The Market-Based Asset Price Probability

Olkhov, Victor

Independent

15 May 2022

Market-Based Asset Price Probability

Victor Olkhov
Independent, Moscow, Russia
victor.olkhov@gmail.com
ORCID: 0000-0003-0944-5113

Dec. 15, 2024

Abstract

We consider volume weighted average price (VWAP) as the 1st market-based statistical moment and derive the dependence of higher statistical moments of price on statistical moments and correlations of the values and volumes of market trades. If all trade volumes are constant during the averaging interval, then the market-based statistical moments equal the frequency-based. We approximate market-based probability of price by a finite number of statistical moments. The use of VWAP results in zero price-volume correlations. We derive the expressions of market-based correlations between prices and squares of trade volumes and between squares of prices and volumes. To forecast market-based averages and volatility of asset prices, one should predict two statistical moments and the correlation of their trade values and volumes. We explain how that limits the number of predicted statistical moments of prices by the first two and limits the accuracy of the forecasts of the probability of asset prices by the accuracy of the Gaussian approximations. To improve the accuracy and reliability of large macroeconomic and market models like those developed by BlackRock's Aladdin, JP Morgan, and the U.S. Fed., the developers should use market-based statistical moments of asset prices.

Keywords : asset price, market trade, statistical moments, correlations, price probability
JEL: G12

This research received no support, specific grant or financial assistance from funding agencies in the public, commercial or nonprofit sectors. We welcome valuable offers of grants, support and positions.

1. Introduction

The history of asset pricing research (Dimson and Mussavian, 1999) tracked price probability up to Bernoulli's studies in 1738, but possibly Bachelier (1900) was the first to really highlight the probabilistic character of price behavior and forecasting. "The probabilistic description of financial prices, pioneered by Bachelier." (Mandelbrot et al., 1997). "in fact, the first author to put forward the idea to use a random walk to describe the evolution of prices was Bachelier." (Shiryaev, 1999). During the last century, countless papers studied models of random prices (Kendall and Hill, 1953; Muth, 1961; Sharpe, 1964; Fama, 1965; Stigler and Kindahl, 1970; Black and Scholes, 1973; Merton, 1973; Tauchen and Pitts, 1983; Mackey, 1989; Friedman, 1990; Cochrane and Hansen, 1992; Campbell, 2000; Heaton and Lucas, 2000; Cochrane, 2001; Poon and Granger, 2003; Andersen et al., 2005; 2006; Cochrane, 2005; Wolfers and Zitzewitz, 2006; DeFusco et al., 2017; Weyl, 2019; Cochrane, 2022). Probabilistic description of asset prices can be found in Shiryaev (1999) and Shreve (2004).

Asset pricing is under the impact of multiple factors. Numerous studies describe the dependence of prices on the market (Fama, 1965; Tauchen and Pitts, 1983; Odean, 1998; Poon and Granger, 2003; DeFusco et al., 2017), on macroeconomics (Cochrane and Hansen, 1992; Heaton and Lucas, 2000; Diebold and Yilmaz, 2008), on business cycles (Mills, 1946; Campbell, 1998), on expectations (Muth, 1961; Malkiel and Cragg, 1980; Campbell and Shiller, 1988; Greenwood and Shleifer, 2014), on trading volumes (Karpoff, 1987; Campbell et al., 1993; Gallant et al., 1992; Brock and LeBaron, 1995; Llorente et al., 2001), and on many other factors that impact price change and fluctuations. The line of factors and references can be continued (Goldsmith and Lipsey, 1963; Andersen et al., 2001; Hördahl and Packer, 2007; Fama and French, 2015).

The usual treatment of a random price $p(t_i)$ times series during the averaging interval Δ is based on the frequency analysis of trades at a price p (Shiryaev, 1999). If m_p is the number of trades at a price p and N is the total number of trades during Δ, then the probability $P(p)$ of a price p is assessed as:

$$P(p) \sim \frac{m_p}{N} \tag{1.1}$$

We denote mathematical expectation as $E[..]$. The finite number N of trades for $n=1,2,..$ gives the estimations of the n-th statistical moments of price $E[p^n(t_i)]$ as:

$$\pi(t;n) = E[\,p^n(t_i)] \sim \frac{1}{N}\sum_{i=1}^{N} p^n(t_i) \tag{1.2}$$

The finite set of the *n-th* statistical moments (1.2) give an approximate description of a random variable (Shiryaev, 1999; Shreve, 2004).

However, asset prices are not independent issues of economics and finance. Asset price is a result of the market trade. We consider the randomness of values and volumes of market trades as the economic origin of price stochasticity and derive market-based price statistical moments that depend on statistical moments and correlations of the values and volumes of trades. That results in significant distinctions of market-based statistical moments of price from the frequency-based relations (2.1). Each market trade at time t_i is described by its trade value $C(t_i)$, volume $U(t_i)$, and price $p(t_i)$, which match a simple equation (1.3):

$$C(t_i) = p(t_i)U(t_i) \tag{1.3}$$

The trivial equation (1.3) states that the given probabilities of the values and volumes of trades determine statistical properties of prices. We derive how the statistical moments and correlations of values and volumes of trades determine the statistical moments of prices.

In Section 2, we describe the dependence of market-based statistical moments of prices on statistical moments and the correlation of the values and volumes of trades. In Section 3, we consider the economic reasons that limit the number of predicted statistical moments of the values, volumes, and prices by the first two and explain the internal problems of the well-known hedge tool Value-at-Risk. In Section 4, we derive the market-based correlations between prices and trade volumes, prices and squares of volumes, squares of prices and squares of volumes. Conclusion in Section 5. We assume that readers are familiar with asset pricing, probability theory, statistical moments, characteristic functions, etc. We assume that all prices are adjusted to current time. We propose that readers already know or can find on their own the notions of terms and models that are not given in the text.

2. Market-based statistical moments of prices

One can equally describe the properties of a random variable by its probability measure, characteristic function, and a set of the *n-th* statistical moments (Shephard, 1991; Shiryaev, 1999; Shreve, 2004). We describe a market price as a random variable using market-based *n-th* statistical moments. The finite set of *n-th* statistical moments of price, $n=1,2,..m$, describes the *m-th* approximation of the price characteristic function and price probability (App.A). Each additional *n-th* statistical moment adds the accuracy to the approximation of market-based price probability.

Let us consider the values $C(t_i)$, volumes $U(t_i)$, and prices $p(t_i)$ of trades at a time t_i. All possible factors that impact asset prices are already imprinted into the time series records of market trades. The times t_i of market deals introduce the initial division of the time axis. For simplicity, we assume that the interval ε between the trades that can be equal to 1 second or a fraction of a second is a constant:

$$t_i - t_{i-1} = \varepsilon \tag{2.1}$$

The time series at t_i (2.1) establishes a time axis division multiple of ε. To describe the price at a time horizon $T >> \varepsilon$, the market time division ε is of little help. To describe the price at a horizon T, which can be equal to weeks, months, or years, one should aggregate or average the initial market time series during a particular averaging interval Δ (2.2):

$$\varepsilon < \Delta < T \quad ; \quad \Delta = \left[t - \frac{\Delta}{2}; t + \frac{\Delta}{2}\right] \quad ; \quad t_i \in \Delta \quad ; \quad i = 1,2,..N \tag{2.2}$$

We assume that the interval Δ contains N of terms of the market trades and take time t as the current time and assume that all prices are adjusted to the current time. N terms of the trade values $C(t_i)$, volumes $U(t_i)$, and prices $p(t_i)$ can behave randomly during Δ. We consider values $C(t_i)$, volumes $U(t_i)$, and prices $p(t_i)$ as random variables during Δ. The averaging during Δ estimates their statistical moments and describes their properties as random variables. The finite number N of market trades estimates only a finite number of statistical moments and describes an approximation of a characteristic function and probability of price as a random variable (App. A).

