# Volatility models versus intensity models: analogy and differences

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28 October 2024

# Volatility models versus intensity models: analogy and differences

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**Abstract**

We consider two popular classes of volatility models, the generalized autoregressive conditional heteroscedastic (GARCH) model and the stochastic volatility (SV) model. We compare these two models with two classes of intensity models, the integer-valued GARCH (INGARCH) model and the integer-valued stochastic volatility/intensity (INSV) model, which are corresponding integer-valued counterparts of the former. We reveal the analogy and differences of the models within the same class of volatility/intensity models, as well as between the two different classes of models.

**Keywords:** GARCH, integer-valued GARCH, integer-valued stochastic intensity, observation-driven models, parameter-driven models, stochastic volatility.

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# 1 GARCH vs INGARCH models: analogy and differences

The integer-valued GARCH process (INGARCH; e.g. Grunwald et al., 2000; Rydberg and Shephard, 2000; Heinen, 2003; Ferland et al., 2006; Fokianos et al., 2009; Zhu, 2011; David and Liu, 2016; Ahmad and Francq, 2016; Aknouche et al., 2018a; Aknouche and Francq, 2021; Aknouche et al., 2022b) has a discrete conditional count distribution (Poisson, negative binomial, Double Poisson, etc.), whose intensity, materialized by the conditional mean, has a similar but not identical equation to that of the GARCH specification (Bollerslev, 1986; Francq and Zakoian, 2019). This is due to the fact that the GARCH volatility equation relates the current volatility to the squared values of the past underlying process, while in the INGARCH model, the intensity equation relates the intensity (which in the Poisson case is exactly the volatility) to the values (and not to the squared values) of the past observed process. To be more precise, let us give the difference between the GARCH and INGARCH model and in order to do that in simple terms, we focus, as far as the GARCH case is concerned, on the conditional Gaussian distribution.

## 1.1 GARCH model

An instance of the conditionally Gaussian GARCH$(1,1)$ model can be written in the following Multiplicative Error Model (MEM, see Engle, 2002; Hausch, 2012; Aknouche et al., 2022a; Aknouche and Francq, 2021, 2023)

$$Y_t = \sqrt{h_t}\eta_t \tag{1a}$$

$$(\eta_t) \text{ is i.i.d. with } \eta_t \sim N(0,1) \tag{1b}$$

$$h_t = \omega + \alpha Y_{t-1}^2 + \beta h_{t-1}, \tag{1c}$$

where $\omega > 0$, $\alpha \geq 0$, and $\beta \geq 0$. Instead of the MEM-GARCH specification $(1a)$, the GARCH model (1) can also be represented by the following distributional representation (e.g. Aknouche and Dimitrakopoulos, 2023)

$$Y_t | \mathscr{F}_{t-1} \sim N(0, h_t) \tag{2a}$$

$$\mathscr{F}_t = \sigma\{Y_t, Y_{t-1}, ...\} \tag{2b}$$

$$h_t = \omega + \alpha Y_{t-1}^2 + \beta h_{t-1}. \tag{2c}$$

Although in the literature on GARCH models, representations (1) and (2) of the GARCH model are generally considered indistinguishable, there is, however, a small but important difference between them. Representation (1) implies (2) in the sense that if a stochastic process $(Y_t)$ satisfies (1), then it necessarily satisfies (2).

However, the converse is not true: if a stochastic process $(Y_t)$ satisfies (2), it does not necessarily satisfy (1). Therefore, representation (2) is more general, since the dependence structure of the process $(Y_t)$ is not explicit in terms of past values. For example, for representation (1), the dependence structure can be manifested via the linear stochastic recurrence equation

$$\mathbf{Y}_t = A_t \mathbf{Y}_t + B_t,$$

where $\mathbf{Y}_t$ is a vector, which depends on $Y_t$ and $(A_t, B_t)$ are given (e.g. Bougerol and Picard, 1992; Francq and Zakoian, 2019).

