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A Novel Differential Dominance Principle based Approach to the Solution of More than Two Persons n Moves General Sum Games

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Abstract

In a previous paper [1] the application of the dominance principle was proposed to find the non-cooperative solution of two persons two by two general sum game with mixed strategies; in this way it was possible to choose the equilibrium point among the classical solutions avoiding the ambiguity due to their non-interchangeability, moreover the non-cooperative equilibrium point was determined by a new approach based on the dominance principle [2]. Starting from that result it is here below proposed the extension of the method to more than two persons general sum games with n by n moves. Generally speaking the dominance principle can be applied to find the equilibrium point both in pure and mixed strategies. In this paper in order to apply the dominance principle to the mixed strategies solution, the algebraic multi-linear forms of the expected payoffs of the players are studied. From these expressions of the expected payoffs the derivatives are obtained and they are used to express the probabilities distribution on the moves after the two definitions as Nash and prudential strategies [1].

The application of the dominance principle allows to choose the equilibrium point between the two equivalent solutions avoiding the ambiguity due to their non-interchangeability and a conjecture about the uniqueness of the solution is proposed in order to solve the problem of the existence and uniqueness of the non-cooperative solution of a many persons n by n game. The uniqueness of the non-cooperative solution could be used as a starting point to find out the cooperative solution of the game too. Some games from the sound literature are discussed in order to show the effectiveness of the presented procedure.

Keywords: *Dominance principle; General sum game; pure strategy, mixed strategy.*

1 Introduction

The main references for the development of the present paper are my previous paper [1] and [2], the master paper by Nash [3] and the texts of Luce and Raiffa [7], Owen [8], Straffin [9], Maschler [13], Dixit and Skeath [11].

It is proposed to use the dominance principle as the only tool to find the non-cooperative solution of a many persons game on the basis that a rational player should never play a dominated move [9, 13]. Straffin [9] argues that there is a conflict between the dominance principle and the Pareto-optimality, but it has to be noted that the dominance principle is cogent for individual rationality whereas the Pareto-optimality is cogent for the group rationality. The individual rationality is here considered suitable for the non-cooperative solution of many persons game, thus the dominance principle is applied to find the solution.

This paper is devoted to the study of the non-cooperative solution of many persons n by n moves game with no dominated moves, therefore it is not considered the trivial case that can be solved by the elimination of all the dominated moves.

On the other hand, the maximin [17] value of any particular player is unaffected by the elimination of his dominated moves, whether those moves are weakly or strictly dominated; moreover the iterated elimination of weakly dominated moves does not lead to the creation of new equilibria (Maschler et al. [13]).

As it is well known the mixed strategies method to find the solution of a game is suitable only for repeatable games. A mixed strategy for a player is defined as the probability distribution on the set of his pure moves [8]; the expected payoff from a mixed strategy is defined as the corresponding probability-weighted average of the payoffs from its constituent pure moves [11]. The search of the non-cooperative solution with the mixed strategies could bring to find more equivalent equilibrium points, but these equilibrium points represent the acceptable non-cooperative solution of the game only if they are interchangeable too [3, 7].

In Part 2 of the paper it is proposed to look for the non-cooperative solution of a three persons n by n game by applying the dominance principle on the mixed strategies and the relationship is studied among the two classical mixed strategies, prudential and Nash strategy [1], and the expected payoff.

Part 3 is a restriction of the application of the dominance principle to the three persons game with 2 by 2 moves in the framework to discuss some literature examples. It has to be reminded that the dominance principle can be applied also when looking for the solution in pure strategies; moreover if the equilibrium point is not existing in pure strategies, the mixed strategies method can be applied only for repetitive games.

In Part 4 four numerical examples are discussed to show the application of the dominance principle and the so found solutions are compared and discussed with respect to the literature solutions. Furthermore in order to show its powerful meaning, the proposed method is shown to be suitable to find the equilibrium point of a game also when the mixed strategies method fails: in this case, if there are more equilibrium points in pure strategies, the dominance principle can be applied in order to find a stable equilibrium; this stable equilibrium is a pure strategy that can be used for repeatable game too. If a game is not repeatable and there is not any stable equilibrium point in the pure strategies, it is not possible to use a stable equilibrium point in the mixed strategies.

Part 5 is devoted to the extension of the method to more than three persons games, the procedure is illustrated by using a four persons game with obvious extension to many persons game.

The conclusion summarizes the main features of the proposed method recognizing it as a powerful tool to find the non-cooperative solution of a three persons general sum game larger than two by two moves.

2 Non-cooperative solution of the normal form of three persons n moves game

2.1 Theory

The normal form of the three persons n by n game can be deployed with a matrix as the following one per each of the n moves of the third player:

Table 1

		... k-th move of player C ...		
		Moves of player B		
		y_1	$...y_j...$	y_n
	x_1	$a_{11k}, b_{11k}, c_{11k}$	$...a_{1jk}, b_{1jk}, c_{1jk}...$	$a_{1nk}, b_{1nk}, c_{1nk}$
	...	..., ...	..., ...	..., ...
Moves of player A	x_i	$a_{i1k}, b_{i1k}, c_{i1k}$	$...a_{ijk}, b_{ijk}, c_{ijk}...$	$a_{ink}, b_{ink}, c_{ink}$
	...	..., ...	..., ...	..., ...
	x_n	$a_{n1k}, b_{n1k}, c_{n1k}$	$...a_{njk}, b_{njk}, c_{njk}...$	$a_{nnk}, b_{nnk}, c_{nnk}$

being

$$\sum_{i=1}^{n} x_i = 1 \tag{1}$$

$$\sum_{j=1}^{n} y_j = 1 \tag{2}$$

and

$$\sum_{k=1}^{n} z_k = 1 \tag{3}$$

with the constraints

$$0 \leq x_i \leq 1 \tag{4}$$

$$0 \leq y_j \leq 1 \tag{5}$$

$$0 \leq z_k \leq 1 \tag{6}$$

the row vectors of the probability distribution on the moves respectively for player A, B and C are following:

$$(x) = (x_1, ...x_i, ..., x_{n-1}, x_n) = (x_1, ..., x_i, ..., x_{n-1}, 1 - \sum_{i=1}^{n-1} x_i) \tag{7}$$

$$(y) = (y_1, ..., y_j, ..., y_{n-1}, y_n) = (y_1, ..., y_j, ..., y_{n-1}, 1 - \sum_{j=1}^{n-1} y_j) \tag{8}$$

and

$$(z) = (z_1, ..., z_k, ..., z_{n-1}, z_n) = (z_1, ..., z_k, ..., z_{n-1}, 1 - \sum_{k=1}^{n-1} z_k) \tag{9}$$

Associated to each possible outcome of the game is a collection of numerical payoffs, one to each player.

The expected payoff for each player is then given by:

$$U_A = \sum_{k=1}^{n} z_k (x)(H)_{Ak}(y)^T \tag{10}$$

$$U_B = \sum_{k=1}^{n} z_k (x)(H)_{Bk}(y)^T \tag{11}$$

$$U_C = \sum_{k=1}^{n} z_k (x)(H)_{Ck}(y)^T \tag{12}$$

where (x) is the vector probability distribution of player A, (H) is the matrix of the payoff of A, B and C corresponding to the k-th move of C, and $(y)^T$ is the transposed of vector (y). These formulas will be used throughout the paper from here on.