We start with definition of the average price or the 1-st statistical moment $a(t;1)$. To highlight the distinction between the market-based and the usual frequency-based probability (1.1; 1.2) of price, we denote $E_m[..]$ market-based mathematical expectation during Δ (2.2):

$$a(t;1) = E_m[p(t_i)] \quad ; \quad i = 1,..N \quad ; \quad t_i \in \Delta \tag{2.3}$$

As usual, investors estimate the average price of the shares in their portfolio as a simple ratio of the total value to the total number of shares in the portfolio. The same meaning has the well-known definition of VWAP (Berkowitz et al., 1988; Buryak and Guo, 2014; Busseti and Boyd, 2015; CME Group, 2020; Duffie and Dworczak, 2021). Using (1.3), one can present VWAP $p(t;1,1)$ (2.4):

$$p(t;1,1) = \frac{\sum_{i=1}^{N} p(t_i)U(t_i)}{\sum_{i=1}^{N} U(t_i)} = \frac{C_\Sigma(t;1)}{U_\Sigma(t;1)} = \frac{C(t;1)}{U(t;1)} \tag{2.4}$$

$$C_\Sigma(t;1) = \sum_{i=1}^{N} C(t_i) \quad ; \quad U_\Sigma(t;1) = \sum_{i=1}^{N} U(t_i) \tag{2.5}$$

$$C(t;1) = \frac{1}{N}\sum_{i=1}^{N} C(t_i) \quad ; \quad U(t;1) = \frac{1}{N}\sum_{i=1}^{N} U(t_i) \tag{2.6}$$

Relations (2.5) define the total value $C_\Sigma(t;1)$ and the total volume $U_\Sigma(t;1)$ of market trades, and (2.6) estimates frequency-based average value $C(t;1)$ and volume $U(t;1)$ by a finite number N of trades during Δ (2.2). Relations between the frequency-based average price (1.2) for $n=1$ and VWAP (2.4) are trivial. VWAP $p(t;1,1)$ (2.4) coincides with the average price (1.2) if all trade volumes $U(t_i)=U$ are constant during Δ (2.2). However, financial markets demonstrate the random behavior in trade volumes. The contribution of our paper is the derivation of the dependence of price statistical moments on the randomness of the volumes and values. We set the market-based average price $a(t;1)$ (2.3) to be equal to VWAP (2.4):

$$a(t;1) = p(t;1,1) \tag{2.7}$$

To derive market-based price n-th statistical moments $a(t;n)$:

$$a(t;n) = E_m[p^n(t_i)] \quad ; \quad n = 2,3,\dots \tag{2.8}$$

let us take the n-th degree of the trade equation (1.3):

$$C^n(t_i) = p^n(t_i)U^n(t_i) \quad ; \quad n = 2,3,.. \tag{2.9}$$

For $n= 2,3,..,$ each equation (2.9) generates the set of price m-th statistical moments $p(t;m,n)$ determined by weight functions $w(t_i;n)$ of the n-th degree of trade volume $U^n(t_i)$:

$$w(t_i;n) = \frac{U^n(t_i)}{\sum_{i=1}^N U^n(t_i)} \quad ; \quad \sum_{i=1}^N w(t_i;n) = 1 \tag{2.10}$$

$$p(t;m,n) = \sum_{i=1}^N p^m(t_i)w(t_i;n) = \frac{1}{\sum_{i=1}^N U^n(t_i)} \sum_{i=1}^N p^m(t_i)U^n(t_i) \tag{2.11}$$

If all $U(t_i)=const.$, than all weight functions $w(t_i;n)=1/N$ and coincide with the frequency-based probability $\sim 1/N$ (1.1; 1.2). For $n=m$ the n-th statistical moments $p(t;n,n)$ of price:

$$p(t;n,n) = \frac{1}{\sum_{i=1}^N U^n(t_i)} \sum_{i=1}^N p^n(t_i)U^n(t_i) = \frac{C_\Sigma(t;n)}{U_\Sigma(t;n)} = \frac{C(t;n)}{U(t;n)} \tag{2.12}$$

$$C_\Sigma(t;n) = \sum_{i=1}^N C^n(t_i) \qquad ; \qquad C(t;n) = \frac{1}{N}\sum_{i=1}^N C^n(t_i) \tag{2.13}$$

$$U_\Sigma(t;n) = \sum_{i=1}^N U^n(t_i) \qquad ; \qquad U(t;n) = \frac{1}{N}\sum_{i=1}^N U^n(t_i) \tag{2.14}$$

Relations (2.13) define the sums $C_\Sigma(t;n)$ of the n-th degrees of trade values $C^n(t_i)$ and estimate the n-th statistical moments $C(t;n)$ of values $C^n(t_i)$. The relations (2.14) define the sums $U_\Sigma(t;n)$ of the n-th degrees of trade volumes $U^n(t_i)$ and the n-th statistical moments $U(t;n)$ by a finite number N of trades during Δ (2.2). From (2.12), obtain the equation (2.15) on the n-th statistical moments of value $C(t;n)$, volume $U(t;n)$, and price $p(t;n,n)$:

$$C(t;n) = p(t;n,n)U(t;n) \quad ; \quad n = 1,2,\dots \tag{2.15}$$

2.1. Market-based volatility of price

Relations (2.9-2.15) establish the basis for the definition of market-based statistical moments $a(t;n)$ (2.8) of prices. The 1-st market-based statistical moment $a(t;1)$ (2.7) is determined by the VWAP $p(t;1,1)$, the weight function $w(t_i;1)$, and is based on the equation (1.3). Equations (2.9) have the same form, and for each $n=1,2,..$, the weight functions $w(t_i;n)$ the define the set of statistical moments of price $p(t;m,n)$, $m=1,2,..$ in a similar way. The 2nd statistical moment $p(t;2,2)$ (2.17) is determined by the weight functions $w(t_i;2)$ (2.16):

$$w(t_i;2) = \frac{U^2(t_i)}{\sum_{i=1}^{N} U^2(t_i)} \quad ; \quad \sum_{i=1}^{N} w(t_i;2) = 1 \tag{2.16}$$

$$p(t;2,2) = \frac{1}{\sum_{i=1}^{N} U^2(t_i)} \sum_{i=1}^{N} p^2(t_i) U^2(t_i) = \frac{C_\Sigma(t;2)}{U_\Sigma(t;2)} = \frac{C(2;t)}{U(2;t)} \tag{2.17}$$

