Strategy-proof Non-dictatorial Social Choice Functions in Clockwise Circular Domain

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Strategy-proof Non-dictatorial Social Choice Functions in Clockwise Circular Domain

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Abstract

Restricting domain had always been an approach to find out strategy proof social choice function in a voting setup where the society would choose one alternative from the individual preferences. By restricting the domain to a single peaked domain, Moulin (1980) [1] found strategy-proof non-dictatorial social choice functions however, Sato (2010) [2] shows that there does not exist any strategy-proof non-dictatorial choice function on a circular domain. Further restricting the circular domain to a clockwise circular domain, here we attempt to find all non-dictatorial social choice functions that are strategy proof on a clockwise circular domain. Many well-known social choice functions like majority rule, plurality rule, Instant runoff, Condorcet winner turns out to be manipulable whereas we find Borda count rule is strategy proof on a such domain for any number of agents and alternatives. We have defined two new SCF pairwise universal winner (PUW) rule and pairwise winner using plurality (PWP) rule which shows interesting properties. Both PUW and PWP are based on pairwise competition between alternatives, but the way a pairwise winner is decided is quite different. We found for two agents, PUW is strategy proof on clockwise circular domain. And PWP satisfies the monotone property on the clockwise circular domain for any number of agents and alternatives.

1 Introduction

Social choice theory is a field of study that deals with the methods and rules for aggregating individual preferences into a collective decision or choice, also known as social choice functions (SCF). We refer to each individual as an agent who has some preference over a set of social alternatives. Social alternatives indicate that the set of alternatives is the same for all agents in society. Social choice theory is applied in various fields such as economics, political science, and philosophy. It has implications for the design of voting systems, the allocation of resources, and the study of welfare economics.

[*]This is a working paper based on master's thesis of the author.

The main concern of social choice theory is how to make social choices that are fair, efficient, and consistent with the agent's preferences. There exist many properties of SCF to ensure these qualities. For example, strategyproof is one such property of SCF which ensures that every agent reports their true preferences only. If somehow, by misreporting they get their preferred outcome as the social outcome then the agents are not incentivized to reveal their true preference. On the other hand, we have the unanimity property of the social choice function which ensures that when all agents agree on what outcome is the best, then that outcome should be the social outcome.

One of the most well-known results in social choice theory is the impossibility theorem published independently by philosopher Allan Gibbard in 1973 [3] and economist Mark Satterthwaite in 1975 [4]. It states that on an unrestricted domain of feasible preferences over more than 2 social alternatives, an SCF is unanimous and strategyproof if and only if it is dictatorial. A dictatorial social choice function is a type of social choice function in which the decision of a single agent, known as the dictator, determines the outcome of the entire group's decision. In other words, the preferences of the dictator override the preferences of all other agents in the group.

According to moral theory, a dictatorial social choice function is considered to be a controversial and problematic form of decision-making; as per the theory of democracy, every individual should have an equal say in the decision-making process. The dictatorial rule is generally seen as morally and ethically wrong, as it involves a concentration of power in the hands of a single individual or a small group of individuals, often at the expense of the basic rights and freedoms of others. That's why we often refer to a dictatorial rule as non-democratic. The Gibbard-Satterthwaite's impossibility result has sparked a continuous search for possibility results. Since the impossibility theorem is defined on an unrestricted domain of feasible preferences, one can find strategyproof, unanimous and non-dictatorial social choice functions by restricting the domain.

The pursuit of possibility results by restricting the domain can be carried out in two ways. The first way involves researchers fixing a suitable set of permissible preferences and investigating/characterizing SCFs that are strategyproof and unanimous within that domain. And the other way to find possibility result focuses on a specific preference aggregation rule and investigate the domains over which these rules satisfies the desirable conditions. Dasgupta and Maskin's (2003)[5] and M.Barbie et al (2006) [6] used this approach to study SCFs like majority rule, plurality rule, Borda count rule and ask the domain on which these satisfies the desirable conditions. In our study, we have opted for the former approach.

Black (1958) [7], Moulin (1980) [1] and Achuthankutty et al. (2017) [8] are some researchers who used the former approach to find the possibility results by investigating the social choice functions on single peaked preference domains and partially single peaked domains. Sato (2010) [2] used the same approach to find the minimal size of the circular domain for the existence of strategyproof SCFs.

Black (1958) [7] in "The Theory of Committees and Elections" represents a seminal work in social choice theory that provides a mathematical framework for analyzing the behavior of committees and voting systems. It introduces the concept of a power index, which measures the influence of each member of the committee in the decision-making process. It analyzes several voting systems, including simple majority voting, approval voting, and the Borda count, and shows how they can be analyzed using the tools of social choice theory. Overall, "The Theory of Committees and Elections" is a foundational work in social choice theory that provides a rigorous mathematical framework for analyzing the behavior of voting systems and committees. It has had a significant impact on the development of modern political science and economics.

Moulin (1980) [1] is the article "On strategyproofness and single peakedness". It discusses the relationship between two important properties of social choice mechanisms: strategyproofness and single-peakedness. Single peakedness refers to the property of preferences where each individual has a peak point, and the farther away from this peak point, an alternative is, the less desirable it becomes. The article shows that if agent preferences are single-peaked, then there exist strategy-proof social choice mechanisms that always select the alternative that is closest to the peak of each individual's preferences. Moreover, the article shows that this social choice mechanism is the unique one that satisfies both strategyproofness and single-peakedness. The article also discusses the implications of these results for the design of voting systems and provides examples of practical applications. Overall, the article makes a significant contribution to the field of social choice theory by providing a deeper understanding of the relationship between strategyproofness and single-peakedness.

Achuthankutty et al. (2017) [8] investigate the existence and characterization of strategy-proof social choice rules for partially single-peaked domains. Partially single-peaked preferences refer to situations where individuals have multiple peak points in their preference rankings, rather than just one. They also establish the existence and uniqueness of strategy-proof social choice rules in this setting. Overall, the paper contributes to the literature on the design of social choice rules by providing a comprehensive analysis of the properties of strategy-proof rules for partially single-peaked domains.

Chatterji et al. [9] study the problem of designing social choice mechanisms on tops-only domains. A tops-only domain is a preference domain where each individual's preferences are defined by a ranking of the alternatives, and where the top-ranked alternative is the only one that matters. In other words, the only alternatives that are considered are those that are ranked first by at least one individual. We defined peaks-only property of SCF which is similar to the tops-only concept. The paper provides a characterization of social choice correspondences on tops-only domains that satisfy a set of desirable properties, including Pareto optimality, strategyproofness, and non-dictatorship. The paper shows that there exists a unique social choice correspondence that satisfies these properties, which selects the alternative that is top-ranked by the largest number of individuals. The paper also discusses the implications of these results for the design of practical social choice mechanisms on tops-only domains and provides examples of applications in the context of resource allocation problems.

In Sato (2010) [2] paper, they restricted the domain to a circular domain. The circular domain is a special case of the one-dimensional Euclidean space, where the set of alternatives forms a closed loop. This domain arises in a variety of real-world applications, including voting, network routing, and job allocation. It proves that in any circular domain, there is no strategy-proof, unanimous, and non-dictatorial social choice function exists. The minimum size of the circular domain which brings about this incompatibility is 2m, where m is the number of alternatives. That is, it is possible to construct a non-dictatorial and strategy-proof social choice function on a domain containing less than 2m circular preferences.