On the other hand, without additional restrictive assumptions, there is no explicit dependence of $Y_t$ in representation (2) a priori. That's why specification (1) is often used when we study the probabilistic structure of a GARCH model. In other words, in this case, (1) is simpler to handle than (2). However, for parameter estimation, the two representations are used interchangeably because they lead to the same conditional distribution. Note that (2) is not a MEM, while (1) it is. However, any GARCH model can be seen as a MEM, whenever it is possible to have a MEM representation with the same conditional distribution. Finally, the GARCH model has the property that the domain of strict stationarity

$$\left\{(\alpha, \beta) \in [0, \infty)^2 : E \log\left(\alpha\eta_1^2 + \beta\right) < 0\right\}$$

is strictly larger than that of the second-order stationarity domain

$$\left\{(\alpha, \beta) \in [0, \infty)^2 : \alpha + \beta < 1\right\}$$

(e.g. Francq and Zakoian, 2019).

### 1.2 INGARCH model

Let us now turn our attention to the INGARCH model which, first of all, is not a MEM because it is defined through the distributional form and thus no corresponding MEM form is possible (e.g. Aknouche and Francq, 2021, 2022; Aknouche and Scotto, 2024). The Poisson INGARCH model has the following distributional representation

$$Y_t | \mathscr{F}_{t-1} \sim \mathcal{P}(\lambda_t) \tag{3a}$$

$$\mathscr{F}_t = \sigma\{Y_t, Y_{t-1}, ...\} \tag{3b}$$

$$\lambda_t = \omega + \alpha Y_{t-1} + \beta\lambda_{t-1}. \tag{3c}$$

Thus, the INGARCH model (3) is defined via a distributional form as in (2), but it differs from (2) in two ways:

i) The conditional distribution of an INGARCH model is discrete (rather than continuous) with a given time-varying conditional mean parameter $\lambda_t$, usually called *intensity*, which, in the Poisson case, coincides with the volatility.

ii) As highlighted above, in the INGARCH model, the intensity equation ($3c$) relates the intensity $\lambda_t$ to the term $Y_{t-i}$ and not to the squared term $Y_{t-i}^2$, as is the case with the volatility $h_t$ of the GARCH model (2) or (1).

The main reason for which model (3) is called Integer-valued GARCH (Ferland et al., 2006) is that, for the conditional Poisson distribution, the intensity

$$\lambda_t := E(Y_t | \mathscr{F}_{t-1}) = Var(Y_t | \mathscr{F}_{t-1})$$

is equal to the conditional variance (also known as volatility). So if $\lambda_t$ is replaced by $h_t$, then

$$Y_t | \mathscr{F}_{t-1} \sim \mathcal{P}(h_t) \tag{3a}$$

$$Var(Y_t | \mathscr{F}_{t-1}) = h_t \tag{3b}$$

$$\mathscr{F}_t = \sigma\left\{Y_t, Y_{t-1}, ...\right\} \tag{3c}$$

$$h_t = \omega + \alpha Y_{t-1} + \beta h_{t-1}, \tag{3d}$$

which is the analog of (2). Nevertheless, as stated above, (3) differs from (2) in terms of ($3a$) and $3d$.

Although the Poisson INGARCH model is not conditionally overdispersed (but conditionally equidispersed), it is unconditionally overdispersed in the sense $Var\left(Y_t\right) > E\left(Y_t\right)$ (e.g. Christou and Fokianos, 2014).

Now, if the conditional distribution of an INGARCH model is negative binomial, that is,

$$Y_t|\,\mathscr{F}_{t-1} \sim \mathcal{NB}\left(\tau, \tfrac{\tau}{\lambda_t+\tau}\right) \tag{4a}$$

$$\mathscr{F}_t = \sigma\left\{Y_t, Y_{t-1}, ...\right\} \tag{4b}$$

$$\lambda_t = \omega + \alpha Y_{t-1} + \beta \lambda_{t-1} \tag{4c}$$

then

$$E\left(Y_t|\,\mathscr{F}_{t-1}\right) = \lambda_t$$

and

$$Var\left(Y_t|\,\mathscr{F}_{t-1}\right) = \left(1 + \tfrac{1}{\tau}\lambda_t\right)\lambda_t$$