The 2nd statistical moment *p(t;2,2)* (2.17) has the same form as VWAP *p(t;1,1)* (2.4), and the economic meaning is similar to the meaning of VWAP *p(t;1,1)* (2.4). The 2nd statistical moment *p(t;2,2)* (2.17) equals the ratio (2.12) of total sum $C_\Sigma(t;2)$ of squares of trade values $C^2(t_i)$ to sum $U_\Sigma(t;2)$ of squares of volumes $U^2(t_i)$ during Δ (2.2). One may suppose that *p(t;2,2)* (2.17) can play the role of the 2nd market-based statistical moment *a(t;2)* (2.8) and define *a(t;2)=p(t;2,2)*. However, *p(t;1,1)* (2.4) and *p(t;2,2)* (2.17) are determined by different weight functions $w(t_i;1)$ and $w(t_i;2)$. Thus, *a(t;2)=p(t;2,2)* may be inconsistent with *a(t;1)=p(t;1,1)*. In particular, the market-based volatility $\sigma^2(t)$ (2.18) of price that is determined by the first two statistical moments should be non-negative:

$$\sigma^2(t) = E_m\left[\left(p(t_i) - a(t;1)\right)^2\right] = a(t;2) - a^2(t;1) \geq 0 \tag{2.18}$$

Meanwhile, *a(t;2)=p(t;2,2)* (2.17) doesn't guarantee (2.18). To derive the 2nd market-based statistical moment *a(t;2)* (2.8) that guarantees (2.18) and is consistent with the 1st market-based statistical moment *a(t;1)* (2.7), we define volatility $\sigma^2(t)$ (2.18) as an average (2.19) over the weight functions $w(t_i;2)$ (2.16):

$$\sigma^2(t) = \sum_{i=1}^{N} (p(t_i) - a(t;1))^2 w(t_i;2) \geq 0 \tag{2.19}$$

Relations (2.19) guarantee that $\sigma^2(t)$ (2.18) is non-negative and determine the 2nd market-based statistical moment $a(t;2) \geq 0$. From (2.11) and (2.19), obtain:

$$\sigma^2(t) = p(t;2,2) - 2p(t;1,2)a(t;1) + a^2(t;1) \geq 0 \tag{2.20}$$

From (2.18; 2.20), obtain the 2nd market-based statistical moment *a(t;2)* (2.21):

$$a(t;2) = p(t;2,2) - 2p(t;1,2)a(t;1) + 2a^2(t;1) \tag{2.21}$$

From (1.3; 2.11), obtain that the statistical moment *p(t;1,2)* (2.22) depends on correlation *corr{C(t)U(t)}* (2.23) between values $C(t_i)$ and volumes $U(t_i)$ of market trades during Δ (2.2):

$$p(t;1,2)U(t;2) = \frac{1}{N} \sum_{i=1}^{N} p(t_i) U^2(t_i) = \frac{1}{N} \sum_{i=1}^{N} C(t_i) U(t_i) \tag{2.22}$$

$$E[C(t_i)U(t_i)] = \frac{1}{N} \sum_{i=1}^{N} C(t_i) U(t_i) = C(t;1)U(t;1) + corr\{C(t), U(t)\} \tag{2.23}$$

From (2.22; 2.23), obtain:

$$p(t;1,2) = \frac{C(t;1)U(t;1) + corr\{C(t), U(t)\}}{U(t;2)} \tag{2.24}$$

Let us substitute (2.17; 2.24) into (2.20; 2.21) and obtain expressions for the market-based volatility $\sigma^2(t)$ (2.18) and the 2nd market-based statistical moment *a(t;2)* (2.21):

$$\sigma^2(t) = \frac{\Omega_C^2(t|1) + a^2(t;1)\Omega_U^2(t|1) - 2a(t;1)corr\{C(t), U(t)\}}{U(t;2)} \tag{2.25}$$

$$a(t;2)=\frac{C(t;2)+2a^2(t;1)\Omega_U^2(t)-2a(t;1)corr\{C(t),U(t)\}}{U(t;2)} \quad (2.26)$$

We denote the volatility $\Omega_C^2(t|1)$ of trade value and volatility $\Omega_U^2(t|1)$ (2.27) of trade volume:

$$\Omega_C^2(t|1)=C(t;2)-C^2(t;1) \quad ; \quad \Omega_U^2(t|1)=U(t;2)-U^2(t;1) \quad (2.27)$$

Relations (2.25; 2.26) highlight the dependence of market-based volatility $\sigma^2(t)$ (2.25) and the 2nd market-based statistical moment $a(t;2)$ of price on statistical moments $C(t;2)$ and $U(t;2)$, volatilities $\Omega_C^2(t|1)$ and $\Omega_U^2(t|1)$ (2.27) and correlation (2.23) of the values and volumes of trades. That reveals the hidden difficulties for predictions of price volatility $\sigma^2(t)$ (2.19; 2.25).

2.2. The 3rd market-based statistical moments

Market-based statistical moments of prices should be non-negative and $a(t;1)\geq 0;\ a(t;2)\geq 0$. We determine 3rd statistical moment $a(t;3)$ similar to VWAP $a(t;1)$ by averaging over the weight function $w(t_i;3)$ (2.10) and define:

$$a(t;3)=E_m[p^3(t_i)]=\sum_{i=1}^N p^3(t_i)w(t_i;3)=\frac{1}{U(t;3)}\frac{1}{N}\sum_{i=1}^N C^3(t_i)=\frac{C(t;3)}{U(t;3)} \quad (2.28)$$

If all trade volumes $U(t_i)$ during Δ (2.2) are constant then 3rd market-based statistical moment $a(t;3)$ takes the form of usual frequency-based statistical moment $\pi(t;3)$ (1.2).

2.3. The 4th market-based statistical moments

To define the 4th statistical moment $a(t;4)\geq 0$ one should check that two even market-based statistical moments: price kurtosis $Ku(t)$ (2.38) and volatility of squares of price $\theta^2(t)$ (2.40), which depend on $a(t;n)$, $n=1,2,3,4$ are non-negative.

$$Ku(t)\sigma^4(t)=E_m[(p(t_i)-a(t;1))^4] \quad (2.38)$$

$$Ku(t)\sigma^4(t)=a(t;4)-4a(t;3)a(t;1)+6a(t;2)a^2(t;1)-3a^4(t;1) \quad (2.39)$$

$$\theta^2(t)=E_m[(p^2(t_i)-a(t;2))^2]=a(t;4)-a^2(t;2) \quad (2.40)$$

Similar to the definition of the 2nd statistical moment $a(t;2)$ through the averaging of volatility (2.19) over the weight function $w(t_i;2)$ (2.16), we determine $a(t;4)$ using volatility $\theta^2(t)$ of squares of prices that is calculated by averaging over the weight function $w(t_i;4)$ (2.16):

$$\theta^2(t)=\sum_{i=1}^N(p^2(t_i)-a(t;2))^2w(t_i;2)=p(t;4,4)-2p(t;2,4)a(t;2)+a^2(t;2) \quad (2.41)$$

From (2.11; 2.12) obtain:

$$p(t;4,4)=\frac{C(t;4)}{U(t;4)} \quad (2.42)$$

$$p(t;2,4)=\frac{1}{U(t;4)}\frac{1}{N}\sum_{i=1}^N p^2(t_i)U^4(t_i)=\frac{1}{U(t;4)}\frac{1}{N}\sum_{i=1}^N C^2(t_i)U^2(t_i) \quad (2.43)$$

$$E[C^2(t_i)U^2(t_i)]=\frac{1}{N}\sum_{i=1}^N C^2(t_i)U^2(t_i)=C(t;2)U(t;2)+corr\{C^2(t),U^2(t)\} \quad (2.44)$$

