.0153	0.1398	0.1353	-0.0725	-0.0164	0.0533
-0.3	-0.0150	0.0353	0.0546	-0.0866	-0.0601	0.0240
-0.2	-0.0534	-0.0390	0.0124	-0.1043	-0.1007	-0.0008
-0.1	-0.0972	-0.0949	-0.0058	-0.1241	-0.1348	-0.0188
0	-0.1414	-0.1396	-0.0114	-0.1414	-0.1585	-0.0292
0.1	-0.1801	-0.1780	-0.0145	-0.1504	-0.1732	-0.0368
0.2	-0.2085	-0.2129	-0.0219	-0.1468	-0.1785	-0.0441
0.3	-0.2250	-0.2539	-0.0447	-0.1301	-0.1777	-0.0547
0.4	-0.2309	-0.3129	-0.0939	-0.1041	-0.1723	-0.0684
0.5	-0.2297	-0.3959	-0.1727	-0.0747	-0.1580	-0.0775
0.6	-0.2250	-0.4851	-0.2606	-0.0472	-0.1300	-0.0740
0.7	-0.2197	-0.5661	-0.3414	-0.0253	-0.0979	-0.0636
0.8	-0.2154	-0.6258	-0.4011	-0.0104	-0.0750	-0.0565
0.9	-0.2129	-0.6590	-0.4345	-0.0024	-0.0724	-0.0619
$\mathbf{n=200}$, $u_t \overset{iid}{\sim}$ *non − central Student − t* with **11** *df, and non − centrality* = 1						
-0.9	0.0350	0.5606	0.5320	-0.0070	0.0117	0.0185
-0.8	0.0336	0.4873	0.4594	-0.0137	0.0067	0.0200
-0.7	0.0308	0.3568	0.3304	-0.0198	0.0082	0.0277
-0.6	0.0261	0.2008	0.1776	-0.0253	0.0058	0.0308
-0.5	0.0187	0.0725	0.0553	-0.0305	-0.0139	0.0164
-0.4	0.0077	0.0144	0.0075	-0.0362	-0.0312	0.0048
-0.3	-0.0075	-0.0066	0.0012	-0.0433	-0.0421	0.0010
-0.2	-0.0267	-0.0256	0.0007	-0.0521	-0.0520	-0.0003
-0.1	-0.0486	-0.0473	0.0003	-0.0620	-0.0623	-0.0011
0	-0.0707	-0.0691	-0.0001	-0.0707	-0.0707	-0.0014
0.1	-0.0901	-0.0883	-0.0004	-0.0752	-0.0751	-0.0016
0.2	-0.1043	-0.1025	-0.0005	-0.0734	-0.0739	-0.0020
0.3	-0.1125	-0.1109	-0.0005	-0.0650	-0.0668	-0.0027
0.4	-0.1154	-0.1178	-0.0040	-0.0521	-0.0575	-0.0056
0.5	-0.1148	-0.1564	-0.0425	-0.0373	-0.0553	-0.0174
0.6	-0.1125	-0.2564	-0.1437	-0.0236	-0.0531	-0.0285
0.7	-0.1098	-0.3976	-0.2859	-0.0127	-0.0406	-0.0269
0.8	-0.1077	-0.5192	-0.4081	-0.0052	-0.0264	-0.0205
0.9	-0.1065	-0.5911	-0.4804	-0.0012	-0.0196	-0.0179

Table 4: Biases of First Order Autocorrelation Estimators

	THETA MM				THETA QML			
	$\mathbf{n = 50}$, $u_t \overset{iid}{\sim} non-central\ Student-t$ with $\mathbf{20}$ df, $and\ non-centrality = 1$							
Theta	**As. Var.**	**AMSE**	**Est. MSE**	**MSE Feas.**	**As. Var.**	**AMSE**	**Est. MSE**	**MSE Feas.**
-0.9	149.4822	45798.7215	9.2603	3.0017×10^6	0.1900	0.6248	1.6791	1.5143
