Fair Division Approach Based on Clustering and Reduction of Inequalities

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ABSTRACT

In this article, we intend to present our Fair Division Approach for Divisible Resources based on Clustering and Reduction of Inequalities (APCR from french for Approche du Partage équitable des ressources divisible basée sur la Classification automatique et la Réduction des inégalités). It is highlighted through the sharing methods that we have proposed, these are: the PRRG method (Resource Distribution Process at Group level) and the PCo method (Corrected Proportional method) alongside the PRRC method (Resource Allocation Process Based on Clustering results) that we proposed in a classic environment in our previous work. We provide an overview of the PRRC process and present the PRRG process in a traditional environment. As for the PCo rule, it is reserved for future work.

Keywords: Approach, Fair Division, Distribution, Divisible resources, Clustering, Reduction of inequalities, Process, Method.

1. INTRODUCTION

The problem of division of an object has led different authors to propose different sharing mechanisms (or methods or even rules), based, for our part, mainly on two elements: the nature of the resource (divisible or indivisible) and the values (preferences or roles to play or demands or demands or characteristics) of individuals who may have this problem ([2], [16], [37], [38]). And yet, there are other elements which can modify the results of one or the other method or even lead to other sharing approaches, if they are taken into account. This is particularly the case for taking into account the plurality and type of variables, and that of the origin of the resource (contributions or non-contributions) [23], [24].

The problem therefore arises of implementing methods for sharing a divisible resource which of course take into account the plurality (several) and the type (homogeneous or heterogeneous) of variables as well as the origin of the resource (contributions or not individuals). Hence the importance of implementing our Fair Sharing Approach based on Classification and Reduction of Inequalities (APCR). We will show how this approach works.

2. MATERIALS AND METHOD

2.1 Fair division

2.1.1 Fair division of a resource

Definition 1 (Fair division problem). Let be a vector $w=(w_1, \ldots, w_i, \ldots, w_n) \in \mathbb{R}_+^n$ respective claims (values) of the n agents 1,, …, i, …, n belonging to the set I and a divisible resource $C \in \mathbb{R}^+$. This is called a fair division problem, the triple (I, C, w) whose solution is a vector of individual parts $c=(c_1, \ldots, c_i, \ldots, c_n) \in \mathbb{R}_+^n$ with $\sum_{i=1}^{n} c_i = C$ [3], [16], [18].

In the case where $C < \sum_{i=1}^{n} w_i$, we speak of a deficit while when $C \geq \sum_{i=1}^{n} w_i$, we are in the case of a surplus.

Definition 2 (Bankruptcy problem). Given a vector $w=(w_1, \ldots, w_i, \ldots, w_n) \in \mathbb{R}_+^n$ of the respective claims of the n agents 1, 2, … , i, …n belonging to the set I and a divisible resource $C \in \mathbb{R}^+$. A bankruptcy problem is a fair division problem in which any allocation is a n-tuple $c=(c_1, \ldots, c_i, \ldots, c_n) \in \mathbb{R}_+^n$ of shares satisfying the following two properties: 1) $\sum_{i=1}^{n} c_i=C$ (Efficiency); 2) $0 < c_i < w_i$ (Reasonableness).

A classic bankruptcy situation consists of a certain sum of currency (resource) which must be shared between a few claimants (individuals) who have claims (values of variables) on the resource and the sum of claims is greater than the resource [28], [29].

A division can be fair or unfair. The division of a resource will be said to be fair when each individual is allocated a share and this is accepted by everyone. Otherwise it is unfair. As for us, a division will be fair if it uses an adequate mechanism which uses several variables (criteria) while taking into account the particularities of the latter and which takes into account the origin of the resource which determines the relationships between individuals, whether or not they are contributors to the creation of the common resource to be shared, thus making it possible to decide whether or not to reduce inequalities between them. We consider that a division is equitable when we allocate to each individual a share corresponding to the value (proportion) that they verify.

The division of resources, whether in the case of bankruptcy or any other case, must be done according to principles of justice.

2.1.2 Principles of justice

As for the principles of justice, Forsé Michel and Parodi Maxime [10] allude to the three principles of justice: 1) absolute equality (or principle of equality) which assigns to the beneficiary individuals an equal share to each, 2) equity (or principle of merit) which shares the resource according to the merit of each in proportion to their merit, 3) the satisfaction of need (at least the basic ones) (or principle of need) which allocates the resource according to the needs of each person.

2.1.3 Methods for divisible (continuous) resources division

2.1.3.1 Non-homogeneous divisible resources division methods

These methods are based on the Cake-cutting model. This model includes among other methods: the “I cut you choose” or Cut-and-choose or Divide-and-choose procedure proposed by Steinhaus Hugo (1949). It is used to share a cake, a piece of land, etc. between two agents [39].

2.1.3.2 Homogeneous divisible resources division methods

These division methods concern resources such as currency (sum of money), electoral seats between candidates, common profit, assets of a bankrupt company, overall cost of common equipment, etc. [5], [16], [22], [36], [37]. These resources are variable or fixed.

In this case, the resource can be variable as well as fixed. The variable resource is determined by individual requests, this is the case of sharing a global cost. Three categories of cost division methods exist, these are: 1) the average cost method, 2) the sequential distribution method and methods inspired by cooperative game theory: the core, the nucleolus and the Value by Shapley. On this subject, for more details see [8], [17], [27], [38]. The fixed resource, as a sum of money, for its part uses different division methods or rules, these include: 1) the proportional method/rule which distributes the resource in proportion to the claims of the individuals [16], 2) the equal surplus or rights-egalitarian method/rule, 3) the losses uniforms or constrained equal loss method/rule, 4) the uniform gains or constrained equal award method/rule, 5) the contested garment method/rule [16]. Its extensions are: (1) the random-priority or random arrival method [1], [2]; (2) the Talmud method/rule [2]; (3) the adjusted proportional rule [20], [33]; 6) the concede-and-divide rule [31], [36].

2.2 Clustering

Here we present the different clustering methods. These methods are useful in the case of resource division where the beneficiaries are not the creators (the contributors). They are part of the large family of multivariate statistical methods called Data Analysis. Data Analysis is a family of descriptive, explanatory and predictive methods which deals with statistical studies relating to several variables. 1) Factor Analysis and 2) Clustering form the two families of descriptive methods.