Simple transformations present the expression for the market-based volatility $\theta^2(t)$ (2.45) of squares of prices and for the 4th statistical moment $a(t;4)\geq 0$ (2.46), which is non-negative due to (2.40) in the form that is almost alike to the form of the expression of price volatility $\sigma^2(t)$ (2.26) and the 2nd statistical moment $a(t;2)$ (2.26):

$$\Theta^2(t) = \frac{\Omega_C^2(t|2)+a^2(t;2)\Omega_U^2(t|2)-2a(t;2)corr\{C^2(t),U^2(t)\}}{U(t;4)} \tag{2.45}$$

$$a(t;4) = \frac{C(t;4)+2a^2(t;2)\Omega_U^2(t|2)-2a(t;2)corr\{C^2(t),U^2(t)\}}{U(t;4)} \tag{2.46}$$

Functions $\Omega_C^2(t|2)$ and $\Omega_U^2(t|2)$ in (2.45; 2.46) denote volatilities of squares of trade values $C^2(t_i)$ and volumes $U^2(t_i)$:

$$\Omega_C^2(t|2) = C(t;4) - C^2(t;2) \quad ; \quad \Omega_U^2(t|2) = U(t;4) - U^2(t;2) \tag{2.27}$$

In App. B we prove that statistical moments of prices $a(t;1)$ (2.7); $a(t;2)$ (2.18; 2.26); $a(t;3)$ (2.28), and $a(t;4)$ (2.40; 2.46) determine non-negative kurtosis $Ku(t)$ (2.38; 2.39) and hence are self-consistent.

2.4. Higher market-based statistical moments

The similarities between the dependence of the 1st and 3rd statistical moments $a(t;1)$ (2.7) and $a(t;3)$ (2.28) and similarities between the dependence of the 2nd and 4th statistical moments $a(t;2)$ (2.26) and $a(t;4)$ (2.46) permit us determine (B.12-B.17) higher market-based statistical moments of price $a(t;n)$, $n= 5,6,\ldots$ We determine odd statistical moments of price $a(t;2k-1)$ (2.28; B.12) alike to the form of VWAP using the averaging over the odd weight function $w(t_i;2k-1)$ (2.10):

$$a(t;2k-1) = \sum_{i=1}^{N} p^{2k-1}(t_i)w(t_i;2k-1) = \frac{C(t;2k-1)}{U(t;2k-1)} \quad ; \quad k = 1,2,3,\ldots \tag{2.28}$$

We determine even statistical moments of price $a(t;2k)$ (2.30; 2.31; B.16) through calculation of the volatility $\sigma^2(t|k)$ (2.29; B.14) of the *k-th* degree of price $p^k(t_i)$.

$$\sigma^2(t|k) = E_m[(p^k(t_i) - a(t;k))^2] = \sum_{i=1}^{N}(p^k(t_i) - a(t;k))^2 w(t_i;2k) \tag{2.29}$$

$$a(t;2k) = \sigma^2(t|k) + a^2(t;k) \quad ; \qquad k = 1,2,\ldots \tag{2.30}$$

$$a(t;2k) = \frac{C(t;2k)+2a^2(t;k)\Omega_U^2(t|k)-2a(t;k)corr\{C^k(t),U^k(t)\}}{U(t;2k)} \tag{2.31}$$

The set of market-based statistical moments $a(t;n)$ of price (2.28-2.31) or (B.12-B.17), for all $n=1,2,3,\ldots q>>1$, defines the finite q-approximation of the characteristic function (A.4-A.6) and the probability of price during the current averaging interval Δ (2.2). The number q of statistical moments $q<N$ of such approximation is limited by the number N of trades during Δ. To prove self-consistency of statistical moments $a(t;n)$, $n=1,\ldots q$, one should approve the non-negativity of all central even statistical moments. We believe that the use of relations similar to (B.2; B.3) and inequalities similar to (B.5-B.7) permit us to prove non-negativity of all even central statistical moments, but we don't present here the general proof.

Meanwhile, we underline that the attempts to describe the approximations of the current probability of price during *Δ* (2.2) with high accuracy by many statistical moments $a(t;n)$, $n=1,\ldots q$, meet the economic-based limitations of predictions of statistical moments of prices by the first two. In turn, these economic-based reasons limit the accuracy of any forecasts of price probability by the accuracy of the Gaussian approximations.

3. The limits on the accuracy of price predictions and Value-at-Risk problem

In this section, we consider the consequences of the dependence of market-based statistical moments of price on statistical moments and correlations of the values and volumes of trades.

3.1 The accuracy of price predictions

The prediction of price as a random variable implies prediction of its probability. The more precise could be the forecast of probability, the more exact could be predictions of the average price, price volatility, etc. The accuracy of probability predictions is determined by the number of predicted statistical moments. The more market-based statistical moments of price could be predicted, the more precise would be the forecast of the probability of price. The dependence of market-based statistical moments $a(t;n)$ of price (2.7; 2.26; 2.28; 2.46) on statistical moments and correlations of the values and volumes of market trades makes the predictions of the price statistical moments very complex. Indeed, to forecast the first n statistical moments of price at a time horizon T, one should predict the same number of statistical moments and correlations of the values and volumes of market trades. Simply speaking, one should forecast the joint probability of n statistical moments of the values and volumes of market trades at the time horizon T.

It is obvious that the interconnections of economic and financial markets make the problem of forecasting the joint probability of the values and volumes of trades with a particular asset, commodity, or service almost equal to the predictions of joint probabilities of all markets that affect the particular trades. That puzzle reveals essential theoretical limits on the number of predicted statistical moments of the values, volumes, and prices. The problem of predictions statistical moments (2.6; 2.33; 2.34) of values and volumes of market trades display the general problem of macroeconomic modeling and forecasting.

Indeed, modern economic models describe relations between macroeconomic variables that are composed by sums of the values or volumes of corresponding market trades during the averaging interval *Δ*. For convenience, we call them the 1st order variables. Macroeconomic investment, credits, and consumption during *Δ* equal to the sums of corresponding investment, credit, and consumption trades. The lack of sufficient data about all trades could make the econometric assessments during the interval Δ almost impossible. In that case, econometrics use comprehensive methodologies (Fox, et al., 2019) and estimate the values of macroeconomic variables using different available data. That creates a gap between the theoretical values of macroeconomic variables and their econometric assessments.

However, the theoretical dependence of macroeconomic variables on the sums of values or volumes of market trades establishes the economic-based obstacles for the predictions of trade statistical moments. The averages of the values or volumes of trades, or their 1st statistical moments, are composed by the sums of values and volumes. But the 2nd statistical moments of the values, volumes, and prices are composed by sums of squares of the values and volumes of market trades. We denote them as the 2nd order variables. Modern economic theories don't describe the evolution of the 2nd order variables. Moreover, modern econometric methodologies don't assess the 2nd order variables, except the frequency-based volatilities of price and returns. Econometric methodologies don't consider assessments of the market-based volatilities of the values $\Omega_C^2(t)$, volumes $\Omega_U^2(t)$ (2.27), and price $\sigma^2(t)$ (2.25). That limits the ability for economic-based predictions of the 2nd market-based statistical moments of price $\sigma^2(t)$ (2.25). Econometric assessments and economic-based predictions of the 3rd, 4th and higher statistical moments of values, volumes, and price are all the more absent. There are no econometric methodologies that can assess and no economic theories that can forecast the market-based skewness $Sk(t)$ (2.30; 2.35) and kurtosis $Ku(t)$ (2.38; 2.47) of price.

All that for many years to come limits the number of predicted statistical moments by the first two and the accuracy of probability forecasts by the Gaussian approximations.

