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ABSTRACT

Risk can be defined as the likelihood that you can deliver your promise. This paper has used the European put option and the European call option to construct the p-index and c-index to measure the risk levels (likelihoods) of owning or short-selling an asset when the asset provides at least δ rate of return. The p-index measures the insurance fees for each insured dollar so that the asset can deliver at least δ rate of return. The c-index measures the insurance fees for each dollar of the insurance deductible if the asset delivers at least δ rate of return. It shows that higher p-index means higher c-index. In the binomial case with up move and down move, (1) assets having lower down move have higher p-index, i.e., higher risk for owning the assets; and (2) assets having higher up move have higher c-index, i.e., higher risk for short-selling the assets. The trinomial example however shows that the rankings of risk levels of assets' providing different rates of returns could reverse.

Keywords: The put-call parity, the p-index, the c-index, risk structures of assets.

JEL Classification: D81, G13, G32.

1. Introduction

In the literature, the beta (from the capital asset pricing model: CAPM) or the variance of an asset's rate of return is defined as the risk level of the asset. Risk can be defined as the likelihood that you can deliver your promise. This paper defines an asset's risk as the likelihood that this asset can deliver at least a specific rate of return. Every asset which provides uncertain payoff has a corresponding put-call parity. This paper has used the European put option and the European call option to construct the p-index and the c-index to measure the risk levels (likelihoods) of owning or short-selling an asset when the asset provides at least δ rate of return. The p-index measures the insurance fees for each insured dollar so that the asset can deliver at least δ rate of return. The c-index measures the insurance fees for each dollar of the insurance deductible if the asset delivers at least δ rate of return. It shows that higher p-index means higher c-index. In the binomial case with up move and down move, (1) assets having lower down move have higher p-index, i.e., higher risk for owning the assets; and (2) assets having higher up move have higher c-index, i.e., higher risk for short-selling the assets. The trinomial example shows that the rankings of risk levels of assets' providing different rates of returns could reverse. These results should be of use to both academics and practicians.

The remainder of this paper is organized as follows. Section 2 derives the p-index and the c-index as well as their upper and lower bounds. It also discusses the properties of the p-index and the c-index under binomial and trinomial models. Concluding remarks appear in Section 3.

2. P-index and C-index for Measuring Risk Structures of Assets

For every asset which provides uncertain payoff at $t = T$, there exists a corresponding put-call parity at $t = 0$:

$$c + \frac{K}{1+r} = S_0 + p \tag{1}$$

where c is the European call option with strike price K, p is the European put option with strike price K, S_0 is the underlying asset, and r is the simple risk-free interest rate.[1] At $t = T$, the payoff of the

[1] Consider two portfolios at $t = 0$:

portfolio (S_0+p) is $Max[S_T, K]$, the payoff of the European put option p is $Max[K-S_T, 0]$, and the payoff of the European call option c is $Max[S_T-K, 0]$. Note that both c and p are also insurances. The put option p can be interpreted as: if at $t=0$ a person buys both the asset S_0 and the insurance p, then at $t=T$ the value of his owning the asset S_T will be worth at least K. The call option c can be interpreted as: if at $t=0$ a person short-sells the asset S_0 and buys the insurance c (where K can be interpreted as the insurance deductible), then at $t=T$ this person will not need to pay more than K to buy the asset back. Also, in eq. (1), if $K>(<)(1+r)S_0$, then $p>(<)c$. If $K=(1+r)S_0$, then $p=c$. That is, for the insurance company, when $K=(1+r)S_0$, the risk of insuring that the asset is worth at least K at $t=T$ is equivalent to the risk of insuring that the insurant needs not to pay more than K to buy the asset at $t=T$.[2]