As mentioned in my previous paper [1], in literature there are two ways to calculate the probability distribution for each player: a prudential strategy [9] and a Nash strategy [3]. These two different strategies can be determined by calculating the first derivatives of the expected payoffs and equating them to zero. First of all the Nash strategies are determined.

$$\partial U_A / \partial x_i = \sum_{k=1}^{n} z_k (\delta_{ij})(H)_{Ak}(y)^T = 0 \tag{13}$$

these partial derivatives, with $i = 1, ..., n$, equated to zero are n equations in $2n$ y_j and z_k unknowns and

$$\partial U_B / \partial y_j = \sum_{k=1}^{n} z_k (x)(H)_{Bk}(\delta_{ij})^T = 0 \tag{14}$$

these partial derivatives, with $i = 1, ..., n$, equated to zero are n equations in $2n$ x_i and z_k unknowns and

$$\partial U_C / \partial z_k = (x)(H)_{Ck}(y)^T = 0 \tag{15}$$

these partial derivatives, with $i = 1, ..., n$, equated to zero are n equations in $2n$ x_i and y_j unknowns and the three groups of equations constitute a system of $3n$ equations in $3n$ x_i, y_j and z_k unknowns. The following definition holds for the vector (δ_{ij}): the term δ_{ij} of the vector is equal to 1 when the index of integration is equal to the position of the element in the vector and it is equal to zero otherwise.

The solution of the system gives the probability distribution after Nash (x_N), (y_N) and (z_N), respectively for player A, B and C.

The meaning of the Nash strategy is that if player B chooses (y_N) and C chooses (z_N) then there is no variation of U_A irrespective of the choice of player A; if player A chooses (x_N) and B chooses (y_N) there is no variation of U_C irrespective of the choice of player C; if player A chooses (x_N) and C chooses (z_N) there is no variation of U_B irrespective of the choice of player B.
Therefore it holds:

$$U_A((x_N),(y_N),(z_N)) = U_A((x_p),(y_N),(z_N))$$
$$U_B((x_N),(y_N),(z_N)) = U_B((x_N),(y_p),(z_N))$$
$$U_C((x_N),(y_N),(z_N)) = U_C((x_N),(y_N),(z_p))$$

The prudential strategies are determined as follows.

$$\partial U_A/\partial y_j = \sum_{k=1}^{n} z_k(x)(H)_{Ak}(\delta_{ij})^T = 0 \tag{16}$$

$$\partial U_A/\partial z_k = (x)(H)_{Ak}(y)^T = 0 \tag{17}$$

these partial derivatives, with j and $k = 1, ..., n$, equated to zero are $2n$ equations in $3n$ x_i, y_j and z_k unknowns and

$$\partial U_B/\partial x_i = \sum_{k=1}^{n} z_k(\delta_{ij})(H)_{Bk}(y)^T = 0 \tag{18}$$

$$\partial U_B/\partial z_k = (x)(H)_{Bk}(y)^T = 0 \tag{19}$$

these partial derivatives, with i and $k = 1, ..., n$, equated to zero are $2n$ equations in $3n$ x_i, y_j and z_k unknowns and

$$\partial U_C/\partial x_i = \sum_{k=1}^{n} z_k(\delta_{ij})(H)_{Ck}(y)^T = 0 \tag{20}$$

$$\partial U_C/\partial y_j = \sum_{k=1}^{n} z_k(x)(H)_{Ck}(\delta_{ij})^T = 0 \tag{21}$$

these partial derivatives, with i and $j = 1, ..., n$, equated to zero are $2n$ equations in $3n$ x_i, y_j and z_k unknowns.

The solution of the system of $6n$ equations, if any, in $3n$ x_i, y_j and z_k unknowns gives the prudential probability distribution (x_p), (y_p) and (z_p), respectively for player A, B and C.

The meaning of the prudential strategy is that if player A chooses (x_p) and player C chooses (z_p) then there is no variation of U_A and U_C irrespective of the choice of player B; if player A chooses (x_p) and player B chooses (y_p) there is no variation of U_A and U_B irrespective of the choice of player C; if player B chooses (y_p) and player C chooses (z_p) there is no variation of U_B and U_C irrespective of the choice of player A.

Therefore it holds:

$$
\begin{aligned}
U_A((x_p),(y_N),(z_p)) &= U_A((x_p),(y_p),(z_p)) \\
U_C((x_p),(y_N),(z_p)) &= U_C((x_p),(y_p),(z_p)) \\
U_A((x_p),(y_p),(z_p)) &= U_A((x_p),(y_p),(z_N)) \\
U_B((x_p),(y_p),(z_p)) &= U_B((x_p),(y_p),(z_N)) \\
U_B((x_p),(y_p),(z_p)) &= U_B((x_N),(y_p),(z_p)) \\
U_C((x_p),(y_p),(z_p)) &= U_C((x_N),(y_p),(z_p))
\end{aligned}
$$

The conclusion is that the use of two players of the prudential strategy makes their payoffs independent from the choice of the third player.

In case of existence of both prudential and Nash strategies, by substituting in the formulas of the expected payoffs of each player respectively the prudential strategies and the Nash's strategies it can easily be verified whether the two triplets of strategies $((x_p), (y_p), (z_p))$ and $((x_N), (y_N), (z_N))$ are equivalent and also interchangeable.

In order to find the dominant strategy the following table should be considered, where the elements of the matrix are the payoffs corresponding to the strategies indicated by the subscripts:

Table 2

		C			
		(z_N)	(z_N)	(z_p)	(z_p)
		B			
		(y_N)	(y_p)	(y_N)	(y_p)
A	(x_N)	$U_{ANNN}, U_{BNNN}, U_{CNNN}$	$U_{ANpN}, U_{BNpN}, U_{CNpN}$	$U_{ANNp}, U_{BNNp}, U_{CNNp}$	$U_{ANpp}, U_{BNpp}, U_{CNpp}$
	(x_p)	$U_{ApNN}, U_{BpNN}, U_{CpNN}$	$U_{AppN}, U_{BppN}, U_{CppN}$	$U_{ApNp}, U_{BpNp}, U_{CpNp}$	$U_{Appp}, U_{Bppp}, U_{Cppp}$

By taking into account the above found equivalences the table can be simplified as follows:

Table 3

		C			
		(z_N)	(z_N)	(z_p)	(z_p)
		B			
		(y_N)	(y_p)	(y_N)	(y_p)
A	(x_N)	$U_{ANNN}, U_{BNNN}, U_{CNNN}$	$U_{ANpN}, U_{BNNN}, U_{CNpN}$	$U_{ANNp}, U_{BNNp}, U_{CNNN}$	$U_{ANpp}, U_{Bppp}, U_{Cppp}$
	(x_p)	$U_{ANNN}, U_{BpNN}, U_{CpNN}$	$U_{Appp}, U_{Bppp}, U_{CppN}$	$U_{Appp}, U_{BpNp}, U_{Cppp}$	$U_{Appp}, U_{Bppp}, U_{Cppp}$

The dominance principle should be applied on Table 3 in order to find dominant equilibrium point.