Chatterji and Sen (2011)[9] independently detects the minimal size 2m of domains that are non-dictatorial and strategy-proof. This ensures that there must exist social choice functions that are strategyproof on the clockwise circular domain. Further, Sato (2014) [10] studies the problem of designing social choice mechanisms for the circular domain. The paper provides a characterization of strategy-proof and unanimous social choice correspondences in the circular domain. It proves that under each strategyproof and unanimous social choice correspondence, there is at least one agent who is decisive and circular sets of preferences over alternatives are sufficient for the existence of a decisive agent. Overall, Shin Sato [2] makes a significant contribution to the field of social choice theory by providing a deeper understanding of the properties of social choice mechanisms in the circular domain, and by providing a practical framework for designing strategy-proof mechanisms in this setting.

There is a lot of work already done on domains like single peaked domains and the characterization strategyproof social choice functions on that domain. However, for a circular domain, not much work has been done so far. For any domain with less than 2m circular preferences, we can find some possible results. Also, Kim and Roush (1980) [11] studied clockwise and counterclockwise circular preferences which is a special form of 2m circular preferences given by Sato (2010) [2], so we have to try restricting the domain to only a clockwise circular domain. Vaguely, a clockwise circular domain with respect to a fixed preference ordering consists of all preference ordering which can be obtained by going in a clockwise fashion on a circle represented by arranging the fixed preference ordering on a circle in a clockwise fashion.

As a clockwise circular domain contains exactly m number of preferences, therefore there must exist a strategy-proof and non-dictatorial SCF. So we have restricted our domain to a clockwise circular domain. Our objective is to find strategy-proof social choice functions on a clockwise circular domain. Hence this motivates our study to find all possible non-dictatorial social choice functions that are strategy-proof under CC domain.

Some results of M. Barbie et al. (2006)[6] align with one of our results. They focused on the Borda count rule and find that it is strategy-proof on a cyclic permutation domain. It uses the second type of approach. They have fixed the Borda count rule for their study and they found that the Borda count is non-manipulable for all tie-breaking rules on cyclic permutation domains.

They have used any scoring method, in general, where scores are inversely proportional to the rank of alternatives and the Borda count is one such rule. Since the Clockwise circular domain is a subset of the cyclic permutation domain, therefore our result aligns with their result. In section 3, we provide a short proof of the same. They have used the equal score difference method and proved that if any domain satisfies the equal score difference condition, then the Borda count rule is non-manipulable on that domain. We didn't use the equal score difference condition for our study.

In section 2, we discuss the notations we used in the whole thesis, and some basic definitions of social choice functions, strategy-proof, etc. Then we will move forward to the definition of a clockwise circular restricted domain and explain it in detail. We will continue with the domain of clockwise circular for our study. Every SCF we will discuss and the examples will be on a clockwise circular domain only. Then, we will give the alternative proof for the strategyproofness of the Borda count rule in the clockwise circular domain in section 3. We first give a proper definition and Borda count rule and explain it properly. We define the peaks-only property of SCFs and proved that the Borda count rule is always peaks-only. Then we have proved that Borda count is strategy-proof for any number of agents and any number of alternatives in a clockwise circular domain.

In section 4, we define a new social choice function. We called the new SCF as pairwise universality winner (PUW) rule. We proved that PUW is strategy proof for 2 agent case. We discussed the similarities and differences between PUW rule and Borda count rule. Then we showed that PUW does not satisfy the strategy proof property for $n = 3$ case. After that, in section 5, we also discussed some already well-known SCFs like majority rule, plurality rule, Condorcet winner, and many such rules only to find that none of them turned out to be strategyproof on our restricted domain. We have also given a new SCF, in section 5.5, which is also based on pairwise competitions between alternatives. We call it a pairwise winner using the plurality rule (PWP) and proved that it satisfies the monotone property. And we have given some insights on strategy-proof of the PWP rule. Finally, we end the discussion in section 6 with the conclusions and provide the future prospects for the study in a clockwise circular domain.

2 Preliminaries

2.1 Notations

Consider a society of n agents who have some strict preference over m social alternatives, where n and m are any finite positive integer. Assume N denotes the set of all n agents and A denotes the set of all m alternatives. Thus, $|N| = n$ and $|A| = m$.

Every agent has a strict preference over the set of alternatives. The preferences are only known to the agents themselves. Let $\succ_i$ represent strict preference ordering of agent i over A $\forall i \in N$. So, $a \succ_i b$ implies that agent i strictly prefers alternative a over alternative b. Let $\mathcal{P}$ denote the set of all strict preference orderings over A. Since $\mathcal{P}$ includes all such orderings, it is also known as an unrestricted domain. Assume a preference profile consists of strict preference orderings of all agents over A is denoted by $\succ \equiv (\succ_1, \succ_2 \ldots \succ_n)$. Since $\succ_i \in \mathcal{P}$, thus $\succ \in \mathcal{P}^n$.

Let $\succ_i (k)$ represent the kth ranked alternative in the preference ordering $\succ_i$. So, the top ranked alternative of $\succ_i$ is denoted by $\succ_i (1)$, the second ranked alternative by $\succ_i (2)$, and so on. For example, consider $A = \{a, b, c\}$ and for some agent $j \in N$, $a \succ_j b \succ_j c$. Then $\succ_j (1) = a$, $\succ_j (2) = b$, and $\succ_j (3) = c$. We call a as the top alternative of agent j and c as the bottom alternative of agent j.

2.2 Definitions

Definition 2.1. ***Social choice function:*** *A mapping $f : \mathcal{P}^n \to A$ that assigns a single social alternative $\in A$ to each preference profile $\succ \equiv (\succ_1, \succ_2 \ldots \succ_n) \in \mathcal{P}^n$ is called a social choice function (SCF).*

Definition 2.2. ***Strategy-proof:*** *A SCF f is strategyproof if $\forall i \in N$, $\succ_i$, $\succ'_i \in \mathcal{P}$ and $\forall \succ_{-i} \in \mathcal{P}^{n-1}$, we have $f(\succ_i, \succ_{-i}) \succ_i f(\succ'_{i'} \succ_{-i})$.*

A strategy-proof SCF is also known as non-manipulable.

Definition 2.3. ***Unanimous:*** *A SCF f is unanimous if for every preference profile $\succ \in \mathcal{P}^n$ with $\succ_i (1) = a \ \forall i \in N$ where $a \in A$, we have $f(\succ) = a$.*

Definition 2.4. ***Dictatorial rule:*** *A SCF f^D is called a dictatorial rule if there exists an agent h, called the dictator, such that for every preference profile $\succ \in \mathcal{P}^n$, we have $f^D(\succ) = \succ_h (1)$.*

2.3 The preference domain: clockwise circular

In this section, we introduce the fundamental concept of this thesis, namely a clockwise circular domain. The definition is presented formally, followed by an explanation of the concept using examples.

Sato (2010) [2] explained the circular domain concept using a Figure 1. Assume 10 social alternatives are arranged on a circle such that for each a_k, the preferences constitute a circular domain if the given two conditions hold:

(i) $\succ_i (1) = a_k$, $\succ_i (2) = a_{k+1}$, $\succ_i (m) = a_{k-1}$
(ii) $\succ'_i (1) = a_k$, $\succ'_i (2) = a_{k-1}$, $\succ'_i (m) = a_{k+1}$.

(Let $a_{m+1} = a_1$ and $a_0 = a_m \cdot$). Where $\succ_i (s)$ represents the sth ranked alternative with respect to preference ordering $\succ$. Note that, all other alternatives can be arranged in any order.

