The relationship between general equilibrium models with infinite-lived agents and overlapping generations models, and some applications

Pham, Ngoc-Sang

EM Normandie Business School, Métis Lab

11 November 2024

The relationship between general equilibrium models with infinite-lived agents and overlapping generations models, and some applications*

Ngoc-Sang PHAM[†]
EM Normandie Business School, Métis Lab (France)

November 11, 2024

Abstract

We prove that a two-cycle equilibrium in a general equilibrium model with infinitely-lived agents also constitutes an equilibrium in an overlapping generations (OLG) model. Conversely, an equilibrium in an OLG model that satisfies additional conditions is part of an equilibrium in a general equilibrium model with infinitely-lived agents. Applying this result, we demonstrate that equilibrium indeterminacy and rational asset price bubbles may arise in both types of models.

Keywords: infinite-horizon, general equilibrium, overlapping generations, asset price bubble, equilibrium indeterminacy.
JEL Classifications:

1 Introduction

General equilibrium models with infinitely-lived agents (GEILA) and overlapping generations (OLG) models are two workhorses in macroeconomics. A vast body of literature explores these two frameworks.[1] This raises a natural question: what is the relationship between these two kinds of models?

The existing literature highlights a connection between standard OLG models and models with infinitely-lived representative agents. Aiyagari (1985) demonstrates that the dynamics of capital in a standard OLG model (Diamond's model) can be derived from a discounted dynamic programming framework. Hou (1987) considers pure exchange economies and establishes an observational equivalence between an OLG model with agents living for two periods and a cash-in-advance economy with

*I would like to thank Stefano Bosi and Cuong Le Van for their helpful comments and discussions.
[†]Email: npham@em-normandie.fr. Phone: +33 2 50 32 04 08. Address: EM Normandie (campus Caen), 9 Rue Claude Bloch, 14000 Caen, France.
[1]See de la Croix and Michel (2002) for an introduction to OLG models and Becker (2006), Magill and Quinzii (2008), and Le Van and Pham (2016), among others, for an introduction to GEILA models.

a single infinitely-lived representative agent. Lovo and Polemarchakis (2010) depart from a model with an infinitely-lived representative agent and show how the qualitative properties of OLG economies can be replicated by introducing a certain level of myopia.[2]

The present paper focuses on general equilibrium models with a finite number of infinitely-lived households, which are more general than models with a single representative household. Our contributions are twofold. First, we prove that (1) a two-cycle equilibrium in a general equilibrium model with infinitely-lived agents is also an equilibrium in an OLG model, and (2) conversely, an equilibrium in an OLG model that satisfies additional conditions is part of an equilibrium in a general equilibrium model with infinitely-lived agents.

The results in Aiyagari (1985) and Hou (1987) cannot be applied to our models because our framework includes endowments, physical capital, and long-lived assets (both with and without dividends), while the model in Aiyagari (1985) features only physical capital (similar to a one-sector optimal growth model), and Hou (1987) considers an exchange economy.

Our paper is related to Woodford (1986), who studies an economy with capital accumulation and money, where there are two classes of agents (capitalists and workers), and workers face a borrowing constraint. He assumes that capitalists have logarithmic utility functions and focuses on the case where capitalists never purchase money and workers never purchase capital. He observes that workers' decisions resemble those in an OLG model with two-period-lived workers.[3][4] Under Woodford's specifications, solving for equilibrium reduces to solving a two-dimensional difference equation. In contrast, our models may involve a three-dimensional system with infinitely many parameters, and we work under general utility functions. Moreover, we work under general utility functions.

Second, we apply our results to show that both equilibrium indeterminacy and rational asset price bubbles can arise in both types of models.

Kehoe and Levine (1985) consider two stationary pure exchange economies: the first involves a finite number of infinitely-lived consumers, and the second-an OLG model-features an infinite number of finitely-lived consumers. They argue that these two models have different implications: in the first model, equilibria are generically determinate, whereas this is not the case in the second model.[5] The models in our paper are more general than those in Kehoe and Levine (1985) because we incorporate capital accumulation and imperfect financial markets (with borrowing constraints). In terms of implications, we demonstrate that in a non-stationary exchange economy with a finite number of infinitely-lived consumers, equilibrium indeterminacy can arise. The intuition is that in such an economy, the equilibrium system can be supported by an OLG model, which creates room for indeterminacy.

---

[2]It is well known that, in some cases, an OLG model with positive bequests can be reformulated as an optimal growth model ‘a la Ramsey (see Barro (1974), Aiyagari (1992), and Michel et al. (2006)).

[3]Budget constrains (1.1b) in Woodford (1986) writes $p_t\big((c^w_t + (k^w_t - dk^w_{t-1})\big) + M^w_{t+1} = M^w_t + r_t k^w_{t-1} + w_t n_t$. He also imposes constraints $k^w_t \geq 0$, $M^w_{t+1} \geq 0$, and borrowing constraint $p_t\big((c^w_t + (k^w_t - dk^w_{t-1})\big) \leq M^w_t + r_t k^w_{t-1}$. He focuses on the case workers choose $k^w_t = 0, \forall t$ in optimal.

[4]In footnote 4 in Kocherlakota (1992), he also notes that "... short sales constraints that bind in alternating periods serve to make the infinite-horizon economy look like an overlapping generations economy." Our paper formalizes this intuition.

[5]See Farmer (2019) for an overview of equilibrium indeterminacy in macroeconomics.

In recent years, the issue of rational asset price bubbles has attracted significant attention from scholars.[6] Since Tirole (1985), it has become relatively straightforward to build OLG models with bubbles. However, in infinite-horizon general equilibrium models, it is well known that constructing a model where rational asset price bubbles exist is more challenging, particularly when assets yield dividends (Tirole, 1982; Kocherlakota, 1992; Santos and Woodford, 1997).[7] A key difficulty is that, in general, the existence of bubbles in such models requires that the asset holdings of at least two agents fluctuate over time and that the borrowing constraints of at least two agents bind at infinitely many points in time (see Proposition 2 in Bosi, Le Van and Pham (2022)). We show that this scenario leads to the notion of a two-cycle equilibrium in GEILA models, as introduced above. Building on our findings, this two-cycle equilibrium can be supported by an equilibrium in an OLG model. Thus, if the latter equilibrium exhibits a bubble, we can apply our results and impose additional conditions (which hold under standard assumptions) to prove that it is part of a bubbly equilibrium in the GEILA model. This insight allows us to recover many examples of rational bubbles found in the literature.

The rest of the paper is organized as follows. Section 2 introduces both GEILA and OLG models. Section 1 formally establishes the connection between these two models. Section 4 presents applications of our results to the study of equilibrium indeterminacy and asset price bubbles. Technical proofs are provided in the Appendix.

2 Two models

2.1 An overlapping generations model

We present an OLG framework based on the models in Tirole (1985), Weil (1990), de la Croix and Michel (2002), and Bosi et al. (2018).

The representative firm (without market power) maximizes its profit $\max_{K_t, L_t \geq 0} \Big\{F(K_t, L_t) - r_t K_t - w_t L_t\Big\}$, where F is assumed to be constant return to scale (CRS). As usual, denote $f(k) \equiv F(k, 1)$.

The consumer born at date t lives for two periods (young and old) and has $e^y_t \geq 0$ units of consumption as endowments at date when young and $e^o_{t+1} \geq 0$ when old. Endowments are exogenous. We assume there is no population growth, and the population size is normalized to 1. Additionally, we assume a single consumption good.

There is a long-lived asset. At period t, if households buy 1 unit of financial asset with price q_t, she will receive ξ_{t+1} units of consumption good as dividend and she will be able to resell the asset with price q_{t+1}. This asset may be land, Lucas' tree (Lucas, 1978), security (Santos and Woodford, 1997), or stock (Kocherlakota, 1992), ...

[6] For detailed surveys, see Brunnermeier and Oehmke (2012), Miao (2014), Martin and Ventura (2018), Hirano and Toda (2024a,b).

[7] Recently, Le Van and Pham (2016), Bosi, Le Van and Pham (2017a,b, 2018a); Bosi et al. (2018); Bosi, Le Van and Pham (2022) construct models where assets with positive dividends exhibit bubbles. Hirano and Toda (2024c) provide conditions in some models under which any equilibrium (if it exists) is bubbly.

Following Tirole (1985), we assume that there is another long-lived asset with a similar structure as Lucas' tree but this asset does not bring any dividend. We refer this asset "fiat money" as in the traditional literature or "pure bubble asset" or fiat money. The only reason why people buy this asset is to be able to resell it in the future

This consumer chooses consumptions c_t^y, c_t^o, investment in physical capital s_t and investment in a long-lived asset a_t (Lucas' tree) and pure bubble asset (or fiat money) b_t in order to maximize her intertemporal utility $u(c_t^y) + \beta u(c_t^o)$ subject to

$$\begin{aligned}
&c_t^y + s_t + q_t a_t + p_t b_t \leq e_t^y + w_t \\
&c_{t+1}^o \leq e_{t+1}^o + (1-\delta + r_{t+1})s_t + (q_{t+1} + \xi_{t+1})a_t + p_{t+1}b_t \\
&s_t, a_t, b_t, c_t^y, c_t^o \geq 0.
\end{aligned}$$

where $\delta \in [0,1]$ is the depreciation rate of physical capital.