-0.8	28.6136	484.4521	5.3034	1.4480×10^8	0.3600	0.7706	1.3503	1.2291
-0.7	10.0950	33.1730	3.1428	2.6871×10^8	0.5100	0.8973	1.0593	0.9926
-0.6	4.7409	6.7677	2.3402	2.3989×10^6	0.6400	1.0051	0.9527	0.9069
-0.5	2.7014	2.8776	2.1187	8.8767×10^8	0.7500	1.0238	0.9517	0.8999
-0.4	1.7958	1.7794	1.9282	95072.049	0.8400	1.0827	1.0117	0.9476
-0.3	1.3564	1.3302	1.5380	400.5876	0.9100	1.1250	1.0847	1.0162
-0.2	1.1355	1.1157	1.2447	1.1304	0.9600	1.1505	1.1306	1.0648
-0.1	1.0309	1.0120	1.0824	1.0504	0.9900	1.1594	1.1455	1.0849
0	1.0000	0.9717	1.0316	1.0170	1.0000	1.1517	1.1420	1.0865
0.1	1.0309	0.9779	1.0612	1.0457	0.9900	1.1272	1.1188	1.0680
0.2	1.1355	1.0315	1.1868	1.1654	0.9600	1.0861	1.0779	1.0316
0.3	1.3564	1.1500	1.4654	128.0959	0.9100	1.0284	1.0163	0.9742
0.4	1.7958	1.3781	1.8267	3558.4082	0.8400	0.9541	0.9270	0.8889
0.5	2.7014	1.8718	2.1320	2.5005×10^7	0.7500	0.8633	0.8386	0.8054
0.6	4.7409	3.7244	2.5099	1.6114×10^7	0.6400	0.7559	0.7542	0.7285
0.7	10.0950	20.8664	3.5297	2.6532×10^9	0.5100	0.6319	0.6671	0.6504
0.8	28.6136	402.3640	5.8394	3.7682×10^7	0.3600	0.4913	0.6040	0.5963
0.9	149.4822	44033.2160	10.0036	6.0477×10^7	0.1900	0.3340	0.6926	0.6972
	$\mathbf{n = 200}$, $u_t \overset{iid}{\sim} non-central\ Student-t$ with $\mathbf{20}$ df, $and\ non-centrality = 1$							
-0.9	149.4822	11561.7921	12.4627	5.0189×10^7	0.1900	0.2980	0.6478	0.6566
-0.8	28.6136	142.5732	6.5330	8.7613×10^8	0.3600	0.4568	0.5673	0.5631
-0.7	10.0950	15.8645	3.8782	3.8452×10^{10}	0.5100	0.5965	0.6426	0.6353
-0.6	4.7409	5.2476	3.2167	9.8300×10^5	0.6400	0.7171	0.7293	0.7141
-0.5	2.7014	2.7454	2.8204	1.6813×10^6	0.7500	0.8185	0.8226	0.7942
-0.4	1.7958	1.7917	2.0722	25.0913	0.8400	0.9007	0.9177	0.8834
-0.3	1.3564	1.3498	1.4596	1.3486	0.9100	0.9637	0.9835	0.9503
-0.2	1.1355	1.1305	1.1697	1.1387	0.9600	1.0076	1.0270	0.9960
-0.1	1.0309	1.0262	1.0414	1.0296	0.9900	1.0324	1.0516	1.0229
0	1.0000	0.9929	1.0014	0.9952	1.0000	1.0379	1.0555	1.0293
0.1	1.0309	1.0177	1.0304	1.0240	0.9900	1.0243	1.0390	1.0150
0.2	1.1355	1.1095	1.1417	1.1271	0.9600	0.9915	1.0028	0.9810
0.3	1.3564	1.3048	1.3960	1.3386	0.9100	0.9396	0.9489	0.9292
0.4	1.7958	1.6914	1.9229	21.9595	0.8400	0.8685	0.8761	0.8583