Risk can be defined as the likelihood that you can deliver your promise. Dividing both sides of eq. (1) by $K=S_0(1+\delta)$, where $\delta>-1$,

$$\frac{c}{(1+\delta)S_0}+\frac{1}{1+r}=\frac{1}{1+\delta}+\frac{p}{(1+\delta)S_0}\ , \tag{2}$$

where

$$\text{p-index:}\quad \frac{p}{K}=\frac{p}{(1+\delta)S_0}$$

$$\text{c-index:}\quad \frac{c}{K}=\frac{c}{(1+\delta)S_0}\ .$$

(i) For an uncertain asset S_0 at $t=0$, the p-index measures the insurance fees for each insured dollar

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Portfolio A: one European call option c with strike price K, and cash $\frac{K}{1+r}$ deposited in a bank;

Portfolio B: one European put option p with strike price K, and one unit of the underlying asset S_0.

On the expiration date $t=T$, both portfolios give exactly the same payoff: $Max[S_T, K]$. Thus, the costs of the two portfolios at $t=0$ must be the same.

[2] Rewrite eq. (1) as: $S_0=c+\left(\frac{K}{1+r}-p\right)$. If S_0 is the market value of a levered firm, then c is the equity, $\left(\frac{K}{1+r}-p\right)$ is the risky debt, and p is the insurance to insure the promised payment K to debtholders. Note that the changes of K will not affect S_0, i.e., the Modigliani-Miller capital structure irrelevancy proposition is an example of 'financial diversification irrelevancy', see Chang (2023). For a given K, because S_0 is a constant, higher p simply means lower $\left(\frac{K}{1+r}-p\right)$ (lower risky debt) and hence, higher c (higher equity).

so that the asset can deliver at least δ rate of return at $t = T$. That is, higher p-index means higher risk (i.e., less likelihood) for the asset S_0 to deliver at least δ rate of return.

(ii) If a person short sells the asset S_0 at $t = 0$, then she needs to buy it back at the price S_T at $t = T$. Thus, the c-index measures the insurance fees for each dollar of the insurance deductible (where $K = (1+\delta)S_0$) if the asset delivers at least δ rate of return at $t = T$. Higher c-index means higher risk for short selling the asset S_0 if the asset delivers at least δ rate of return.

As shown by Chang (2020, 2023), upper and lower bounds for the put option p is: $Max\left[\frac{K}{1+r} - S_0, 0\right] \leq p < \frac{K}{1+r}$, i.e., an asset cannot sell for more than or equal to the present value of a sure payment of its maximum payoff; and upper and lower bounds for the call option c is: $Max\left[S_0 - \frac{K}{1+r}, 0\right] \leq c < S_0$. Hence, upper and lower bounds for the p-index is: $Max\left[\frac{1}{1+r} - \frac{1}{1+\delta}, 0\right] \leq \frac{p}{S_0(1+\delta)} < \frac{1}{1+r}$; and upper and lower bounds for the c-index is: $Max\left[\frac{1}{1+\delta} - \frac{1}{1+r}, 0\right] \leq \frac{c}{(1+\delta)S_0} < \frac{1}{1+\delta}$. Also, $\frac{\partial}{\partial\delta}\left[\frac{c}{(1+\delta)S_0}\right] < 0$ because $\frac{\partial c}{\partial \delta} < 0$, and $\frac{\partial}{\partial\delta}\left[\frac{p}{(1+\delta)S_0}\right] > 0$ if $\frac{\partial p/p}{\partial\delta/\delta} > 1$.[3]

For any pair of uncertain assets i and j, we have:

$$\frac{c_i}{(1+\delta)S_{0i}} + \frac{1}{1+r} = \frac{1}{1+\delta} + \frac{p_i}{(1+\delta)S_{0i}}$$

$$\frac{c_j}{(1+\delta)S_{0j}} + \frac{1}{1+r} = \frac{1}{1+\delta} + \frac{p_j}{(1+\delta)S_{0j}}$$

and hence,

$$\frac{c_i}{(1+\delta)S_{0i}} - \frac{c_j}{(1+\delta)S_{0j}} = \frac{p_i}{(1+\delta)S_{0i}} - \frac{p_j}{(1+\delta)S_{0j}} \ . \tag{3}$$

That is, for any given δ rate of return, the asset having higher p-index must have higher c-index. There is a one-to-one correspondence between the p-index and the c-index.