The clustering method consists of finding classes which are such that the individuals of the same class are as similar as possible (intra-class homogeneity) and those of different classes the most dissimilar (inter-class heterogeneity) [7], [8].

2.2.1 Clustering subdivision

We distinguish between supervised clustering (classification) and unsupervised clustering (clustering).

In classification, the classes are defined in advance and new individuals must be classified one after the other into one or other of these already labeled classes.

With regard to unsupervised clustering, the classes are not fixed in advance but are determined progressively. This includes: hierarchical clustering (hierarchical approach) and non-hierarchical clustering (partitive approach). [7], [11], [25], [32].

Hierarchical (unsupervised) clustering consists of grouping individuals by constructing a hierarchy. It includes Hierarchical Ascending Clustering and Hierarchical Descending Clustering. The Hierarchical Ascending Clustering (HAC) or Clustering by aggregation is a method used to group individuals into a certain number of classes emerging from a hierarchy of partitions while the Hierarchical Descending Clustering (HDC) or Clustering by division consists of splitting a given group into two others. At each stage, a group is designated to be split into two. The hierarchical tree representation and the procedure for its division are the same, both by the "Ascending" approach and by the "Descending" approach [7], [35].

Non-hierarchical (unsupervised) clustering consists of dividing a set of data into different homogeneous subsets without a hierarchical link in the groupings of individuals. It includes the following methods: The k-means method (or moving centers) and the dynamic swarm method [7], [12], [35].

The clustering method can be applied directly to the data submitted for study in cases where the variables are homogeneous and there is no interest in bringing the origin of the axes to the center of gravity of the cloud of points. But when we are interested in the latter case, we transform the initial data into centered data, and if the variables are heterogeneous, we determine the reduced data.

2.2.2 Hierarchical Ascending Clustering Method

Clustering is a method of Statistics which consists of finding classes which are such that the individuals of the same class are as similar as possible (intra-class homogeneity) while those of different classes the most dissimilar (inter-class heterogeneity). Among the Clustering methods, there is the Ascending Hierarchical Clustering (AHC) which we will use in the following. The AHC makes it possible to group individuals into a certain number of classes derived from a hierarchy of partitions [7].

The creation of a AHC involves the following stages: 1) Constitution of the data table, 2) calculation of the distances between individuals in pairs, 3) calculation of the distances between groups of individuals (a group can be made up of a single individual), 4) construction and cutting of the dendrogram, 5) interpretation of the AHC results.

2.2.2.1 Constitution of the data table

It is a question of constituting the data from those collected from individuals based on the variables retained. This can be preceded by the transformation of the initial data into reduced (or centered-reduced) data in the case of heterogeneous data or into centered data. Centered data makes it possible to bring the origin of the axes back to the center of gravity while reduced data makes it possible to cancel the influence of one unit of measurement on that of the others.

2.2.2.2 Calculation of distances between individuals two by two

This step of the CAH consists of calculating the distances between individuals two by two leading to the grouping of two individuals having the smallest distance (1st grouping).

At this level, it is appropriate to choose a function or distance index [4], [23], [26].

Definition 3 (Distance). Let I be a set of points. A distance on I is a map d: I x I $\to \mathbb{R}^+$ satisfying $\forall$ i, k, l $\in$ I the three following properties: (1) d(i, k) = 0 if i = k (The separation) (If a distance is zero, it means that the two points are at the same place); (2) d(i, k) = d(k, i) (Symmetry) (The distance from i to k is equal to the distance from k to i); (3) d(i, l) $\leq$ d(i, k) + d(k, l) (The triangular inequality).

Different expressions of distances exist [7], these are: 1) Manhattan (or city-block) distance, 2) Euclidean distance, 3) Minkowski distance (distance to the power), 4) Chebyshev distance, 5) Squared Euclidean distance, 6) Canberra distance.

In particular, the Euclidean distance is given by:

$$d(i,k) = \sqrt{\sum_{j=1}^{m}\left(w_{ij} - w_{kj}\right)^2}$$

This is the 2-distance.

2.2.2.3 Calculation of distances between groups of individuals

This step consists of calculating the distances between the 1st group and the rest of the isolated individuals with a view to a new grouping. Therefore, it is necessary to choose an aggregation criterion (an index) [12]. The two closest objects (individuals or group of individuals) are merged. We will repeat this procedure until all the individuals find themselves grouped in the same group. This step makes HAC an iterative method. [4], [34].

Speaking of calculating the distance between two groups of individuals, one of them can consist of a single individual (isolated individual). We choose an aggregation index among many others allowing us to calculate the distance between the groups two by two. The closest ones are merged into a new group for the current iteration. This operation will continue until finally finding a single group containing all the individuals. Isolated individuals keep their distances already calculated in the previous step [4], [7], [34].

The different aggregation indices (or aggregation methods) are as follows: 1) minimum distance or simple deviation or minimum jump, 2) maximum distance or complete deviation or diameter, 3) average distance or average deviation, 4) median distance, 5) Ward's method, 6) distance of barycenters.

For the latter case, the distance between two groups I_1 and I_2 is that which is defined between their respective barycenters G_1 and G_2: $d(I_1, I_2) = d(G_1, G_2)$

Definition 4 (Barycenter). Let I be a cloud of n points 1, …, i, …, n of $\mathbb{R}^m$ with $i = (x_{i1}, \ldots, x_{ij}, \ldots, x_{im})$, x_{ij} being the value of i for the variable X_j. We call the barycenter (or the center of gravity) of I, the point $G = (\bar{X}_1, \ldots, \bar{X}_j, \ldots, \bar{X}_m) \in \mathbb{R}^m$ with $\bar{X}_j = \frac{x_{1j} + \cdots + x_{ij} + \cdots + x_{nj}}{n}$ the arithmetic mean of the variable X_j.

2.2.2.4 Construction and cutting of the dendrogram

To obtain a better partition, we cut the dendrogram at the level where the aggregation index makes a significant jump when we go from one partition to another using a horizontal line. It is then that we will obtain the best partition made up of classes which are 1) each non-empty, 2) two by two disjoint and 3) their union gives the whole made up of all their elements [6], [7].