3.2 The risks of Value-at-Risk

The limitations of the accuracy of predictions of asset price probability determine the accuracy and reliability of Value-at-Risk (VaR) – one of the most widespread tools to hedge the risks of a random price change. The foundation for VaR was developed more than 30 years ago (Longerstaey and Spencer, 1996; CreditMetrics™, 1997; Choudhry, 2013). "Value-at-Risk is a measure of the maximum potential change in value of a portfolio of financial instruments with a given probability over a pre-set horizon" (Longerstaey and Spencer, 1996). Despite the progress in VaR performance since then, the core features of VaR remain the same. To assess VaR at horizon T one should forecast the integral of the left tail of the probability of prices or returns.

Such projections with a given probability could limit the possible capital loss due to market price random variations at a time horizon T. VaR is used by the largest banks and investment funds to hedge their AUM and portfolios valued at billions of USD from the risk of a random market price change. However, the large portfolios and AUM under risk requires consider the impact of large trade volumes on price probability. The largest banks and funds should use probability that is determined by the market-based statistical moments of price.

As we show above, the predictions of market-based statistical moments of price depend on the forecasts of statistical moments and correlations of the values and volumes of trades. Hence, VaR as a method to hedge large AUM from risks of market price change at horizon T is based on the forecasts of the statistical moments and correlations of the values and volumes at the same horizon T. The accuracy of the market-based assessments of VaR at horizon T is determined by the accuracy of forecasts of the statistical moments and correlations of the values and volumes of trades. The more statistical moments of market trade are predicted, the higher the accuracy of VAR. Simply put, VaR assessments almost equal the predictions of the joint probability of the values and volumes of trades.

As we discussed above, the economic-based factors limit the number of predicted statistical moments of the values and volumes of market trades by the first two. The accuracy of predictions of market-based probabilities of price is limited by Gaussian approximations. The forecasting of the volatilities and correlations of the values and volumes, which determine market-based volatility of price, require development of econometric methods for their assessments and economic models that could describe mutual evolution of the 1st and 2nd order variables. All of that is absent now.

Our theoretical consideration reveals the hidden economic obstacles that can lead for unexpected losses of use VaR.

4. Market-based price-volume correlations

Price-volume correlations have been studied in numerous papers (Tauchen and Pitts, 1983; Karpoff, 1987; Campbell et al., 1993; Llorente et al., 2001; DeFusco et al., 2017). These researchers investigate the frequency-based correlations of price-volume time series.

Actually, the correlations of two random variables are determined by their joint probabilities. The choice of probabilities determines the value of mutual correlations. The choice of frequency-based probabilities of price and trade volume determines the results of the above authors. However, the consideration of price-volume correlations as a result of market trade randomness and the use of market-based price statistical moments give results that differ from those presented by (Tauchen and Pitts, 1983; Karpoff, 1987; Campbell et al., 1993; Llorente et al., 2001; DeFusco et al., 2017).

The use of VWAP as the market-based average price $a(t;1)$ (2.4; 2.7) reveals that the market-based price-volume correlation equals zero:

$$corr\{p(t), U(t)\} = E[p(t_i)U(t_i)] - E_m[p(t_i)]E[U(t_i)] \tag{4.1}$$

Indeed, from (2.4-2.6; 2.15), obtain:

$$E[p(t_i)U(t_i)] = \frac{1}{N}\sum_{i=1}^{N} p(t_i)U(t_i) = \frac{1}{\sum_{i=1}^{N} U(t_i)}\sum_{i=1}^{N} p(t_i)U(t_i) \cdot \frac{1}{N}\sum_{i=1}^{N} U(t_i) = E_m[p(t_i)]E[U(t_i)]$$

The market-based correlation $corr\{p(t),U(t)\}$ (4.1) between price $p(t_i)$ and volume $U(t_i)$ is zero:

$$corr\{p(t), U(t)\} = 0 \tag{4.2}$$

Let us assess correlation $corr\{p(t),U^2(t)\}$ between price $p(t_i)$ and squares of volumes $U^2(t_i)$.

$$corr\{p(t), U^2(t)\} = E[p(t_i)U^2(t_i)] - a(t;1)U(t;2) \tag{4.3}$$

$$E[p(t_i)U^2(t_i)] = E[C(t_i)U(t_i)] = C(t;1)U(t;1) + corr\{C(t), U(t)\}$$

From (2.4) and (2.27), obtain:

$$corr\{p(t), U^2(t)\} = corr\{C(t), U(t)\} - a(t;1)\Omega_U^2(t) \tag{4.4}$$

Let us derive correlation $corr\{p^2(t),U^2(t)\}$ between squares of prices and trade volumes:

$$corr\{p^2(t), U^2(t)\} = E[p^2(t_i)U^2(t_i)] - a(t;2)U(t;2)$$

$$E[p^2(t_i)U^2(t_i)] = E[C^2(t_i)] = C(t;2)$$

After simple transformations, obtain the form of $corr\{p^2(t)U^2(t)\}$:

$$corr\{p^2(t), U^2(t)\} = 2a(t;1)U(t;2)[p(t;1,2) - a(t;1)] \tag{4.5}$$

From (2.24), obtain:

$$corr\{p^2(t), U^2(t)\} = 2a(t;1)[corr\{C(t), U(t)\} - a(t;1)\Omega_U^2(t)] \tag{4.6}$$

From (4.4) and (4.6), obtain:

$$corr\{p^2(t), U^2(t)\} = 2a(t;1)corr\{p(t), U^2(t)\}$$

The different probabilities give different correlations between the same random variables.

5. Conclusion

The usual frequency-based price probability describes random market prices under the implicit assumption that all trade volumes are constant during the averaging interval. That assumption is rather far from the reality of current random market trade.

As opposed to the frequency-based approach, market-based statistical moments of prices directly depend on the statistical moments and correlations of random values and volumes of trades. That highlights the complex impact of market trade randomness on price stochasticity and reveals the dependence of price variations on the randomness of large volumes of market deals. Market-based statistical moments describe the dependence of price-volume correlations on statistical moments and correlations of random trade values and volumes. The use of market-based statistical moments of asset prices would improve the accuracy and reliability of large macroeconomic and market models like BlackRock's Aladdin, JP Morgan, and the U.S. Fed.

The forecasts of market-based statistical moments of price are limited by the first two. The accuracy of predictions of probability is limited by the Gaussian approximations. Market-based price probability reveals the economic limits on the accuracy of Value-at-Risk.

The development of market-based assessments of statistical moments of price would benefit financial markets, economic modeling, and management.

Appendix A.

Approximations of the price characteristic function and probability measure

We consider price as a random variable during the averaging interval *Δ* (2.2). One can equally describe a random variable by its characteristic function *F(t;x)* (A.1), probability measure *μ(t;p)* (A.2), and a set of the *n-th* statistical moments *a(t;n)* (2.8) (Shephard, 1991; Shiryaev, 1999; Shreve, 2004). The Taylor series expansion of the market-based characteristic function *F(t;x)* presents it through the set of the *n-th* statistical moments *a(t;n)*:

$$F(t;x) = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} a(t;n)\, x^n \tag{A.1}$$

$$\mu(t;p) = \frac{1}{\sqrt{2\pi}} \int F(t;x) \exp(-ixp)\, dx \tag{A.2}$$

$$a(t;n) = \frac{d^n}{(i)^n dx^n} F(t;x)|_{x=0} = \int p^n \mu(t;p)\, dp \quad ; \quad \int \mu(t;p)\, dp = 1 \tag{A.3}$$

In (A.1;A.2), *i* is the imaginary unit. For simplicity, we take price as a continuous random variable during *Δ* (2.2). Any predictions of the *market-based* price probability *μ(t;p)* and characteristic function *F(t;x)* at a horizon *T* should match the forecasts of price *n-th* statistical moments *a(t;n).* The direct dependence of market-based price *n-th* statistical moments *a(t;n)* on statistical moments of market trade values and volumes and their correlations highlights the dependence of forecasts of price probability *μ(t;p)* and characteristic function *F(t;x)* on predictions of market trade statistical moments at the same horizon *T*.