The Binomial Case

The binomial option pricing model may be presented as the follows.

[3] See also Chang (2021).

$$c_u = Max[S_0 \cdot u - K, 0]$$

$$p_u = Max[K - S_0 \cdot u, 0]$$

$$\begin{array}{ccc} & \xrightarrow{\ \pi\ } & S_0 \cdot u \\ S_0 & & \\ & \xrightarrow[1-\pi]{} & S_0 \cdot d \end{array}$$

$$c = ? \qquad\qquad c_d = Max[S_0 \cdot d - K, 0]$$

$$p = ? \qquad\qquad p_d = Max[K - S_0 \cdot d, 0]$$

where $u > 1 + r,\ 0 < d < 1 + r,\ r$ is the simple risk-free interest rate, and K is the strike price.

From the Gordan theory we have:[4]

$$\begin{cases} \text{Money market:} \quad 1 = \frac{1}{1+r}[\pi(1+r) + (1-\pi)(1+r)] \\ \text{The asset:} \quad S_0 = \frac{1}{1+r}[\pi \cdot S_0 u + (1-\pi) \cdot S_0 d] \\ \text{Call option:} \quad c = \frac{1}{1+r}[\pi \cdot (S_0 u - K) + (1-\pi) \cdot 0] \\ \text{Put option:} \quad p = \frac{1}{1+r}[\pi \cdot 0 + (1-\pi) \cdot (K - S_0 d)] \end{cases} \tag{4}$$

where $S_0 d < K < S_0 u,\ \pi = \frac{(1+r)-d}{u-d}$ and $1 - \pi = \frac{u-(1+r)}{u-d}$. The p-index and the c-index for the asset S_0 are:

[4] Chang (2015, p. 41) has shown the Gordan theory:

Let A be an $m \times n$ matrix. Then, exactly one of the following systems has a solution:

System 1: $Ax > 0$ for some $x \in R^n$

System 2: $A^T\pi = 0$ for some $\pi \in R^m,\ \pi \geq 0,\ e^T\pi = 1$ where $e = \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$.

Cox et al.'s (1979) binomial option pricing model is System 2 when System 1 does not hold, i.e., when there is no arbitrage.

$$\frac{p}{K}=\frac{\frac{1-\pi}{1+r}[S_0(1+\delta)-S_0 d]}{(1+\delta)S_0}=\frac{(1+\delta)-d}{1+\delta}\cdot\frac{1-\pi}{1+r}\ ;\qquad \frac{c}{K}=\frac{\frac{\pi}{1+r}[S_0 u-S_0(1+\delta)]}{(1+\delta)S_0}=\frac{u-(1+\delta)}{1+\delta}\cdot\frac{\pi}{1+r}\ . \tag{5}$$

It shows that for any given δ rate of return, the asset which has lower down move d means higher risk level for the owner of the asset, i.e., $\frac{\partial}{\partial d}\left[\frac{p}{(1+\delta)S_0}\right]=\frac{-1}{1+\delta}\cdot\frac{1-\pi}{1+r}<0$, $\frac{\partial^2}{\partial d^2}\left[\frac{p}{(1+\delta)S_0}\right]=0$; and $\frac{\partial}{\partial \delta}\left[\frac{p}{(1+\delta)S_0}\right]=\frac{d}{(1+\delta)^2}\cdot\frac{1-\pi}{1+r}>0$, $\frac{\partial^2}{\partial \delta^2}\left[\frac{p}{(1+\delta)S_0}\right]<0$. Also, assets having $d=0$ will have exactly the same constant p-index: $\frac{p}{K}=\frac{p}{(1+\delta)S_0}=\frac{1-\pi}{1+r}$. For any given δ rate of return, the asset which has higher up move u means higher risk level for the investor who short-sells the asset, i.e., $\frac{\partial}{\partial u}\left[\frac{c}{(1+\delta)S_0}\right]=\frac{1}{1+\delta}\cdot\frac{1-\pi}{1+r}>0$, $\frac{\partial^2}{\partial u^2}\left[\frac{c}{(1+\delta)S_0}\right]=0$; and $\frac{\partial}{\partial \delta}\left[\frac{c}{(1+\delta)S_0}\right]=\frac{-u}{(1+\delta)^2}\cdot\frac{\pi}{1+r}<0$, $\frac{\partial^2}{\partial \delta^2}\left[\frac{c}{(1+\delta)S_0}\right]>0$.