Therefore in order to find the dominant strategy it should be verified:

for A

$$U_A((x_N),(y_p),(z_N)) \geq U_A((x_p),(y_p),(z_p))$$
$$U_A((x_N),(y_N),(z_p)) \geq U_A((x_p),(y_p),(z_p))$$
$$U_A((x_N),(y_p),(z_p)) \geq U_A((x_p),(y_p),(z_p))$$

all inequalities should be verified and at least one in strong way for (x_N) to be dominant, otherwise

$$U_A((x_N),(y_p),(z_N)) \leq U_A((x_p),(y_p),(z_p))$$
$$U_A((x_N),(y_N),(z_p)) \leq U_A((x_p),(y_p),(z_p))$$
$$U_A((x_N),(y_p),(z_p)) \leq U_A((x_p),(y_p),(z_p))$$

all inequalities should be verified and at least one in strong way for (x_p) to be dominant;

for B

$$U_B((x_N),(y_N),(z_p)) \geq U_B((x_p),(y_p),(z_p))$$
$$U_B((x_p),(y_N),(z_N)) \geq U_B((x_p),(y_p),(z_p))$$
$$U_B((x_p),(y_N),(z_p)) \geq U_B((x_p),(y_p),(z_p))$$

all inequalities should be verified and at least one in strong way for (y_N) to be dominant, otherwise

$$U_B((x_N),(y_N),(z_p)) \leq U_B((x_p),(y_p),(z_p))$$
$$U_B((x_p),(y_N),(z_N)) \leq U_B((x_p),(y_p),(z_p))$$
$$U_B((x_p),(y_N),(z_p)) \leq U_B((x_p),(y_p),(z_p))$$

all inequalities should be verified and at least one in strong way for (y_p) to be dominant;

for C

$$U_C((x_N),(y_p),(z_N)) \geq U_C((x_p),(y_p),(z_p))$$
$$U_C((x_p),(y_N),(z_N)) \geq U_C((x_p),(y_p),(z_p))$$
$$U_C((x_p),(y_p),(z_N)) \geq U_C((x_p),(y_p),(z_p))$$

all inequalities should be verified and at least one in strong way for (z_N) to be dominant, otherwise

$$U_C((x_N),(y_p),(z_N)) \leq U_C((x_p),(y_p),(z_p))$$
$$U_C((x_p),(y_N),(z_N)) \leq U_C((x_p),(y_p),(z_p))$$
$$U_C((x_p),(y_p),(z_N)) \leq U_C((x_p),(y_p),(z_p))$$

all inequalities should be verified and at least one in strong way for (z_p) to be dominant.
In case a dominance is found for a player, this dominance could be used to exclude some of the choices of the other players and it could help to find the dominances of the other players. In case no dominance is found, the Nash equilibria could be searched for on the Table 3; if there is not any Nash equilibrium, there is not any equilibrium triplet in the mixed strategy.

If all inequalities are verified as equalities the Nash and prudential strategies are indifferent for all the players, but they are not interchangeable because, generally speaking, it could happen that:

$$U_A((x_N),(y_N),(z_N)) \neq U_A((x_p),(y_p),(z_p))$$
$$U_B((x_N),(y_N),(z_N)) \neq U_B((x_p),(y_p),(z_p))$$
$$U_C((x_N),(y_N),(z_N)) \neq U_C((x_p),(y_p),(z_p))$$

It is worth noting that, if the Nash and prudential strategies are equivalent, it holds:

$$U_A((x_N),(y_N),(z_N)) = U_A((x_p),(y_p),(z_p)) = U_A((x_p),(y_N),(z_N)) = U_A((x_p),(y_N),(z_p)) = $$
$$= U_A((x_p),(y_p),(z_N)) = U_A^*$$
$$U_B((x_N),(y_N),(z_N)) = U_B((x_p),(y_p),(z_p)) = U_B((x_N),(y_p),(z_N)) = U_B((x_N),(y_p),(z_p)) = $$
$$= U_B((x_p),(y_p),(z_N)) = U_B^*$$
$$U_C((x_N),(y_N),(z_N)) = U_C((x_p),(y_p),(z_p)) = U_C((x_N),(y_N),(z_p)) = U_C((x_p),(y_N),(z_p)) = $$
$$= U_C((x_N),(y_p),(z_p)) = U_C^*$$

Table 4

		C			
		(z_N)	(z_N)	(z_p)	(z_p)
		B			
		(y_N)	(y_p)	(y_N)	(y_p)
A	(x_N)	U_A^*, U_B^*, U_C^*	$U_{ANpN}, U_B^*, U_{CNpN}$	U_{ANNp}, U_{BNNp}, U_C*	U_{ANpp}, U_B^*, U_C^*
	(x_p)	$U_A^*, U_{BpNN}, U_{CpNN}$	U_A^*, U_B^*, U_{CppN}	U_A^*, U_{BpNp}, U_C^*	U_A^*, U_B^*, U_C^*

The dominance principle should be applied on Table 4 in order to find the dominant equilibrium point by applying the same above described procedure.

If all inequalities are verified as equalities the Nash and prudential strategies are interchangeable too: in this case there are two stable equilibria of the game.

2.2 Remarks about the solution of three persons n moves games

The remarks in reference [2] to the solution of two persons and n by m moves games are totally applicable to the solution of a three persons game: it is known that a solution of the game exists [8], but there are a lot of different ways to find out that solution. The proposed procedure is very simple for finding the solution also if in some cases it fails and some other ways should be used such as the search for the Nash equilibria. It has to be reminded that the Nash equilibria can be used as mixed strategy too, but the mixed strategy cannot be used as equilibrium point for an unrepeatable game.

The theory is developed for n by n moves, but it can be applied also for n by m moves with suitable adjustment to the equations; in this case it has to be verified whether the conditions for the existence of the solution are satisfied.
The application of the geometric approach, proposed to find the non-cooperative solution of the two by two general sum game with mixed strategies [1], is not recommended in the case of n by m moves games, because it becomes too much troublesome in the n by m dimensions space.

3 Non-cooperative solution of the normal form of three persons 2 moves game

3.1 Theory

The normal form of three persons 2 by 2 game is the following one:

Table 5

		Moves of player C			
		z_1	z_1	z_2	z_2
		Moves of player B			
		y_1	y_2	y_1	y_2
Moves of player A	x_1	$a_{111}, b_{111}, c_{111}$	$a_{121}, b_{121}, c_{121}$	$a_{112}, b_{112}, c_{112}$	$a_{122}, b_{122}, c_{122}$
	x_2	$a_{211}, b_{211}, c_{211}$	$a_{221}, b_{221}, c_{221}$	$a_{212}, b_{212}, c_{212}$	$a_{222}, b_{222}, c_{222}$

$$(x) = (x_1, x_2) = (x, 1 - x) \tag{22}$$

$$(y) = (y_1, y_2) = (y, 1 - y) \tag{23}$$

$$(z) = (z_1, z_2) = (z, 1 - z) \tag{24}$$

are the vectors of the probability distribution on the moves respectively for player A, B and C, with the constraints

$$0 \leq x_i \leq 1 \tag{25}$$

$$0 \leq y_j \leq 1 \tag{26}$$

$$0 \leq z_k \leq 1 \tag{27}$$

Associated to each possible outcome of the game is a collection of numerical payoffs, one to each player.
The expected payoff for each player is then given by:

$$U_A = \sum_{k=1}^{2} z_k (x)(H)_{Ak}(y)^T \tag{28}$$

$$U_B = \sum_{k=1}^{2} z_k (x)(H)_{Bk}(y)^T \tag{29}$$

$$U_C = \sum_{k=1}^{2} z_k (x)(H)_{Ck}(y)^T \tag{30}$$

$$U_A = \sum_{k=1}^{2} z_k(x)(H)_{Ak}(y)^T =$$
$$= z_1(x_1, x_2)(H)_{A1}(y_1, y_2)^T + z_2(x_1, x_2)(H)_{A2}(y_1, y_2)^T =$$
$$= a_{111}xyz + a_{112}xy(1-z) + a_{121}x(1-y)z + a_{122}x(1-y)(1-z) + a_{211}(1-x)yz +$$
$$+ a_{212}(1-x)(1-z)y + a_{221}(1-x)(1-y)z + a_{222}(1-x)(1-y)(1-z) \quad (31)$$

$$U_B = \sum_{k=1}^{2} z_k(x)(H)_{Bk}(y)^T =$$
$$= z_1(x_1, x_2)(H)_{B1}(y_1, y_2)^T + z_2(x_1, x_2)(H)_{B2}(y_1, y_2)^T =$$
$$= b_{111}xyz + b_{112}xy(1-z) + b_{121}x(1-y)z + b_{122}x(1-y)(1-z) + b_{211}(1-x)yz +$$
$$+ b_{212}(1-x)(1-z)y + b_{221}(1-x)(1-y)z + b_{222}(1-x)(1-y)(1-z) \quad (32)$$

$$U_C = \sum_{k=1}^{2} z_k(x)(H)_{Ck}(y)^T =$$
$$= z_1(x_1, x_2)(H)_{C1}(y_1, y_2)^T + z_2(x_1, x_2)(H)_{C2}(y_1, y_2)^T =$$
$$= c_{111}xyz + c_{112}xy(1-z) + c_{121}x(1-y)z + c_{122}x(1-y)(1-z) + c_{211}(1-x)yz +$$
$$+ c_{212}(1-x)(1-z)y + c_{221}(1-x)(1-y)z + c_{222}(1-x)(1-y)(1-z) \quad (33)$$

These formulas will be used throughout the paper from here on.

