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Kbiri, Mohammed Alaoui and Ghassan, Hassan Belkacem

Department of Mathematics, King Khalid University, KSA,
Department of Economics, Umm Al-Qura University, KSA
(Previously)

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Mohammed Kbiri Alaoui [1]   Hassan Belkacem Ghassan [2]

[1] Department of Mathematics, King Khalid University, KSA, mka_la@yahoo.fr

[2] Department of Economics, Umm Al-Qura University, KSA (Previously), hbghassan@yahoo.com

Abstract

In this paper, we propose an alternative method that applies the model reduction techniques to the VAR framework when the number of variables is sufficiently large. The relevance of the model reduction in the VAR and $SVAR$ systems comes from that the new trajectories preserve the qualitative properties of the initial trajectories. In economic analysis, this allow to apprehend the underlying phenomena. Also, the resulting model is accurate, computationally less expensive and based on the real meaning of the system.

Keywords. Complexity, Dynamic system, Dimension reduction, Economic analysis, Reduced VAR, Reduced structural VAR.

1. Introduction

Recently high-dimensional models gained considerable importance in several areas of economics as monetary forecasting and home-price data. Even if the standard VAR models (Sims 1980, Stock and Watson 2001) usually include no more than ten variables, the number of parameters grows quadratically with the size of the model. The reduction dimension is more interesting than the principal component analysis (PCA), because the latter drops many important information between Y and X. The reduction method operates a paradigm of reduction before modeling the regression between Y and X, and does not have the disadvantage of losing information in X necessary for estimating Y, and it does not limit the choice of regression methodology (Adragni and Cook 2009, Cook 2007). The direct numerical simulations of dynamical systems have been an extremely successful means for studying complex phenomena in sciences. However, the dimensionality of such simulations may increase to unmanageable levels of storage and computational requirements. One approach to overcoming this problem is through model reduction. Model reduction of large-scale dynamical systems has become very popular (Antoulas 1998, 1999, 2005; Grimme 1995, 1997; and a good survey by Snowden et al. 2017). The idea is to construct a "simple" lower order model that approximates well the behavior of a "complex" larger dynamical model. A complex system is essentially a mathematical model which describes a real-world process. This mathematical model is often characterized by partial differential equations (PDEs) or ordinary differential equations (ODEs). Since improved accuracy (using e.g. a very fine discretization) leads to models of high complexity, this may become prohibitive for certain computations (as control process and optimization). Therefore, it is essential to design models of reduced complexity without sacrificing too much accuracy.[1]

Our goal is to give a first step in using the model reduction technics particularly in economics modeling to apprehend complex phenomena that induce many endogenous variables. We will apply the model order reduction to the VAR framework that preserve the properties of the contracted structural models. Such model reduction is based on the real meaning of the underlying system and its mathematical modeling. The suggested approach permits to retain the significant variable functions in the model. It allows that the resulting model is accurate yet computationally less expensive. In Section 2, we introduce the notion of model reduction in dynamical system and we display the methodology of the VAR and $SVAR$ models. The focus of Section 3 is the presentation of the methodology adopted to define the reduced model $RVAR$ and the associated algorithm. Section 4 addresses the experimental results and the comparison with the classical methods. We conclude in Section 5.

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[1] For more details on the model reduction and the related technics see Antoulas (1999, 2005), Grimme (1997), and Moore (1981).

2. Preliminaries

Let consider the following linear dynamical system:

$$\begin{cases} \sigma X(t) = AX(t) + Bu(t) \\ y(t) = CX(t) + Du(t) \end{cases} \qquad (S)$$

Where σ is either the derivative operator $\sigma f(t) = \dot{f}(t)$ or the shift operator $\sigma f(t) = f(t+1)$, $t \in Z$ and the matrices A, B, C and D are such that

$$\Sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{R}^{(n+p)\times(n+m)}$$

We identify the complexity of the system (S) with its order n. The motivation for this definition is that the simulation time of the system (S) is strongly correlated to the number n of differential equations. A reduced order system i.e. an approximation of (S) is a system of the following form:

$$\begin{cases} \sigma X_k(t) = A_k X_k(t) + B_k u_k(t) \\ y_k(t) = C_k X_k(t) + D_k u_k(t) \end{cases} \qquad (S_k)$$

with $\quad \Sigma_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix} \in \mathbb{R}^{(k+p)\times(k+m)}$

Where $X_k(t) \in \mathbb{R}^k$, and $k \ll n$ such that the following properties are satisfied:

(i)   The approximation error is small, and there exists a global error bound.

(ii)   System properties, like stability, passivity, are preserved.

(iii)   The procedure is computationally stable and efficient.