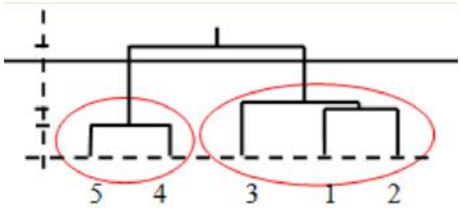

Fig 1. Cutting the dendrogram into two classes.

In this illustration, the dendrogram is divided into two classes: individuals 5 and 4 form one class, the first one, while 3, 1, 2 the second class. These two classes form a partition.

2.2.2.5 Interpretation of a partition

After having determined the best partition after cutting the dendrogram, we proceed with its interpretation of this partition. Which amounts to 1) calculating the position and dispersion parameters for each class: the mean, variance, standard deviation of each variable; 2) to characterize (or describe) the classes: by their individuals by determining those which are the most typical (Paragons and extremes) or by their variables by determining those which are the most important and finally 3) to graphically represent the classes of individuals .

We intend to use clustering to highlight the closest individuals in order to reduce inequalities between them. Because, as for us, the idea is that the closest individuals, that is to say belonging to the same class, should help each other before seeing others who are more distant from them can provide them with help. In addition, individuals belonging to the same population should help each other before another population comes to their aid. Thus, individuals show solidarity in their respective class and in the population as a whole. Which justifies in certain cases, the double reduction of inequalities: at the class level then at the level of the entire population. This concerns individuals who did not contribute to create the resource to be shared. Unlike the case where individuals contributed to create the resource. In the latter case, there is no question, for our part, of reducing inequalities [23].

3. RESULTS

3.1 Fair Division Approach based on Clustering and Reduction of Inequalities (APCR)

We intend to present our approach of divisible resources division called the Fair Division Approach based on Clustering and Reduction of Inequalities (APCR). It is obviously based on the clustering and reduction of inequalities. It takes into account, in addition to the plurality and type of variables, the origin of the resource, more particularly the case where it does not come from the contributions of individuals (unlike the case where individuals contributed to create the resource to share, which prevents the reduction of inequalities), which allows the reduction of inequalities between individuals belonging to the same class following their proximity or even between all of them at the population level following their belonging to the same population of where the relationships of proximity (belonging to the same class) and belonging to the same population.

This new approach is informed by our division methods that we have proposed based of course on the clustering and the reduction of inequalities (both in classic and fuzzy environments). Clustering makes it possible to find the most similar individuals (proximity) which will form a class. This will allow individuals of the same class to unite by reducing inequalities between them either at the level of their respective classes, or at the level of the entire population considered as a single class, or even at the level of classes and the population successively.

Regarding the plurality and type of variables, see below.

The sharing methods that we have put in place and on which we will focus in this chapter are: 1) the corrected proportional rule (PCo), 2) the Process for the Distribution of Resources at the Group (Class) level (PRRG) and 3) the Resource Allocation Process based on Classification Results (PRRC). We presented the latter in our previous work [23], [24].

These three methods stand out from the existing division rules in the literature and from the PRRS process (Process for the Distribution of Resources Without Reduction of Inequalities) that we proposed which of course uses the Proportional rule (P).

As a reminder, the PRRS method takes into account the plurality and type of variables that can modify the sharing results. It also takes into account the origin of the resource, more particularly the case where the resource comes from the contributions of individuals, which prevents the reduction of inequalities and pushes the use of P rule (without being able to first reduce inequalities). By proposing this process, we wanted to show that if there are several variables, before thinking of arriving at a single value for each individual, we must first check the type of the variables. These can either be homogeneous with a unit of measurement expressed in a unique way (for example for three variables the units are: kg, kg, kg), in which case they are used directly; either homogeneous but for which the same unit of measurement is expressed differently for each individual (for example: kg, g, hg), in which case it is necessary first to convert this unit of measurement into a single expression for all the variables and finally; or be heterogeneous (for example: kg, number of inhabitants, Congolese francs), in which case it is necessary to transform the initial data into reduced data by dividing each value by the standard deviation of the corresponding variable as is done in Statistics. Transforming the initial data into reduced data makes it possible to eliminate the influence of the units of measurement on each other. Centering the data is not important at this stage because it is useful for better visualization of a graphical representation.

In the PRRS process and therefore in P rule, individuals are selfish in the sense that each only aims for their own interests and they are independent of each other, each belongs to their own class which is confused with their own population, which justifies the calculation of proportional shares without reducing inequalities. These rules are more suitable for individuals such as shareholders of a company who have contributed a given amount and who expect to receive in return a share corresponding to their respective contribution. They are more oriented towards an economic aspect.

But in the PCo, PRRG and PRRC methods which are part of the family of APCR methods that we propose, individuals are altruistic and dependent on each other due to belonging to the same class or the same population. This justifies the reduction of inequalities at the class level or at the level of the population considered in particular as a single class or even at the successive levels of classes and of the entire population. They are better suited to the case where individuals have not contributed to creating the resource to be shared: sharing a donation, revenue from a state entity, an inheritance, to name but a few. These methods are more socially oriented.

3.1.1 Origin of the APCR

We intend to explain the elements which are the basis of the proposition of our approach of fair division, APCR, which involves the notions of classification and reduction of inequalities. Thus, we will talk about: 1) prescription of the law, 2) different problems raised by this law, 3) problem of taking into account the plurality and type of variables, 4) problem of taking into account the origin of the resource.

3.1.1.1 Prescribed by law

The main idea of our approach starts from the organic law of the Democratic Republic of Congo, still in force, relating to the composition, organization and functioning of Decentralized Territorial Entities (ETDs) and their relationships with the State and the Provinces which stipulates that the ETDs distribute the 40\% of the share of national revenue allocated to them by the province according to three criteria: production capacity, surface area and population. It reserves the establishment of the distribution mechanism for an edict [13], [14]

3.1.1.2 Various problems raised by this law

From this law, we have identified some problems, notably: (1) For ETDs, no mechanism for distributing these revenues using the three criteria proposed by the legislator is given. Moreover, these criteria are heterogeneous variables whose values cannot be used directly. And in practice, instead of three criteria, only one, “Population”, is used, assigning to each a share proportional to its population. Which is unfair, from the point of view of the law. (2) In addition, no mechanism taking into account the relationships between individuals, in the sense of reducing inequalities between them, is proposed. This is supported by Kapya Jean Salem [15] and Punga Paulin [30], who noticed the injustice in this distribution following the absence of an adequate mechanism for correcting inequalities and the use of a single variable. No solution has been proposed [15], [23], [30].