In this setup, the long-lived asset having dividend is similar to Lucas' tree (Lucas, 1978). The stream of real dividends (ξ_t) are exogenous.

Denote

$$R_t \equiv 1 - \delta + r_t.$$

Definition 1. *Given $k_0 > 0, a_{-1} > 0, b_{-1} > 0$, $e_t^y \geq 0$, $e_t^o \geq 0$. An intertemporal equilibrium of the two-period OLG economy is a list $(s_t, a_t, b_t, c_t^y, c_t^o \geq 0, K_t, L_t, N_t, w_t, R_t, q_t, p_t)$ satisfying three conditions: (1) given $R_{t+1}, (q_t, q_{t+1}), (p_t, p_{t+1}), w_t$, the allocation $(s_t, a_t, b_t, c_t^y, c_t^o)$ is a solution to the household's problem and the allocation (K_t, L_t) is a solution to the firm's problem, (2) markets clear: $L_t = 1, K_{t+1} = N_t s_t$, $a_t = 1$, $b_t = 1$ and $s_t + c_t^y + c_t^o = f(k_t) + (1-\delta)k_t + e_t^y + e_t^o + \xi_t$, and (3) $w_t > 0, R_t > 0, q_t > 0, p_t \geq 0, \forall t$.*

Let us denote this two-period OLG economy by

$$\mathcal{E}_{OLG} \equiv \mathcal{E}_{OLG}(u, \beta, f(\cdot), \delta, (\xi_t)_t, (e_t^y, e_t^o)_t).$$

Standard assumptions are required.

Assumption 1. *(1) u_i is in C^1, $u_i'(0) = +\infty$, and u_i is strictly increasing, concave, continuously differentiable.*

(2) $f(\cdot)$ is strictly increasing, concave, continuously differentiable, $f(0) = 0$. The depreciation rate $\delta \in [0,1]$.

(3) $0 < \xi_t < \infty$ $\forall t$.

Let us focus on interior solutions in the sense that $K_t > 0, \forall t$ (this is ensured by, for instance, the Inada condition $f'(0) = +\infty$).

Denote $f(x) \equiv G(x, 1)$, $k_t \equiv K_t/N_t$. Since we consider $N_t = 1$, we have $k_t = K_t$. The first order conditions (FOC) of firm give

$$w_t = f(k_t) - k_t f'(k_t) \text{ and } r_t = f'(k_t). \tag{1}$$

We have the FOCs of households:

$$u'(c_t^y) = \beta R_{t+1} u'(c_{t+1}^o) \tag{2}$$

$$q_t R_{t+1} = q_{t+1} + \xi_{t+1} \tag{3}$$

$$p_t R_{t+1} = p_{t+1}, \tag{4}$$

Market clearing conditions are $K_{t+1} = s_t, L_t = 1, a_t = 1, b_t = 1$. FOC (2) becomes

$$u'(e^y_t + w_t - K_{t+1} - q_t - p_t) = \beta R_{t+1} u'\Big(e^o_{t+1} + R_{t+1}(K_{t+1} + q_t + p_t)\Big). \tag{5}$$

We can redefine equilibrium as follows.

Definition 2. *Given $k_0 > 0, a_{-1} = 1, b_{-1} = 1$, $e^y_t \geq 0$, $e^o_t \geq 0$. An interior inter-temporal equilibrium of the two-period OLG economy is a list $(q_t, p_t, K_{t+1})_{t\geq 0}$ of asset prices and capital stocks, satisfying the following conditions.*[8]

$$u'(e^y_t + f(K_t) - K_t f'(K_t) - K_{t+1} - q_t - p_t) = \beta R_{t+1} u'\Big(e^o_{t+1} + R_{t+1}(K_{t+1} + q_t + p_t)\Big) \tag{6a}$$

$$q_t R_{t+1} = (q_{t+1} + \xi_{t+1}) \tag{6b}$$

$$p_t R_{t+1} = p_{t+1} \tag{6c}$$

$$K_{t+1} > 0, q_t \geq 0, p_t \geq 0 \tag{6d}$$

2.2 A general equilibrium model with infinitely-lived agents

We develop the model in Le Van and Pham (2016) by adding two ingredients: endowments and pure bubble asset, allowing us to cover both exchange and production economies. Consider an infinite-horizon general equilibrium model without uncertainty and discrete time $t = 0, \ldots, \infty$. There are a representative firm without market power and m heterogeneous households. Each household having an endowment (which can be zero) at each date invests in physical asset and/or financial asset, and consumes.

There is a single consumption good which is the numéraire. At each period t, agent i consumes $c_{i,t}$ units of consumption good. At time t, if agent i buys $k_{i,t+1} \geq 0$ units of capital, she will receive $(1-\delta)k_{i,t+1}$ units of old capital at period $t+1$, after being depreciated (δ is the depreciation rate), and $k_{i,t+1}$ units of old capital can be sold at price r_{t+1} .

As in our OLG model above, there are a long-lived asset bringing dividends and fiat money. Each household i takes the sequence $(q, p, r) = (q_t, p_t, r_t)_{t=0}^{\infty}$ as given and chooses the sequences of capital $(k_{i,t})$, of the long-lived asset $a_{i,t}$, of fiat money $(b_{i,t})$ and of consumption $(c_{i,t})$ in order to maximizes her intertemporal utility.

$$(P_i(q, r)) : \max_{(c_{i,t}, k_{i,t+1}, a_{i,t})_{t=0}^{+\infty}} \left[\sum_{t=0}^{+\infty} \beta_i^t u_i(c_{i,t}) \right] \tag{7}$$

subject to constraints $k_{i,t+1}, a_{i,t}, b_{i,t} \geq 0$,[9] and budget constraint

$$c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t} + q_t a_{i,t} + p_t b_{i,t} \leq r_t k_{i,t} + (q_t + \xi_t)a_{i,t-1} + p_t b_{i,t-1} + \theta^i_t \pi_t + e_{i,t}. \tag{8}$$

where $e_{i,t} \geq 0$ is endowment of agent i at date t.

[8] See Bosi et al. (2018) for an equilibrium analysis for the case $p_t = 0, \forall t$.

[9] We may eventually introduce a short-sale constraint as in Bosi, Le Van and Pham (2022) but it is the main aim of the present paper.

For each period t, the representative firm takes prices (r_t) as given and maximizes its profit by choosing physical capital amount K_t.

$$(P(r_t)): \qquad \pi_t \equiv \max_{K_t \geq 0} \Big[f(K_t) - r_t K_t \Big] \tag{9}$$

$(\theta_t^i)_{i=1}^m$ is the share of profit at date t. $\theta_i \equiv (\theta_t^i)_t$ is exogenous, $\theta_t^i \geq 0\ \forall i$ and $\sum_{i=1}^m \theta_t^i = 1$.

Denote $\mathcal{E}_{GEILA}$ the economy characterized by a list

$$\mathcal{E}_{GEILA} = \Big((u_i, \beta_i, (e_{i,t})_t, k_{i,0}, a_{i,-1}, b_{i,-1}, \theta^i)_{i=1}^m, f, (\xi_t)_t, \delta \Big).$$

Definition 3. *A sequence of prices and quantities* $\left(\bar{q}_t, \bar{r}_t, (\bar{c}_{i,t}, \bar{k}_{i,t+1}, \bar{a}_{i,t})_{i=1}^m, \bar{K}_t\right)_{t=0}^{+\infty}$ *is an intertemporal equilibrium of the economy* $\mathcal{E}_{GEILA}$ *if the following conditions are satisfied: (i) Price positivity:* $\bar{q}_t, \bar{r}_t > 0, p_t \geq 0\ \forall t \geq 0$*; (ii) Market clearing:* $\bar{K}_t = \sum\limits_{i=1}^m \bar{k}_{i,t},\ \sum\limits_{i=1}^m \bar{a}_{i,t} = 1,\ \sum\limits_{i=1}^m \bar{b}_{i,t} = 1,$ *and*

$$\sum_{i=1}^m (\bar{c}_{i,t} + \bar{k}_{i,t+1} - (1-\delta)\bar{k}_{i,t}) = e_t + f(\bar{K}_t) + \xi_t, \forall t \geq 0,$$

where $e_t \equiv \sum_{i=1}^m e_{i,t}$ *is the aggregate endowment; (iii) Optimal consumption plans: for all* i, $(\bar{c}_{i,t}, \bar{k}_{i,t+1}, \bar{a}_{i,t})_{t=0}^\infty$ *is a solution of the problem* $(P_i(\bar{q}, \bar{r}))$*; (iv) Optimal production plan: for all* $t \geq 0$, $\bar{K}_t$ *is a solution of the problem* $(P(\bar{r}_t))$.

Standard assumptions are required.