We rewrite the system (S) in some appropriate coordinates with k components. There are many sets of methods which are currently in use as Singular Values Decomposition (SVD) based methods, moments matching based methods, truncated balanced model reduction. This paper discusses some technical methods of economic analysis founded on the Structural Vector Auto Regression models $(SVAR)$. It corresponds to an alternative to the existing methods of VAR models in the case of large scale. The canonical innovations, associated to a VAR model, represent the shocks whose propagation results in the fluctuations of the studied dynamic system. Under the assumption that the innovations are not instantaneously correlated, the contribution of each impulse on the various series of a given system is measurable. If the shocks are not independent, Sims (1980), proceeded by an orthogonalization of the Cholesky type. But, this orthogonalization is purely statistical and is not associated to an economic theory. Moreover, it skews the economic interpretation of the obtained shocks.

VAR **Methodology and hypotheses**

Let's consider an economic system composed of a vector X and let u_t be the canonical innovation which corresponds to the anticipated part of the series observed between the dates t and $t-1$. The estimate of these innovations is carried out according to the Sims principle given in its founder article (1980), starting from the dynamic canonical vector auto regression representation given by:

$$X_t = A_1 X_{t-1} + A_2 X_{t-2} + \cdots + A_p X_{t-p} + u_t \tag{2.1}$$

At each date t, the errors are estimated by the residues of the regression corresponding to the individual estimate of each equation of the VAR. If the shocks are not independent, Sims (1980) proceeded by an orthogonalization of the Cholesky type, which constitutes a statistical constraint. The disadvantage of this approach is that it does not allow for an economic interpretation. The orthogonalization obtained by the decomposition of Cholesky is largely criticized by the partisans of the $SVAR$ methodology, who recommend an orthogonalization based on identifying constraints resulting from the economic theory (Shapiro and Watson 1988, Blanchard and Quah 1989, King and al. 1991). The methodology of identification assumes the existence of a linear relation between the structural shocks noted ε_t and the canonical shocks u_t of the form:

$$\varepsilon_t = H u_t \tag{2.2}$$

where H is a passage matrix. This method supposes that the components of ε_t are not correlated between them and have a unit variance:

$$E(\varepsilon_t \varepsilon_t') = I \tag{2.3}$$

3. *RVAR* **methodology**

In the reduction procedure, the model (S_k) should be of lower order than the model (S), but its trajectories should preserve some qualitative properties of the trajectories of (S) and allow to capture the underlying phenomena. Precisely, we will proceed as follows:

Step 1: Change the coordinates $X(t)$. This consists to find a suitable invertible matrix $T \in \mathbb{R}^{n \times n}$ and then transform the state-space model.

Step 2: Use the truncated balanced realization method (see section 3.2).

Globally, we need to find "good" coordinate transformations T, and a suitable approximation order k of the model (S).

3.1 Singular values decomposition method (SVD)

3.1.1 Presentation of the SVD

Let A be an $n \times n$ matrix. Let the ordered non-negative numbers $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ be the positive square roots of the eigenvalues of AA^*; let I_n denotes the $n \times n$ identity matrix. There exist unitary matrices $n \times n$, U and V such that

$$UU^* = I_n , \qquad VV^* = I_n , \quad A = U\Sigma V^* \tag{3.1}$$

where Σ is an $n \times n$ matrix with $\Sigma_{ii} = \sigma_i$ and zero elsewhere. The columns vectors of U and V are the eigenvectors of AA^* and A^*A, respectively. The decomposition form of A in (3.1) is called the SVD of A. Let's remark that:

1. The SVD is unique when the matrix A is square and the singular values have multiplicity one.

2. One can use a faster algorithm introduced by Higham (2000) to obtain the singular value decomposition of the matrix A.

3.1.2 Why do we need the SVD in the statistical model?

This type of decomposition can also be interpreted in the optic of the statistical study of a set of data. Then, the main columns of U represent the trends of the set considered (the vectors of U represent the "directions of high variation" of the set). The diagonal values of Σ are then analogous to the "energy" or the "representativeness" which will be considered as a weight of these behaviors. They decrease more rapidly if the statistical set is ordered. We can consider, for example from the perspective of data mining, that the "important" information of the set is that which corresponds to a more marked structure. Then, by vanishing the diagonal of Σ after some value, then by reconstituting the initial matrix, one obtains filtered data, representing the dominant information of the initial set. Equivalently, data of energy after a certain threshold can be considered equal to zero.

Thus, the SVD makes possible the construction of an empirical model. It is also possible to reconstruct, using a basis of singular vectors from a first dataset, another dataset with more or less precision, in order to determine the similarity between the two. The effectiveness of the method depends in particular on the way in which the information is presented.