From the above, also taking into account the problems that arise in Mathematics on the division of resources, we have generally retained four problems which are reasons for injustice in the sharing, in particular, of a sum of money. These are: (1) Using a single variable instead of several. This results in: imbalance between individuals. (2) The direct use of initial data from homogeneous variables for which the same unit of measurement is expressed differently for each variable. Consequence: the influence of the scale of one variable on those of the others. (3) Direct use of initial data from heterogeneous variables. Consequence: the influence of one unit of measurement on others. (4) The absence of reduction of inequalities between individuals, in certain cases. Consequence: the absence of solidarity between individuals. These causes concern on the one hand the individuals who did not contribute to the creation of the resource to be shared ((1), (2), (3) and (4)), and those who contributed to the creation of the resource ((2) and (3)), on the other [23].

We maintain that for a division to be fair it must use an adequate mechanism which uses several variables while taking into account the particularities (types) of the latter and the relationships between individuals allowing, if necessary, the reduction of inequalities between them. A question arose: What processes should be put in place to resolve these problems? We considered the possibility of implementing these processes using several homogeneous or heterogeneous variables, taking into account the relationships between individuals in the sense of reducing or not the inequalities between them [23].

3.1.1.3 Problem of taking into account the plurality and type of variables

We based ourselves on the principle of equity which brings together philosophers, mathematicians, economists and even lawyers, and which uses the proportionality rule to allocate shares to individuals as well as the notions of data transformation, clustering and inequalities index used in Statistics.

The different fair division methods in the literature do not pay attention to the problems of types of variables (homogeneous or heterogeneous) and the origin of the resource (whether it comes from the contributions of individuals or not). Also, the literature is less rich regarding the plurality of variables. These problems can at a certain level involve problems of proximity of individuals and their belonging to the same population thus justifying the reduction of inequalities. The plurality and types of variables as well as the origin of the resource can negatively impact the division results if they are not taken into account.

Regarding the taking into account of a plurality of variables, it comes after data transformation but even before using the chosen fair division rule. For each individual, we must have a single value that we call total value corresponding to the terminology better known in the literature: claim. This is the amount claimed by an individual in a division, more particularly, the case of bankruptcy problem (bankruptcy). According to our approach, to arrive at this total value, it is necessary after conversion of the unit of measurement or transformation of the data, add the values of each individual. The total value found for each individual will constitute their claim and will allow their share to be calculated.

As for the types of variables, before carrying out the clustering or applying the fair division rule, it is necessary to check whether the variables are homogeneous or heterogeneous. In the case where they are homogeneous and for which the same unit of measurement is expressed in a unique way, we maintain the initial data and use them directly. If the variables are homogeneous and for which the same unit of measurement is expressed differently for each variable, we proceed by converting these different expressions into a single expression. And finally, if the variables are heterogeneous, we transform the initial data into centered data or reduced data or even centered-reduced data. The data are centered to bring the origin of the axes to the center of gravity (or barycenter) of the cloud of individuals (Case of homogeneous or heterogeneous variables) for better visibility of the graphic representation while they are reduced in order to annihilate the influence of units of measurement (Case of heterogeneous variables). Heterogeneous variables are those which are expressed on different units of measurement. Otherwise, they are said to be homogeneous. Clustering carried out without taking into account the problem of prior data transformation [9], [19] can lead to false results in our opinion [23].

The transformation of the data is done as follows: Let X_j be a variable and x_{ij} the value of individual i relative X_j. We have : 1) The centered variable: $\check{X}_j$ = x_{ij} - $\bar{X}_j$ where $\bar{X}_j = \frac{\sum_{i=1}^{n} x_{ij}}{n}$ is the arithmetic mean of the variable X_j . 2) The reduced variable: $t_j = \frac{x_{ij}}{\sigma_j}$ where $\sigma_j = \sqrt{\frac{\sum_{i=1}^{n}(x_{ij}-\bar{x}j)^2}{n}}$ is the Pearson deviation of $\bar{X}_j$ (Calculated for the entire population) or $\sigma_j = \sqrt{\frac{\sum_{i=1}^{n}(x_{ij}-\bar{x}j)^2}{n-1}}$, the Standard Deviation (Calculated for a sample of a population). 3) The centered-reduced variable: $\hat{X}_j = \frac{x_{ij}-\bar{x}j}{\sigma_j}$.

More particularly, the transformation of the initial data into reduced data resolves, for our part, the problem of injustice in the distribution of resources due to the direct use of heterogeneous variables [23].

At the time, we took into account the plurality and types of variables and the origin (contribution or non-contribution) through our two processes: PRRS and PRRC. The two differ according to the following: The PRRS does not admit the reduction of inequalities considering individuals as each belonging to their own group due to the fact that they contributed to create the resource to be shared. Everyone therefore aims for their own interests. It is a process which uses the proportional rule from several variables or claims taking into account their type. As for the PRRC process, it admits the reduction of inequalities at two levels, at the class level following their proximity and at the level of the entire population following their belonging to the same population.

3.1.1.4 Problem taking into account the origin of the resource

Speaking of the origin of the resource, it is a question of checking whether the resource to be shared comes from the contributions of its beneficiaries or if it comes from sources other than contributions. Considering this last case, that is to say of non-contribution, we consider that the individuals are first of all similar at the level of their class (ratio of proximity of individuals) and then between them all forming the same population (ratio belonging to the population). Which will justify the reduction of inequalities at the class and population level.

1) Absence of resemblance relationship.

The case where there is no relationship of resemblance (proximity) between individuals. Each is in a different set from another. The degree of membership (characteristic function) of each to its own set is 1 and to others 0. Individuals are assumed to belong to different populations. Which corresponds, in our opinion, to the case of selfish individuals for whom each benefits from their share entirely according to their verified value. The resource is shared directly according to each person's claim using an existing fair division rule, in particular the proportional rule (P), to name but a few. We say this is suitable for the case where individuals contributed to create the resource.