As shown in section 1, in literature there are two ways to calculate the probability distribution for each player: a prudential strategy and a Nash strategy.

First of all the Nash strategies are determined.

$$\partial U_A/\partial x = (a_{111} - a_{211} - a_{121} + a_{221} - a_{112} + a_{212} + a_{122} - a_{222})yz +$$
$$+ (a_{112} - a_{212} - a_{122} + a_{222})y + (a_{121} - a_{221} - a_{122} + a_{222})z + a_{122} - a_{222} =$$
$$= A_1yz + A_2y + A_3z + A_4 \quad (34)$$

$$\partial U_B/\partial y = (b_{111} - b_{211} - b_{121} + b_{221} - b_{112} + b_{212} + b_{122} - b_{222})xz +$$
$$+ (b_{112} - b_{212} - b_{122} + b_{222})x + (b_{211} - b_{221} - b_{212} + b_{222})z + b_{212} - b_{222} =$$
$$= B_1xz + B_2x + B_3z + B_4 \quad (35)$$

$$\partial U_C/\partial z = (c_{111} - c_{211} - c_{121} + c_{221} - c_{112} + c_{212} + c_{122} - c_{222})xy +$$
$$+ (c_{211} - c_{212} - c_{221} + c_{222})y + (c_{121} - c_{221} - c_{122} + c_{222})x + c_{221} - c_{222} =$$
$$= C_1xy + C_2x + C_3y + C_4 \quad (36)$$

Equating the three partial derivatives to zero the following system of three equations in three unknowns should be solved to find the Nash probability distribution:

$$\partial U_A/\partial x = A_1yz + A_2y + A_3z + A_4 = 0 \tag{37}$$

$$\partial U_B/\partial y = B_1xz + B_2x + B_3z + B_4 = 0 \tag{38}$$

$$\partial U_C/\partial z = C_1xy + C_2x + C_3y + C_4 = 0 \tag{39}$$

To find the solution the following equations should be solved:

$$y = -(C_2x + C_4)/(C_1x + C_3) = y_N \tag{40}$$

$$z = -(B_2x + B_4)/(B_1x + B_3) = z_N \tag{41}$$

$$ax^2 + bx + c = 0 \tag{42}$$

with the following constraints:

$$x \neq -B_3/B_1 \tag{43}$$

$$x \neq -C_3/C_1 \tag{44}$$

$$y \neq -A_3/A_1 \tag{45}$$

being

$$a = A_4B_1C_1 – A_2B_1C_2 – A_3B_2C_1 + A_1B_2C_2 \tag{46}$$

$$b = A_4B_1C_3 – A_2B_1C_4 + A_4B_3C_1 - A_2B_3C_2 - A_3B_2C_3 + A_1B_2C_4 - A_3B_4C_1 + A_1B_4C_2 \tag{47}$$

$$c = A_4B_3C_3 – A_2B_3C_4 – A_3B_4C_3 + A_1B_4C_4 \tag{48}$$

The two solutions of the second degree equation, if any, allow to find the probability distribution for player A, B and C after Nash.

The prudential strategies are determined here below.

In order to find x and z the following equations should be solved:

$$\begin{aligned}\partial U_A/\partial y = {} & (a_{111} - a_{211} - a_{121} + a_{221} - a_{112} + a_{212} + a_{122} - a_{222})xz + \\ & + (a_{112} - a_{212} - a_{122} + a_{222})x + (a_{211} - a_{221} - a_{212} + a_{222})z + a_{212} - a_{222} = \\ & = A_1xz + A_2x + A_5z + A_6 = 0\end{aligned}$$

$$\begin{aligned}\partial U_C/\partial y = {} & (c_{111} - c_{211} - c_{121} + c_{221} - c_{112} + c_{212} + c_{122} - c_{222})xz + \\ & + (c_{211} - c_{212} - c_{221} + c_{222})z + (c_{112} - c_{212} - c_{122} + c_{222})x + c_{212} - c_{222} = \\ & = C_1xz + C_5x + C_3z + C_6 = 0\end{aligned}$$

$$\partial U_A/\partial y = A_1xz + A_2x + A_5z + A_6 = 0 \tag{49}$$

$$\partial U_C/\partial y = C_1xz + C_5x + C_3z + C_6 = 0 \tag{50}$$

To find the solution the following equations should be solved:

$$z = -(C_5x + C_6)/(C_1x + C_3) = z_p \tag{51}$$

$$ax^2 + bx + c = 0 \tag{52}$$

with the following constraints:

$$x \neq -C_3/C_1 \tag{53}$$

$$x \neq -A_5/A_1 \tag{54}$$

being

$$a = A_1C_5 – A_2C_1 \tag{55}$$

$$b = A_5C_5 + A_1C_6 - A_2C_3 - A_6C_1 \tag{56}$$

$$c = A_5C_6 – A_6C_3 \tag{57}$$

In order to find x and y the following equations should be solved:

$$\partial U_A/\partial z = (a_{111} - a_{211} - a_{121} + a_{221} - a_{112} + a_{212} + a_{122} - a_{222})xy+$$
$$+ (a_{211} - a_{221} - a_{212} + a_{222})y + (a_{121} - a_{221} - a_{122} + a_{222})x + a_{221} - a_{222} =$$
$$= A_1xy + A_5y + A_3x + A_7 = 0$$
$$\partial U_B/\partial z = (b_{111} - b_{211} - b_{121} + b_{221} - b_{112} + b_{212} + b_{122} - b_{222})xy+$$
$$+ (b_{121} - b_{221} - b_{122} + b_{222})x + (b_{211} - b_{221} - b_{212} + b_{222})y + b_{212} - b_{222} =$$
$$= B_1xy + B_5x + B_3y + B_4 = 0$$

$$\partial U_A/\partial z = A_1xy + A_5y + A_3x + A_7 = 0 \tag{58}$$

$$\partial U_B/\partial z = B_1xy + B_5x + B_3y + B_4 = 0 \tag{59}$$

To find the solution the following equations should be solved:

$$y = -(B_5x + B_6)/(B_1x + B_3) = y_p \tag{60}$$

$$ax^2 + bx + c = 0 \tag{61}$$

with the following constraints:

$$x \neq -B_3/B_1 \tag{62}$$

$$x \neq -A_5/A_1 \tag{63}$$

being

$$a = A_1B_5 – A_3B_1 \tag{64}$$

$$b = A_5B_5 + A_1B_6 - A_3B_3 - A_7B_1 \tag{65}$$

$$c = A_5B_6 – A_7B_3 \tag{66}$$

In order to find y and z the following equations should be solved:

$$\partial U_B/\partial x = (b_{111} - b_{211} - b_{121} + b_{221} - b_{112} + b_{212} + b_{122} - b_{222})yz+$$
$$+ (b_{112} - b_{212} - b_{122} + b_{222})y + (b_{121} - b_{221} - b_{122} + b_{222})z + b_{122} - b_{222} =$$
$$= B_1yz + B_2y + B_5z + B_7 = 0$$
$$\partial U_C/\partial x = (c_{111} - c_{211} - c_{121} + c_{221} - c_{112} + c_{212} + c_{122} - c_{222})yz+$$
$$+ (c_{112} - c_{212} - c_{122} + c_{222})y + (c_{121} - c_{221} - c_{122} + c_{222})z + c_{221} - c_{222} =$$
$$= C_1yz + C_2z + C_5y + C_4 = 0$$

$$\partial U_B/\partial x = B_1yz + B_2y + B_5z + B_7 = 0 \tag{67}$$

$$\partial U_C/\partial x = C_1yz + C_2z + C_5y + C_4 = 0 \tag{68}$$

To find the solution the following equations should be solved:

$$z = -(B_2y + B_7)/(B_1y + B_5) = z_p \tag{69}$$

$$ay^2 + by + c = 0 \tag{70}$$

with the following constraints:

$$y \neq -B_5/B_1 \tag{71}$$

$$y \neq -C_2/C_1 \tag{72}$$

being

$$a = C_1B_2 – C_5B_1 \tag{73}$$

$$b = B_2C_2 + C_1B_7 - C_5B_5 - C_7B_1 \tag{74}$$

$$c = C_2B_7 – C_7B_5 \tag{75}$$

The six equations should be simultaneously satisfied and in this case this is the prudential probability distribution for player A, B and C.

By substituting in the formulas of the expected payoffs of each player respectively the prudential strategies and the Nash's strategies it can easily be verified whether the two triplets of strategies (x_p, y_p, z_p) and (x_N, y_N, z_N) are equivalent and interchangeable. The same procedure of paragraph 2.1 should be used in order to find the equilibrium triplet by applying the dominance principle.

3.2 Remarks about the solution of three persons 2 moves games

It is worth to note that the proposed procedure to determine the equilibrium strategies of a game does not depend upon the value of the payoffs of the tri-matrix, nevertheless the resulting equilibrium strategies depend totally upon those values.

Moreover, as already said in my previous paper [1], there is a possible flaw in the proposed procedure. The prudential strategy is calculated for each player on the basis of the knowledge of the payoff matrices of the other players. Something similar happens for the Nash's way because the strategy of each player is based on the matrices of the payoffs of the other players, so the expected payoff of a player is depending upon the matrix of the payoffs of the others. In both cases there is a possible flaw of the method because also if a player should be able to state precisely his payoffs matrix corresponding to each of his own pure strategies, he could not be able to state precisely the payoffs matrices of the competitors. This flaw is overcome by the theorem that every finite n-person game with perfect information has an equilibrium n-tuple of strategies [8]. Nevertheless the theorem gives a demonstration of the existence of a solution, but it does not give the way to find it.

The proposed procedure could not work both in the case two by two moves and in the case of different number of moves between the three players: it depends whether the algebraic requirements for the existence of a solution of the system of equations are satisfied or not. If the requirements are not satisfied another procedure should be adopted: this situation will be presented in the following numerical examples.

4 Numerical solutions of a three persons 2 moves game in normal form

4.1 Example 1

As a first example a general sum game published and solved by Dixit and Skeath [11] is shown in Table 6. There are three players, A, B and C, all live on the same small street. Each is asked to contribute toward the creation of a flower garden at the intersection of their small street with the main highway. The ultimate size and splendour of the garden depends on how many of them contribute. Furthermore, although each player is happy to have the garden, each is reluctant to contribute because of the cost he must incur to do so.

Table 6

		Moves of player C			
		z_1	z_1	z_2	z_2
		Moves of player B			
		y_1	y_2	y_1	y_2
Moves of player A	x_1	3, 3, 3	3, 4, 3	3, 3, 4	1, 2, 2
	x_2	4, 3, 3	2, 2, 1	2, 1, 2	2, 2, 2

$$(x) = (x_1, x_2) = (x, 1-x)$$
$$(y) = (y_1, y_2) = (y, 1-y)$$
$$(z) = (z_1, z_2) = (z, 1-z)$$

are the vectors of the probability distribution on the moves respectively for player A, B and C.

Associated to each possible outcome of the game is a collection of numerical payoffs, one to each player.

The expected payoff for each player is then given by formulas 28, 29 and 30.

First of all the Nash strategies are determined by applying formulas 34, 35, 36 and 37, 38, 39.

$$\partial U_A/\partial x = -4yz + 2y + 2z - 1 = 0$$
$$\partial U_B/\partial y = -4xz + 2x + 2z - 1 = 0$$
$$\partial U_C/\partial z = -4xy + 2x + 2y - 1 = 0$$

To find the solution the following equations are obtained:

$$y = -(2x-1)/(-4x+2)$$
$$z = -(2y-1)/(-4y+2)$$
$$z = -(2x-1)/(-4x+2)$$

with the following constraints:

$$x \neq 1/2$$
$$y \neq 1/2$$

therefore the solution is given by the following equations:

$$y = -(2x-1)/(-4x+2)$$
$$z = -(2x-1)/(-4x+2)$$
$$16x^2 + 0x + 0 = 0$$

The two solutions of the second degree equation are both null and it results:

$$x = 0$$
$$y = 1/2$$
$$z = 1/2$$

This solution is not acceptable due to the constraints, thus there is not the probability distribution for player A, B and C after Nash.

The prudential strategies are searched here below by applying formulas 49, 50, 58, 59, 67, 68.

In order to find x and z the following equations should be solved:

$$\partial U_A/\partial y = -4xz + 2x + 2z + 0 = 0$$
$$\partial U_C/\partial y = -4xz + 2x + 2z + 0 = 0$$

therefore the solution is given by the following equations:

$$z = -(2x+0)/(-4x+2)$$
$$0x^2 + 0x + 0 = 0$$

with the following constraints:

$$x \neq 1/2$$

The two solutions of the second degree equation are undefined.

In order to find x and y the following equations should be solved:

$$\partial U_A/\partial z = -4xy + 2y + 2x + 0 = 0$$
$$\partial U_B/\partial z = -4xy + 2x + 2y - 1 = 0$$

therefore the solution is given by the following equations:

$$y = -(2x + 0)/(-4x + 2)$$
$$0x^2 + 0x + 0 = 0$$

with the following constraints:

$$x \neq 1/2$$

The two solutions of the second degree equation are undefined.

In order to find y and z the following equations should be solved:

$$\partial U_B/\partial x = -4yz + 2y + 2z + 0 = 0$$
$$\partial U_C/\partial x = -4yz + 2z + 2y - 1 = 0$$

therefore the solution is given by the following equations:

$$z = -(2y + 0)/(-4y + 2)$$
$$0y^2 + 0y + 0 = 0$$

with the following constraints:

$$y \neq 1/2$$

The two solutions of the second degree equation are undefined.

The six equations have undefined solutions, thus there is not the prudential probability distribution for the game.