Assumption 2. *(1) At initial period 0,* $k_{i,0}, a_{i,-1}, b_{i,-1} \geq 0$*, and* $(k_{i,0}, a_{i,-1}) \neq (0,0)$ *for* $i = 1, \ldots, m$*. Moreover, we assume that* $\sum_{i=1}^m a_{i,-1} = 1$, $\sum_{i=1}^m b_{i,-1} = 1$*, and* $K_0 \equiv \sum_{i=1}^m k_{i,0} > 0$.

(2) For each agent i*, her utility is finite:* $\sum\limits_{t=0}^\infty \beta_i^t u_i(D_t) < \infty$*, where* $(D_t)_t$ *is defined by* $D_0 \equiv f(K_0) + \xi_0 + \sum\limits_{i=1}^m e_{i,0}, D_t = f(D_{t-1}) + \xi_t + \sum\limits_{i=1}^m e_{i,t}$.

By adopting the proof in Le Van and Pham (2016), under the above assumptions, we can prove that there exists an equilibrium in the infinite-horizon economy $\mathcal{E}_{GEILA}$.

We now introduce the notion of two-cycle economy and two-cycle equilibrium.

Definition 4 (two-cycle economy)**.** *The economy* $\mathcal{E}$ *is called a two-cycle economy if (1) there are 2 consumers, called* H *and* F*, with* $u_i = u$, $\beta_i = \beta \in (0,1)\ \forall i = \{H, F\}$*, (2) their endowments are* $k_{H,0} = 0,\ a_{H,-1} = 0, b_{H,-1} = 0, \forall t \geq 0\ k_{F,0} > 0,\ a_{F,-1} = 1, b_{F,-1} = 1,\ \forall t \geq 0$ *and (3) their shares of the profits are: For* $t \geq 0$, $\theta_{2t}^H = 1,\ \theta_{2t+1}^H = 0,\quad \theta_{2t}^F = 0,\ \theta_{2t+1}^F = 1$.

Denote this two-cycle economy

$$\mathcal{E}_{GEILA2} \equiv \mathcal{E}_{GEILA2}(u, \beta, (e_{i,t})_t, f(\cdot), \delta, (\xi_t)_t).$$

Definition 5. *An intertemporal equilibrium* $\left(q_t, r_t, (c_{i,t}, k_{i,t+1}, a_{i,t})_{i \in I}, K_t\right)_t$ *of the economy* $\mathcal{E}_{GEILA2}$ *is called a two-cycle equilibrium if*

$$k_{H,2t} = 0, a_{H,2t-1} = 0, b_{H,2t-1} = 0, \quad k_{H,2t+1} = K_{2t+1}, \quad a_{H,2t} = 1, b_{H,2t} = 1 \tag{10a}$$
$$k_{F,2t} = K_{2t}, a_{F,2t-1} = 1, b_{F,2t-1} = 1, \quad k_{F,2t+1} = 0, a_{F,2t} = 0, b_{F,2t} = 0. \tag{10b}$$

Observe that in a two-cycle equilibrium, we have that

$$c_{H,2t-1} = e_{H,2t-1} + R_{2t-1}K_{2t-1} + q_{2t-1} + \xi_{2t-1} + p_{2t-1} \tag{11a}$$
$$c_{H,2t} = e_{H,2t} + \pi_{2t} - K_{2t+1} - q_{2t} - p_{2t} \tag{11b}$$
$$c_{F,2t-1} = e_{F,2t-1} + \pi_{2t-1} - K_{2t} - q_{2t-1} - p_{2t-1} \tag{11c}$$
$$c_{F,2t} = e_{F,2t} + R_{2t}K_{2t} + q_{2t} + \xi_{2t} + p_{2t}, \tag{11d}$$

where we denote $R_t = r_t + (1-\delta)$.

We have the following key result characterizing the two-cycle equilibrium.

Proposition 1. *Denote*

$$e^o_{2t} \equiv e_{F,2t}, e^o_{2t+1} \equiv e_{H,2t+1}, \quad e^y_{2t} \equiv e_{H,2t}, e^y_{2t+1} \equiv e_{F,2t+1}. \tag{12}$$

A positive list $\left(q_t, p_t, r_t, (c_{i,t}, k_{i,t+1}, a_{i,t})_{i\in I}, K_t\right)_t$ *is a two-cycle equilibrium of the economy* $\mathcal{E}_{GEILA2}(u, \beta, (e_{i,t})_t, f(\cdot), \delta, (\xi_t)_t, (e_t)_t)$ *if and only if conditions (10a, 10b, 11a, 11c) hold and*

$$q_t R_{t+1} = (q_{t+1} + \xi_{t+1}), \quad p_t R_{t+1} = p_{t+1} \tag{13a}$$
$$\frac{1}{R_{t+1}} = \frac{\beta u'(e^o_{t+1} + R_{t+1}K_{t+1} + q_{t+1} + \xi_{t+1} + p_{t+1})}{u'(e^y_t + \pi_t - K_{t+1} - q_t - p_t)} \tag{13b}$$
$$\geq \frac{\beta u'(e^y_{t+1} + \pi_{t+1} - K_{t+2} - q_{t+1} - p_{t+1})}{u'(e^o_t + R_t K_t + q_t + \xi_t + p_t)} \tag{13c}$$
$$\lim_{t\to\infty} \beta^{2t} u'(e_{H,2t} + \pi_{2t} - K_{2t+1} - q_{2t})(K_{2t+1} + q_{2t}) = 0 \tag{13d}$$
$$\lim_{t\to\infty} \beta^{2t-1} u'(e_{F,2t-1} + \pi_{2t-1} - K_{2t} - q_{2t-1})(K_{2t} + q_{2t-1}) = 0. \tag{13e}$$

where recall that $\pi_t \equiv f(K_t) - f'(K_t)K_t$.

Proof. See Appendix A. □

3 Relationship between GEILA vs OLG models

We now present our main result which shows the connection between the equilibrium in an OLG model and that in a two-cycle economy.

Theorem 1. *Let* $(u, \beta, f(\cdot), \delta, (\xi_t)_t)$ *be a list of fundamentals.*

1. *(GEILA ⇒ OLG) If* $\left(q_t, p_t, r_t, (c_{i,t}, k_{i,t+1}, a_{i,t}, b_{i,t})_{i\in I}, K_t\right)_t$ *is a two-cycle equilibrium of the economy* $\mathcal{E}_{GEILA2} \equiv \mathcal{E}_{GEILA2}(u, \beta, (e_{i,t})_t, f(\cdot), \delta, (\xi_t)_t)$*, then the sequence* $(K_{t+1}, q_t, p_t)_{t\geq 0}$ *is an equilibrium of the OLG economy*

$$\mathcal{E}_{OLG} \equiv \mathcal{E}_{OLG}(u, \beta, f(\cdot), \delta, (\xi_t)_t, (e^y_t, e^o_t)_t)$$

where the sequence $(e^y_t, e^o_t)_t$ *is defined by (12).*

2. *(OLG ⇒ GEILA) Assume that the positive sequence* $(q_t, p_t, K_{t+1})_{t\geq 0}$ *is an equilibrium of the two-period OLG economy* $\mathcal{E}_{OLG} \equiv \mathcal{E}_{OLG}(u, \beta, f(\cdot), \delta, (\xi_t)_t, (e^y_t, e^o_t)_t)$ *(see Definition 2). Then, we have that: the list*

$$\Big(q_t, p_t, r_t, (c_{i,t}, k_{i,t+1}, a_{i,t})_{i=1,2}, K_t\Big)_t \tag{14}$$

where $r_t = f'(K_t)$ *and* $(c_{i,t}, k_{i,t+1}, a_{i,t}, b_{i,t})_{i=1,2}$ *is given by (10a, 10b, 11a, 11c)*

is a two-cycle equilibrium of the economy $\mathcal{E}_{GEILA2} \equiv \mathcal{E}_{GEILA2}(u, \beta, (e_{i,t})_t, f(\cdot), \delta, (\xi_t)_t)$*, where the endowments* $(e_{i,t})_t$ *is defined by (12), if and only if the following conditions are satisfied:*

$$u'(e^o_t + R_tK_t + q_t + \xi_t + p_t) \geq \beta R_{t+1}u'(e^y_{t+1} + w_{t+1} - K_{t+2} - q_{t+1} - p_{t+1}) \tag{15a}$$

$$\lim_{t\to\infty} \beta^{2t}u'(e^y_{2t} + w_{2t} - K_{2t+1} - q_{2t} - p_{2t})(K_{2t+1} + q_{2t} + p_{2t}) = 0 \tag{15b}$$

$$\lim_{t\to\infty} \beta^{2t-1}u'(e^y_{2t-1} + w_{2t-1} - K_{2t} - q_{2t-1} - p_{2t-1})(K_{2t} + q_{2t-1} + p_{2t-1}) = 0, \tag{15c}$$

where $w_t \equiv f(K_t) - K_tf'(K_t)$*.*

Proof. This is a consequence of Definition 2 and Proposition 1. □

Our result leads to interesting implications. First, point 1 shows that analyzing two-cycle equilibria requires us to understand the properties of equilibrium in a two-period OLG model. Point 2 provides a way to construct an two-cycle equilibria from an equilibrium in a two-period OLG model. However, we need to impose additional conditions (15a-15c).