Finite number *q* of price statistical moments *a(t;n)*, *n=1,2,..q* determines finite q-approximation of price characteristic function $F_q(t;x)$ (A.4):

$$F_q(t;x) = 1 + \sum_{n=1}^{q} \frac{i^n}{n!} a(t;n)\, x^n \tag{A.4}$$

We present a simple example of approximation. Statistical moments determined by $F_q(t;x)$ for *n>q* can be different, but the first *q* moments are equal to *a(t;n), n=1,2,..q*. Taylor expansion (A.4) is not too useful to derive Fourier transform (A.2) and to obtain *q-*

approximation of the price probability measure $\mu_q(t;p)$. Let us consider the price characteristic function $G_q(t;x)$ (A.5):

$$G_q(t;x) = exp\left\{\sum_{n=1}^{q} \frac{i^n}{n!}\, b(t;n)\, x^n - B\, x^{2Q}\right\} \;;\; q = 1,2,\ldots;\; q < 2Q \;;\; B > 0 \quad \text{(A.5)}$$

and require that $G_q(t;x)$ (A.5) obey relations (A.3):

$$a(t;n) = \frac{d^n}{(i)^n dx^n} G_q(t;x)|_{x=0} \quad ; \quad n \le q \qquad \text{(A.6)}$$

Relations (A.6) define terms $b(t;n)$ in (A.5) through price statistical moments $a(t;n)$, $n \le q$. The term Bx^{2Q}, $B>0$, $2Q>q$ don't impact relations (A.3; A.6) but guarantee the existence of the price probability measures $\mu_q(t;p)$ as the Fourier transform (A.2) of the characteristic functions $G_q(t;x)$ (A.5). The uncertainty of $B>0$ and power $2Q>q$ in (A.5) highlights the well-known fact that the first q statistical moments don't explicitly determine the characteristic function and probability measure of a random variable. Relations (A.5) describe the set of characteristic functions $G_q(t;x)$ with different $B>0$ and $2Q>q$ and the corresponding set of probability measures $\mu_q(t;p)$ that match (A.2; A.5; A.6).

For $q=1$ the approximate price characteristic function $G_1(t;x)$ and probability $\mu_q(t;p)$ are trivial:

$$G_1(t;x) = exp\{i\, b(t;1)x\} \;\; ; \; a(t;1) = -i\frac{d}{dx} G_1(t;x)|_{x=0} = b(t;1) \qquad \text{(A.7)}$$

$$\mu_1(t;p) = \int dx\;\, G_1(t;x)\, exp(-ipx) = \delta\big(p - \, b(t;1)\big) \qquad \text{(A.8)}$$

For $q=2$ approximation $G_2(t;x)$ describes Gaussian probability measure $\mu_2(t;p)$:

$$G_2(x;t) = exp\left\{i\, b(t;1)x - \frac{b(t;2)}{2} x^2\right\} \qquad \text{(A.9)}$$

It is easy to show that

$$a(t;2) = -\frac{d^2}{dx^2} G_2(t;x)|_{x=0} = b(t;2) + b^2(t;1)$$

$$b(t;2) = a(t;2) - a^2(t;1) = \sigma^2(t) \qquad \text{(A.10)}$$

Coefficient $b(t;2)$ equals price volatility $\sigma^2(t)$ (2.16; 2.20) and the Fourier transform (A.2) for $G_2(t;x)$ gives Gaussian price probability measure $\mu_2(t;p)$:

$$\mu_2(t;p) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma(t)} exp\left\{-\frac{(p-b(t;1))^2}{2\sigma^2(t)}\right\} \qquad \text{(A.11)}$$

For $q=3$ approximation $G_3(t;x)$ has form:

$$G_3(t;x) = exp\left\{i\, b(t;1)x - \frac{\sigma^2(t)}{2} x^2 - i\,\frac{b(t;3)}{6} x^3 - B\, x^{2Q}\right\} \qquad \text{(A.12)}$$

$$a(t;3) = i\frac{d^3}{dx^3} G_3(t;x)|_{x=0} = b(t;3) + 3b(t;1)\sigma^2(t) + \, b^3(t;1)$$

$$b(t;3) = E_m\left[\big(p - b(t;1)\big)^3\right] = Sk(t)\sigma^3(t) \qquad \text{(A.13)}$$

Coefficient *b(t;3)* (A.13) depends on price skewness *Sk(t),* which describes the asymmetry of the market-based price probability from the normal distribution.

For the *q=4* approximation $G_4(t;x)$ depends on the choice of *B>0* and power *2Q>4*:

$$G_4(t;x) = exp\left\{i\, b(t;1)x - \frac{\sigma^2(t)}{2}x^2 - i\,\frac{b(t;3)}{6}x^3 + \frac{b(t;4)}{24}x^4 - Bx^{2Q}\right\} \;\; ; \;\; 2Q > 4 \quad (A.14)$$

Simple, but long calculations give:

$$b(t;4) = a(t;4) - 4a(t;3)a(t;1) + 12a(t;2)a^2(t;1) - 6a^4(t;1) - 3a^2(t;2)$$

$$b(t;4) = E_m\left[\left(p - b(t;1)\right)^4\right] - 3E_m^2\left[\left(p - b(t;1)\right)^2\right]$$

Price kurtosis *Ku(p)* (B.11) describes how the tails of the price probability measure $\eta_K(t;p)$ differ from the tails of a normal distribution.

$$Ku(t)\sigma_p^4(t;p) = E_m\left[\left(p - b(t;1)\right)^4\right] \quad (A.15)$$

$$b(t;4) = [Ku(t) - 3]\sigma^4(t)$$

Even the simplest Gaussian approximation $G_2(t;x)$, $\mu_2(t;p)$ (A.9; A.11) highlights the direct dependence of the market-based volatility $\sigma^2(t)$ (2.19; 2.25; A.10) of price on the first two statistical moments *C(t;1), C(t;2)* of the values and *U(t;1), U(t;2)* of volumes and their correlations (2.23). Thus, prediction of price volatility $\sigma^2(t)$ for Gaussian measure $\mu_2(t;p)$ (A.9) should follow non-trivial forecasting of the first two statistical moments (2.13; 2.14) and correlations (2.23) of the market trade value and volume.

Appendix B.