so that

$$\lambda_t = E\left(Y_t|\,\mathscr{F}_{t-1}\right) < Var\left(Y_t|\,\mathscr{F}_{t-1}\right),$$

which means that the intensity $\lambda_t$ is no longer equal to the volatility. In this case, the analogy with the GARCH specification is lost and this is the main reason for which there is a strong controversy around the INGARCH notation (Davis et al., 2016, Davis et al., 2021). Nevertheless, by an abuse of notation we continue to name model (4) the (negative binomial) INGARCH.

It can be observed that the negative binomial INGARCH process is conditionally overdispersed ($Var\left(Y_t|\,\mathscr{F}_{t-1}\right) > E\left(Y_t|\,\mathscr{F}_{t-1}\right)$) and therefore necessarily unconditionally overdispersed ($Var\left(Y_t\right) > E\left(Y_t\right)$), where the latter magnitude of overdispersion is greater than that of the Poisson INGARCH process. Overdispersion is an important concept in count data and it is the analogue of the heavy tail property for real-valued distributions (Aknouche and Scotto, 2024).

As for the GARCH specification in (2), the study of path properties (such as ergodicity, geometric ergodicity, mixing etc.) of (3) and (4) is more tedious than that of a MEM representation, as in (1). Typically, a nonlinear stochastic equation driven by an i.i.d. innovation is associated to model (3), based on which, the path properties of (3) are studied. See, for example, the Poisson/mixed-Poisson process representations (e.g. Fokianos et al., 2009; Doukhan et al., 2012; Christou and Fokianos, 2014; Aknouche et al., 2018b; Aknouche and Demmouche, 2019), and the quantile function representation (Neumann, 2011; Davis and Liu, 2016; Aknouche and Francq, 2021, 2022).

For example, the Poisson INGARCH model can be represented by the following stochastic (Poisson process) equation (Fokianos et al., 2009; Doukhan et al., 2012)

$$Y_t = N_t\left(\lambda_t\right) \tag{5a}$$

$$\left(N_t\right) \text{ is an i.i.d. sequence of Poisson processes with intensity } 1 \tag{5b}$$

$$\lambda_t = \omega + \alpha Y_{t-1} + \beta \lambda_{t-1}. \tag{5c}$$

As is the case with the relationship between (1) and (2), representation (5) implies (3) but the converse is not true. However, (3) and (5) have the same conditional distribution but the Poisson INGARCH is not a MEM.

Also, the negative binomial INGARCH model can be represented by the following stochastic (mixed-Poisson) equation (Christou and Fokianos, 2014)

$$Y_t = N_t \left( Z_t \lambda_t \right) \tag{6a}$$

$$\left( Z_t \right) \text{ is an i.i.d. with mean 1 and variance } \rho^2 = \frac{1}{\tau} \tag{6b}$$

$$Z_t \sim G \left( \tau, \tau \right) \tag{6c}$$

$$\left( N_t \right) \text{ is an i.i.d. sequence of Poisson processes with intensity 1} \tag{6d}$$

$$\left( N_t \right) \text{ and } \left( Z_t \right) \text{ are independent} \tag{6e}$$

$$\lambda_t = \omega + \alpha Y_{t-1} + \beta \lambda_{t-1}. \tag{6f}$$

When $Z_t$ is degenerate at 1 (i.e. $\rho^2 = 0$), equation (6) reduces to the Poisson INGARCH (4). When the condition ($6c$) is dropped (or unspecified), model (6) does not necessarily have the negative binomial distribution.

As is the case with the relationship between (1) and (2), representation (6) implies (4) but the converse is not true. However, (4) and (6) have the same conditional distribution but the Negative binomial INGARCH is not a MEM.

Finally, in contrast with the GARCH model, the strict and second-order stationarity domains for the INGARCH$(1, 1)$ model coincide and are both given by

$$\left\{ (\alpha, \beta) \in [0, \infty)^2 : \alpha + \beta < 1 \right\}.$$

(e.g. Aknouche and Francq, 2021).