Now, let us focus on two particular cases. First, consider an exchange economy, i.e., productions do not take into account and agents have endowment, we have the following result.

Corollary 1 (exchange economy)**.** *Let* $(u, \beta, (\xi_t)_t), (e_{i,t})_t, (e^y_t, e^o_t)_t$*, where (12) holds, be a list of fundamentals.*

1. *(GEILA ⇒ OLG) If* $\Big(q_t, p_t, (c_{i,t}, a_{i,t}, b_{i,t})_{i\in I}\Big)_t$ *is a two-cycle equilibrium of the economy* $\mathcal{E}_{GEILA2} \equiv \mathcal{E}_{GEILA2}(u, \beta, (e_{i,t})_t, (\xi_t)_t)$*, then the sequence* $(K_{t+1}, q_t, p_t)_{t\geq 0}$ *is an equilibrium of the OLG economy* $\mathcal{E}_{OLG} \equiv \mathcal{E}_{OLG}(u, \beta, (\xi_t)_t, (e^y_t, e^o_t)_t)$*.*

2. *(OLG ⇒ GEILA) Assume that the positive sequence* $(q_t, p_t)_{t\geq 0}$ *is an equilibrium of the two-period OLG economy* $\mathcal{E}_{OLG} \equiv \mathcal{E}_{OLG}(u, \beta, (\xi_t)_t, (e^y_t, e^o_t)_t)$*. Then, we have that: the list*

$$\Big(q_t, p_t, (c_{i,t}, a_{i,t})_{i=1,2}\Big)_t \tag{16}$$

where $(c_{i,t}, a_{i,t}, b_{i,t})_{i=1,2}$ *is given by (10a, 10b, 11a, 11c)*

is a two-cycle equilibrium of the economy $\mathcal{E}_{GEILA2} \equiv \mathcal{E}_{GEILA2}(u, \beta, (e_{i,t})_t, (\xi_t)_t)$ *if and only if the following conditions are satisfied:*

$$u'(e^o_t + q_t + \xi_t + p_t) \geq \beta R_{t+1}u'(e^y_{t+1} - q_{t+1} - p_{t+1}) \tag{17a}$$

$$\lim_{t\to\infty} \beta^{2t}u'(e^y_{2t} - q_{2t} - p_{2t})(q_{2t} + p_{2t}) = 0 \tag{17b}$$

$$\lim_{t\to\infty} \beta^{2t-1}u'(e^y_{2t-1} - q_{2t-1} - p_{2t-1})(q_{2t-1} + p_{2t-1}) = 0, \tag{17c}$$

Corollary 2 (production economy)**.** *Let $(u, \beta, f(\cdot), \delta, (\xi_t)_t)$ be a list of fundamentals.*

1. *(GEILA ⇒ OLG) If $\Big(q_t, p_t, r_t, (c_{i,t}, k_{i,t+1}, a_{i,t}, b_{i,t})_{i\in I}, K_t\Big)_t$ is a two-cycle equilibrium of the economy economy $\mathcal{E}_{GEILA2} \equiv \mathcal{E}_{GEILA2}(u, \beta, f(\cdot), \delta, (\xi_t)_t)$, then the sequence $(K_{t+1}, q_t, p_t)_{t\geq 0}$ is an equilibrium of the OLG economy $\mathcal{E}_{OLG} \equiv \mathcal{E}_{OLG}(u, \beta, f(\cdot), \delta, (\xi_t)_t)$.*

2. *(OLG ⇒ GEILA) Assume that the positive sequence $(q_t, p_t, K_{t+1})_{t\geq 0}$ is an equilibrium of the two-period OLG economy $\mathcal{E}_{OLG} \equiv \mathcal{E}_{OLG}(u, \beta, f(\cdot), \delta, (\xi_t)_t)$. Then, we have that: the list*

$$\Big(q_t, p_t, r_t, (c_{i,t}, k_{i,t+1}, a_{i,t})_{i=1,2}, K_t\Big)_t \tag{18}$$

where $r_t = f'(K_t)$ and $(c_{i,t}, k_{i,t+1}, a_{i,t}, b_{i,t})_{i=1,2}$ is given by (10a, 10b, 11a, 11c)

is a two-cycle equilibrium of the economy $\mathcal{E}_{GEILA2} \equiv \mathcal{E}_{GEILA2}(u, \beta, f(\cdot), \delta, (\xi_t)_t)$ if and only if the following conditions are satisfied:

$$u'(R_t K_t + q_t + \xi_t + p_t) \geq \beta R_{t+1} u'(w_{t+1} - K_{t+2} - q_{t+1} - p_{t+1}) \tag{19a}$$

$$\lim_{t\to\infty} \beta^{2t} u'(w_{2t} - K_{2t+1} - q_{2t} - p_{2t})(K_{2t+1} + q_{2t} + p_{2t}) = 0 \tag{19b}$$

$$\lim_{t\to\infty} \beta^{2t-1} u'(w_{2t-1} - K_{2t} - q_{2t-1} - p_{2t-1})(K_{2t} + q_{2t-1} + p_{2t-1}) = 0, \tag{19c}$$

where $w_t \equiv f(K_t) - K_t f'(K_t)$.

4 Applications: Indeterminacy and asset price bubbles

In this section, we present some applications of our results for studying the issue of indeterminacy and asset price bubble. First, we provide a formal definition of asset price bubble (Tirole, 1982, 1985; Kocherlakota, 1992; Santos and Woodford, 1997; Huang and Werner, 2000). Assume that we have an asset pricing equation

$$q_t = \frac{q_{t+1} + \xi_{t+1}}{R_{t+1}}. \tag{20}$$

Solving recursively (20), we obtain an asset price decomposition in two parts

$$q_t = Q_{t,t+\tau} q_{t+\tau} + \sum_{s=1}^{\tau} Q_{t,t+s}\xi_{t+s}, \text{ where } Q_{t,t+s} \equiv \frac{1}{R_{t+1}\dots R_{t+s}}$$

is the discount factor of the economy from date t to $t + s$.

Definition 6. 1. *The Fundamental Value of 1 unit of asset at date t is the sum of discounted values of dividends:*

$$FV_t \equiv \sum_{s=1}^{\infty} Q_{t,t+s}\xi_{t+s}.$$

2. *We say that there is a bubble at date t if $q_t > FV_t$.*

3. *When $\xi_t = 0$ for any $t \geq 0$ (the Fundamental Value is zero), we say that there is a pure bubble (or the fiat money's price is strictly positive) if $q_t > 0$ for any t.*

Lemma 1 (Montrucchio (2004))**.** *Consider the case $\xi_t > 0, \forall t$. There is a bubble if and only if $\sum_{t=1}^{\infty} \frac{\xi_t}{q_t} < \infty$.*

Clearly, we have $q_t = FV_t + \lim_{\tau\to\infty} Q_{t,t+\tau} q_{t+\tau}$. Thus, condition $q_t - FV_t > 0$ does not depend on t. Therefore, if a bubble exists at date 0, it exists forever. Moreover, we also see that $q_{t+1} - FV_{t+1} = R_{t+1}(q_t - FV_t)$.

We now apply our results in Section 1 to study the issue of rational asset price.

4.1 Exchange economy

First, we focus on the exchange economy. Let us summarize our equilibrium system in Corollary 1.

$$q_t R_{t+1} = (q_{t+1} + \xi_{t+1}), \quad p_t R_{t+1} = p_{t+1} \tag{21a}$$

$$\frac{1}{R_{t+1}} \equiv \frac{\beta u'(e^o_{t+1} + q_{t+1} + \xi_{t+1} + p_{t+1})}{u'(e^y_t - q_t - p_t)} \tag{21b}$$

$$\frac{1}{R_{t+1}} \geq \frac{\beta u'(e^y_{t+1} - q_{t+1} - p_{t+1})}{u'(e^o_t + q_t + \xi_t + p_t)} \tag{21c}$$

$$\lim_{t\to\infty} \beta^{2t} u'(e^y_{2t} - q_{2t} - p_{2t})(q_{2t} + p_{2t}) = 0 \tag{21d}$$

$$\lim_{t\to\infty} \beta^{2t-1} u'(e^y_{2t-1} - q_{2t-1} - p_{2t-1})(q_{2t-1} + p_{2t-1}) = 0. \tag{21e}$$

According to Corollary 1, conditions (21a) and (21a) characterize the intertemporal equilibrium in an OLG model. All conditions (21a-21e) characterize the two-cycle equilibrium of the economy $\mathcal{E}_{GEILA2} \equiv \mathcal{E}_{GEILA2}(u, \beta, (e_{i,t})_t, (\xi_t)_t)$. We will use the system (21a-21e) to show that equilibrium indeterminacy and asset price bubbles can exist along a two-cycle equilibrium.[10]

Example 1 (unique equilibrium)**.** *Assume that $u(c) = ln(c), \forall c$, and $e^o_t = 0, \forall t$. Consider a particular case where there is no fiat money, i.e., $p_t = 0, \forall t$. In this case, conditions (21a) and (21a) implies that there is a unique equilibrium in the OLG model. Moreover, the asset price is $q_t = \frac{\beta}{1+\beta} e^y_t$. This is also part of a two-cycle equilibrium in the economy $\mathcal{E}_{GEILA2}$ because FOCs and TCVs (21c-21e) are satisfied.*

According to Lemma 1, the equilibrium is bubbly if and only if $\sum_t \frac{\xi_t}{q_t} < \infty$, or, equivalently, $\sum_t \frac{\xi_t}{e^y_t} < \infty$. In words, this requires that the dividend would be very small with respect to the endowment of the economy.