3.2 Truncated balanced realization (TBR)

Before introducing the TBR method, let's give the following result which is the key ingredient of the last method. Let consider the following type of equation called Sylvester equation (see Bartels, Stewart 1972 and Bhatia, Rosenthal 1997):

$$AX + XB = C \quad \text{(or also as } AX - XB = C\text{)}$$

Then given matrices A, B, and C, the problem is to find the possible matrices X that obey this equation. All matrices are assumed to have coefficients in the complex numbers. For the equation to make sense, the matrices must have appropriate sizes, A and B must be square matrices of sizes n and m respectively, and then X and C both have n rows and m columns.

A Sylvester equation has a unique solution for X exactly when there are no common eigenvalues of A and $-B$. The Truncated Balanced Realization (TBR) (See Enns 1984, Moore 1981, Pernebo et al. 1982) produces a guaranteed stable reduced model. It produces a reduced model with globally accurate frequency response approximation.

Let A be an $n \times n$ stable matrix. Let P and Q be the unique Hermitian positive definite solutions to

$$AP + PA^t = -BB^t, \qquad A^tQ + QA = -CC^t \tag{3.2}$$

The $\sigma_i(\Sigma)$ are the square roots of the eigenvalues of the product PQ: $\sigma_i(\Sigma) = \sqrt{\lambda_i(PQ)}$.

We give below the algorithm for the k-truncated balanced realization which will be used in the reduction model (See Safonov et al. 1989, Tombs et al. 1987):

Algorithm

1. Solve the equations (3.2)   (3.3)

2. Find the Cholesky decomposition

$$P = Z^B(Z^B)^t, \quad Q = Z^C(Z^C)^t \tag{3.4}$$

3. Calculate the singular value decomposition of $(Z^C)^t Z^B$

$$U^L \Sigma (U^R)^t = (Z^C)^t Z^B \tag{3.5}$$

where,

$$U^R = [u_1^R \; ... \; u_n^R], \qquad U^L = [u_1^L \; ... \; u_n^L], \qquad \Sigma = \text{diag}(\sigma_1, ..., \sigma_n)$$

4. If $\sigma_k > \sigma_{k+1}$, let

$$S^B = Z^B[u_1^R \; ... \; u_k^R]\text{diag}\left(\frac{1}{\sqrt{\sigma_1}}, ..., \frac{1}{\sqrt{\sigma_k}}\right) \tag{3.6}$$

and

$$S^C = Z^C [u_1^L \ \dots \ u_k^L] \text{diag} \left(\frac{1}{\sqrt{\sigma_1}}, \dots, \frac{1}{\sqrt{\sigma_k}} \right) \tag{3.7}$$

5. The order k-truncated balanced realization is given by

$$A_k = (S^C)^t A S^B, \qquad B_k = (S^C)^t B, \qquad C_k = C S^B \tag{3.8}$$

3.3 Linear transformation of the dynamical system

Our goal is to express the solution $X(t) \in \mathbb{R}^n$ of the model (S) in only k variables. Such a solution can be written as

$$X(t) = S^B X_k(t)$$

where $X_k(t) \in \mathbb{R}^k$ and $S^B \in \mathbb{R}^{n \times k}$. S^B is chosen such that its columns span a subspace where the solution $X(t)$ belongs.

If the believed solution is inserted into the original state-space model (S), we obtain

$$\sigma X(t) = S^B \sigma X_k(t) = A S^B X_k(t) + Bu + E \tag{3.9}$$

where E is a residual. From (3.9), $X(t) = S^B X_k(t)$ is a solution to the original problem if, and only if, $E(t) = 0$ for all t. There are n equations in (3.9), but only k unknowns X_k. The system is generally over-determined. To find a unique solution, we can require that the projection of the residual $E(t)$ onto the subspace spanned by S^B is zero. This projection is given by $(S^C)^t$. Hence, we add the condition

$$(S^C)^t E(t) = 0, \forall \, t \tag{3.10}$$

We then obtain the reduced model:

$$\sigma X_k(t) = (S^C)^t A S^B X_k(t) + (S^C)^t Bu \tag{3.11}$$

Remark. Let's remark that the relation $X(t) = S^B X_k(t)$ permit us to pass from the reduced vector X_k to the original vector X.

3.4 Some useful results

3.4.1 Approximation with low rank

Low-rank approximation is a minimization problem, in which the cost function measures the fit between a given matrix and an approximating matrix, subject to a constraint that the approximating matrix has

low rank. The problem is used for mathematical modeling and data compression. The rank constraint is related to a constraint on the complexity of a model that fits the data. The low-rank approximation is closely related to: (i) principal component analysis, (ii) factor analysis, (iii) total least squares, (iv) latent semantic analysis, (v) orthogonal regression.