# 2 SV vs INSI models: analogy and differences

The analogy between the stochastic volatility (SV) model and the integer-valued stochastic volatility/intensity (INSI) model is more pronounced than that between the GARCH and INGARCH.

## 2.1 The SV model

To simplify the analysis, consider the (conditionally) Gaussian SV model given by the following equation

$$Y_t = \sqrt{h_t} \eta_t \tag{7a}$$

$$\left( \eta_t \right) \text{ is i.i.d. with } \eta_t \sim N \left( 0, 1 \right) \tag{7b}$$

$$\log \left( h_t \right) = \phi_0 + \phi_1 \log \left( h_{t-1} \right) + \sigma e_t \tag{7c}$$

$$(e_t) \text{ is i.i.d. with } e_t \sim N(0,1) \tag{7d}$$

$$(\eta_t) \text{ and } (e_t) \text{ are independent,} \tag{7e}$$

where $\phi_0, \phi_1 \in \mathbb{R}$ and $\sigma > 0$.

Due to $(7a)$, the conditionally Gaussian SV representation (7) is a MEM, as is the case with (1). The SV model can also be represented by the following distributional form

$$Y_t | h_t \sim N(0, h_t) \tag{8a}$$

$$\log(h_t) = \phi_0 + \phi_1 \log(h_{t-1}) + \sigma e_t \tag{8b}$$

$$(e_t) \text{ is i.i.d. with } e_t \sim N(0,1). \tag{8c}$$

As for the relation between (1) and (2), the SV representations (7) and (8) are not equivalent, as (7) implies (8) but the converse is not true, so (8) is more general. However, (7) and (8) have the same conditional distribution and are used interchangeably for parameter estimation. By an abuse of notation, we say that the SV model (7) or (8) is a MEM (as in the GARCH case).

Note that the SV model (7)-(8) is parameter-driven in the sense of Cox (1981). The reason is that, due to the presence of the term $e_t$ in the dynamic equation (8b), the volatility $h_t$ in (7)-(8) is unobserved, even with perfect knowledge of the parameters $(\phi_0, \phi_1)$, and is driven by its past latent values. Unobservability for random variables is the analog of "unknowness" for (non-random) real parameters. In fact, the term "unknown" cannot be used for latent variables, since the values taken even by observed random variables are always unknown before observing them. That's why models driven by latent variables are called parameter-driven (Cox, 1981) or also state space models, as opposed to parameter spaces for unknown parameters.

On the contrary, the GARCH model (2)-(3) is observation-driven, since the volatility $h_t$ is a deterministic function of past observations $\mathscr{F}_{t-1} = \sigma\{Y_{t-1}, Y_{t-2}, ...\}$, and therefore is observed under perfect knowledge of the parameters $(\omega, \alpha, \beta)$. The opposition "observation-driven" vs "parameter-driven" is expressed by Cox (1981) in another but an equivalent way. It should be noted that the term "stochastic volatility" is also very controversial because the volatility in both the SV and GARCH models is already stochastic. The correct terminology, regarding "stochastic volatility" would be, in reference to Cox (1981), "unobserved conditional volatility". On the contrary, the term "GARCH" could be replaced by the term "observed conditional volatility".

For the SV model, the domain of strict stationarity coincides with that of second-order stationarity and is given by

$$\{\phi_1 \in \mathbb{R} : |\phi_1| < 1\},$$

(see e.g. Taylor, 1982; Aknouche, 2017).

## 2.2 The INSI model

Aknouche et al. (2024) introduced an integer-valued model for count data, which is an analog to the SV model. It was called "integer-valued stochastic intensity" (INSI) model. The authors of this paper considered both the Poisson and the negative binomial conditional distribution cases. In particular, the Poisson INSI model is given by the following distributional form (e.g. Aknouche et al., 2024)

$$Y_t|\lambda_t \sim \mathcal{P}(\lambda_t) \tag{9a}$$

$$\log(\lambda_t) = \phi_0 + \phi_1 \log(\lambda_t) + \sigma e_t \tag{9b}$$

$$(e_t) \text{ is i.i.d. with } e_t \sim N(0,1), \tag{9c}$$

where $\phi_0$, $\phi_1$ and $\sigma$ are defined as in the SV specification (7)-(8).