The key condition for the existence of bubble, i.e., $\sum_t \frac{\xi_t}{e^y_t} < \infty$, is presented in Section 9.3.2 in Bosi, Le Van and Pham (2017b), and Section 5.1.1, Section 5.2 in Bosi, Le Van and Pham (2018a).[11] Hirano and Toda (2024c)'s Proposition 1 gives a similar condition.

[10] Solving the system (21a-21e) is far from trivial (see Bosi, Le Van and Pham (2022)'s Section 4 for an analysis with more details in the case $p_t = 0, \forall t$.)

[11] Bosi, Le Van and Pham (2022)'s Proposition 7 focuses on the case $p_t = 0, \forall t$, and provide conditions under which there is a continuum equilibria of the long-lived asset. Note that their analyses still apply for the case with only fiat money (their Section 4.1.1.

We now consider the case where the fiat money may have the strictly positive price $p_t > 0$. Let us focus on the case where there is only the fiat money.[12]

Example 2 (continuum of equilibria)**.** *Consider an economy with only fiat money. Assume that $u'(c) = c^{-\sigma}$, where $\sigma > 0$.*

Any sequence (p_t) satisfying the following conditions (A.7a-A.7c) is the sequence of prices of a two-cycle equilibrium is a two-cycle equilibrium of of the economy $\mathcal{E}_{GEILA2} \equiv \mathcal{E}_{GEILA2}(u, \beta, (e_{i,t})_t, (\xi_t)_t)$, where the endowments $(e_{i,t})_t$ is defined by (12),

$$e^y_t - e^o_t \geq 2p_t \geq 0 \tag{22a}$$

$$\lim_{t\to\infty} \beta^t (e^y_t)^{1-\sigma} = 0 \tag{22b}$$

$$p_t = \beta p_{t+1}\Big(\frac{e^y_t - p_t}{e^o_{t+1} + p_{t+1}}\Big)^\sigma. \tag{22c}$$

Proof. See Appendix. □

Let us consider two particular cases of Example 2.

1. Let $e^y_t - e^o_t \geq 0$ and $\lim_{t\to\infty} \beta^t (e^y_t)^{1-\sigma} = 0$. Then, $p_t = 0, \forall t$ is a solution of the system (22). This is a no trade equilibrium.

2. Let $e^y_t = ye^t, e^o_t = de^t$ where $y, d, e > 0$ satisfying $y > d, 1 < \beta e(\frac{y}{de})^\sigma$, and $\beta e^{1-\sigma} < 1$. Let p be determined by $1 = \beta e(\frac{y-p}{(d+p)e})^\sigma$. Then the sequence (p_t) defined by $p_t = pe^t, \forall t \geq 0$, is a two-cycle equilibrium. Note that this is an equilibrium where the fiat money has a strictly positive price. Note that Example 1 in Kocherlakota (1992) corresponds to the case $\sigma = 2, \beta = 7/8, e = 8/7, p = 14, y = 70, d = 35$.

   By combining with point 1, we observe that two sequences ((0) (i.e., $p_t = 0, \forall t$, and $(pe^t)_t$) are two solutions to the system (22). By using the same argument in the proof of Proposition 5 in Bosi, Le Van and Pham (2022), we can prove that any sequence $(p_t)_{t\geq 0}$ defined by $0 < p_0 < p$ and $p_t = \beta p_{t+1}\Big(\frac{e^y_t - p_t}{e^o_{t+1} + p_{t+1}}\Big)^\sigma, \forall t$, is a solution to the system (22). By consequence, there is a continuum of two-cycle equilibria whose fiat money's price is strictly positive.[13] This is an added value with respect to Example 1 in Kocherlakota (1992) where he only provides 1 equilibrium.

4.2 Production economy with financial assets

Applying Proposition 2 for a particular where $u(c) = ln(c), \forall c$, we obtain the following result.

Corollary 3. *Let $u(c) = ln(c)$, $\beta \in (0, 1)$. Assume that there is no endowment, i.e., $e_{i,t} = 0, \forall i, t$. Assume that $(q_t, p_t, K_{t+1})_{t\geq 0}$ is an equilibrium of the two-period OLG*

[12]See also Weil (1990) for an excellent analysis of fiat money in a stochastic OLG model.

[13]Section 4.1.1 in Bosi, Le Van and Pham (2022) for a full characterization in the case $\sigma = 1$.

economy, i.e.,

$$K_{t+1} + q_t + p_t = \frac{\beta}{1+\beta} w_t = \frac{\beta}{1+\beta}\Big(f(K_t) - K_t f'(K_t)\Big) \tag{23a}$$

$$q_t R_{t+1} = (q_{t+1} + \xi_{t+1}) \tag{23b}$$

$$p_t R_{t+1} = p_{t+1} \tag{23c}$$

$$K_{t+1} > 0, q_t \geq 0, p_t \geq 0. \tag{23d}$$

If

$$w_{t-1}\beta^2\Big(1 - \delta + f'(K_t)\Big)\Big(1 - \delta + f'(K_{t+1})\Big) \leq w_{t+1} \quad \forall t \tag{24}$$

then $(q_t, K_{t+1})_t$ are asset prices and aggregate capital stocks of a two-cycle equilibrium of the two-cycle economy.

Proof. Under logarithmic utility function, the Euler equation (6a) becomes (23a). By consequence, the TVCs (19b, (19c are satisfied. Lastly, condition (19a) becomes (24). □

We now apply this result to construct two-cycle equilibria with bubbles in general equilibrium models with two agents $\mathcal{E}_{GEILA2} \equiv \mathcal{E}_{GEILA2}(u, \beta, f(\cdot), \delta, (\xi_t)_t)$. We consider two standard cases: Linear and Cobb-Douglas production functions.

4.2.1 Cobb-Douglas production function

The following result is an application of Corollary 3.

Example 3 (pure bubble in a model with Cobb-Douglas production function)**.** *Let $u(c) = ln(c)$, $\beta \in (0,1)$, $\delta = 1$, the Cobb-Douglas production function $f(k) = Ak^\alpha$, where $\alpha \in (0,1)$. Let us focus on the model with only the pure bubble asset and physical capital.*

Denote K^* the capital intensity in the bubbleless steady state, that is the steady state without pure bubble asset.

$$K^* = \rho^{1/(1-\alpha)}, \text{ where } \rho \equiv \gamma\alpha A \tag{25}$$

Denote $\gamma \equiv \frac{\beta}{1+\beta}\frac{1-\alpha}{\alpha}$. Observe that $\gamma \equiv \frac{\beta}{1+\beta}\frac{1-\alpha}{\alpha} = \frac{n}{f'(k^*_x)} = \frac{1}{f'(k^*_x)}$.

Assume that $\gamma > 1$ (i.e., $f'(K^*) < 1$; this is so-called low interest rate condition).

There exists a two-cycle equilibrium with bubble of the general equilibrium model with two agents $\mathcal{E}_{GEILA2} \equiv \mathcal{E}_{GEILA2}(u, \beta, f(\cdot), \delta, (\xi_t)_t)$. In such an equilibrium, the aggregate capital and the asset price are determined by

$$K_t = (\alpha A)^{\frac{1-\alpha^{t-1}}{1-\alpha}} K_1^{\alpha^{t-1}}, \forall t \geq 2, \quad K_1 = \frac{\alpha w_0}{(1-\alpha)(1+\beta)}, w_0 = f(K_0) - K_0 f'(K_0) \tag{26}$$

$$p_t = (\gamma - 1)K_{t+1}, \forall t \geq 0. \tag{27}$$

Moreover, in this equilibrium, we have that

$$\lim_{t\to\infty} K_t = (\alpha A)^{1/(1-\alpha)} < K^* \text{ and } \lim_{t\to\infty} p_t = (\gamma - 1)(\alpha A)^{1/(1-\alpha)} > 0. \tag{28}$$

In terms of implications, Example 3 shows that a standard model with pure bubble asset as in Tirole (1985) can be embedded in a general equilibrium model with infinitely-lived agents. Note that under specifications in Example 3, as we prove in Lemma 3 in Appendix, the equilibrium (26-27) is the unique solution satisfying the system (23) and the asset price does not converge to zero.