Nevertheless it is easy to see that there are four Nash equilibria:
- first

$$(x) = (0, 1)$$
$$(y) = (1, 0)$$
$$(z) = (1, 0)$$

- second

$$(x) = (1, 0)$$
$$(y) = (0, 1)$$
$$(z) = (1, 0)$$

- third

$$
\begin{aligned}
(x) &= (1, 0) \\
(y) &= (1, 0) \\
(z) &= (0, 1)
\end{aligned}
$$

- forth

$$
\begin{aligned}
(x) &= (0, 1) \\
(y) &= (0, 1) \\
(z) &= (0, 1)
\end{aligned}
$$

Looking at the payoff matrices it can be seen that A chooses (0, 1) because it could give him the maximum payoff of 4 and the same choice is suitable for B and for C, therefore the stable Nash equilibrium will be:

$$
\begin{aligned}
(x) &= (0, 1) \\
(y) &= (0, 1) \\
(z) &= (0, 1)
\end{aligned}
$$

with a payoff of (2, 2, 2) respectively for A, B, C. This result is different from the result obtained by Skeath [11] because she is solving the game in the extensive form (sequential moves) by using the rollback analysis (backward induction) making player A choosing first, player B choosing second and player C choosing as third. In this way the stable equilibrium becomes:

$$
\begin{aligned}
(x) &= (0, 1) \\
(y) &= (1, 0) \\
(z) &= (1, 0)
\end{aligned}
$$

with a payoff of (4, 3, 3) respectively for A, B, C.

4.2 Example 2

As a second example a general sum game published and solved by Dixit and Skeath [11] is shown in Table 7. A modified version of the first example is discussed: a somewhat richer variety of possible outcomes and payoff is considered. The size and splendour of the garden will now differ according to the exact number of contributors: three contributors will produce the best garden, two contributors will produce a medium garden, and one contributor will produce a small garden.

$$
\begin{aligned}
(x) &= (x_1, x_2) = (x, 1 - x) \\
(y) &= (y_1, y_2) = (y, 1 - y) \\
(z) &= (z_1, z_2) = (z, 1 - z)
\end{aligned}
$$

Table 7

		Moves of player C			
		z_1	z_1	z_2	z_2
		Moves of player B			
		y_1	y_2	y_1	y_2
Moves of player A	x_1	$5, 5, 5$	$3, 6, 3$	$3, 3, 6$	$1, 4, 4$
	x_2	$6, 3, 3$	$4, 4, 1$	$4, 1, 4$	$2, 2, 2$

are the vectors of the probability distribution on the moves respectively for player A, B and C.

The expected payoff for each player is then given by formulas 28, 29 and 30.

First of all the Nash strategies are determined by applying formulas 34, 35, 36 and 37, 38, 39.

$$\partial U_A/\partial x = 0yz + 0y + 0z - 1 = 0$$
$$\partial U_B/\partial y = 0xz + 0x + 0z - 1 = 0$$
$$\partial U_C/\partial z = 0xy + 0x + 0y - 1 = 0$$

the solution should be given by the following equation:

$$0x^2 + 0x + 0 = 0$$

The system of equations is impossible, thus there is not any solution and the probability distribution after Nash for player A, B and C does not exist.

The prudential strategies are searched here below by applying formulas 49, 50, 58, 59, 67, 68.

In order to find x and z the following equations should be solved:

$$\partial U_A/\partial y = 0xz + 0x + 0z + 2 = 0$$
$$\partial U_C/\partial y = 0xz + 0x + 0z + 2 = 0$$

the solution should given by the following equation:

$$0x^2 + 0x + 0 = 0$$

The system of equations is impossible, thus there is not any solution.

In order to find x and y the following equations should be solved:

$$\partial U_A/\partial z = 0xy + 0y + 0x + 2 = 0$$
$$\partial U_B/\partial z = 0xy + 0x + 0y + 2 = 0$$

the solution should given by the following equation:

$$0x^2 + 0x + 0 = 0$$

The system of equations is impossible, thus there is not any solution.

In order to find y and z the following equations should be solved:

$$\partial U_B/\partial x = 0yz + 0y + 0z + 2 = 0$$
$$\partial U_C/\partial x = 0yz + 0z + 0y + 2 = 0$$

the solution should given by the following equation:

$$0y^2 + 0y + 0 = 0$$

The system of equations is impossible, thus there is not any solution.

The six equations do not have any solution, thus there is not the prudential probability distribution for player A, B and C.

Looking at the payoff matrices it can be seen that there is a Nash equilibrium:

$$(x) = (0, 1)$$
$$(y) = (0, 1)$$
$$(z) = (0, 1)$$

with a payoff of (2, 2, 2) respectively for A, B, C.
Moreover it easy to see that there are dominances: for player A the move x_2 is dominating x_1, for player B the move y_2 is dominating y_1, for player C the move z_2 is dominating z_1, therefore the solution is the above found Nash equilibrium.
This result is equal to the result obtained by Skeath [11].

4.3 Example 3

As a third example a general sum game published as exercise by Maschler [13] is shown in Table 8.

Table 8

		Moves of player C			
		z_1	z_1	z_2	z_2
		Moves of player B			
		y_1	y_2	y_1	y_2
Moves of player A	x_1	$1, 1, 1$	$0, 1, 3$	$3, 0, 1$	$1, 1, 0$
	x_2	$1, 3, 0$	$1, 0, 1$	$0, 1, 1$	$0, 0, 0$

$$(x) = (x_1, x_2) = (x, 1 - x)$$
$$(y) = (y_1, y_2) = (y, 1 - y)$$
$$(z) = (z_1, z_2) = (z, 1 - z)$$

are the vectors of the probability distribution on the moves respectively for player A, B and C.

The expected payoff for each player is then given by formulas 28, 29 and 30.

First of all the Nash strategies are determined by applying formulas 34, 35, 36 and 37, 38, 39.

$$\partial U_A/\partial x = -1yz + 2y - 2z + 1 = 0$$
$$\partial U_B/\partial y = -1xz - 2x + 2z + 1 = 0$$
$$\partial U_C/\partial z = -1xy + 2x - 2y + 1 = 0$$

To find the solution the following equations are obtained:

$$y = -(2x + 1)/(-x - 2)$$
$$z = -(2y + 1)/(-y - 2)$$
$$z = -(-2x + 1)/(-x + 2)$$

with the following constraints:

$$x \neq -2$$
$$x \neq 2$$
$$y \neq -2$$

therefore the solution is given by the following equations:

$$y = -(2x + 1)/(-x - 2)$$
$$z = -(-2x + 1)/(-x + 2)$$
$$13x^2 - 6x - 11 = 0$$

Both the solutions of the second degree equation are not acceptable and it results:

$$x' = (3 + 2\sqrt{38})/13 = 1,18 > 1$$
$$x'' = (3 - 2\sqrt{38})/13 = -0,717 < 0$$

These solutions are not acceptable, thus there is not the probability distribution after Nash for player A, B and C.

The prudential strategies are searched here below by applying formulas 49, 50, 58, 59, 67, 68.

In order to find x and z the following equations should be solved:

$$\partial U_A/\partial y = -xz + 2x + 0z + 0 = 0$$
$$\partial U_C/\partial y = -xz + 0x + 2z + 1 = 0$$

therefore the solution is given by the following equations:

$$z = (0x + 1)/(x - 2)$$
$$2x^2 + 3x + 0 = 0$$

with the following constraints:

$$x \neq 2$$

One of the two solutions of the second degree equation is acceptable, but the solution for z is negative and thus it is not acceptable.

In order to find x and y the following equations should be solved:

$$\partial U_A/\partial z = -xy + 2y + 0x + 0 = 0$$
$$\partial U_B/\partial z = -xy - 2x + 0y + 1 = 0$$

therefore the solution is given by the following equations:

$$y = -(2x - 1)/(x + 0)$$
$$2x^2 - 5x + 2 = 0$$

with the following constraints:

$$x \neq 0$$

One of the two solutions of the second degree equation is acceptable and the solution of these two equations gives:

$$(x) = (x, 1 - x) = (1/2, 1/2)$$
$$(y) = (y, 1 - y) = (0, 1)$$
$$(z) = (z, 1 - z)$$

are the vectors of the probability distribution on the moves respectively for player A, B and C.