Which led us to talk about the Process of Resource Distribution Without Reduction of Inequalities (PRRS) between individuals based on several particularly heterogeneous variables [23].

2) Relationship of resemblance and belonging to the same population.

If individuals are distributed into different classes, there is the relationship of resemblance between those belonging to the same class and, for all of them, the idea of belonging to the same population. Those belonging to the same class look alike.

Considering this aspect, we proposed in our previous work the PRRC method for individuals belonging to different classes and are assumed to belong to the same population. As a result, he admits the reduction of inequalities for two reasons: the fact that individuals belong to the same class (that is to say they are most similar) and, all of them, to the same population (considered a single class). It shares the resource to individuals proportionally, not to their starting values, but to their values corrected using the Mputu index, at the level of their classes and the population. It therefore reduces inequalities at two levels: at the class level due to their proximity and at the population level (considered as a single class) following their belonging to the same population. Regarding the reduction of inequalities, the logic is this: individuals are altruistic and supportive, they help each other among those closest to them in a restricted group before helping each other, all together, at the population level. which considered as a unique class [23], [24].

Indeed, we judge that, in certain situations of life and more particularly in the case where individuals are similar or form the same population or in a social or state framework or more simply, if the individuals have not contributed to create the resource to be shared, we must also take into account the relationships between them at the level of their classes, that is to say among those who are closest to them as well as in the population. These are relationships of proximity and belonging to the same population. This allows solidarity between individuals in the sense of reducing inequalities between them to guarantee social peace. This leads us to consider the principle of justice that we have called multidimensional reduced (or corrected) equity, the sharing rule of which is multidimensional reduced (or corrected) proportionality [23].

As for the degree of belonging of individuals to each class, if the classification is deterministic (or classical), this is 1 if the individual belongs to a class and 0 if he does not belong to it. If, on the other hand, the environment is fuzzy, each individual belongs to each class partially, to a degree of membership found in the interval [0, 1] and the sum of its degrees of membership to all classes gives 1. In the case where a single class must be assigned to each individual, the latter will belong to the class to which its degree of membership is greatest. If there are many individuals in a class, their degrees of belonging stand out from those of others by their magnitude.

3.1.2 APCR Extension

When talking about the extension of APCR, it is a question of alluding, on the one hand, to other methods, in a classical environment, in addition to the PRRS and PRRC processes presented in our work previous ones [23], [24] and all these methods in fuzzy environment, on the other hand. Thus, as for us, due to the resemblance between them, individuals should show solidarity through the reduction of inequalities based on their values or demands in their respective classes and in the entire population due to their belonging to the same population. This led us to distinguish different fair division methods in their own right, completely different from those that exist in the literature, in particular the proportional rule (P) and the adjusted proportional rule (AP).

We consider the PRRC process as part of the family of fair division methods based on clustering and reduction of inequlities which form APCR, that is to say on the clustering and reduction of inequalities (using the Mputu index). . To this process, we add two other PRRG and PCo, which will make a total of three methods that we will compare with the proportional rule (P) and the adjusted proportional rule (AP) which already exist in the literature. We will approach these methods in both classic and fuzzy environments.

3.1.2.1 Four situations of individuals in a sharing and choice of method to use

From the above, four situations of individuals can arise for sharing a given resource. To each we can associate a given sharing rule. Let us illustrate these four situations. In the figures that follow, the square represents a population (a universe) and the circle a class. It should be noted that the details given there make it possible to distinguish our methods from each other. However, when the beneficiaries of a resource decide to use one or another rule or method this would not pose any problem provided that they have agreed beforehand.

1. Each individual belongs to its own class combined with a population in its own right

Each individual belongs to its own population different from that of others. He will be alone in his class or population. This means that by reducing inequalities, the inequality reduction index will give 0. Which means total equality and therefore there is no need to reduce inequalities.

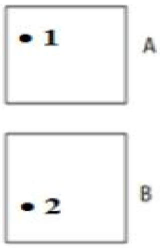

Fig 2 Each individual belongs to its own population

Considering the clustering (with the HAC method), this situation corresponds to that of the cut of the dendrogram is done at the bottom, at the level of the leaves, that is to say at the level of simple individuals.

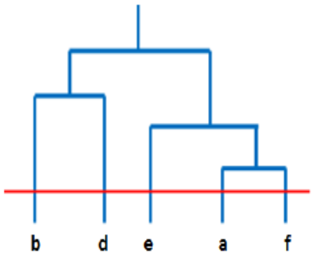

Fig. 3 Dendrogram (Each belongs to its class)

This situation is therefore likened to the case of selfish individuals who feel alone without belonging to any group (population) with others. The rules applied are PRRS (which we proposed in our previous work) and P. The PRRS method applies the P rule to multidimensional data. We used this method in Mputu

Denis-Robert (2022) [23], [24]. She does not interest us in this work.

This case does not even require the application of the clustering.

2. Each individual belongs to a class notably with others and forming with all the others the same population

This is the case of several classes each containing one or more individuals (the case where each has a single individual is excluded) and the latter feel like they belong to the same population.

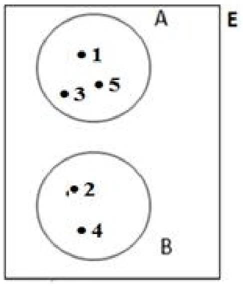

Fig. 4 Individuals each belonging to their class and all forming the same population

The application of the clustering is mandatory in this situation.

Considering the HAC, the cutting of the dendrogram is done at the level of the branches, that is to say at a level other than the trunk and the leaves. We therefore find several classes with one or more individuals (the case where each has a single individual is excluded)

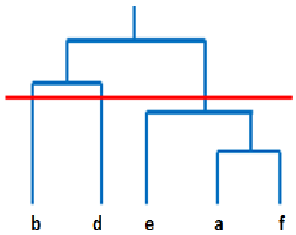

Fig. 5 Dendrogram (Different classes and the same population)

The fair division method used here is: PRRC (proposed in our previous work) which reduces inequalities by using the Mputu index at the class level then at the population level where it calculates the shares.

3. Each individual belongs to a class notably with others and each of the classes forms a population in its own right

This is the case where several classes each contain one or more individuals (the case where all of them each have a single individual is excluded) and each of the classes constitutes a population in its own right. Each class is therefore confused with a population in its own right.