4.2.2 Linear technology

Consider a linear production function: $G(K, L) = AK + BL$, we have that: an equilibrium $(q_t, p_t, K_{t+1})_{t\geq 0}$ of the two-period OLG economy are asset prices and aggregate capital stocks of a two-cycle equilibrium of the two-cycle economy if and only if $\beta(1-\delta+A) \leq 1$.[14]

According to (23b) and (23c), we can compute that

$$p_t = R^t p_0$$
$$q_0 = \sum_{s=1}^{t} \frac{\xi_s}{R^s} + \frac{q_t}{R^t}, \text{ which implies } q_t = R^s\Big(q_0 - \sum_{s=1}^{t} \frac{\xi_s}{R^s}\Big)$$

To sum up, we get the following result.

Example 4. *Assume that (1) $u(c) = ln(c)$, $\beta \in (0, 1)$, (2) there is no endowment, i.e., $e_{i,t} = 0, \forall i, t$, (3) $G(K, L) = AK + BL$, and*

$$R \leq 1 \tag{29}$$
$$\frac{\beta}{1+\beta} w > \sum_{s=1}^{t} \frac{\xi_s}{R^s} \tag{30}$$
$$\frac{\beta}{1+\beta} w > R^s\Big(\frac{\beta}{1+\beta} w - \sum_{s=1}^{t} \frac{\xi_s}{R^s}\Big). \tag{31}$$

Then, any sequence $(k_{t+1}, b_t)_{t\geq 0}$ determined by the following conditions

$$p_0 \geq 0 \tag{32}$$
$$\sum_{s=1}^{\infty} \frac{\xi_s}{R^s} \leq q_0 < \frac{\beta}{1+\beta} w - p_0 \tag{33}$$
$$q_t = R^t\left(q_0 - \sum_{s=1}^{t} \frac{\xi_s}{R^s}\right) \tag{34}$$
$$p_t = R^t p_0 \tag{35}$$
$$nk_{t+1} + q_t + p_t = \frac{\beta}{1+\beta} w \tag{36}$$

is part of intertemporal in the two-cycle economy. Moreover, we have that:

1. *Fiat money has a positive price if $p_0 > 0$. The supremum value of initial fiat price p_0 such that $p_t > 0, \forall t$ is $\frac{\beta}{1+\beta} w - \sum_{s=1}^{\infty} \frac{\xi_s}{R^s}$.*

[14] Le Van and Pham (2016)'s Section 6.1 corresponds to this model with $p_t = 0, \forall t$. This case is also related to Proposition 5 in Bosi et al. (2018).

2. *If* $q_0 = \sum_{s=1}^{\infty} \frac{\xi_s}{R^s}$*, then there is no bubble of the long-lived asset. In this case, we have* $p_0 \geq 0$*. There is a continuum of equilibria with pure bubble, indexed by* p_0*.*

3. *If* $q_0 > \sum_{s=1}^{\infty} \frac{\xi_s}{R^s}$*, then there is a bubble of the long-lived asset. Moreover, in this case,* $\lim_{t\to\infty} b_t > 0$ *if and only if* $R = 1$*.*

Example 4 shows that there exist a continuum of equilibria with a strictly positive price of fiat money (pure bubble asset) and/or with bubbles of the long-lived assets. Our three points above suggests that there is no causal relationship between two kinds of bubbles.

5 Conclusion

This paper bridges two foundational macroeconomic models: the infinite-horizon general equilibrium model with infinitely-lived agents (GEILA) and the overlapping generations (OLG) model. By establishing the connection between the two models, we have provided a unified view that deepens our understanding of phenomena like equilibrium indeterminacy and rational asset price bubbles in both models.

A Appendix

Before proving Proposition 1, we present the following result which can be proved by adopting the proof in Bosi, Le Van and Pham (2018a).

Lemma 2. *A sequence* $\left(q_t, p_t, r_t, (c_{i,t}, k_{i,t+1}, a_{i,t})_{i\in I}, K_t\right)_t$ *is an equilibrium if and only if there exists non-negative sequences* $\left((\sigma_{i,t}, \mu_{i,t}, \nu_{i,t})_{i\in I}\right)_t$ *such that*

(i) $\forall t,\ \forall i,\ c_{i,t} > 0, k_{i,t+1} \geqslant 0,\ a_{i,t} \geqslant 0,\ \sigma_{i,t} \geqslant 0, \nu_{i,t} \geqslant 0,$
$\forall t, K_t \geqslant 0, q_t > 0, r_t > 0$

(ii) First order conditions

$$\frac{1}{r_{t+1} + 1 - \delta} = \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})} + \sigma_{i,t}, \quad \sigma_{i,t} k_{i,t+1} = 0$$
$$\frac{q_t}{q_{t+1} + \xi_{t+1}} = \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})} + \mu_{i,t}, \quad \mu_{i,t} a_{i,t} = 0$$
$$p_t = \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})} p_{t+1} + \nu_{i,t}, \quad \nu_{i,t} b_{i,t} = 0.$$

(iii) Transversality conditions

$$\lim_{t\to\infty} \beta_i^t u_i'(c_{i,t}) k_{i,t+1} = \lim_{t\to\infty} \beta_i^t u_i'(c_{i,t}) q_t a_{i,t} = \lim_{t\to\infty} \beta_i^t u_i'(c_{i,t}) p_t b_{i,t} = 0.$$

(iv) $f(K_t) - r_t K_t = \pi_t = \max\{f(K) - r_t K : k \geqslant 0\}, \forall t.$

(v) $c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t} + q_t a_{i,t} + p_t b_{i,t} = r_t k_{i,t} + (q_t + \xi_t) a_{i,t-1} + p_t b_{i,t-1} + \theta_t^i \pi_t + e_{i,t}.$

(vi) $K_t = \sum_{i\in I} k_{i,t},\ \sum_{i\in I} a_{i,t} = 1,\ \sum_{i\in I} b_{i,t} = 1.$

Proof of Proposition 1. According to Lemma 2, first order conditions become

$$\frac{1}{r_{2t}+1-\delta} = \frac{q_{2t-1}}{q_{2t}+\xi_{2t}} = \frac{\beta_F u'_F(c_{F,2t})}{u'_F(c_{F,2t-1})} \geq \frac{\beta_H u'_H(c_{H,2t})}{u'_H(c_{H,2t-1})} \tag{A.1a}$$

$$\frac{1}{r_{2t+1}+1-\delta} = \frac{q_{2t}}{q_{2t+1}+\xi_{2t+1}} = \frac{\beta_H u'_H(c_{H,2t+1})}{u'_H(c_{H,2t})} \geq \frac{\beta_F u'_F(c_{F,2t+1})}{u'_F(c_{F,2t})} \tag{A.1b}$$

$$p_{2t-1} = \frac{\beta_F u'_F(c_{F,2t})}{u'_F(c_{F,2t-1})} p_{2t} \geq \frac{\beta_H u'_H(c_{H,2t})}{u'_H(c_{H,2t-1})} p_{2t} \tag{A.1c}$$

$$p_{2t} = \frac{\beta_H u'_H(c_{H,2t+1})}{u'_H(c_{H,2t})} p_{2t+1} \geq \frac{\beta_F u'_F(c_{F,2t+1})}{u'_F(c_{F,2t})} p_{2t+1}. \tag{A.1d}$$

Recall that

$$c_{H,2t-1} = e_{H,2t-1} + R_{2t-1}K_{2t-1} + q_{2t-1} + \xi_{2t-1} + p_{2t-1} \tag{A.2a}$$

$$c_{H,2t} = e_{H,2t} + \pi_{2t} - K_{2t+1} - q_{2t} - p_{2t} \tag{A.2b}$$

$$c_{F,2t-1} = e_{F,2t-1} + \pi_{2t-1} - K_{2t} - q_{2t-1} - p_{2t-1} \tag{A.2c}$$

$$c_{F,2t} = e_{F,2t} + R_{2t}K_{2t} + q_{2t} + \xi_{2t} + p_{2t}, \tag{A.2d}$$

According to (11a-11c) and $\beta_H = \beta_F = \beta, u_H = u_F = u$, the inequalities in FOCs are rewritten as follows:

$$\frac{\beta u'(e_{F,2t} + R_{2t}K_{2t} + q_{2t} + \xi_{2t} + p_{2t})}{u'(e_{F,2t-1} + \pi_{2t-1} - K_{2t} - q_{2t-1} - p_{2t-1})} \geq \frac{\beta u'(e_{H,2t} + \pi_{2t} - K_{2t+1} - q_{2t} - p_{2t})}{u'(e_{H,2t-1} + R_{2t-1}K_{2t-1} + q_{2t-1} + \xi_{2t-1} + p_{2t-1})}$$

$$\frac{\beta u'(e_{H,2t+1} + R_{2t+1}K_{2t+1} + q_{2t+1} + \xi_{2t+1} + p_{2t+1})}{u'(e_{H,2t} + \pi_{2t} - K_{2t+1} - q_{2t} - p_{2t})} \geq \frac{\beta u'(e_{F,2t+1} + \pi_{2t+1} - K_{2t+2} - q_{2t+1} - p_{2t+1})}{u'(e_{F,2t} + R_{2t}K_{2t} + q_{2t} + \xi_{2t} + p_{2t})}.$$