0.5	2.7014	2.4940	2.7137	3.7218×10^5	0.7500	0.7783	0.7795	0.7639
0.6	4.7409	4.4867	3.2293	2.1971×10^7	0.6400	0.6690	0.6760	0.6644
0.7	10.0950	12.7878	4.1397	5.6913×10^9	0.5100	0.5405	0.5753	0.5693
0.8	28.6136	122.0512	6.9133	1.5682×10^{10}	0.3600	0.3928	0.4505	0.4509
0.9	149.4822	11120.4157	13.0629	8.0211×10^8	0.1900	0.2260	0.3895	0.4041

Table 5: MSEs of the MA Coefficient Estimators

	THETA MM				THETA QML			
	$\mathbf{n = 50}$, $u_t \overset{iid}{\sim}$ $non-central\ Student-t$ with $\mathbf{11}$ df, $and\ non-centrality = 1$							
Theta	**As. Var.**	**AMSE**	**Est. MSE**	**MSE Feas.**	**As. Var.**	**AMSE**	**Est. MSE**	**MSE Feas.**
-0.9	149.4822	45796.7977	6.7796	3.1337×10^{9}	0.1900	0.6223	4.8244	3.8564
-0.8	28.6136	484.0577	4.2198	4.3013×10^{11}	0.3600	0.7427	4.6397	3.7568
-0.7	10.0950	33.0270	2.6322	4.3913×10^{8}	0.5100	0.8496	3.7673	3.1505
-0.6	4.7409	6.6975	1.8382	3.5795×10^{7}	0.6400	0.9402	2.8431	2.4473
-0.5	2.7014	2.8377	1.5646	2.6715×10^{10}	0.7500	1.0145	2.1713	1.9058
-0.4	1.7958	1.7535	1.5122	3.3288×10^{7}	0.8400	1.0724	1.7860	1.5748
-0.3	1.3564	1.3113	1.4555	5542.5185	0.9100	1.1139	1.5719	1.3869
-0.2	1.1355	1.1004	1.3234	3.4207×10^{6}	0.9600	1.1389	1.4998	1.3301
-0.1	1.0309	0.9985	1.1855	8.5623×10^{6}	0.9900	1.1473	1.4464	1.2934
0	1.0000	0.9591	1.1132	1.0635	1.0000	1.1391	1.3999	1.2629
0.1	1.0309	0.9653	1.1329	148.5750	0.9900	1.1143	1.3414	1.2186
0.2	1.1355	1.0182	1.2347	47993.0271	0.9600	1.0730	1.2802	1.1687
0.3	1.3564	1.1350	1.3926	1.8318×10^{6}	0.9100	1.0155	1.1987	1.0970
0.4	1.7958	1.3605	1.5400	1.5703×10^{6}	0.8400	0.9419	1.1269	1.0336
0.5	2.7014	1.8514	1.7182	6.3816×10^{5}	0.7500	0.8522	1.0895	1.0041
0.6	4.7409	3.7094	2.1307	9.0900×10^{8}	0.6400	0.7466	1.0673	0.9908
0.7	10.0950	20.9308	3.0811	3.0584×10^{7}	0.5100	0.6247	1.1649	1.0859
0.8	28.6136	403.3154	4.8125	6.5571×10^{7}	0.3600	0.4864	1.4213	1.3209
0.9	149.4822	44059.5724	7.5403	3.2970×10^{10}	0.1900	0.3315	1.9286	1.7865
	$\mathbf{n = 200}$, $u_t \overset{iid}{\sim}$ $non-central\ Student-t$ with $\mathbf{11}$ df, $and\ non-centrality = 1$							
-0.9	149.4822	11561.3110	12.4152	5.8496×10^{10}	0.1900	0.2974	0.6451	0.6527
-0.8	28.6136	142.4745	6.5224	1.1308×10^{8}	0.3600	0.4557	0.5528	0.5494