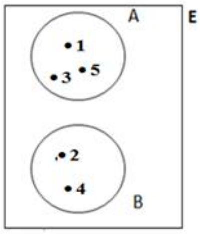

Fig. 6 Individuals belonging to different classes, each forming a population in its own right

The application of the clustering is mandatory in this situation. The fair division method used is: PRRG which reduces inequalities by the Mputu index at the class level, each forming a population and which calculates the shares at the class level. Considering the HAC, the cutting of the dendrogram is done at the level of the branches, that is to say at a level other than the trunk and the leaves. We therefore find several classes with one or more individuals (the case where each has a single individual is excluded)

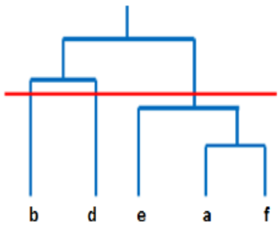

Fig. 7 Dendrogram (Different classes and the same population)

4. All individuals belong to a single class combined with the population

This situation corresponds to that where all individuals form the same class which is confused with the population. Which leads to the reduction of inequalities at a single level, which is the population.

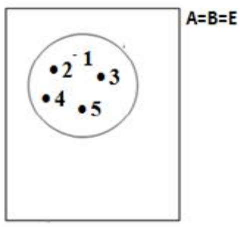

Fig. 8 Individuals all belonging to the same class combined with the population

Considering the HAC, this situation corresponds to the one which leads to the cutting of the dendrogram at the trunk, in a single class. The rule applied is PCo which uses a formula allowing proportional sharing after reducing inequalities to a level by the Mputu index. This situation does not even require the application of the clustering.

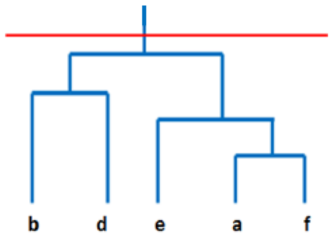

Fig. 9 Dendrogram (All belong to the same population

3.1.2.2 Other elements of extension of this approach: PCo, PRRG and PRRC methods in fuzzy environment

Another important element in extending the fair division approach through clustering and reduction of inequalities consists of considering the PCo, PRRG and PRRC methods in a fuzzy environment. It is therefore a question of bringing the notion of fuzzy clustering into a fair division process. Since the case of the PCo method does not require the application of clustering, it will therefore not be affected by the fuzzy environment of clustering. Only the PRRG and PRRC methods are affected.

3.2 Presentation of APCR methods

It is a question of presenting our three sharing methods which underpin the APCR approach that we propose. We will limit ourselves to the case of the classical environment. As for the fuzzy environment, it will be addressed in our subsequent work.

In the literature, most authors address the classification of classical data, that is to say real numbers, although there are some recent works which attempt to approach the case of data in the form of intervals. of $\mathbb{R}$. As for classification under fuzzy intervals, it seems to us that the field is still virgin. We will limit ourselves to discussing the classification of classic data (without interval).

The methods of the APCR approach are: 1) The Resource Distribution Process based on Classification results (PRRC) which was addressed in our previous work, we will only present the stages [23], [24]. 2) The Process of Distribution of Resources at the Class (Group) Level (PRRG). IT will be discussed in detail in this article. 3) The Corrected Proportional Rule (PCo). It will be addressed in our subsequent work.

The PRRC (PRRC/D for deterministic PRRC), PRRG (PRRG/D) and PCo methods constitute the classic aspect of our approach. The fuzzy PRRC (PRRC/F) and fuzzy PRRG (PRRG/F) methods form the fuzzy aspect of our approach. They will be discussed later. The PCo rule produces the same result in a classic environment and in a fuzzy environment. This will also be discussed later.

Indeed, in a classic clustering environment, individuals each belong to one and only one class and their degree of membership to their class belongs to the pair {0, 1}. The degree is 0 if the individual does not belong to the class and 1 otherwise. In a fuzzy environment, each individual partially belongs to classes and its degree of membership to a class is in the closed interval $[0, 1] \subset \mathbb{R}$.

3.2.1 Resource Distribution Process based on Classification Results (PRRC)

This method is to be used whenever individuals belong to k different classes and consider themselves to belong to the same population at the same time. As a result, they want to show solidarity on two levels: at the level of their classes and throughout the population. The classic aspect of PRRC (PRRC/D or PRRC) was addressed in our previous work while the fuzzy aspect (PRRC/F) is reserved for our later work.

To calculate the shares of individuals using the PRRC method, we proceed through the following detailed steps: 1) presentation of the table of data maintained, converted or reduced; 2) carrying out the CAH and presenting its results; 3) presentation of the table of degrees of belonging of individuals to each class; 4) calculation of total values (and their product with the degrees of belonging of individuals); 5) calculation of the inequality index at the level of each class; 6) calculation of corrected values of individuals at the class level; 7) calculation of the inequality index at the population level, 8) calculation of the corrected values at the population level, 9) calculation of the shares of individuals from the corrected values at the population level and 10) graphical representation of the shares of individuals.

The development of these stages as well as the related example is taken up in [23], [24].

3.2.2 Resource Distribution Process at the Group (Class) Level (PRRG)

This method is to be used whenever individuals belong to k different classes (classification) which they consider as k different populations. Solidarity is between individuals in their classes considering themselves separate from others in other classes.

To calculate the shares of individuals using the PRRG method, we proceed through the following detailed steps: 1) presentation of the table of data maintained, converted or reduced; 2) carrying out the CAH and presenting its results; 3) presentation of the table of degrees of belonging of individuals to each class; 4) calculation of total values and their product with the degrees of belonging of individuals; 5) calculation of the inequality index at the level of each class; 6) calculation of corrected values of individuals at the class level; 7) calculation of the shares of the classes in proportion to their corrected (global) values, 8) calculation of the shares of individuals in their respective classes using the share of the class, in proportion to the corrected value of each at the class level.