With our notations $e^o_{2t} \equiv e_{F,2t}, e^o_{2t+1} \equiv e_{H,2t+1}$ and $e^y_{2t} \equiv e_{H,2t}, e^y_{2t+1} \equiv e_{F,2t+1}$, these inequalities become

$$\frac{\beta u'(e^o_{t+1} + R_{t+1}K_{t+1} + q_{t+1} + \xi_{t+1} + p_{t+1})}{u'(e^y_t + \pi_t - K_{t+1} - q_t - p_t)} \geq \frac{\beta u'(e^y_{t+1} + \pi_{t+1} - K_{t+2} - q_{t+1} - p_{t+1})}{u'(e^o_t + R_tK_t + q_t + \xi_t)}. \tag{A.3}$$

Transversality conditions become

$$\lim_{t\to\infty} \beta_H^{2t} u'_H(c_{H,2t}) k_{H,2t+1} = \lim_{t\to\infty} \beta_H^{2t} u'_H(c_{H,2t}) q_{2t} a_{H,2t} \tag{A.4a}$$

$$= \lim_{t\to\infty} \beta_H^{2t} u'_H(c_{H,2t}) p_{2t} b_{H,2t} = 0 \tag{A.4b}$$

$$\lim_{t\to\infty} \beta_H^{2t+1} u'_H(c_{H,2t+1}) k_{H,2t+2} = \lim_{t\to\infty} \beta_H^{2t+1} u'_H(c_{H,2t+1}) q_{2t+1} a_{H,2t+1} \tag{A.4c}$$

$$= \lim_{t\to\infty} \beta_H^{2t+1} u'_H(c_{H,2t+1}) p_{2t+1} b_{H,2t+1} = 0 \tag{A.4d}$$

$$\lim_{t\to\infty} \beta_F^{2t} u'_F(c_{F,2t}) k_{F,2t+1} = \lim_{t\to\infty} \beta_F^{2t} u'_F(c_{F,2t}) q_{2t} a_{F,2t} \tag{A.4e}$$

$$= \lim_{t\to\infty} \beta_F^{2t} u'_F(c_{F,2t}) p_{2t} b_{F,2t} = 0 \tag{A.4f}$$

$$\lim_{t\to\infty} \beta_F^{2t+1} u'_F(c_{F,2t+1}) k_{F,2t+2} = \lim_{t\to\infty} \beta_F^{2t+1} u'_F(c_{F,2t+1}) q_{2t+1} a_{F,2t+1} \tag{A.4g}$$

$$= \lim_{t\to\infty} \beta_F^{2t+1} u'_F(c_{F,2t+1}) p_{2t+1} b_{F,2t+1} = 0. \tag{A.4h}$$

These are rewritten as follows:

$$\lim_{t\to\infty} \beta_H^{2t} u'_H(c_{H,2t})(K_{2t+1} + q_{2t} + p_{2t}) = 0 \tag{A.5a}$$

$$\lim_{t\to\infty} \beta_F^{2t+1} u'_F(c_{F,2t+1})(K_{2t+2} + q_{2t+1} + p_{2t+1}) = 0. \tag{A.5b}$$

Since $\beta_H = \beta_F = \beta, u_H = u_F = u$, TVCs become

$$\lim_{t\to\infty} \beta^{2t} u'(e_{H,2t} + \pi_{2t} - K_{2t+1} - q_{2t} - p_{2t})(K_{2t+1} + q_{2t}) = 0 \tag{A.6a}$$

$$\lim_{t\to\infty} \beta^{2t-1} u'(e_{F,2t-1} + \pi_{2t-1} - K_{2t} - q_{2t-1} - p_{2t-1})(K_{2t} + q_{2t-1}) = 0. \tag{A.6b}$$

□

Proof of Example 2. The system (21a-21e) becomes.

$$p_{t+1} = p_t R_{t+1} \geq 0 \tag{A.7a}$$

$$\frac{1}{R_{t+1}} \equiv \frac{\beta u'(e^o_{t+1} + p_{t+1})}{u'(e^y_t - p_t)} \geq \frac{\beta u'(e^y_{t+1} - p_{t+1})}{u'(e^o_t + p_t)} \tag{A.7b}$$

$$\lim_{t\to\infty} \beta^{2t} u'(e^y_{2t} - p_{2t})p_{2t} = \lim_{t\to\infty} \beta^{2t-1} u'(e^y_{2t-1} - p_{2t-1})p_{2t-1} = 0. \tag{A.7c}$$

Then, we can verify these conditions under assumptions in the Example. □

Proof of Example 3. According to (3), it suffices to show that our sequence $(K_{t+1}, p_t)_{t\geq 9}$ satisfies the equilibrium system

$$\begin{cases} & w_{t-1}\beta^2 f'(K_t) f'(K_{t+1}) \leq w_{t+1} \\ K_1 + b_0 & = \frac{\beta}{1+\beta} w_0 \\ K_{t+1} + p_t & = \gamma\alpha A K_t^\alpha, \forall t \geq 0, \text{ where } \gamma \equiv \frac{\beta}{1+\beta}\frac{1-\alpha}{\alpha} \\ p_{t+1} & = \alpha A K_{t+1}^{\alpha-1} p_t \\ K_{t+1} & > 0, p_t \geq 0. \end{cases} \tag{A.8}$$

It is easy to verify the last four conditions. Let us check the first condition. Note that $K_{t+1} = \rho_1 K_t^\alpha$ where $\rho_1 \equiv \alpha A$. Since $\delta = 1$, condition (24) becomes

$$w_{t-1}\beta^2 f'(K_t) f'(K_{t+1}) \leq w_{t+1} \quad \forall t \tag{A.9}$$

$$(1-\alpha)AK_{t-1}^\alpha \beta^2 \alpha A K_t^{\alpha-1} \alpha A K_{t+1}^{\alpha-1} \leq (1-\alpha)AK_{t+1}^\alpha, \forall t \tag{A.10}$$

$$\beta^2 A^2 \alpha^2 K_{t-1}^\alpha K_t^{\alpha-1} \leq K_{t+1}, \forall t \tag{A.11}$$

$$\beta \leq \frac{K_{t+1}}{\alpha A K_t^\alpha} \frac{K_t}{\alpha A K_{t-1}^\alpha} = 1 \tag{A.12}$$

which is satisfied because $\beta < 1$

□

Lemma 3 (solving the system (A.13)). *Consider the following system (A.13).*

$$K_1 + b_0 = \frac{\beta}{1+\beta} w_0 \tag{A.13a}$$

$$K_{t+1} + p_t = \gamma\alpha A K_t^\alpha, \forall t \geq 0, \text{ where } \gamma \equiv \frac{\beta}{1+\beta}\frac{1-\alpha}{\alpha} \tag{A.13b}$$

$$p_{t+1} = \alpha A K_{t+1}^{\alpha-1} p_t \tag{A.13c}$$

$$K_{t+1} > 0, p_t \geq 0, \tag{A.13d}$$

1. *If $\gamma \leq 1$ (i.e., $f'(K^*) \geq 1$), the system has a unique solution*

$$p_t = 0, \quad K_t = \rho^{\frac{1-\alpha^{t-1}}{1-\alpha}} K_1^{\alpha^{t-1}} \quad \forall t \geq 2, \quad K_1 = \frac{\beta}{(1+\beta)} w_0 \tag{A.14}$$

where $\rho \equiv \gamma\alpha A$. Moreover, $\lim_{t\to\infty} K_t = K^*$.

2. *If $\gamma > 1$ (i.e., $f'(K^*) < 1$), the system is indeterminate: The set of solutions $(K_{t+1}, p_t)_{t\geq 0}$ is defined by (A.13b), (A.13c), and $p_0 \in \left[0, \bar{b}\right]$, where the bubble critical value $\bar{b}$ is defined by*

$$\bar{b} \equiv w_0 \frac{\beta}{1+\beta}\frac{\gamma-1}{\gamma} = w_0\left[1 - \frac{1+\alpha\beta}{(1-\alpha)(1+\beta)}\right] \tag{A.15}$$

which is positive if $\gamma > 1$.

Moreover,

(a) *(bubbleless solution) If $p_0 = 0$, and, thus, $p_t = 0$ forever. The sequence (K_t) is given by (A.14).*

(b) *(bubbly solution) If $p_0 > 0$, then $p_t > 0$ for any t.*
When $p_0 < \bar{b}$, we have $\lim_{t\to\infty} p_t = 0$ and $\lim_{t\to\infty} K_t = K^*$.
When $p_0 = \bar{b}$, we have $\lim_{t\to\infty} p_t > 0$. We also have

$$p_t = \frac{\gamma-1}{\sigma}K_{t+1}\forall t \geq 0 \tag{A.16}$$

$$K_t = \rho_1^{\frac{1-\alpha^{t-1}}{1-\alpha}} K_1^{\alpha^{t-1}} \quad \forall t \geq 2, \quad K_1 = \frac{\alpha w_0}{(1-\alpha)(1+\beta)} \tag{A.17}$$

and $\rho_1 \equiv \alpha A$. Moreover,

$$\lim_{t\to\infty} K_t = \rho_1^{1/(1-\alpha)} < K^* \text{ and } b \equiv \lim_{t\to\infty} p_t = \gamma - 1(\alpha A)^{1/(1-\alpha)} > 0. \tag{A.18}$$

Proof. The proof here is similar to the proof in the literature (see Proposition 4 in Bosi et al. (2018) among others).