In order to find y and z the following equations should be solved:

$$\partial U_B/\partial x = -yz - 2y + 0z + 1 = 0$$
$$\partial U_C/\partial x = -yz + 2z + 0y + 0 = 0$$

therefore the solution is given by the following equations:

$$z = -(-2y + 1)/(-y + 0) = (1 - 2y)/y$$
$$2y^2 - 5y + 2 = 0$$

with the following constraints:

$$y \neq 0$$

One of the two solutions of the second degree equation is acceptable and the solution of these two equations gives:

$$(x) = (x, 1 - x)$$
$$(y) = (y, 1 - y) = (1/2, 1/2)$$
$$(z) = (z, 1 - z) = (0, 1)$$

are the vectors of the probability distribution on the moves respectively for player A, B and C.

The six equations do not have compatible solutions, thus there is not the prudential probability distribution for player A, B and C.

It is easy to see that there are not dominances. Looking at the payoff matrices it can be seen that there is a Nash equilibrium:

$$(x) = (1, 0)$$
$$(y) = (1, 0)$$
$$(z) = (1, 0)$$

with a payoff of (1, 1, 1) respectively for A, B, C.

This result is equal to the result obtained by Maschler [13].

4.4 Example 4

As a fourth example a general sum game published as exercise by Maschler [13] is shown in Table 9.

$$(x) = (x_1, x_2) = (x, 1 - x)$$
$$(y) = (y_1, y_2) = (y, 1 - y)$$
$$(z) = (z_1, z_2) = (z, 1 - z)$$

Table 9

		Moves of player C			
		z_1	z_1	z_2	z_2
		Moves of player B			
		y_1	y_2	y_1	y_2
Moves of player A	x_1	$0,0,0$	$1,0,0$	$0,1,0$	$0,0,1$
	x_2	$0,0,1$	$0,1,0$	$1,0,0$	$0,0,0$

are the vectors of the probability distribution on the moves respectively for player A, B and C.

The expected payoff for each player is then given by formulas 28, 29 and 30.

First of all the Nash strategies are determined by applying formulas 34, 35, 36 and 37, 38, 39.

$$\partial U_A/\partial x = 0yz - y + z + 0 = 0$$
$$\partial U_B/\partial y = 0xz + x - z + 0 = 0$$
$$\partial U_C/\partial z = 0xy - x + y + 0 = 0$$

To find the solution the following equations are obtained:

$$y = -(-x + 0)/(0x + 1)$$
$$z = -(-y + 0)/(0y + 1)$$
$$z = -(x + 0)/(0x - 1)$$

therefore the solution is given by the following equations:

$$y = x$$
$$z = x$$
$$0x^2 + 0x + 0 = 0$$

The system of three equations have undefined solutions, thus there is not the Nash probability distribution for the game.

The prudential strategies are searched here below by applying formulas 49, 50, 58, 59, 67, 68.

In order to find x and z the following equations should be solved:

$$\partial U_A/\partial y = 0xz - x - z + 1 = 0$$
$$\partial U_C/\partial y = 0xz - x - z + 0 = 0$$

therefore the solution is given by the following equations:

$$z = -(-x + 0)/(0x + 1)$$
$$0x^2 + 2x - 1 = 0$$

The solution of the second equation is acceptable and the solution of these two equations gives:

$$(x) = (x, 1-x) = (1/2, 1/2)$$
$$(y) = (y, 1-y)$$
$$(z) = (z, 1-z) = (1/2, 1/2)$$

are the vectors of the probability distribution on the moves respectively for player A, B and C.

In order to find x and y the following equations should be solved:

$$\partial U_A/\partial z = 0xy - y + x + 0 = 0$$
$$\partial U_B/\partial z = 0xy - x - y + 1 = 0$$

therefore the solution is given by the following equations:

$$y = -(-x+1)/(0x-1)$$
$$0x^2 + 2x - 1 = 0$$

The solution of the second equation is acceptable and the solution of these two equations gives:

$$(x) = (x, 1-x) = (1/2, 1/2)$$
$$(y) = (y, 1-y) = (1/2, 1/2)$$
$$(z) = (z, 1-z)$$

are the vectors of the probability distribution on the moves respectively for player A, B and C.

In order to find y and z the following equations should be solved:

$$\partial U_B/\partial x = 0yz + y - z + 0 = 0$$
$$\partial U_C/\partial x = 0yz - z - y + 1 = 0$$

therefore the solution is given by the following equations:

$$z = -(y+0)/(0y-1)$$
$$0y^2 + 2y - 1 = 0$$

The solution of the second equation is acceptable and the solution of these two equations gives:

$$(x) = (x, 1-x)$$
$$(y) = (y, 1-y) = (1/2, 1/2)$$
$$(z) = (z, 1-z) = (1/2, 1/2)$$

are the vectors of the probability distribution on the moves respectively for player A, B and C.

The six equations have compatible solutions, thus there is the prudential probability distribution for player A, B and C:

$$(x) = (x, 1 - x) = (1/2, 1/2)$$
$$(y) = (y, 1 - y) = (1/2, 1/2)$$
$$(z) = (z, 1 - z) = (1/2, 1/2)$$

with a payoff of (1/4, 1/4, 1/4) respectively for A, B, C.

Nevertheless it is easy to see that there are two Nash equilibria:
- first

$$(x) = (1, 0)$$
$$(y) = (1, 0)$$
$$(z) = (1, 0)$$

- second

$$(x) = (0, 1)$$
$$(y) = (0, 1)$$
$$(z) = (0, 1)$$

with a payoff of (0, 0, 0) respectively for A, B, C, but the two Nash equilibria are equivalent and not interchangeable; moreover the prudential strategy is dominating both Nash equilibria, therefore there is no solution with pure strategies and the only solution is the mixed prudential strategy.
The conclusion for this game is that there is not any equilibrium point if the game should be played only once; but if the game could be repeated, there is an equilibrium point with the mixed strategies.

5 Non-cooperative solution of the normal form of four persons 2 moves game

5.1 Theory

The normal form of four persons 2 by 2 game is the following one:

$$(x) = (x_1, x_2) = (x, 1 - x) \tag{76}$$

$$(y) = (y_1, y_2) = (y, 1 - y) \tag{77}$$

$$(z) = (z_1, z_2) = (z, 1 - z) \tag{78}$$

$$(w) = (w_1, w_2) = (w, 1 - w) \tag{79}$$

Table 10

		Moves of player D			
		w_1	w_1	w_2	w_2
		Moves of player C			
		z_1	z_2	z_1	z_2
		Moves of player B			
		y_1,y_2	y_1,y_2	y_1,y_2	y_2,y_2
Moves of player A	x_1	$(a,b,c,d)_{1j11}$	$(a,b,c,d)_{1j21}$	$(a,b,c,d)_{1j12}$	$(a,b,c,d)_{1j22}$
	x_2	$(a,b,c,d)_{2j11}$	$(a,b,c,d)_{2j21}$	$(a,b,c,d)_{2j12}$	$(a,b,c,d)_{2j22}$

are the vectors of the probability distribution on the moves respectively for player A, B, C and D, with the constraints

$$0 \leq x_i \leq 1 \tag{80}$$

$$0 \leq y_j \leq 1 \tag{81}$$

$$0 \leq z_k \leq 1 \tag{82}$$

$$0 \leq w_l \leq 1 \tag{83}$$

Associated to each possible outcome of the game is a collection of numerical payoffs a_i,b_j,c_k,d_l, one to each player, collected in sixteen 2 by 2 matrices $(a,b,c,d)_{ijkl}$, four per each player, $(A)_{11}$, $(A)_{21}$,$(A)_{12}$,$(A)_{22}$, and so on.