The non-negativity of kurtosis *Ku(t)* and higher statistical moments of prices

Let us consider the kurtosis *Ku(t)* (2.38; 2.39):

$$Ku(t)\sigma^4(t) = a(t;4) - 4a(t;3)a(t;1) + 6a(t;2)a^2(t;1) - 3a^4(t;1) \quad (B.1)$$

Below we show that market-based statistical moments of prices *a(t;1)* (2.7); *a(t;2)* (2.18; 2.26); *a(t;3)* (2.28), and *a(t;4)* (2.40; 2.46) determine non-negative kurtosis *Ku(t)* (B.1). Let us consider volatility of squares $\theta^2(t)$ (2.45) of prices $p^2(t_i)$ and present *a(t;4)* as:

$$a(t;4) = \Theta^2(t) + a^2(t;2) \quad (B.2)$$

We use (2.18) and present *a(t;2)* as (B.3):

$$a(t;2) = \sigma^2(t) + a^2(t;1) \quad (B.3)$$

Let us substitute (B.2; B.3) into (B.1) and obtain:

$$Ku(t,\tau)\sigma^4(t,\tau) = \Theta^2(t) + \sigma^4(t) + 8a^2(t;1)\sigma^2(t) + 4a^4(t;1) - 4a(t;3)a(t;1) \quad (B.4)$$

Let us present 3rd statistical moment *a(t;3)* as:

$$a(t;3) = E_m[p^3(t_i)] = E_m[p(t_i)p^2(t_i)] = a(t;1)a(t;2) + corr\{p(t_i), p^2(t_i)\} \quad (B.5)$$

$$corr\{p(t_i), p^2(t_i)\} = \sigma(t)\Theta(t)\frac{corr\{p(t_i),p^2(t_i)\}}{\sigma(t)\Theta(t)} \quad (B.6)$$

Coefficient of correlation of prices is less than a unit (Shiryaev, 1999, p.123):

$$-1 \leq \frac{corr\{p(t_i),p^2(t_i)\}}{\sigma(t)\Theta(t)} \leq 1 \qquad ; \qquad corr\{p(t_i),p^2(t_i)\} \leq \sigma(t)\Theta(t)$$

Hence, obtain inequality for the 3rd statistical moment *a(t;3)*:

$$a(t;3) \leq a(t;1)a(t;2) + \sigma(t)\Theta(t) \tag{B.7}$$

Substitute (B7) into (B.4) and obtain:

$$Ku(t)\sigma^4(t) \geq \Theta^2(t) + \sigma^4(t) + 8a^2(t;1)\sigma^2(t) + 4a^4(t;1) - 4a(t;2)a^2(t;1) \\ - 4a(t;1)\sigma(t)\Theta(t)$$

From (B.3), obtain:

$$Ku(t)\sigma^4(t) \geq \Theta^2(t) + \sigma^4(t) + 8a^2(t;1)\sigma^2(t) + 4a^4(t;1) - 4a^2(t;1)\sigma^2(t) - 4a^4(t;1) \\ - 4a(t;1)\sigma(t)\Theta(t)$$

$$Ku(t)\sigma^4(t) \geq \sigma^4(t) + \Theta^2(t) - 4a(t;1)\sigma(t)\Theta(t) + 4a^2(t;1)\sigma^2(t)$$

One can easily transform above inequality into the form:

$$Ku(t)\sigma^4(t) \geq \sigma^4(t) + [\Theta(t) - 4a(t;1)\sigma(t)]^2 \geq 0 \tag{B.8}$$

Inequality (B.8) proves that market-based statistical moments of prices *a(t;1)* (2.7); *a(t;2)* (2.18; 2.26); *a(t;3)* (2.28), and *a(t;4)* (2.40; 2.46) determine non-negative kurtosis *Ku(t)* (B.1). Hence, statistical moments *a(t;1)*, *a(t;2), a(t;3),* and *a(t;4)* are self-consistent and determine finite approximation of characteristic function (A.4; A.5) and probability of prices.

To derive higher market-based statistical moments *a(t;n), n=5,6,…* of price let us consider again the dependence of the first four statistical moments *a(t;n), n=1,2,3,4*.

$$a(t;1) = \frac{C(t;1)}{U(t;1)} \quad ; \quad a(t;3) = \frac{C(t;3)}{U(t;3)} \tag{B.9}$$

$$a(t;2) = \frac{C(t;2) + 2a^2(t;1)\Omega_U^2(t|1) - 2a(t;1)corr\{C(t),U(t)\}}{U(t;2)} \tag{B.10}$$

$$a(t;4) = \frac{C(t;4) + 2a^2(t;2)\Omega_U^2(t|2) - 2a(t;2)corr\{C^2(t),U^2(t)\}}{U(t;4)} \tag{B.11}$$

One can see obvious alikeness between the dependence of *a(t;1)* and *a(t;3)* and between the dependence of *a(t;2)* and *a(t;4)*. We use these parallels and define odd statistical moments *a(t;2k-1)* (B.12) alike to VWAP (2.4; 2.12) by the odd weight functions $w(t_i;2k-1)$ (2.10):

$$a(t;2k-1) = \sum_{i=1}^{N} p^{2k-1}(t_i)w(t_i;2k-1) = \frac{C(t;2k-1)}{U(t;2k-1)} \quad ; \quad k = 1,2,3,\ldots \tag{B.12}$$

We define even statistical moments *a(t;2k)* through the even weight functions $w(t_i;2k)$ (2.10):

$$a(t;2k) = \sigma^2(t|k) + a^2(t;k) \quad ; \qquad k = 1,2,\ldots \tag{B.13}$$

$$\sigma^2(t|k) = E_m[(p^k(t_i) - a(t;k))^2] = \sum_{i=1}^{N}(p^k(t_i) - a(t;k))^2 w(t_i;2k) \tag{B.14}$$

The functions $\sigma^2(t|k)$ have the meaning of the volatility of the *k-th* degree of price $p^k(t_i)$. Similar to (2.40-2.46) one easily obtains:

$$\sigma^2(t|k) = \frac{\Omega_C^2(t|k)+a^2(t;k)\Omega_U^2(t|k)-2a(t;k)corr\{C^k(t),U^k(t)\}}{U(t;2k)} \quad ; \quad k = 1,2,\dots \tag{B.15}$$

$$a(t;2k) = \frac{C(t;2k)+2a^2(t;k)\Omega_U^2(t|k)-2a(t;k)corr\{C^k(t),U^k(t)\}}{U(t;2k)} \tag{B.16}$$

Functions $\Omega_C^2(t|k)$ and $\Omega_U^2(t|k)$ in (B.15; B.16) denote volatilities (B.17) of k-degrees of trade values $C^k(t_i)$ and volumes $U^k(t_i)$:

$$\Omega_C^2(t|k) = C(t;2k) - C^2(t;k) \quad ; \quad \Omega_U^2(t|k) = U(t;2k) - U^2(t;k) \tag{B.17}$$

The relations (B.12-B.17) determine the set of market-based statistical moments of price $a(t;n)$ for all $n=1,2,\dots q$, and thus for any q define the finite q-approximation of the current characteristic function (A.4; A.5) and the probability of price during the averaging interval Δ (2.2). The number N of market trades during the interval Δ (2.2) limits the accuracy of q-approximations, $q<N$. We believe that the use of expressions similar to (B.2; B.3) and transformations alike (B.5-B.7) permit to prove that all even statistical moments of returns that are determined by (B.12-B.17) are non-negative, but we don't give a general proof.