If $p_0 > 0$, or, equivalently, $p_t > 0, \forall t$. Combining (A.13b) and (A.13c), we have that

$$\frac{K_{t+1}}{p_t} + 1 = \frac{\gamma\alpha A K_t^{\alpha}}{p_t} = \frac{\gamma\alpha A K_t^{\alpha}}{\alpha A K_t^{\alpha-1} p_{t-1}} = \gamma\frac{K_t}{p_{t-1}}, \forall t \geq 1. \tag{A.19}$$

Denote $z_t \equiv nk_{t+1}/(\sigma b_t)$. We get a single dynamic equation:

$$z_{t+1} = \gamma z_t - 1 \quad \forall t \geq 0. \tag{A.20}$$

If $\gamma \neq 1$, the solution of the difference equation (A.20) is given by

$$z_t = \gamma^t z_0 - \frac{1-\gamma^t}{1-\gamma}, \forall t \geq 1$$

1. When $\gamma \leq 1$, there is no bubble. Indeed, suppose that there is a pure bubble. Since $\gamma \leq 1$, condition (A.20) implies that z_t becomes negative soon or later: this leads to a contradiction. In this case, capital transition becomes $k_{t+1} = \rho k_t^{\alpha}$, where $\rho \equiv \gamma\alpha A$. Solving recursively, we find the explicit solution (A.14).

2. Let $\gamma > 1$.

   If $p_t = 0$, then (A.14) follows immediately.

   If $p_t > 0$. Then, we obtain

$$z_t = \frac{\left[(\gamma-1)z_0 - 1\right]\gamma^t + 1}{\gamma - 1}. \tag{A.21}$$

A positive solution exists if and only if $z_0 \geq 1/(\gamma - 1)$. Hence, the existence of a positive solution requires

$$b_0 \leq (\gamma - 1)K_1 = (\gamma - 1)\left[\frac{\beta}{1+\beta}w_0 - b_0\right].$$

Solving this inequality for b_0, we find $0 < b_0 \leq \bar{b}$.

Now, given $b_0 \in \left(0, \bar{b}\right]$, the sequence (K_{t+1}, p_t) constructed by (A.13b) and (A.13c) is a solution with $p_t > 0$ for any t.

When $b_0 < \bar{b}$ (that is $z_0 > 1/(\gamma - 1)$), because of (A.21), we get $\lim_{t\to\infty} z_t = \infty$. According to (A.13b), K_t is uniformly bounded from above, which implies that $\lim_{t\to\infty} p_t = 0$. Thus, $\lim_{t\to\infty} K_t = K^*$.

When $b_0 = \bar{b}$, we have $z_t = 1/(\gamma - 1)$ for any $t \geq 0$. In this case, $k_{t+1} = \rho_1 k_t^\alpha$ where $\rho_1 \equiv \alpha A/n$ for any $t > 0$ and $b_t = (\gamma - 1)\, nk_{t+1}$. Solving recursively, we get the explicit solution (A.16).

□

References

Aiyagari, S.R., 1985. *Observational equivalence of the overlapping generations and the discounted dynamic programming frameworks for one-sector growth.* Journal of Economic Theory 35, Issue 2, p. 201-221.

Aiyagari, S.R., 1992. *Co-existence of a representative agent type equilibrium with a non-representative agent type equilibrium.* Journal of Economic Theory, Volume 57, Issue 1, June 1992, Pages 230-236

Arrow, J. K., Debreu, G., 1954. *Existence of an equilibrium for a competitive economy.* Econometrica 22, 265-290.

Barro, R. J., 1974. *Are government bonds net wealth.* Journal of Political Economy 82, 1095-1117.

Bosi, S., Ha-Huy, T., Le Van, C., Pham, C.T., Pham, N.-S., 2018a. Financial bubbles and capital accumulation in altruistic economies, *Journal of Mathematical Economics*, vol. 75, p. 125-139.

Bosi, S., Le Van, C., Pham, N.-S., 2017a. Asset bubbles and efficiency in a generalized two-sector model, *Mathematical Social Sciences*, vol. 88, p. 37-48.

Bosi, S., Le Van, C., Pham, N.-S., 2017b. Rational land and housing bubbles in infinite-horizon economies, in: K. Nishimura, A. Venditti and N. Yannelis (Eds.), *Sunspots and Non-Linear Dynamics - Essays in honor of Jean-Michel Grandmont*, Series "Studies in Economic Theory", Springer-Verlag (2017).

Bosi, S., Le Van, C., Pham, N.-S., 2018. *Intertemporal equilibrium with heterogeneous agents, endogenous dividends and collateral constraints*, Journal of Mathematical Economics, vol. 76, p. 1-20.

Bosi, S., Le Van, C., Pham, N.-S., 2022. Real indeterminacy and dynamics of asset price bubbles in general equilibrium. *Journal of Mathematical Economics*, Volume 100, 102651.

Brunnermeier, M.K., Oehmke, M., 2012. Bubbles, financial crises, and systemic risk. *Handbook of the Economics of Finance*, vol. 2.

de la Croix, D., Michel, P., 2002. *A Theory of Economic Growth : Dynamics and Policy in Overlapping Generations, 1st edition.* Cambridge University Press.

Farmer, R. E. A. 2019. *The indeterminacy agenda in macroeconomics.* NIESR Discussion Paper No. 507.

Hirano, T., Toda, A. A., (2024) Bubble economics. *Journal of Mathematical Economics*, vol. 111, 2024.

Hirano, T., Toda, A. A., (2024) Note on Bubbles Attached to Real Assets. arXiv:2410.17425v1.

Hirano, T., Toda, A. A., (2024) Bubble necessity theorem. *Journal of Political Economy*, forthcoming (2024)

Hou, T. M. 1987. Observational equivalence of the overlapping-generations and the cash-in-advance economies. Economics Letters Volume 25, Issue 1, 1987, Pages 9-13

Huang, K. X. D., Werner, J., 2000. Asset price bubbles in Arrow-Debreu and sequential equilibrium. Economic Theory 15, 253-278.

Kehoe T. and D. Levine, *Comparative Statics and Perfect Foresight in Infinite Horizon Economies*, Econometrica, 53, 433-453.

Kocherlakota, N. R., 1992. *Bubbles and constraints on debt accumulation.* Journal of Economic Theory 57, 245 - 256.

Martin, A., Ventura, J., 2018. The Macroeconomics of Rational Bubbles: A User's Guide. *Annual Review of Economics*, 10(1), 505-539.

Miao, J., 2014. *Introduction to economic theory of bubbles. Journal of Mathematical Economics* 53, p. 130-136.

Montrucchio, L., 2004. Cass transversality condition and sequential asset bubbles. *Economic Theory* 24, p. 645-663.

Le Van, C., Pham, N.S., 2016. *Intertemporal equilibrium with financial asset and physical capital.* Economic Theory 62, 155-199.

Lovo, S., Polemarchakis, H. 2010 *Myopia and monetary equilibria*, Journal of Mathematical Economics, Volume 46, Issue 5, 20 September 2010, Pages 925-936.

Lucas, R., 1978. *Asset Prices in an Exchange Economy.* Econometrica 46,1429-45.

Magill M. and Quinzii M., *Incomplete markets*, volume 2, infinite horizon economies, Edward Elgar Publishing Company, 2008.

Michel, F., Thibault, E., Vidal, J.-P., 2006. *Intergenerational altruism and neoclassical growth models.* In: Kolm, S.-C., Ythier, J. M., (Eds.), Handbook of the Economics of Giving, Altruism and Reciprocity, vol. 2, Chap. 15.

Becker, R. A., Dana, R.-A., Le Van, C., Mitra, T., Nishimura, K., 2006. *Equilibrium Dynamics with Many Agents*. In: Becker, R. A., Dana, R.-A., Le Van, C., Mitra, T., Nishimura, K., (Eds.), Handbook on Optimal Growth 1: Discrete Time, p. 385-442.

Santos, S., Woodford, M., 1997. *Rational asset pricing bubbles*. Econometrica 65, 19-57.

Stephan, G., Muller-Furstenberger, G., Previdoli, P., 1997. *Overlapping Generations or Infinitely-Lived Agents: Intergenerational Altruism and the Economics of Global Warming*. Environmental and Resource Economics 10, Issue 1, pp. 27?40.

Tirole, J., 1982. On the possibility of speculation under rational expectations. Econometrica 50, 1163-1181.

Tirole, J., 1985. *Asset bubbles and overlapping generations*. Econometrica 53, 1499-1528.

Weil, F., 1987. *Confidence and the real value of money in an overlapping generations economy*. Quarterly Journal of Economics, 102, 1-22.

Weil, F., 1990. *On the possibility of price decreasing bubbles*. Econometrica, 58, 1467-1474.

Woodford, M., 1986. *Stationary sunspot equilibria in a finance constrained economy*. Journal of Economic Theory 40, Issue 1, pp 128-137.