3.2.2.1 Presentation of the table of maintained, converted or reduced data

Let the resource C be shared between n individuals x_1, …, x_i, …, x_n (or 1, …, i, …, n) of I from the values of m variables (in units of measurement maintained, converted or transformed (reduced data)) $y_1, \ldots, y_j, \ldots, y_m$ and verified on the n individuals for which y_{ij} is the value checked by individual x_i for the variable y_j. Suppose they are presented in an array $T_{(n,m)}$. We have: $T_{(n,m)} = (y_{ij})_{1 \le i \le n,\ 1 \le j \le m}$

3.2.2.2 Creation of the AHC and presentation of its results

The determination of the clustering results is done using one of the clustering methods such as k-means, moving center, AHC, DHC, etc. In the case of AHC, the partition of the classes found after cutting, at the desired level, of the dendrogram is presented: each individual belonging to one and only one class (deterministic clustering approach). Each class will be presented as well as the individuals belonging to it.

We carry out the clustering, by the AHC method, from the table $T_{(n,m)}$. We thus determine a partition made up of r classes

3.2.2.3 Presentation of the table of degrees of belonging of individuals to each class

Each individual belongs to a unique class with a membership degree of either 0 (if it belongs to the class) or 1 (if it belongs to it). We present the table of degrees of belonging of individuals to each class as follows: $T_{(n,p)} = \left[u_{K_p}(x_i)\right]_{1 \le p \le r}$ with $u_{K_p}(x_i) = \begin{cases} 0 & si\ x_i \notin K_p \\ 1 & si\ x_i \in K_p \end{cases}$ the degree of belonging of an individual x_i to the class K_p (the p[th] class), $K_p \subset I$.

As a result, we determine the matrix of degrees of membership of q individuals belonging either totally or not at all to each of the r classes. For any individual x_i, $\sum_{p=1}^{r} u_{K_p}(x_i) = 1$

3.2.2.4 Calculation of total values and their product with the degrees of belonging of the individuals

We calculate for each i, its total (starting) value: $w_i = \sum_{j=1}^{m} y_{ij}$, the product of its total value with its degree of belonging to its class:

$w_{ip} = u_{K_p}(x_i).w_i$, as well as the overall value of the population: $V = \sum_{i=1}^{n} w_i$.

3.2.2.5 Calculation of the inequality index at the level of each class

Consider the class K_p with q individuals of products of total values with the respective membership degrees $w_{1p}, \ldots, w_{ip}, \ldots, w_{qp}$, then the index of inequalities at the level of the next class is given by: $J_{Mp} = \frac{\sum_{i=1}^{q}(w_{ip} - w_{1p})}{\sum_{i=1}^{q} w_{ip}}$ where q is the number of individuals belonging to the class K_p , $\sum_{i=1}^{q} w_{ip} = V_p$ the global value of the class K_p and $w_{1p} = min\{w_{ip}\}_{1 \le i \le q}$ with $w_{1p} \le \ldots \le w_{ip} \le \ldots \le w_{qp}$.

3.2.2.6 Calculation of corrected values of individuals at class level

We proposed the following expression to calculate the corrected values of individuals. For individuals of class K_p, the corrected value of an individual i is given by: $Z_{ip} = w_{1p} + w_{ip}.J_{Mp}, \forall i \in K_p$

Theorem 1 (The corrected value of an individual [24])

Let y={y_1, …, y_j, .., y_m} be the set of m variables; K_p={ x_1, …, x_i, …, x_q }, the p[th] class has q individuals;

$x_i = (y_{i1}, \ldots, y_{ij}, .., y_{im})$ the vector of m values verified by individual i, the possible transformation of the data starting point having been carried out, then the corrected value of an individual $i \in K_p$ of total value Z_{ip} is also written: $Z_{ip} = \frac{W_{1p}V_p + W_{ip}(V_p - qW_{1p})}{V_p}, \forall i \in K_p$

Proof see [24]

Corollary 1 (Total value of an individual based on its corrected value [24])

The total value of an individual i belonging to the class K_p to q individuals is determined after reduction of inequalities from its corrected value calculated: $w_{ip} = \frac{V_p (z_{ip} - w_{1p})}{V_p - q w_{1p}}$

Proof see [24]

3.2.2.7 Calculation of class shares in proportion to their corrected overall values

It is first necessary to share the resource C to the r classes formed proportionally to their corrected values.

Thus, the share of the pth class K_p of global value V_p is given by: $C_p = \frac{V_p}{V} . C$ where $V = \sum_{p=1}^{r} V_p$ is the overall population value.

3.2.2.8 Calculation of the proportional shares of individuals in their respective classes

The share of individual i belonging to class K_p (by the PRRG method) is given by: $c_i^{PRRG} = \frac{Z_{ip}}{V_p} . C_p$.
We do the same thing for all the individuals in their respective classes and then we gather the results.

4. APPLICATION

4.1 Problem Statement

We give a scholarship of C=50MFC (Fifty million Congolese Francs) to five students a, b, d, e and f who obtained, out of 10 points in Maths and French, the following respective grades: (6, 4) , (7, 8), (2, 3), (8, 9), (2, 6). Knowing that 1 point obtained corresponds to a bonus of 1 MFC (One million Congolese Francs), we ask to calculate the shares of individuals by the PRRG process (PRRG/D)

4.2 Problem Focus

Before calculating the shares, let us emphasize that the values verified by these five individuals come from two homogeneous variables which are “Points obtained in Maths” and “Points obtained in French”. There is therefore no need, for this case, to convert these variables to the same expression of the unit of measurement or to transform them into reduced variables. In addition, these students did not contribute to create the 50MFc resource that they must share. Which gives the idea of solidarity between the closest beneficiaries. Hence, according to our theory, the importance of clustering and therefore of reducing inequalities between those belonging to the same class and/or the same population. Also, the points obtained by the students constitute their demands.