However, we will only consider domain which puts restriction on all alternatives rather than just the top and its adjacent alternatives like Sato (2010) [2] did. Imagine all indexed alternatives from a_1 to a_m arranged on a circle such that for each alternative a_i on the circle, $a_{(i+1)}$ is sitting just

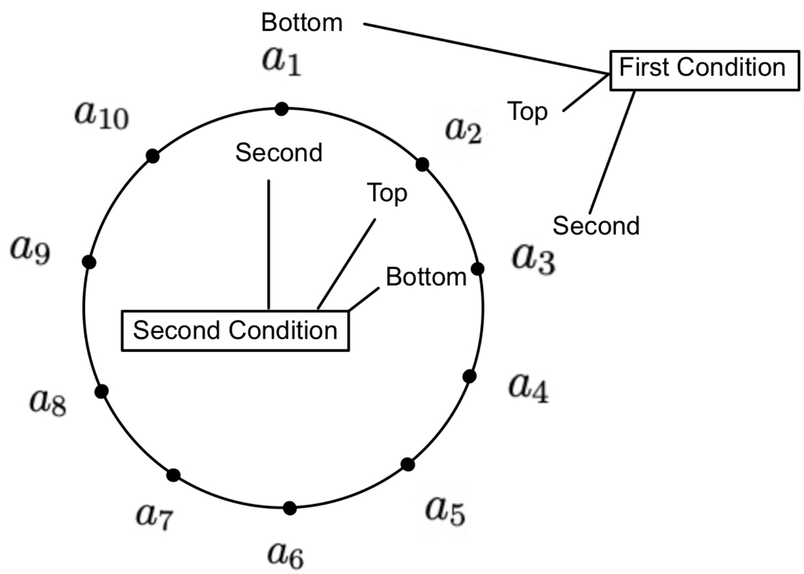

Figure 1: A circular domain with 10 alternatives [2]

next to a_i on the right side as shown in Figure 2. Thus, just next to a_m, we have a_1.

From this, we can observe the most intriguing property of the clockwise circular domain which is that the top and bottom alternatives in one preference ordering are actually neighboring alternatives in some other preference ordering in a given domain.

Definition 2.5. ***Clockwise Circular(CC):*** *A clockwise circular domain ($\mathcal{D}$) with respect to a fixed preference ordering $a_1 \succ a_2 \succ \ldots \succ a_m$ is a collection of all preference orderings of the form $a_i \succ a_{(i+1)} \succ \ldots \succ a_{(m-1)} \succ a_m \succ a_1 \succ \ldots \succ a_{(i-1)}$, where $a_i \in A = (a_1, a_2, \ldots a_m)$.*

Note that in the above definition, $a_0 = a_m$ and $a_{m+1} = a_1$. The domain is rich in the sense that each alternative is topped at least in one preference ordering. And since each alternative is topped in only one preference ordering in the admissible domain, hence the total cardinality of the clockwise circular domain is exactly equal to m.

Since the counter clockwise domain shows the similar properties to the clockwise circular domain, therefore all our study will apply to counter clockwise domain as well. Hence, from now on, our discussion will be focussed on a clockwise circular domain $\mathcal{D}$ only.

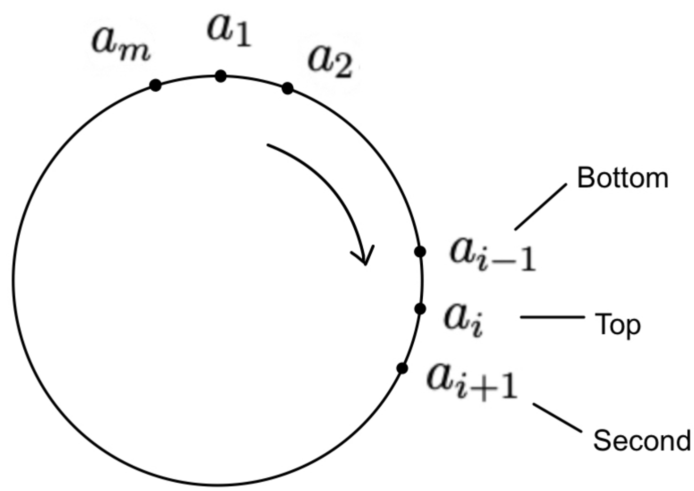

Figure 2: A clockwise circular domain over A with $|A| = m$.

3 Alternative proof: strategyproofness of Borda count rule

3.1 Notations and Definitions

Let R$[x, \succ_i]$ denote the rank of alternative x in the preference ordering $\succ_i$ $\forall i \in N$. So, for any $x \in A$ and any positive integer $k \leq m$, if $\succ_i (k) = x$ then R$[x, \succ_i] = k$. Consider an example of 3 agents and 3 alternatives. Let $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$. Consider a preference profile $\succ \equiv (a \succ_1 b \succ_1 c,\ b \succ_2 a \succ_2 c,\ c \succ_3 a \succ_3 b) \in \mathcal{P}^n$. Then R$[a, \succ_1] = 1$, R$[b, \succ_2] = 1$, R$[c, \succ_3] = 1$, and so on.

Definition 3.1. ***Borda Count Rule:*** *For any preference profile $\succ \in \mathcal{P}^n$, the Borda Count rule $f^B : (\mathcal{P}^n) \to A$ will select an outcome $x \in A$ such that sum of all ranks of the alternative x in all n preference orderings in $\succ$, that is, $\sum_{i \in N} R(x, \succ_i)$, is minimum. In case there are multiple alternatives with the minimum rank, the unique outcome is decided using any tie-breaking rule.*

So, in the previous example, for alternative a, $\sum_{i \in N} R(a, \succ_i) = \text{R}[a, \succ_1] + \text{R}[a, \succ_2] + \text{R}[a, \succ_3] = 1 + 2 + 2 = 5$. Similarly one can check that for alternative b and alternative c, we get $\sum_{i \in N} R(b, \succ_i) = 6$ and $\sum_{i \in N} R(c, \succ_i) = 7$ respectively. So, we get $f^B(\succ) = a$.

One advantage of Borda count SCF is that it takes into account the preferences of all agents, rather than just the top preference of each agent as will see in section 6. However, there is one disadvantage as it can sometimes lead to strategic manipulation, where agents rank the alternatives based on their perceived chances of winning, rather than their true preferences.

We can see in the above example if agent 2 misreports his preference as $b \succ_2' c \succ_2' a$ while agents

1 and 3 stick to their true preferences. Then for alternatives a, b, and c, we get the sum of their ranks in preference profile $\succ'$ as $\sum_{i\in N} R(a, \succ_i') = 6$, $\sum_{i\in N} R(b, \succ_i') = 6$, $\sum_{i\in N} R(c, \succ_i') = 6$ respectively. So we have a tie between alternatives a, b, and c. To resolve this tie, we need a tie-breaking rule. Consider a tie breaking rule as $b \succ_{\mathcal{T}} a \succ_{\mathcal{T}} c$. Then, $f^B(\succ') = b$. Since $b \succ_2 a$, hence agent 2 can successfully manipulate.

Note that, in the above example, both clockwise and counterclockwise circular preferences are permissible in the domain. However, if we restrict the preferences to be only clockwise circular or just counterclockwise circular then such manipulations can be prevented. Turns out, the Borda count rule is strategy-proof in a clockwise circular domain for any m and n.

In this section, we discuss a slightly different approach to proving Borda count is strategy-proof in a clockwise circular domain from that of M. Barbie et al.(2006).

Definition 3.2. ***Peaks only.***
A social choice function defined on a preference domain $\mathcal{P}$, $f : \mathcal{P}^n \longmapsto A$ is peaks only if there exists $i \in N$ such that $f(\succ) = \succ_i (1)$ where $\succ_i (1)$ is the top alternative in agent i's preference ordering $\succ_i \in \mathcal{P}$.