As for the INGARCH model, specification (9) is not a MEM. It is defined through the conditional distribution, as is the case with the SV representation (8). However, (9) differs from (8) in only one aspect, unlike the difference between the INGARCH and the GARCH:

- The conditional distribution of the Poisson INSI is discrete rather than continuous, as is the case with the real-valued SV model.

Nonetheless, the log-intensity equation ($9b$) is exactly the same as that of the Gaussian SV model. The main reason for which model (9) could be called Integer-valued SV is that, in the conditional Poisson distribution, the intensity

$$\lambda_t = E(Y_t|\lambda_t) = Var(Y_t|\lambda_t) := h_t$$

is equal to the conditional variance.

Thus, at first glance, model (9) should be simply named Poisson SV and not Poisson INSV. However, the latter name is kept for reasons that become apparent in the case of the negative binomial INSV model. Indeed, if the conditional distribution of an INSV/INSI model is the negative binomial, that is,

$$Y_t|\lambda_t \sim \mathcal{NB}\left(\tau, \tfrac{\tau}{\lambda_t+\tau}\right) \tag{10a}$$

$$\log(\lambda_t) = \phi_0 + \phi_1 \log(\lambda_t) + \sigma e_t \tag{10b}$$

$$(e_t) \text{ is i.i.d. with } e_t \sim N(0,1), \tag{10c}$$

then $E(Y_t|\lambda_t) = \lambda_t$ but

$$Var(Y_t|\lambda_t) = \left(1 + \tfrac{1}{\tau}\lambda_t\right)\lambda_t.$$

Hence, the intensity

$$\lambda_t = E(Y_t|\lambda_t) \neq VarE(Y_t|\lambda_t)$$

is no longer equal to the volatility. In this case, we lose the analogy with the SV and this is the main reason for which Aknouche et al. (2024) highlighted that there may be a controversy regarding the INSV notation. Nevertheless, by an abuse of notation we continue to name model (4) (negative binomial) INSV or INSI. Note finally that model (10) cannot be called negative binomial SV, as is the case with the Poisson INSV, which could be called Poisson SV, as emphasized above. This is because in the negative binomial case, the intensity $\lambda_t$ is no longer the volatility although the INSV (10) and the SV (8) have exactly the same log-(intensity/volatility) equation. In the sequel, model (9) or (10) could be called INSI. It is clear that, like the SV, the INSI is parameter-driven. On the contrary, the INGARCH is, like the GARCH, observation-driven model.

It is difficult to study the probabilistic path properties of the Poisson or negative binomial INSV models due to the non-MEM property. So Aknouche et al. (2024) wrote (9) and (10) as stochastic equation representations, involving the Poisson and Mixed Poisson processes, respectively (e.g. Aknouche et al., 2024). A more general quantile function representation could also be used (e.g. Aknouche and Francq, 2021; Aknouche et al., 2024).

In the Poisson INSI case, it is possible to represent (9) with the following stochastic equation, involving the Poisson process:

$$Y_t = N_t(\lambda_t) \tag{11a}$$

$$(N_t) \text{ is an i.i.d. sequence of Poisson processes with intensity 1} \tag{11b}$$

$$\log(\lambda_t) = \phi_0 + \phi_1 \log(\lambda_t) + \sigma e_t \tag{11c}$$

$$(e_t) \text{ is i.i.d. with } e_t \sim N(0,1) \tag{11d}$$

$$(N_t) \text{ and } (e_t) \text{ are independent.} \tag{11e}$$

As for (1) and (2), specification (11) implies (9) and the converse is not true, so (9) is more general. Moreover, the Poisson INSI model (9) or (11) is not a MEM. As for the Poisson INGARCH model, the Poisson INSI model is not conditionally overdispersed since $E(Y_t|\lambda_t) = Var(Y_t|\lambda_t)$. But it is, in fact, unconditionally overdispersed as $Var(Y_t) > E(Y_t)$ (e.g. Aknouche et al., 2024, Proposition 2.3).