-0.7	10.0950	15.8279	3.9224	8.6988×10^{6}	0.5100	0.5949	0.6340	0.6278
-0.6	4.7409	5.2300	3.2107	1.8720×10^{8}	0.6400	0.7151	0.7278	0.7142
-0.5	2.7014	2.7354	2.8316	1.4926×10^{6}	0.7500	0.8161	0.8262	0.7980
-0.4	1.7958	1.7852	2.0949	9949.5922	0.8400	0.8981	0.9222	0.8886
-0.3	1.3564	1.3450	1.4791	1.3617	0.9100	0.9610	0.9909	0.9579
-0.2	1.1355	1.12672	1.1855	1.1542	0.9600	1.0047	1.0355	1.0047
-0.1	1.0309	1.0228	1.0562	1.0443	0.9900	1.0293	1.0614	1.0328
0	1.0000	0.9897	1.0161	1.0100	1.0000	1.0348	1.0663	1.0400
0.1	1.0309	1.0145	1.0458	1.0397	0.9900	1.0211	1.0509	1.0268
0.2	1.1355	1.1061	1.1591	1.1446	0.9600	0.9883	1.0153	0.9934
0.3	1.3564	1.3010	1.4205	1.3530	0.9100	0.9364	0.9616	0.9417
0.4	1.7958	1.6870	1.9883	2.1583×10^{8}	0.8400	0.8655	0.8869	0.8689
0.5	2.7014	2.4889	2.6981	5142.8915	0.7500	0.7756	0.7847	0.7691
0.6	4.7409	4.4829	3.2454	3.7048×10^{6}	0.6400	0.6666	0.6833	0.6717
0.7	10.0950	12.8039	4.1560	1.9465×10^{6}	0.5100	0.5387	0.5780	0.5719
0.8	28.6136	122.2890	6.9785	3.1515×10^{6}	0.3600	0.3916	0.4568	0.4572
0.9	149.4822	11127.0047	13.1387	2.3224×10^{7}	0.1900	0.2254	0.3954	0.4100

Table 6: MSEs of the MA Coefficient Estimators

	RHO MM				RHO QML			
	$\mathbf{n=50}$, $u_t \overset{iid}{\sim}$ *non − central Student − t* with **20** *df*, *and non − centrality* = 1							
Theta	**As. Var.**	**AMSE**	**Est. MSE**	**MSE Feas.**	**As. Var.**	**AMSE**	**Est. MSE**	**MSE Feas.**
-0.9	0.5028	0.5046	0.5528	0.5240	0.0006	0.0019	0.0043	0.0055
-0.8	0.5126	0.5135	0.5121	0.4894	0.0064	0.0125	0.0126	0.0151
-0.7	0.5327	0.5319	0.4569	0.4447	0.0269	0.0431	0.0402	0.0445
-0.6	0.5676	0.5637	0.4383	0.4413	0.0766	0.1094	0.0985	0.1050
-0.5	0.6224	0.6130	0.4896	0.5097	0.1728	0.2272	0.1990	0.2075
-0.4	0.6998	0.6819	0.6155	0.6515	0.3273	0.4031	0.3580	0.3686
-0.3	0.7957	0.7667	0.7486	0.7964	0.5338	0.6225	0.5763	0.5906
-0.2	0.8945	0.8556	0.8596	0.9148	0.7563	0.8461	0.8094	0.8277
-0.1	0.9710	0.9303	0.9348	0.9910	0.9324	1.0190	0.9912	1.0089
0	1.0000	0.9717	0.9692	1.0176	1.0000	1.0917	1.0656	1.0743
0.1	0.9710	0.9679	0.9531	0.9858	0.9324	1.0398	1.0058	0.9989
0.2	0.8945	0.9209	0.8922	0.9052	0.7563	0.8764	0.8338	0.8118
0.3	0.7957	0.8455	0.8011	0.7949	0.5338	0.6485	0.6054	0.5758