So for the preference profile like $\succ \equiv (a \succ_1 b \succ_1 c \succ_1 d,\ b \succ_2 d \succ_2 a \succ_2 c,\ c \succ_3 a \succ_3 d \succ_3 b)$, if a social choice function f satisfies peaks only, then $f(\succ) \in \{a, b, c\}$ as alternative a, b, and c are the top alternative of agent 1, agent 2 and agent 3 respectively. We now show that the outcome selected by the Borda count rule is peaks only in a clockwise circular domain $\mathcal{D}$.

3.2 Borda count is Peaks-only

Theorem 3.1. *The social outcome based on Borda rule $f^B : (\mathcal{D}^n) \longmapsto A$ is always peaks only, where $\mathcal{D}^n$ represents the clockwise circular preference profile domain.*

Proof: Let's assume for contradiction that $f^B(\succ_1, \succ_2) = a_r \in A$ such that $\succ_i (1) \neq a_r$ $\forall$ i $\in N$.

We define $C(a_r) = \{y \in A \mid \forall k \in N, R(y) < R(a_r)\}$.

Note that, $C(a_r)$ is nonempty (Since $a_{r-1} \in C(a_r)$ always). Since for any alternative $t \in C(a_r)$, $\sum_{k\in N} R(t, \succ_k) < \sum_{k\in N} R(a_r, \succ_k)$. This contradicts the fact that $f^B(\succ_1, \succ_2) = a_r$. Hence proved.

3.3 Borda count rule is always strategy proof in $\mathcal{D}$

Theorem 3.2. *The social outcome based on Borda rule $f^B : \mathcal{D}^n \longmapsto A$ is always strategy-proof, where $\mathcal{D}^n$ represents the clockwise circular preference profile domain*

Proof: Note that for $m \leq 2$ or $n = 1$, the proof is trivial. So, we assume $n \geq 2$ and $m \geq 3$. Consider any profile $\succ$ in $\mathcal{P}^n$. Suppose, without loss of generality, $f^B(\succ) = y$ where $y \in A$.

Now, consider an agent i such that $\succ_i (1) = x$. If $x = y$, then agent i have no incentive to misreport as the social outcome selected by the Borda rule is its top preferred outcome. Let x and y be different. We need to check if agent i can gain from misreporting or not.

For any preference ordering $\succ \in \mathcal{P}$, we define:

$$S\,(x \longmapsto y) = \{j \in A \,|\ni x \succ j \succ y\} \tag{1}$$

Consider any profile $\succ_i' (1) = z \in \mathcal{P}$. If $z = x$ or $z = y$, the agent does not gain since the social outcome remains the same. Assume $z \in S(x \longmapsto y)$. Since the total rank of z and y decreases by the same amount from $\succ_i (1)$ to $\succ_i' (1)$. Hence the total rank of z with respect to y does not change from $\succ$ to $\succ'$, that is, their rank has decreased by $|S(x \longmapsto z)|$ +1. Since z was not chosen before, thus the outcome remains the same, which was y. Similarly, the rank of all alternatives in $S(z \longmapsto y)$ does not change with respect to y, so the social outcome cannot be from $S(z \longmapsto y)$ also. Thus, agent i does not gain.

Now let us assume that $z \in S(y \longmapsto x)$. Although misreporting as such actually increases the rank of all alternatives in $\succ_i (1)$ to $\succ_i' (1)$. And thus the total rank of all alternatives in $S(x \longmapsto y)$ with respect to y degrades from $\succ$ to $\succ'$. Since any alternative from $S(x \longmapsto y)$ was not chosen earlier, hence the outcome can not be from $S(x \longmapsto y)$. Thus, agent i can not gain in this case too.

Hence proved.

4 Pairwise Universal Winner Rule

4.1 Notations and Definitions

In this section, we define a new social choice function called Pairwise Universal Winner Rule.

Definition 4.1. ***Pairwise Winner:*** *For any alternative x and $y \in A$, alternative x is called the pairwise winner in a pairwise competition of x and y, denoted by $W(x,y) = x$, if $|S(x \longmapsto y)| > |S(y \longmapsto x)|$, where $S(x \longmapsto y) = \{j \in A \,|\ni x \succ_k j \succ_k y\}$. If $|S(x \longmapsto y)| = |S(y \longmapsto x)|$, then any tie-breaking rule can be used.*

Consider an example when there are 2 agents, namely agent 1 and agent 2, and three social alternatives a, b, and c. Consider a clockwise circular domain of preferences as $\{a \succ_i b \succ_i c,\ b \succ_i c \succ_i a,\ c \succ_i a \succ_i b\}$. Then, for each possible pair (a,b), (a, c), and (b, c), we get a pairwise winner as:
$\mathrm{W}(a,b) = b$ as $S(b \longmapsto a) > S(a \longmapsto b)$
$\mathrm{W}(a,c) = a$ as $S(a \longmapsto c) > S(c \longmapsto a)$

W$(b, c) = c$ as $S(c \longmapsto b) > S(b \longmapsto c)$

Since our focused domain is the clockwise circular domain, hence each alternative pair x and y are arranged on a circle. A clockwise circular preference can be imagined as all alternatives uniformly arranged on a circle, then the pairwise winner of pair (x, y) can be imagined as the one which is closer to the other alternative. For example, in Figure 3, x is closer to y, therefore x is going to be the pairwise winner between alternative x and y. Note that, it satisfies the condition $S(x \longmapsto y) > S(y \longmapsto x)$ of pairwise winner.

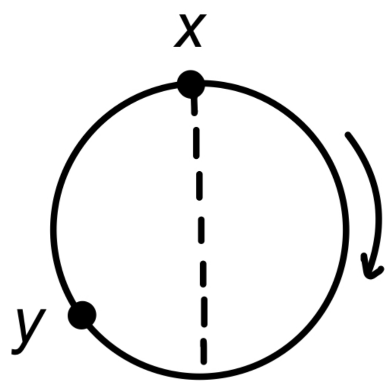

Figure 3: Pairwise winner using circle diagram

Now we define our new social choice function as follows:

Definition 4.2. ***Pairwise Universal Winner Rule (PUW):*** *A social choice function $f^U : \mathcal{D}^n \longmapsto A$ is defined on a clockwise circular domain. A pairwise universal winner refers to an alternative who is a pairwise winner the maximum number of times when pairwise competition is taken out for all agent's top alternatives. If there is a tie, a tie-breaking rule is used.*

In the above example, for $n = 2$, consider a preference profile $\in \mathcal{D}^n$ as $\succ \equiv (a \succ_1 b \succ_1 c$, $c \succ_2 a \succ_2 b)$. Since there are only two agents with top preferences as a and c respectively, therefore we need to take only one pairwise competition which is for the pair (a, c). Since we have only one pair, the winner of this pair is going to be the pairwise universal winner. Therefore, $f^U(\succ) = a$.

For better understanding, consider a case of 3 agents with 15 social alternatives. Assume that their true preference orderings are $\succ_1, \succ_2, \succ_3 \in \mathcal{D}$ with top preferences as $\succ_1 (1) = a_1$, $\succ_2 (1) = a_5$ and $\succ_3 (1) = a_{13}$ respectively. Since we have 3 agents, thus we have a total of 3 top alternatives for

pairwise competition. We have
$W(a_1, a_5) = a_5$
$W(a_1, a_{13}) = a_1$
$W(a_5, a_{13}) = a_5$
We can see that a_5 is winning the maximum times in all pairwise competitions, therefore a_5 will be the social outcome using the pairwise universal winner rule.

4.2 Pairwise Universal Winner Rule vs Borda Count Rule

We can check that the pairwise universal winner rule and Borda count rule are both different social choice functions as they both may give different results for the same preference profile.