The negative binomial INSI model (10) can also be represented through the following stochastic equation, involving the mixed Poisson process:

$$Y_t = N_t(Z_t \lambda_t) \tag{12a}$$

$$(Z_t) \text{ is an i.i.d. with mean 1 and variance } \rho^2 = \frac{1}{\tau} \tag{12b}$$

$$Z_t \sim G(\tau, \tau) \tag{12c}$$

$$(N_t) \text{ is an i.i.d. sequence of Poisson processes with intensity 1} \tag{12d}$$

$$(e_t) \text{ is i.i.d. with } e_t \sim N(0,1) \tag{12e}$$

$$(N_t) \text{ and } (e_t) \text{ are independent.} \tag{12f}$$

Of course, (12) implies (10) but the converse is not true. However, (10) and (12) have the same conditional distribution.

It is possible to drop the Gamma assumption (12$c$) and the Gaussian assumption (12$e$) to get the following more general mixed Poisson INSI representation.

$$Y_t = N_t(Z_t \lambda_t) \tag{13a}$$

$$(Z_t) \text{ is an i.i.d. with mean 1 and variance } \rho^2 = \frac{1}{\tau} \tag{13b}$$

$$(N_t) \text{ is an i.i.d. sequence of Poisson processes with intensity 1} \tag{13c}$$

$$(e_t) \text{ is i.i.d. with mean zero and variance } \sigma^2 \tag{13d}$$

$$(N_t) \text{ and } (e_t) \text{ are independent.} \tag{13e}$$

This is the model, whose path properties were studied in Aknouche et al., (2024). For parameter estimation, Aknouche et al. (2024) considered two particular cases of (13): i) The Poisson INSI with a Gaussian innovation and $Z_t$, being degenerate at 1. ii) The negative Binomial INSI with Gaussian innovation and $Z_t \sim G(\tau, \tau)$.

As for the negative binomial INGARCH model, the negative binomial INSI model is conditionally overdispersed ($Var(Y_t|\lambda_t) > E(Y_t|\lambda_t)$) and is therefore necessarily unconditionally overdispersed ($Var(Y_t) > E(Y_t)$, e.g. Aknouche et al., 2024), with this overdispersion being greater than the one obtained by the Poisson INSI model.

Finally, as with the GARCH and SV models, the INSI model could be called "integer-valued unobserved intensity model" as opposed to the INGARCH model, which, in turn, could be called "integer-valued observed intensity model".

# 3 Conclusion

Some remarks can be drawn.

- The mixed Poisson INSI model (13) encompasses the Poisson conditional distribution ($Z_t = 1$ a.s.), the negative binomial distribution ($Z_t \sim G(\tau, \tau)$) and other conditional distributions, depending on the distribution of $Z_t$. The path properties of model (13) have been revealed without any assumption on the distribution of $Z_t$ or that of $e_t$.

- Model (13) is parameter-driven like the SV model. In contrast, both the GARCH and INGARCH are observation-driven.

- The INGARCH and the INSI model (13) are not MEM, in contrast with the GARCH and SV. Since the INSI is not MEM, it does not admit an ARMA representation (Aknouche et al., 2024).

- The INSI model is an interesting alternative with better formal correlation properties than the mixed Poisson INGARCH model (e.g. Aknouche et al., 2024) for which the intensity depends only on past observations. In particular, unlike the INGARCH model, the INSI model incorporates a contemporary innovation term in the conditional mean and also allows for negative autocorrelations, which can gain further flexibility.

- Finally, as a consequence of the Cox (1981) dichotomy, it is possible to oppose "observed conditional volatility" with "unobserved conditional volatility" for volatility models (GARCH and SV). Likewise, "observed conditional intensity" is opposed to "unobserved conditional intensity" for intensity models (INGARCH and INSI).

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