4.3 Solution by the PRRG method

4.3.1 Clustering (classical) of individuals

4.3.1.1 Presentation of classification results

Using the R software and starting from the values of the individuals, we determine the results of the Ascending Hierarchical Clustering (AHC):

```
> H=matrix(c(6, 4, 7, 8, 2, 3, 8, 9, 2, 6), nrow=5, byrow=TRUE)

> H

     [,1] [,2]
[1,]    6    4
[2,]    7    8
[3,]    2    3
[4,]    8    9
[5,]    2    6

F=hclust(dist(H[,1 :2]))

> plot(F)

> rect.hclust(F,2)
```

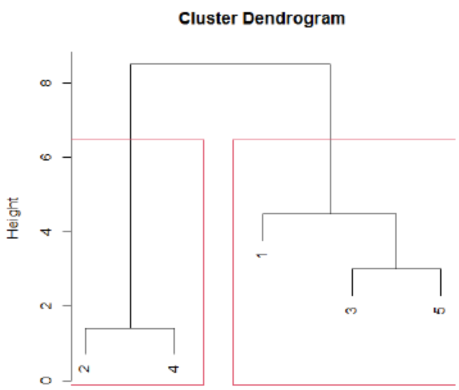

Fig. 10 Dendrogram divided into two classes from the logiciel R

Individuals are divided into two classes K_1={dfa}=dfa and K_2={b, e}=be. The total values of these classes are respectively V_1=5+8+10=23 and V_2=15+17=32.

4.3.1.2 Presentation of the matrix of membership degrees (characteristic function) after classic clustering

The degree of membership of individual a to class K_1 is $\mu_a(K_1)=1$ and to class K_2 is $\mu_a(K_2)=0$. Likewise we determine the degrees of membership of the other individuals: $\mu_b(K_1)=0$, $\mu_b(K_2)=1$; $\mu_d(K_1)=1$, $\mu_d(K_2)=0$; $\mu_e(K_1)=0$, $\mu_e(K_2)=1$; $\mu_f(K_1)=1$, $\mu_f(K_2)=0$.

We will verify that: $\sum_{j=1}^{q}\mu_i(K_j)=1$ where q is the number of individuals of a class K_j.

4.3.1.3 Calculation of the inequality index

(1) Sum of product of total value of each and its degree of belonging to each of the classes (Classic case)

$$\sum_{j=1}^{q}\mu_i(K_j)w_i=w_i, \forall i \in I$$

For individual a, we have: $w_1.\mu_a(K_1)+w_1.\mu_a(K_2)$ =10.1+10.0=10=w_1

(2) Calculation of the inequality reduction index}}

For the 1st class K_1={d, f, a}={5, 8, 10}, the minimum of the total values is: 5, that of d. We are: $J_{M_1}=\frac{\sum_{i=1}^{n}(w_{i1}-w_{11})}{V_1}=\frac{(5-5)+(8-5)+(10-5)}{23}=$ 0,347826087

and for the 2nd class K_2={b, e} = {15, 17}, the minimum is 15, that of b. We have: J_{M_2} = 0.0625.

4.3.1.4 Calculation of corrected values at the class level (Reduction of inequalities at the class level)

We use the inequality indices calculated above. Using the formula $Z_{ij}=w_{1j}+w_{ij}J_{M_j}$ to calculate the corrected values of individuals in their classes. We have :

- for the 1st class, the corrected values of individuals d, f and a are respectively $Z_{11}=w_{11}+w_{11}J_{M_1}$ =5+5.(0,347826087)=5+1,75=6,74; Z_{21}=5+8.(0,347826087)=7,78 and Z_{31}=5+10.(0,347826087)=8,48

So the corrected values of the individuals in the 1st class are: (d,f,a)=(6.74;7.78;8.48) .

- for the 2nd class, the corrected values of individuals b and e are respectively Z_{12}=15+15.(0.0625)=15.94 and Z_{22}=15+17.(0.0625)=16.06.

So the corrected values of the individuals in the 2nd class are: (b, e)=(15.94;16.06)

Thus, the corrected values at the class level are:

Table 1. Values corrected at class level

Individuals	K_1	Individuals	K_2
d	6.74	b	15.94
f	7.78	e	16.06
a	8.48	-	-
Total	23	-	32

4.3.1.5 Calculation of proportional shares of classes

Knowing that the resource to share is 50MFC, we calculate the shares of the classes proportionally to their corrected values in their respective classes.

Table 2. Shares of classes proportional to their corrected class values

Classes	Corrected values	Proportional shares
K_1	23	20.91
K_2	32	29.09
Total	55	50

4.3.1.6 Calculation of shares of individuals using the PRRG method (Calculation of shares at class level)

Knowing that the shares accruing to the classes proportionally to their corrected values are respectively 20.91MFC and 29.09MFC, then the shares of individuals by the PRRG method are proportional to their respective values in their class.

Table 3. Shares of individuals by the PRRG method

Indivi-duals	K_1	Individual shares	Indivi-duals	K_2	Individ ual shares
d	6.74	6.13	b	15.9 4	14.49
f	7.78	7.07	e	16.0 6	14.6
a	8.48	7.71	-	-	-

Total	23	20.91	-	32	29.09

4.3.1.7 Gathering of the shares of individuals calculated in the previous step

c_i^{PRRG}=(d, f, a, b, e)=(6,13; 7,07; 7,71; 14,49; 14,6)

5. CONCLUSION

We have just presented our equitable sharing approach, APCR, based on the clustering and reduction of inequalities as well as its different methods. Also, we presented and applied one of its methods called the Resource Distribution Process at the Group Level (PRRG), the different stages of which were compared to that of the PRRC method (Resource Distribution Process based on results of the Clustering) also part of this approach to equitable sharing. The PRRC method was the subject of an article that we published in 2023 [24].

For a better understanding of the APCR approach, we have successively presented its origin, its extension and its methods.

As for the PRRG method, it is to be used each time individuals belong to k different classes that they consider as k different populations. Solidarity is between individuals in their classes considering themselves separate from others in other classes. To calculate the shares of individuals using this method, we proceed through the following detailed steps: 1) presentation of the table of data maintained, converted or reduced; 2) carrying out the CAH and presenting its results; 3) presentation of the table of degrees of belonging of individuals to each class; 4) calculation of total values and their product with the degrees of belonging of individuals; 5) calculation of the inequality index at the level of each class; 6) calculation of corrected values of individuals at the class level; 7) calculation of the shares of the classes in proportion to their corrected (global) values, 8) calculation of the shares of individuals in their respective classes using the share of the class, in proportion to the corrected value of each at the class level.

The results produced by the PRRC method are preferred by the poor to those calculated using the PRRG method.

As for the Corrected Proportional Rule (PCo), a separate article is reserved for it.

6. ACKNOWLEDGMENTS

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