For example, consider a case of three agents with 15 social alternatives. Consider a profile $\succ \equiv (\succ_1, \succ_2, \succ_3)$ with top preferences as $\succ_1 (1) = a_5$, $\succ_2 (1) = a_{11}$ and $\succ_3 (1) = a_{13}$ respectively. Using Borda rule, we get $f^B(\succ) = a_{13}$.

By using the pairwise universal winner rule, we have
$W(a_5, a_{11}) = a_{11}$
$W(a_5, a_{13}) = a_5$
$W(a_{11}, a_{13}) = a_{13}$
Since we do not have a single alternative who is winning maximum times in the pairwise competitions, therefore we have a tie. Consider a tie-breaking rule as $a_1 \succ_\mathcal{T} a_2 \ldots \succ_\mathcal{T} a_m$. Then, $f^U(\succ) = a_5$.

Therefore, we get different social outcomes by the pairwise universal winner rule and Borda count rule. Hence, the pairwise universal winner rule is different from the Borda count rule.

4.3 Pairwise universal winner rule is strategy-proof for 2 agents

Theorem 4.1. *When $n = 2$, the social outcome based on pairwise universal winner rule $f^U : \mathcal{D}^n \longmapsto A$ is strategy proof, where $\mathcal{D}^n$ represents the clockwise circular preference profile domain.*

Proof: Note that for $m \leq 2$ or $n = 1$, the proof is trivial. So, we assume $n \geq 2$ and $m \geq 3$. Consider any profile $\succ \equiv (\succ_1, \succ_2)$ in $\mathcal{P}^n$. Let $\succ_1 (1) = x$ and $\succ_2 (1) = y$. Since we have only two alternatives, namely x and y, who are the top alternatives in profile $(\succ_1, \succ_2)$, therefore $f^U (\succ_1, \succ_2) = W(x, y)$. If $y = x$ then $W(x, x) = x$. Therefore, PUW is unanimous. Since both agents are getting their top alternative as the social outcome, no one has the incentive to misreport. Assume, let x and y are different.

Since $m \geq 3$, both $S\,(x \longmapsto y)$ and $S\,(y \longmapsto x)$ can not be empty. Now it must be $|S\,(x \longmapsto y)| < |S(y \longmapsto x)|$ or $|S\,(x \longmapsto y)| \geq |S\,(y \longmapsto x)|$.

Case I: $|S(x \longmapsto y)| > |S(y \longmapsto x)| \Rightarrow f^U(\succ_1, \succ_2) = x$. For Agent 1, no incentive to misreport. For agent 2, need to check if agent 2 can gain by misreporting or not. Let $\succ_2'(1) = z$. If $z = y$, the social outcome remains the same. If $z = x$, again $f^U(\succ_1, \succ_2') = W(x, x) = x$, so no manipulation is possible.

Assume $z \in S(y \longmapsto x)$. Then $|S(z \longmapsto x)| < |S(y \longmapsto x)|$ and $|S(x \longmapsto z)| > |S(x \longmapsto y)|$. Since $|S(x \longmapsto y)| > |S(y \longmapsto x)|$, thus we have $|S(x \longmapsto z)| > |S(z \longmapsto x)|$. Thus, we get $f^U(\succ_1, \succ_2') = W(x, z) = x$. So, agent 2 does not gain.

Assume $z \in S(x \longmapsto y)$. Since $f^U(\succ_1, \succ_2') = W(x, z) \in \{x, z\}$, that is, either the social outcome, in this case, remains the same or becomes z which was worse than alternative x. Therefore such misreporting does not benefit agent 2.

Thus, in this case, when $|S(x \longmapsto y)| > |S(y \longmapsto x)|$, we can say that agent 2 cannot manipulate the social outcome.

Case II: $|S(x \longmapsto y)| < |S(y \longmapsto x)| \Rightarrow f^U(\succ_1, \succ_2) = y$. For agent 2, no incentive to misreport. For agent 1, need to check if agent 1 can gain by misreporting or not. By the same argument we used in Case I, we can say that agent 1 can not manipulate the social outcome in this case too.

Case III: $|S(x \longmapsto y)| = |S(y \longmapsto x)|$.

If $x \succ_T y$ then $f^U(\succ_1, \succ_2) = x$. For agent 1, no incentive to misreport. For agent 2, need to check if agent 2 can gain by misreporting or not. Let $\succ_2'(1) = z$. If $z = y$, the social outcome remains the same. If $z = x$, again $f^U(\succ_1, \succ_2') = W(x, x) = x$, so no manipulation is possible.

Assume $z \in S(y \longmapsto x)$. Then, since $|S(z \longmapsto x)| < |S(y \longmapsto x)|$ and $|S(x \longmapsto z)| > |S(x \longmapsto y)|$ and since we have $|S(x \longmapsto y)| = |S(y \longmapsto x)|$, thus we have $|S(x \longmapsto z)| > |S(z \longmapsto x)|$. Thus, we get $f^U(\succ_1, \succ_2') = W(x, z) = x$. So, agent 2 does not gain.

Assume $z \in S(x \longmapsto y)$. Then, since $|S(z \longmapsto x)| > |S(y \longmapsto x)|$ and $|S(x \longmapsto z)| < |S(x \longmapsto y)|$ and since we have $|S(x \longmapsto y)| = |S(y \longmapsto x)|$, thus we have $|S(x \longmapsto z)| < |S(z \longmapsto x)|$. Thus, we get $f^U(\succ_1, \succ_2') = W(x, z) = z$. But since, $x \succ_2 z$, therefore agent 2 get even worse outcome than before.

If $y \succ_T x$ then $f^U(\succ_1, \succ_2) = y$. For agent 2, no incentive to misreport. For agent 1, need to check if agent 1 can gain by misreporting or not. By using similar arguments as in the case when $x \succ_T y$, we can say that agent 1 cannot manipulate.

Hence proved.

4.4 Pairwise universal winner rule is not strategy-proof for n agents

As pairwise universal winner rule gives a positive result for $n = 2$ agents, which motivates us to check for $n > 2$ case. But we found a counter example for $n = 3$ agents case.

Consider a case of $n = 3$ with 15 alternatives. Assume their individual true preference ordering are $\succ_1, \succ_2, \succ_3 \in \mathcal{D}$ with top preferences as $\succ_1 (1) = a_5$, $\succ_2 (1) = a_{11}$ and $\succ_3 (1) = a_{13}$ respectively. Then,
W$(a_5, a_{11}) = a_{11}$
W$(a_5, a_{13}) = a_5$
W$(a_{11}, a_{13}) = a_{13}$

Since every top alternative is winning equal times in all pairwise competitions, we have a tie between a_5, a_{11} and a_{13}. Let the tie-breaking rule be $a_1 \succ_{\mathcal{T}} a_2 \ldots \succ_{\mathcal{T}} a_m$. Then, $f^U(\succ) = a_5$. If agent 3 decides to misreport its preference as $\succ'_3$ such that $\succ'_3 (1) = a_1$. Then,
W$(a_5, a_{11}) = a_{11}$
W$(a_5, a_1) = a_5$
W$(a_{11}, a_1) = a_1$

And now using tie-breaking rule over a_5, a_{11} and a_1, we get a_1 as the social outcome i.e., $f^U(\succ) = a_1$. But $a_1 \succ_3 a_5$. Hence, agent 3 can successfully manipulate. Therefore, the Pairwise universal winner rule is prone to strategic manipulation for $n > 2$ agents.