References

Andersen, T., Bollerslev, T., Diebold, F.X, Ebens, H. (2001). The Distribution of Realized Stock Return Volatility, Journal of Financial Economics, 61, 43-76

Andersen, T.G, Bollerslev, T., Christoffersen, P.F. and F.X. Diebold, (2005). Volatility Forecasting, CFS WP 2005/08, 1-116

Andersen, T.G., Bollerslev, T., Christoffersen, P.F., and Diebold, F.X. (2006). Volatility and Correlation Forecasting, in G. Elliot, C.W.J. Granger, and Allan Timmermann (eds.), Handbook of Economic Forecasting. Amsterdam: North-Holland, 778-878

Bachelier, L., (1900). Théorie de la speculation, Annales scientifiques de l'É.N.S. 3e série, **17**, 21-86

Berkowitz, S.A., Dennis, E., Logue, D.E., Noser, E.A. Jr. (1988). The Total Cost of Transactions on the NYSE, *The Journal of Finance*, 43, (1), 97-112

Black, F. and M. Scholes, (1973). The Pricing of Options and Corporate Liabilities, J. Political Economy, 81 (3), 637-654

Brock, W.A. and B.D. LeBaron, (1995). A Dynamic structural model for stock return volatility and trading volume. NBER, WP 4988, 1-46

Buryak, A. and I. Guo, (2014). Effective And Simple VWAP Options Pricing Model, Intern. J. Theor. Applied Finance, 17, (6), 1450036, https://doi.org/10.1142/S0219024914500356

Busseti, E. and S. Boyd, (2015). Volume Weighted Average Price Optimal Execution, 1-34, arXiv:1509.08503v1

Campbell, J.Y. and R.J. Shiller, (1988). Stock Prices, Earnings And Expected Dividends. NBER, WP 2511, 1-42

Campbell, J.Y., Grossman, S.J. and J.Wang, (1993). Trading Volume and Serial Correlation in Stock Return. Quatr. Jour. Economics, 108 (4), 905-939

Campbell, J.Y. (1998). Asset Prices, Consumption, and the Business Cycle. NBER, WP6485

Campbell, J.Y. (2000). Asset Pricing at the Millennium. Jour. of Finance, 55(4), 1515-1567

Choudhry, M., (2013). An Introduction to Value-at-Risk, 5th Edition. Wiley, 1-224

CME Group, (2020). https://www.cmegroup.com/search.html?q=VWAP

Cochrane, J.H. and L.P. Hansen, (1992). Asset Pricing Explorations for Macroeconomics. Ed., Blanchard, O.J., Fischer, S. NBER Macroeconomics Annual 1992, v. 7, 115 – 182

Cochrane, J.H. (2001). Asset Pricing. Princeton Univ. Press, Princeton, US

Cochrane, J.H. (2005). Time Series for Macroeconomics and Finance. Graduate School of Business, Univ. Chicago, 1-136

Cochrane, J.H. (2022). Portfolios For Long-Term Investors, Rev. Finance, 26(1), 1–42

CreditMetrics™, (1997). Technical Document. J.P. Morgan & Co, NY. 1-212
DeFusco, A.A., Nathanson, C.G. and E. Zwick, (2017). Speculative Dynamics of Prices and Volume, Cambridge, MA, NBER WP 23449, 1-74
Diebold, F.X. and K. Yilmaz, (2008). Macroeconomic Volatility And Stock Market Volatility, Worldwide, NBER WP 14269, 1-35
Dimson. E. and M. Mussavian, (1999). Three centuries of asset pricing. J.Banking&Finance, 23(12) 1745-1769
Duffie, D., Dworczak, P. (2021). Robust Benchmark Design. Journal of Financial Economics, 142(2), 775–802
Fama, E.F. (1965). The Behavior of Stock-Market Prices. J. Business, 38 (1), 34-105
Fama, E.F. and K.R. French, (2015). A five-factor asset pricing model. J. Financial Economics, 116, 1-22
Fox, D.R., et al. (2019). Concepts and Methods of the U.S. National Income and Product Accounts, BEA, Dep. Commerce, US, Chapters 1-13, 1- 449
Friedman, D.D. (1990). Price Theory: An Intermediate Text. South-Western Pub. Co., US
Gallant, A.R., Rossi, P.E. and G. Tauchen, (1992). Stock Prices and Volume, The Review of Financial Studies, 5(2), 199-242
Goldsmith, R.W. and R.E. Lipsey, (1963). Asset Prices and the General Price Level, NBER, 166 – 189, in Studies in the National Balance Sheet of the United States, Ed. Goldsmith, R.W. and R. E. Lipsey
Greenwood, R. and A. Shleifer, (2014). Expectations of Returns and Expected Returns. The Review of Financial Studies, 27 (3), 714–746
Heaton, J. and D. Lucas, (2000). Stock Prices and Fundamentals. Ed. Ben S. Bernanke, B.S and J. J. Rotemberg, NBER Macroeconomics Annual 1999, v. 14., 213 – 264
Hördahl, P. and F. Packer, (2007). Understanding asset prices: an overview. Bank for International Settlements, WP 34, 1-38
Karpoff, J.M. (1987). The Relation Between Price Changes and Trading Volume: A Survey. The Journal of Financial and Quantitative Analysis, 22 (1), 109-126
Kendall, M.G and A.B. Hill, (1953). The Analysis of Economic Time-Series-Part I: Prices. Jour. Royal Statistical Soc., Series A, 116 (1), 11-34
Llorente, G., Michaely R., Saar, G. and J. Wang. (2001). Dynamic Volume-Return Relation of Individual Stocks. NBER, WP 8312, Cambridge, MA., 1-55
Longerstaey, J., and M. Spencer, (1996). RiskMetrics -Technical Document. J.P.Morgan & Reuters, N.Y., Fourth Edition, 1-296

Mackey, M.C. (1989). Commodity Price Fluctuations: Price Dependent Delays and Nonlinearities as Explanatory Factors. J. Economic Theory, 48, 497-509

Malkiel, B. and J.G. Cragg, (1980). Expectations and the valuations of shares, NBER WP 471, 1-70

Mandelbrot, B., Fisher, A. and L. Calvet, (1997). A Multifractal Model of Asset Returns, Yale University, Cowles Foundation Discussion WP1164, 1-39

Merton, R.C. (1973). An Intertemporal Capital Asset Pricing Model, Econometrica, 41, (5), 867-887

Mills, F.C. (1946). Price-Quantity Interactions in Business Cycles. NBER, Prins.Univ., NY

Muth, J.F. (1961). Rational Expectations and the Theory of Price Movements, Econometrica, 29, (3) 315-335

Odean,T. (1998). Volume, Volatility, Price, And Profit When All Traders Are Above Average, The Journal of Finance, LIII, (6), 1887-1934

Poon, S-H. and C.W.J. Granger, (2003). Forecasting Volatility in Financial Markets: A Review, J. of Economic Literature, 41, 478–539

Sharpe, W.F. (1964). Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk. The Journal of Finance, 19 (3), 425-442

Shephard, N.G. (1991). From Characteristic Function to Distribution Function: A Simple Framework for the Theory. *Econometric Theory*, 7 (4), 519-529

Shiryaev, A.N. (1999). Essentials Of Stochastic Finance: Facts, Models, Theory. World Sc. Pub., Singapore. 1-852

Shreve, S. E. (2004). Stochastic calculus for finance, Springer finance series, NY, USA

Stigler, G.J. and J.K. Kindahl, (1970). The Dispersion of Price Movements, NBER, 88 - 94 in Ed. Stigler, G.J, Kindahl, J.K. The Behavior of Industrial Prices

Tauchen, G.E. and M. Pitts, (1983). The Price Variability-Volume Relationship On Speculative Markets, Econometrica, 51, (2), 485-505

Weyl, E.G. (2019). Price Theory, AEA J. of Economic Literature, 57(2), 329–384

Wolfers, J. and E. Zitzewitz, (2006). Interpreting Prediction Market Prices As Probabilities, NBER WP12200, 1-22