If the matrix A is with full rank, we can approximate it with another matrix with lower rank by using the following result due to Schmidt-Eckart-Young-Mirsky theorem (see Antoulas 2005):

Theorem 3.1: Let A be a matrix in $\mathbb{R}^{n \times n}$ such that the corresponding singular values are ordered such that $\sigma_k > \sigma_{k+1}$. Then

$$\min_{X \in R^{n \times n}, rank(X)=k} \|A - X\| = \sigma_{k+1}$$

If σ_{k+1} is very small, then the matrix X approximated well the matrix A.

3.4.2 Nearest correlation matrix problem

Given a symmetric matrix, the nearest correlation matrix designed the nearest symmetric positive matrix with unit diagonal. The term correlation matrix comes from statistics, since a matrix whose (i,j) entry is the correlation coefficient between two random variables X_i, X_j is symmetric positive semidefinite with unit diagonal. For more details about this type of approximation one can see Anjos et al (2003) and Higham (2002). Correlation matrices appear naturally in different fields:

- In graph theory: certain matrix completion problems are modeled using graphs. In this type of modeling, the correlation matrices play an important role.

- In Statistics and Finance: these are matrices which collect the different correlation coefficients which exist for a finite number of random variables. In the case of Finance, for example these random variables are the prices of different stocks listed on the stock market.

We also find the correlation matrices in optimal control, when we apply a method of "proper orthogonal decomposition" where it collects the different scalar products two by two of an orthonormal base, called base POD, obtained from the classical base given by a decomposition into finite elements.

Let now give the following theorem of Higham (2002) which we will use below.

Theorem 3.2: Let A be a symmetric square matrix, there exist an approximate matrix B (in some sense) symmetric positive semidefinite matrix with unit diagonal.

In the last references (as in Anjos et al. 2003 and Higham 2002), many algorithms for computing fastly the nearest correlation matrix are presented.

4. Experimental results

The first objective is to reduce the classical VAR model to the reduced vectoral autoregressive model ($RVAR$). In our case, we can consider the case of the system (S) with $B = I, C = I$, and $D = 0$. We have then the following VAR model

$$X_t = AX_{t-1} + u_t \tag{4.1}$$

where the matrix A is large. Let define a tolerance parameter a^* acceptable in economic theory for which if $\sigma_k/\sigma_1 < a^*$ the truncation at order k of the system can be done. Then, the reduced model (3.11) becomes:

$$\hat{X}_t = \hat{A}\hat{X}_{t-1} + \hat{u}_t \tag{4.2}$$

It is clear that the new error vector satisfies in general $E(\hat{u}_t\hat{u}_t') \neq I_k$. The steps used in the experimental results will be as the following:

1. We consider the VAR model with matrix A.[2]

2. If A is with full rank, we give its approximation $\tilde{A}$ with low rank k and in the model (4.1) we replace the matrix A by $\tilde{A}$.

3. Consider the variance covariance matrix of the canonical innovations u_t, $E(u_t u_t') = E$.

4. Consider the reduced VAR model as described above by $RVAR$.

5. Give the Structural model $SRVAR$ by the Blanchard and Quah (1989) approach or by Amisano and Giannini (1997) known as AB approach

    5.a. Use the SVD method to determine the structural shocks: supposing $\hat{\varepsilon}_t = \hat{H}\hat{u}_t$ and
       $\hat{E} = E(\hat{u}_t\hat{u}_t'), \Phi = E(\hat{\varepsilon}_t\hat{\varepsilon}_t')$, then $\Phi = \hat{H}\hat{E}\hat{H}^t$.

    5.b. Use the restriction coming from the economics a priori.

    5.c. Compare the two results (from 5.a and 5.b) and give an interpretation.

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[2] If the matrix A is not stable, we approximate it with a stable matrix by a projection as in Higham (2002).

6. In the system (2.2), we use the SVD method to determine the matrix H and without supposing that $E(\varepsilon_t \varepsilon_t') = I$. If not, we use the nearest correlation matrix which is mathematically possible, but did not impose any conditions on the structural innovation.

7. Come back to the original VAR and $SVAR$ and compare its results.

We remark that if we use the identification shocks introduced by Ghassan, Kbiri and Souissi (2009), we do not need to use the result of Higham (2002), since we do not need the last approximation by a correlation matrix. The relation (2.2) was replaced by a differential inclusion.

5. Conclusion

We have suggested a model order reduction for the large VAR and SVAR modeling such that the new trajectories preserve the qualitative properties of the initial trajectories. This allows for capturing particularly in economics the complex phenomena that induce many endogenous variables. Moreover, the resulting models are accurate, computationally less expensive and based on the real meaning of the system and its mathematical modeling. In this paper, the main ingredients are the singular value decomposition and the truncated balanced realization.

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