Pairwise universal winner rule with constant tie-breaking rule: If there exists a pairwise winner then that alternative will be the social outcome, otherwise, a constant function is used as a tie-breaking rule. We can see that the pairwise universal winner rule with a constant tie-breaking rule is also prone to strategic manipulation. Consider the case of 3 agents and 15 alternatives to choose from. Assume their individual true preference ordering are $\succ_1, \succ_2, \succ_3 \in \mathcal{D}$ with top preferences as $\succ_1 (1) = a_1$, $\succ_2 (1) = a_5$ and $\succ_3 (1) = a_{13}$ respectively. Then,
W$(a_1, a_5) = a_5$
W$(a_1, a_{13}) = a_1$
W$(a_5, a_{13}) = a_5$

Since a_5 is winning the maximum times in all pairwise competitions, thus we get $f^U(\succ) = a_5$. Let agent 3 tries to misreport its preference as $\succ'_3$ such that $\succ'_3 (1) = a_{12}$. Then,
W$(a_1, a_5) = a_5$
W$(a_1, a_{12}) = a_1$
W$(a_5, a_{12}) = a_{12}$

Since now we have a tie, and if we have a constant tie-breaking rule such that it always chooses a_1, then $f^U(\succ') = a_1$. But since $a_1 \succ_3 a_5$, thus agent 3 can successfully manipulate.

So, we have found that the PUW rule is strategy proof for $n = 2$ agent case, but for $n = 3$ agent case we found a counter example. Since the Borda count rule is strategy proof for any n agents, it makes us curious to find if there exists some similarity between the Borda count rule and the PUW rule, and yes, in fact, these two results gives the same results when $n = 2$ case.

Remark. For $n = 2$, the pairwise universal winner rule and Borda count rule select the same social alternative given any preference profile $\succ \equiv (\succ_1, \succ_2, \ldots, \succ_n) \in \mathcal{D}^n$, provided same tie breaking rule is used to resolve ties.

Proof: For any x and $y \in A$, consider two agents with their top preferences as $\succ_1 (1) = x$ and $\succ_2 (1) = y$. If x and y are the same, then by unanimity, the social outcome selected by the Borda count rule and pairwise universal winner rule will be the same. Let x and y be different. To calculate the social outcome based on the pairwise universal winner rule, we need $\mathrm{W}(x, y)$. Suppose, without loss of generality, take $\mathrm{W}(x, y) = x$. It implies that $f^U (\succ_1, \succ_2) = x$. It must be that $|S(x \longmapsto y)| \geq |S(y \longmapsto x)|$. It implies that $\mathrm{R}[y, \succ_1] \geq \mathrm{R}[x, \succ_2]$. Therefore, we get $f^B (\succ_1, \succ_2) = x$.

5 Search for other strategy-proof SCFs

In this section, we will discuss already well-known social choice functions like majority rule, plurality rule, and some more already existing social choice functions in the literature like the instant run-off method, and Condorcet winner which gives us negative results in the clockwise circular domain. Turns out, none of them satisfies strategy-proof property even in such a restricted domain of clockwise circular preferences. We will first discuss the definitions and then gives a contradictory example for each SCF.

5.1 Majority rule

Definition 5.1. ***Majority rule:*** *Majority rule is a decision-making process in which a group of agents votes for their top alternative to determine a social outcome, and the alternative that receives more than* 50% *of the votes is considered the social outcome.*

In other words, the choice that is favored by the most people in the group is the one that is selected. Majority voting is widely used in democratic elections, where voters choose their preferred candidate or party, and the candidate or party with the most votes is declared the winner. It is also used in corporate governance, where shareholders vote on company policies or elect members of the board of directors. In some cases, majority voting can also be used to resolve disputes or make decisions in non-election contexts, such as in jury deliberations or committee meetings.

In the simplest form of majority voting, each agent is allowed to vote for only one alternative, and the alternative that receives the most votes is chosen as the winner.

Majority rule can be manipulated through strategic voting, where voters cast their votes in a way that is not consistent with their true preferences, in order to maximize the likelihood of their preferred candidate winning. Strategic voting can also occur when voters strategically vote against a candidate who they believe is more likely to win, in order to prevent that candidate from winning. The majority voting rule is prone to strategic manipulation, especially in situations where voters have preferences over multiple alternatives. However, even for a very small society and very less social alternatives, the majority voting rule can still be manipulable.

Consider a case when alternatives make the Condorcet cycle. Consider a three-agent society and three alternatives to choose from. Consider their true preferences as $a \succ_1 b \succ_1 c$, $b \succ_2 c \succ_2 a$ and $c \succ_3 a \succ_3 b$. In this case, no alternative has a majority, since every alternative got 1 vote. Let the tie-breaking rule be $a \succ_{\mathcal{T}} b \succ_{\mathcal{T}} c$. Then, a will be the social outcome. Now, since Agent 2 got its worst preferred outcome, Agent 2 may try to manipulate. If agent 2 misreports his preference as $c \succ'_2 a \succ'_2 b$, then c has 2 votes, a has 1, and b has 0. So in this case, c has a majority of votes, and hence c becomes the social outcome and since $c \succ_2 a$, hence agent 2 can successfully manipulate. Hence, the majority voting rule is not strategy-proof on $\mathcal{D}$.

5.2 Plurality rule

Definition 5.2. ***Plurality rule:*** *Plurality rule is a voting system in which the alternative with the most votes, but not necessarily a majority (i.e., more than 50% of the votes), is selected as a social outcome. We assume the agents vote for their top alternative.*

In other words, in a plurality system, the alternative who receives the highest number of votes, regardless of whether they have a majority or not, becomes the social outcome. This is different from a majority rule, in which the winning alternative must receive more than 50% of the total votes cast.

For example, if there are three candidates in an election and candidate A receives 40% of the votes, candidate B receives 35%, and candidate C receives 25%, candidate A would be declared the winner under a plurality rule because they received the highest number of votes, even though they did not receive a majority of the votes.

The plurality rule is not strategy-proof on the clockwise circular domain. It can be manipulated in the same way as the majority voting rule. As we have seen in the example of 3 agents 3 alternative case with Condorcet cycles. When the individuals preferences are $a \succ_1 b \succ_1 c$, $b \succ_2 c \succ_2 a$ and $c \succ_3 a \succ_3 b$, the outcome is chosen by a tie-breaking rule. If the tie-breaking rule is such that $a \succ_{\mathcal{T}} b \succ_{\mathcal{T}} c$, then the social outcome will be a. But agent 2 can successfully manipulate with another preference ordering $c \succ'_2 a \succ'_2 b$ and since now c has the most votes, therefore c will be the social outcome which was more preferred than the previous social outcome from his true preference. Note that the plurality rule actually chooses the majority winner in this case.

5.3 Instant run off

Definition 5.3. ***Instant run-off voting (IRV) rule:*** *If an alternative is a top alternative for more than 50% of the agents, then that alternative is going to be the social outcome. Otherwise, the alternative with the fewest first-choice votes is eliminated in each round until one alternative has a majority of votes. In the scenario where all alternatives have equal votes in any round of an instant run-off, a tie-breaking rule can be used to determine which alternative has to be eliminated in that round.*

Instant Runoff Voting rule is a ranked-choice voting system where instead of selecting only one alternative, agents rank the alternatives in order of preference which is different from majority and plurality rule.

IR is often used in political elections and can provide a more accurate representation of the electorate's preferences than other voting systems, such as first-past-the-post, where the candidate with the most votes, even if not a majority, wins.