The expected payoff for each player is then given by:

$$U_A = \sum_{l=1}^{2} w_l \sum_{k=1}^{2} z_k(x)(A)_{kl}(y)^T \tag{84}$$

$$U_B = \sum_{l=1}^{2} w_l \sum_{k=1}^{2} z_k(x)(B)_{kl}(y)^T \tag{85}$$

$$U_C = \sum_{l=1}^{2} w_l \sum_{k=1}^{2} z_k(x)(C)_{kl}(y)^T \tag{86}$$

$$U_D = \sum_{l=1}^{2} w_l \sum_{k=1}^{2} z_k(x)(D)_{kl}(y)^T \tag{87}$$

First of all the Nash strategies are determined.

$$\partial U_A/\partial x_i = \sum_{l=1}^{2} w_l \sum_{k=1}^{2} z_k(\delta_{ij})(A)_{kl}(y)^T = 0 \tag{88}$$

these partial derivatives, with $i = 1$, 2, equated to zero are 2 equations in 6 y_j, z_k and w_l unknowns and

$$\partial U_B/\partial y_j = \sum_{l=1}^{2} w_l \sum_{k=1}^{2} z_k(x)(B)_{kl}(\delta_{ij})^T = 0 \tag{89}$$

these partial derivatives, with $i = 1, 2$, equated to zero are 2 equations in 6 x_i, z_k and w_l unknowns and

$$\partial U_C/\partial z_k = \sum_{l=1}^{2} w_l(x)(C)_{kl}(y)^T = 0 \tag{90}$$

these partial derivatives, with $i = 1, 2$, equated to zero are 2 equations in 6 x_i, y_j and w_l unknowns and

$$\partial U_D/\partial w_l = \sum_{k=1}^{2} z_k(x)(D)_{kl}(y)^T = 0 \tag{91}$$

these partial derivatives, with i, j, k and $l = 1, 2$, equated to zero a system of 8 equations in 8 x_i, y_j, z_k and w_l unknowns. The following definition holds for the vector (δ_{ij}): the term δ_{ij} of the vector is equal to 1 when the index of integration is equal to the position of the element in the vector and it is equal to zero otherwise.

By using equations 76, 77, 78, 79, the unknowns are reduced to four and the equations are reduced to four too.

The solution of the system, if any, gives the probability distribution for player A, B, C and D after Nash.

The prudential strategies are determined as follows.

$$\partial U_A/\partial y_j = \sum_{l=1}^{2} w_l \sum_{k=1}^{2} z_k(x)(A)_{kl}(\delta_{ij})^T = 0 \tag{92}$$

$$\partial U_A/\partial z_k = \sum_{l=1}^{2} w_l(x)(A)_{kl}(y)^T = 0 \tag{93}$$

$$\partial U_A/\partial w_l = \sum_{k=1}^{2} z_k(x)(A)_{kl}(y)^T = 0 \tag{94}$$

these partial derivatives, with j, k and $l = 1, 2$, equated to zero are 6 equations in 8 x_i, y_j, z_k and w_l unknowns and

$$\partial U_B/\partial x_i = \sum_{l=1}^{2} w_l \sum_{k=1}^{2} z_k(\delta_{ij})(B)_{kl}(y)^T = 0 \tag{95}$$

$$\partial U_B/\partial z_k = \sum_{l=1}^{2} w_l(x)(B)_{kl}(y)^T = 0 \tag{96}$$

$$\partial U_B/\partial w_l = \sum_{k=1}^{2} z_k(x)(B)_{kl}(y)^T = 0 \tag{97}$$

these partial derivatives, with i, k and $l = 1, 2$, equated to zero are 6 equations in 8 x_i, y_j, z_k and w_l unknowns and

$$\partial U_C/\partial x_i = \sum_{l=1}^{2} w_l \sum_{k=1}^{2} z_k(\delta_{ij})(C)_{kl}(y)^T = 0 \tag{98}$$

$$\partial U_C/\partial y_j = \sum_{l=1}^{2} w_l \sum_{k=1}^{2} z_k(x)(C)_{kl}(\delta_{ij})^T = 0 \tag{99}$$

$$\partial U_C/\partial w_l = \sum_{k=1}^{2} z_k(x)(C)_{kl}(y)^T = 0 \tag{100}$$

these partial derivatives, with i, j and $l = 1, 2$, equated to zero are 6 equations in 8 x_i, y_j, z_k and w_l unknowns and

$$\partial U_D/\partial x_i = \sum_{l=1}^{2} w_l \sum_{k=1}^{2} z_k(\delta_{ij})(D)_{kl}(y)^T = 0 \tag{101}$$

$$\partial U_D/\partial y_j = \sum_{l=1}^{2} w_l \sum_{k=1}^{2} z_k(x)(D)_{kl}(\delta_{ij})^T = 0 \tag{102}$$

$$\partial U_D/\partial z_k = \sum_{l=1}^{2} w_l(x)(D)_{kl}(y)^T = 0 \tag{103}$$

these partial derivatives, with i, j and $k = 1, 2$, equated to zero are 6 equations in 8 x_i, y_j, z_k and w_l unknowns.

The solution of the system of 24 equations, if any, gives the prudential probability distribution for player A, B, C and D. The twenty four equations should be simultaneously satisfied and in this case this is the prudential probability distribution for player A, B, C and D.

By using equations 76, 77, 78, 79, the unknowns are reduced to four and the equations are reduced to twelve.

By substituting in the formulas of the expected payoffs of each player respectively the prudential strategies and the Nash's strategies it can easily be verified whether the two strategies (x_p, y_p, z_p, w_p) and (x_N, y_N, z_N, w_N) are equivalent and interchangeable. The same procedure of paragraph 2.1 should be used in order to find the equilibrium strategy by applying the dominance principle.

5.2 Remarks about the solution of four persons 2 moves games

It is worth to note that the proposed procedure to determine the equilibrium strategies of a game does not depend upon the value of the payoffs of the bi-matrix, nevertheless the resulting equilibrium strategies depend totally upon those values.

With suitable adjustment of the formulas the procedure can be applied to n persons 2 moves game. All the remarks of paragraph 3.2 are still holding.

6 Conclusions

The proposed non-cooperative solution of three persons n by n games is based on the application of the dominance principle, therefore the paper is dealing only with games with no dominances on the pure strategies and the dominance principle is applied to find the solution on the mixed strategies.

The main conclusions holding independently from the specific values of the payoff matrix are following:

A) The value of the expected payoff corresponding to the prudential distribution for a player is not only independent either from the prudential or the Nash's distribution of the other players, but it is independent from every distribution of the other players; moreover when two players choose the Nash's distribution the expected payoff of the third player is not depending upon his own strategy distribution;

B) Generally speaking the n-tuples of prudential and Nash's strategies are not interchangeable, but by applying the dominance principle it is possible to choose the right equilibrium strategies avoiding the bad consequences due to the non-interchangeability of the strategies;

C) It is worth noting that in the case of zero sum game the prudential and the Nash strategy are coincident and they are the unique mixed strategies solution of the game; as it can easily be understood, the zero sum game is a special case of the general sum games;

D) On the basis of the dominance principle the dominant mixed strategy is given by the equilibrium point that has the greatest expected payoff: on the basis of point B) the so found equilibrium pair is candidate to be a perfect equilibrium pair [8];

E) A conjecture of the proposed way of solution is that the so found solution is unique (Nash [5]). In this case the so found equilibrium three n-tuples of the non-cooperative solution gives the perfect equilibrium triplet of the game and the corresponding expected payoff could be the starting point for finding the cooperative solution of the game too [8].

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