Example 1. Let $K=(1+\delta)S_0$ and $S_0 d<(1+\delta)S_0<S_0 u$.

$$\begin{cases}
\text{Money market:}\quad 1=\dfrac{1}{1+0.25}\left[\dfrac{3}{4}\cdot(1+0.25)+\dfrac{1}{4}\cdot(1+0.25)\right]\\
\text{Asset A:}\quad S_A=48=\dfrac{1}{1+0.25}\left[\dfrac{3}{4}\cdot(70)+\dfrac{1}{4}\cdot(30)\right]\\
\text{Asset B:}\quad S_B=60=\dfrac{1}{1+0.25}\left[\dfrac{3}{4}\cdot(85)+\dfrac{1}{4}\cdot(45)\right]\\
\text{Call option for Asset A:}\quad c_A=\dfrac{1}{1+0.25}\left[\dfrac{3}{4}\cdot\left(48\cdot\left(\dfrac{70}{48}\right)-48(1+\delta)\right)+\dfrac{1}{4}\cdot 0\right]\\
\text{Put option for Asset A:}\quad p_A=\dfrac{1}{1+0.25}\left[\dfrac{3}{4}\cdot 0+\dfrac{1}{4}\cdot\left(48(1+\delta)-48\cdot\left(\dfrac{30}{48}\right)\right)\right]\\
\text{Call option for Asset B:}\quad c_B=\dfrac{1}{1+0.25}\left[\dfrac{3}{4}\cdot\left(60\cdot\left(\dfrac{85}{60}\right)-60(1+\delta)\right)+\dfrac{1}{4}\cdot 0\right]\\
\text{Put option for Asset B:}\quad p_B=\dfrac{1}{1+0.25}\left[\dfrac{3}{4}\cdot 0+\dfrac{1}{4}\cdot\left(60(1+\delta)-60\cdot\left(\dfrac{45}{60}\right)\right)\right]
\end{cases}$$

		p-index and c-index				
		$\delta=-25\%$	$\delta=r=25\%$	$\delta=30\%$	$\delta=40\%$	$\delta=500\%$
Asset A	$\frac{p_A}{(1+\delta)S_A}$	0.0333	0.1	0.1038	0.1107	0.6333

	$\frac{c_A}{(1+\delta)S_A}$	0.5666	0.1	0.0731	0.0249	0.0
Asset B	$\frac{p_B}{(1+\delta)S_B}$	0.0	0.08	0.0846	0.0929	0.6333
	$\frac{c_B}{(1+\delta)S_B}$	0.5333	0.08	0.0538	0.0072	0.0

Since asset A's $d = 0.625$ is less than asset B's $d = 0.75$, the p-index shows that owning asset A is riskier than owning asset B for both the assets to provide at least δ rate of return. Since asset A's $u = 1.4583$ is larger than asset B's $u = 1.4167$, the c-index shows that short-selling asset A is riskier than short-selling asset B if both the assets provide at least δ rate of return. As shown in eq. (3), the asset which has higher p-index must have higher c-index.