For example, consider a case where voters have to choose from 3 candidates A, B, and C. Let 358 voters vote for candidate A, 278 voters vote for candidate B and 214 voters vote for candidate C. In the first round since no candidate has more than 50% of the total voters, hence we need to eliminate the candidate with the fewest votes and redistribute those votes to those voter's 2nd choice. Due to the restriction of our clockwise circular domain, the domain contains only 3 types of preference orderings namely, $A \succ B \succ C$, $B \succ C \succ A$ and $C \succ A \succ B$. Thus, all votes for C are now redistributed to A. Now, A has 572 votes, B has 278 votes and C has zero. Since A has more than 50% of the votes now, therefore A is the winner.

To check if the instant-runoff rule is strategy-proof or not consider the same case of 3 agents and 3 alternatives making Condorcet cycles to check for manipulation. Since no alternative has more than 50% of the votes and each alternative has 1 vote, there we need a tie-breaking rule to clear a tie. Let the tie-breaking rule is $a \succ_{\mathcal{T}} b \succ_{\mathcal{T}} c$. Then, a is the social outcome chosen by instant runoff. Since again agent 2 can manipulate by misrepresenting his/her preference as $c \succ'_2 a \succ'_2 b$. In this way, c has a majority by 2 votes which is greater than 50% of the total votes and c wins which was more preferred than the previous outcome a by agent 2. Therefore, the instant run-off rule can be strategically manipulated.

Instant-runoff with plurality rule: Since instant run-off with majority rule can be strategically prone to manipulation and we have seen that plurality rule also sometimes contains a majority winner. Therefore instant-runoff with plurality rule is also prone to strategic manipulation. This can be verified using the same example as above. In the first case, due to the tie-breaking rule, we get the social outcome. Now, if an agent tries to misreport his/her preference as $c \succ'_2 a \succ'_2 b$, then now c has the most votes and there is no need for redistribution of votes and c will be the social outcome. Thus, agent 2 can successfully manipulate.

5.4 Condorcet winner

Definition 5.4. ***Condorcet winner:*** *The Condorcet winner is a concept that refers to an alternative that would win in a head-to-head or pairwise competition against any other alternative. In other words, if there is a Condorcet winner, this alternative would beat all the other alternatives in one-on-one pairwise competitions.*

To explain a Condorcet winner, the majority of graphs can become quite handy.

Definition 5.5. ***Majority Graph:*** *In a Majority Graph, vertices represent all alternatives in A, and for any alternative $a,b \in A$, we have an arrow from a to b if a is winning by majority of votes over alternative b. The number on the arrow represents the number of votes by which a is winning. If a and b have equal votes, then it is represented by a line.*

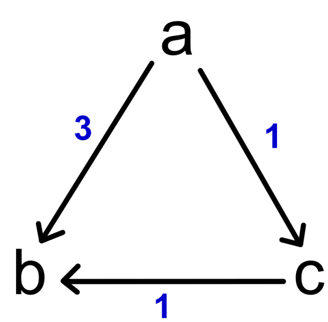

Figure 4: Majority graph example

For example, consider a case of 3 agents with 3 alternatives. Assume their preferences as $a \succ_1 b \succ_1 c$, $a \succ_2 c \succ_2 b$ and $c \succ_3 a \succ_3 b$. Then, between a and b, a is preferred over b by all 3 agents, thus a is winning over b by 3 votes. We can show this by an arrow from alternative a to b with a label as 3 in the majority graph as shown in figure 4. Similarly, a is winning over c by 1 more vote and c is winning over b by 1 vote, therefore the majority graph for this example will look like Figure 4. In this example, we can see that alternative a is the Condorcet winner as it is winning over the other two alternatives.

However, many a time there is no clear Condorcet winner because no candidate can beat all the others in pairwise head-to-head competitions. As we can see in the example of 3 agents 3 alternatives case, with the preferences as $a \succ_1 b \succ_1 c$, $b \succ_2 c \succ_2 a$ and $c \succ_3 a \succ_3 b$, there is no Condorcet winner, and we get a Condorcet cycle as shown in Figure 5. Hence we need a tie-breaking rule to resolve this tie. Already there exist different methods to resolve this issue, we have used some of them for our study, but they all turn out to be manipulable under the clockwise circular domain. If we take a tie-breaking rule as $a \succ_{\mathcal{T}} b \succ_{\mathcal{T}} c$, then a will be the social outcome.

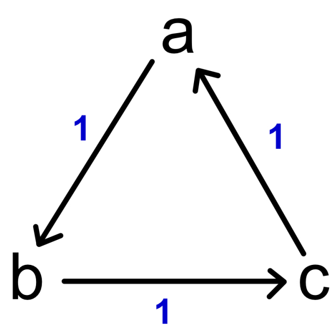

Figure 5: Majority graph for Condorcet cycle in 3 agents 3 alternative case

If agent 2 tries to manipulate by misreporting its preference as $c \succ'_2 a \succ'_2 b$, then the majority graph for profile $\succ'$ becomes like Figure 6. Now, we have c as the Condorcet winner as it is winning over the other two alternatives. Therefore, now c is the social outcome which was more preferred over alternative a by agent 3. Hence, agent 3 can successfully manipulate. Thus, we can conclude that Condorcet winner is not strategy-proof in a clockwise circular domain.

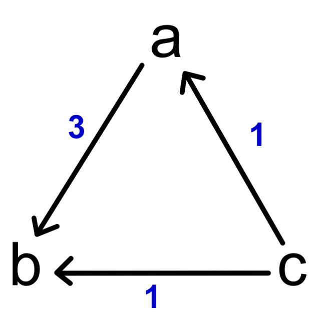

Figure 6: Majority graph for profile ($a \succ'_1 b \succ'_1 c$, $c \succ'_2 a \succ'_2 b$, $c \succ'_3 a \succ'_3 b$)

5.5 Pairwise winner using plurality rule

Here, we introduce a new SCF called pairwise winner using the plurality rule (PWP). This SCF is also based on pairwise competition like in PUW, however, it takes pairwise competition for all alternatives in A rather than just the top alternatives. Also, one major difference between PUW and PWP is that the pairwise winner is decided according to the number of total votes an alternative has over the other rather than comparing the size of $S(x \longmapsto y)$ and $S(y \longmapsto x)$, just like we do for finding Condorcet winners.

Definition 5.6. ***Pairwise winner using plurality rule (PWP):*** *A pairwise winner using plu-*

rality rule $f^P : \mathcal{P}^n \longmapsto A$ selects the alternative which wins by the maximum number of total vote differences in a pairwise competition with all other alternatives in A. In case of a tie, any tie-breaking rule can be used to resolve the ties.

To understand better, let's discuss a case of 2 agents, namely agent 1 and agent 2, and 4 alternatives a, b, c, and d. Consider a preference profile $\succ \equiv (\succ_1, \succ_2) \in \mathcal{D}^n$ such that $a \succ_1 b \succ_1 c \succ_1 d$ and $c \succ_2 d \succ_2 a \succ_2 b$. We can get a better picture with the majority graph for calculating a social outcome using the PWP rule. In this case, a majority graph is given in Figure 7. Since alternative a and c both have equal maximum total vote differences which is 2, therefore we have a tie. Let the tie-breaking rule be $a \succ_{\mathcal{T}} b \succ_{\mathcal{T}} c \succ_{\mathcal{T}} d$. Then, using the PWP rule, we get $f^P(\succ) = a$.