0.4	0.6998	0.7627	0.6816	0.6532	0.3273	0.4174	0.3840	0.3551
0.5	0.6224	0.6894	0.5697	0.5113	0.1728	0.2307	0.2173	0.1947
0.6	0.5676	0.6341	0.5191	0.4243	0.0766	0.1069	0.1095	0.0955
0.7	0.5327	0.5971	0.5373	0.4061	0.0269	0.0395	0.0458	0.0394
0.8	0.5126	0.5753	0.5864	0.4270	0.0064	0.0103	0.0152	0.0130
0.9	0.5028	0.5645	0.6280	0.4526	0.0006	0.0013	0.0050	0.0043
	$\mathbf{n=200}$, $u_t \overset{iid}{\sim}$ *non − central Student − t* with **20** *df*, *and non − centrality* = 1							
-0.9	0.5028	0.5032	0.5272	0.5010	0.0006	0.0009	0.0021	0.0026
-0.8	0.5126	0.5128	0.4731	0.4526	0.0064	0.0080	0.0094	0.0104
-0.7	0.5327	0.5325	0.4187	0.4081	0.0269	0.0310	0.0333	0.0352
-0.6	0.5676	0.5666	0.4350	0.4367	0.0766	0.0848	0.0853	0.0880
-0.5	0.6224	0.6201	0.5460	0.5588	0.1728	0.1864	0.1818	0.1848
-0.4	0.6998	0.6954	0.6749	0.6947	0.3273	0.3463	0.3403	0.3442
-0.3	0.7957	0.7884	0.7760	0.8006	0.5338	0.5560	0.5499	0.5561
-0.2	0.8945	0.8848	0.8689	0.8968	0.7563	0.7787	0.7735	0.7822
-0.1	0.9710	0.9608	0.9415	0.9696	0.9324	0.9541	0.9516	0.9603
0	1.0000	0.9929	0.9714	0.9953	1.0000	1.0229	1.0222	1.0266
0.1	0.9710	0.9702	0.9484	0.9638	0.9324	0.9593	0.9583	0.9547
0.2	0.8945	0.9011	0.8809	0.8860	0.7563	0.7863	0.7842	0.7729
0.3	0.7957	0.8082	0.7903	0.7860	0.5338	0.5625	0.5607	0.5458
0.4	0.6998	0.7155	0.6939	0.6819	0.3273	0.3499	0.3496	0.3359
0.5	0.6224	0.6392	0.5808	0.5571	0.1728	0.1873	0.1890	0.1789
0.6	0.5676	0.5842	0.4808	0.4329	0.0766	0.0842	0.0889	0.0828
0.7	0.5327	0.5488	0.4689	0.3900	0.0269	0.0301	0.0352	0.0324
0.8	0.5126	0.5283	0.5145	0.4106	0.0064	0.0074	0.0101	0.0093
0.9	0.5028	0.5182	0.5697	0.4494	0.0006	0.0008	0.0024	0.0022

Table 7: MSEs of the First Order Autocorrelation Estimators

	RHO MM				RHO QML			
$\mathbf{n = 50}$, $u_t \overset{iid}{\sim} non-central\ Student-t$ with **11** df, $and\ non-centrality = 1$								
Theta	**As. Var.**	**AMSE**	**Est. MSE**	**MSE Feas.**	**As. Var.**	**AMSE**	**Est. MSE**	**MSE Feas.**
-0.9	0.5028	0.5004	0.5794	0.5585	0.0006	0.0013	0.0236	0.0223
-0.8	0.5126	0.5090	0.5505	0.5361	0.0064	0.0105	0.0260	0.0307
-0.7	0.5327	0.5269	0.5074	0.5052	0.0269	0.0391	0.0560	0.0677
-0.6	0.5676	0.5581	0.4731	0.4890	0.0766	0.1034	0.1283	0.1470
-0.5	0.6224	0.6067	0.4812	0.5205	0.1728	0.2200	0.2469	0.2723