The Trinomial Case

Example 2. Let $K = (1+\delta)S_0$.

$$\begin{cases} \text{Money market:} \quad 1 = \dfrac{1}{1+0.25}\left[\dfrac{1}{4}\cdot(1+0.25) + \dfrac{3}{8}\cdot(1+0.25) + \dfrac{3}{8}\cdot(1+0.25)\right] \\ \text{Asset A:} \quad S_A = 48 = \dfrac{1}{1+0.25}\left[\dfrac{1}{4}\cdot(21) + \dfrac{3}{8}\cdot(56) + \dfrac{3}{8}\cdot(90)\right] \\ \text{Asset B:} \quad S_B = 48 = \dfrac{1}{1+0.25}\left[\dfrac{1}{4}\cdot(18) + \dfrac{3}{8}\cdot(70) + \dfrac{3}{8}\cdot(78)\right] \\ c_A = \dfrac{1}{1+0.25}\left[\dfrac{1}{4}\cdot Max(21-48(1+\delta),0) + \dfrac{3}{8}\cdot Max(56-48(1+\delta),0) + \dfrac{3}{8}\cdot Max(90-48(1+\delta),0)\right] \\ p_A = \dfrac{1}{1+0.25}\left[\dfrac{1}{4}\cdot Max(48(1+\delta)-21,0) + \dfrac{3}{8}\cdot Max(48(1+\delta)-56,0) + \dfrac{3}{8}\cdot Max(48(1+\delta)-90,0)\right] \\ c_B = \dfrac{1}{1+0.25}\left[\dfrac{1}{4}\cdot Max(18-48(1+\delta),0) + \dfrac{3}{8}\cdot Max(70-48(1+\delta),0) + \dfrac{3}{8}\cdot Max(78-48(1+\delta),0)\right] \\ p_B = \dfrac{1}{1+0.25}\left[\dfrac{1}{4}\cdot Max(48(1+\delta)-18,0) + \dfrac{3}{8}\cdot Max(48(1+\delta)-70,0) + \dfrac{3}{8}\cdot Max(48(1+\delta)-78,0)\right] \end{cases}$$

		p-index and c-index				
		$\delta = -50\%$	$\delta = r = 25\%$	$\delta = 30\%$	$\delta = 50\%$	$\delta = 500\%$
Asset A	$\frac{p_A}{(1+\delta)S_A}$	0.025	0.15	0.1635	0.2083	0.6333

	$\frac{c_A}{(1+\delta)S_A}$	1.225	0.15	0.1327	0.0750	0.0
Asset B	$\frac{p_B}{(1+\delta)S_B}$	0.05	0.14	0.1423	0.1583	0.6333
	$\frac{c_B}{(1+\delta)S_B}$	1.25	0.14	0.1115	0.0250	0.0

For providing at least -50% rate of return, owning asset B is riskier than owning asset A (i.e., $\frac{p_B}{(1-0.5)S_B} = 0.05 > \frac{p_A}{(1-0.5)S_A} = 0.025$), and short-selling asset B is riskier than short-selling asset A (i.e., $\frac{c_B}{(1-0.5)S_B} = 1.25 > \frac{c_A}{(1-0.5)S_A} = 1.225$). But for providing at least 25% or 30% rate of return, owning asset A is riskier than owning asset B; and short-selling asset A is riskier than short-selling asset B. This shows that the rankings of risk levels of assets' providing different rates of returns could reverse.

3. Concluding Remarks

An asset's risk can be defined as the likelihood that the asset can deliver at least δ rate of return. This paper has used the European put option and the European call option to construct the p-index and c-index to measure the risk levels (likelihoods) of owning or short-selling an asset when the asset provides at least δ rate of return. The p-index measures the insurance fees for each insured dollar so that the asset can deliver at least δ rate of return. The c-index measures the insurance fees for each dollar of the insurance deductible if the asset delivers at least δ rate of return. It shows that higher p-index means higher c-index. In the binomial case with up move and down move, (1) assets having lower down move have higher p-index, i.e., higher risk for owning the assets; and (2) assets having higher up move have higher c-index, i.e., higher risk for short-selling the assets. The trinomial example shows that the rankings of risk levels of assets' providing different rates of returns could reverse.

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