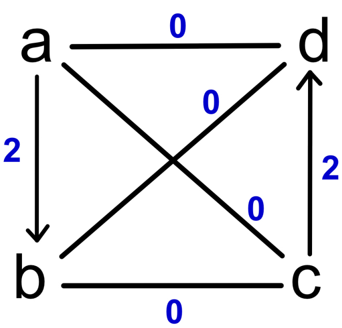

Figure 7: Majority graph for $\succ \equiv (a \succ_1 b \succ_1 c \succ_1 d,\ c \succ_2 d \succ_2 a \succ_2 b)$

Now since agent 2 is not getting its top preference as the social outcome by reporting its true preference, let agent 2 misreports $\succ'_2$ *as* $d \succ'_2 a \succ'_2 b \succ'_2 c$. Then, the majority graph for preference profile $\succ' \equiv (a \succ'_1 b \succ'_1 c \succ'_1 d,\ d \succ'_2 a \succ'_2 b \succ'_2 c)$ will look like Figure 8. We can see that a is winning a total of 4 vote differences, b is winning by 2 vote differences and c and d are winning zero times. So, a is winning by maximum total vote differences, hence $f^P(\succ') = a$. We can see that Agent 2 cannot manipulate in this case.

Now we will define a new concept we haven't discussed so far. For any alternative $x \in A$, let $B(x, \succ_i)$ be the set of alternatives below x in preference ordering $\succ_i$. Formally, $B(x, \succ_i) := \{y \in A : x \succ_i y\}$. We also call $B(x, \succ_i)$ as the below set of alternative x in preference ordering $\succ_i$.

In the above example of 2 agents 4 alternative case, for a preference profile like $\succ \equiv (\succ_1, \succ_2) \in \mathcal{D}^n$ such that $a \succ_1 b \succ_1 c \succ_1 d$ and $c \succ_2 d \succ_2 a \succ_2 b$, we can verify that $B(a, \succ_1) = \{b, c, d\}$, while $B(a, \succ_2) = \{b\}$.

Definition 5.7. ***Monotone Property:*** *A social choice function f is monotone if $\forall \succ, \succ' \in \mathcal{P}^n$ with $B(f(\succ), \succ_i) \subseteq B(f(\succ), \succ'_i)\ \forall i \in N$, we have $f(\succ) = f(\succ')$.*

So in the example we discussed above with two profiles as another preference profile as $\succ \equiv (a \succ_1 b \succ_1 c \succ_1 d,\ c \succ_2 d \succ_2 a \succ_2 b) \in \mathcal{D}^n$ $\succ' \equiv (a \succ'_1 b \succ'_1 c \succ'_1 d,\ d \succ'_2 a \succ'_2 b \succ'_2 c)$, since

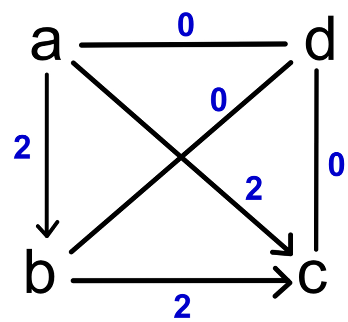

Figure 8: Majority graph for $\succ \equiv (a \succ'_1 b \succ'_1 c \succ'_1 d,\, d \succ'_2 a \succ'_2 b \succ'_2 c)$

$f^P(\succ) = a$ and we can see that $B(a, \succ_1) = \{b, c, d\}$, $B(a, \succ'_1) = \{b, c, d\}$, $B(a, \succ_2) = \{b\}$, and $B(a, \succ'_2) = \{b, c\}$. So, we can say that $B(a, \succ_i) \subseteq$ B(a,$\succ'_i$) $\forall i \in \{1, 2\}$ and we got $f^P(\succ') = a$ also. Therefore, in this example monotone property is satisfied.

Turns out, the PWP property always satisfies the monotone property.

Theorem 5.1. *The pairwise winner using plurality voting is monotone under a clockwise circular domain.*

Proof: For any preference profile $\succ \equiv (\succ_1, \succ_2, \ldots, \succ_n) \in \mathcal{D}^n$,without loss of generality, suppose $f^P(\succ) = x$, where $x \in A$. Consider agent j $\in N$ such that $\succ_j (1) = y \neq x$. Let agent j misreports his preference as $\succ'_j (1) = z$. To check monotone property, only those $z \in A$ are allowed such that the following condition is satisfied:

B(x,$\succ_j$) $\subseteq$ B(x,$\succ'_j$). This restrict the condition on z, therefore only such $\succ'_j$ are allowed such that $z \in S\,(y \longmapsto x)$.

Let w be all alternatives in A such that $w \in S\,(y \longmapsto z)$ and t be all other alternatives in A other than w, that is $t \in A/S\,(y \longmapsto z)$.

Since $\forall w \in S\,(y \longmapsto z)$, in profile $\succ$, $w \succ_j t$ whereas in profile $\succ'$, $t \succ'_j w$. Therefore the chance of winning alternatives w has actually decreased from profile $\succ$ to $\succ'$. Thus, $f^P(\succ') \neq w$. And now, all t have 1 more vote of agent j in profile $\succ'$ with respect to profile $\succ$, but since x was winning earlier and no other alternative has an advantage over x in profile $\succ'$, therefore $f^P(\succ') = x = f^P(\succ)$. Hence proved.

Debasis Mishra [12] has proved that on an unrestricted domain, an SCF is strategy-proof if and only if it satisfies the monotone property. While on a restricted domain, the necessity condition of the monotone property holds but the sufficiency condition requires a domain where we can rank any pair of alternatives first and second. But our restricted domain of the clockwise circular domain does not satisfy this property, hence PWP is not strategy-proof on a clockwise circular domain.

6 Results

So we discussed different types of social choice functions which depend upon the ranking of alternatives, the relative ranking of alternatives, or just the number of times an alternative is topped and their properties on a clockwise circular domain. We mainly focussed on strategy-proof property. We found counter-examples for the well-known social choice functions like majority rule, plurality rule, instant runoff voting rule, and Condorcet winner rule that exists in the literature.

We know that the Borda count rule is prone to strategic manipulation under any domain of preferences, however restricting the domain makes Borda count less prone to strategic manipulation. And under the clockwise circular domain, it became completely strategy-proof.

So far, Borda count is the only rule that is strategy-proof in the clockwise circular domain for any number of agents and any number of alternatives. We gave an alternate proof for the same as this result aligns with M. Barbie et al(2006)[6].

We introduced a new social choice function called the pairwise universal winner rule which is based on the ranking of alternatives with respect to the other alternative. We proved that PUW is strategy-proof for 2 agent case. Also, for the 2 agent case, PUW and Borda give the same social outcome. But for n $\geq$ 3, PUW and Borda may give different social outcomes so they are not the same. We give a counter-example for PUW to show that it is not strategy-proof for n $>$ 2 case.

We also introduced a different social choice function called pairwise winner using the plurality rule (PWP) which is based on pairwise competition of all alternatives using the plurality voting rule. We proved that on a clockwise circular domain, PWP satisfies the monotone property. Though since the clockwise circular domain does not satisfy the sufficiency requirement of monotone property for strategy-proofness given in Debasis Mishra's notes, we can conclude that PWP is not strategy-proof on a clockwise circular domain.

These all results are equally valid for a counterclockwise circular domain also as both clockwise and counterclockwise circular domains can be defined in the same way and exhibit the same properties.

7 Future scope

Till now, we have covered already well-known social choice functions in social choice theory and introduced and analyzed some new SCFs like PUW and PWP in the given time. We couldn't find a counter-example to show PWP is not strategy-proof thus far. There is a possibility (although not a necessity) that PWP SCF may turn out to be strategy-proof as it satisfies the monotone property.

Also by tweaking already existing SCFs, we can find more SCFs, if not then we can declare that Borda is the only strategy-proof social choice function that exists on a clockwise circular domain. Like using a suitable tie-breaking rule for PUW, it can become strategy-proof in a clockwise circular domain. One can also find a pattern in all strategy-proof social choice functions in the clockwise circular domain after a deep analyzing the properties of the clockwise circular domain.

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