-0.4	0.6998	0.6747	0.5521	0.6171	0.3273	0.3970	0.4146	0.4455
-0.3	0.7957	0.7582	0.6823	0.7713	0.5338	0.6213	0.6332	0.6685
-0.2	0.8945	0.8456	0.8205	0.9255	0.7563	0.8538	0.8734	0.9108
-0.1	0.9710	0.9188	0.9220	1.0291	0.9324	1.0375	1.0707	1.1013
0	1.0000	0.9591	0.9691	1.0612	1.0000	1.1191	1.1615	1.1727
0.1	0.9710	0.9550	0.9604	1.0223	0.9324	1.0704	1.1167	1.0988
0.2	0.8945	0.9083	0.8990	0.9214	0.7563	0.9035	0.9542	0.9079
0.3	0.7957	0.8340	0.7986	0.7769	0.5338	0.6679	0.7241	0.6610
0.4	0.6998	0.7524	0.6893	0.6209	0.3273	0.4283	0.4912	0.4272
0.5	0.6224	0.6803	0.6198	0.5017	0.1728	0.2351	0.3023	0.2502
0.6	0.5676	0.6260	0.6035	0.4388	0.0766	0.1077	0.1663	0.1325
0.7	0.5327	0.5899	0.6297	0.4258	0.0269	0.0390	0.0799	0.0621
0.8	0.5126	0.5687	0.6663	0.4344	0.0064	0.0098	0.0342	0.0263
0.9	0.5028	0.5584	0.6920	0.4449	0.0006	0.0011	0.0199	0.0156
$\mathbf{n = 200}$, $u_t \overset{iid}{\sim} non-central\ Student-t$ with **11** df, $and\ non-centrality = 1$								
-0.9	0.5028	0.5022	0.5236	0.4974	0.0006	0.0008	0.0022	0.0027
-0.8	0.5126	0.5117	0.4722	0.4516	0.0064	0.0075	0.0095	0.0105
-0.7	0.5327	0.5313	0.4213	0.4106	0.0269	0.0300	0.0333	0.0352
-0.6	0.5676	0.5652	0.4389	0.4404	0.0766	0.0833	0.0859	0.0887
-0.5	0.6224	0.6185	0.5487	0.5614	0.1728	0.1846	0.1832	0.1862
-0.4	0.6998	0.6935	0.6802	0.7002	0.3273	0.3447	0.3435	0.3475
-0.3	0.7957	0.7863	0.7845	0.8094	0.5338	0.5557	0.5554	0.5618
-0.2	0.8945	0.8823	0.8796	0.9080	0.7563	0.7806	0.7809	0.7899
-0.1	0.9710	0.9579	0.9541	0.9828	0.9324	0.9587	0.9606	0.9696
0	1.0000	0.9898	0.9851	1.0095	1.0000	1.0298	1.0322	1.0368
0.1	0.9710	0.9670	0.9619	0.9779	0.9324	0.9669	0.9687	0.9653
0.2	0.8945	0.8980	0.8933	0.8988	0.7563	0.7931	0.7940	0.7827
0.3	0.7957	0.8053	0.8014	0.7975	0.5338	0.5674	0.5687	0.5537
0.4	0.6998	0.7130	0.7032	0.6914	0.3273	0.3526	0.3546	0.3408
0.5	0.6224	0.6369	0.5834	0.5591	0.1728	0.1884	0.1910	0.1807
0.6	0.5676	0.5822	0.4825	0.4345	0.0766	0.0844	0.0900	0.0837
0.7	0.5327	0.5470	0.4675	0.3884	0.0269	0.0299	0.0351	0.0323
0.8	0.5126	0.5267	0.5188	0.4134	0.0064	0.0073	0.0102	0.0094
0.9	0.5028	0.5167	0.5732	0.4522	0.0006	0.0008	0.0026	0.0024

Table 8: MSEs of the First Order Autocorrelation Estimators