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Abstract

This paper presents a fairly general treatment of recursive infinite horizon forward looking optimizing systems on infinite dimensional spatial domains. It includes optimal control, an analysis of local stability of spatially flat optimal steady states and development of techniques to compute spatially heterogeneous optimal steady states. The paper also develops a concept of rational expectations equilibrium, a local stability analysis for spatially homogeneous rational expectations steady states, and computational techniques for spatially heterogeneous rational expectations steady states.

Keywords: Pattern formation, spatial spillovers, optimal control, spillover induced instability, growth models

JEL Classification: C61, 041.

1. Introduction

When modeling spatial interactions, where "space" is a general concept which is wide enough to include social interactions, many researchers in mathematical biology and economics have used kernel type expressions of the form

$$X(t,z) = \int_{z' \in \mathcal{Z}} w(z-z')\, x(z',t)\, dz' \tag{1}$$

where $x(t,z)$ is a state variable at time $t \in (0,\infty)$ and spatial point $z \in \mathcal{Z}$, where $\mathcal{Z}$ is the spatial domain over which the influence kernel $w(z-z')$ is defined. The main emphasis of the literature is to study dynamical systems forces that cause agglomeration of economic activity in economics and general agglomeration phenomena in biology (e.g. Murray (2002, 2003)). Turing (1952) type analysis and Fourier series (Krugman (1996)) play an important role in this approach.

Some examples of recent papers that study spatial interactions in dynamical systems with tools like Turing analysis and Fourier analysis in forward looking contexts are Quah (2002), Boucekkine, Camacho, and Zou (2006), and Mossay (2006). We offer what we believe to be the first relatively general treatment of pattern formation in infinite horizon recursive forward looking dynamical systems models with spatial kernels that is suitable for use in economic modeling.[1] Brock and Xepapadeas (2008a,b), hereafter BX, study infinite horizon forward looking systems where the spatial interactions are of diffusion type. That is, BX (2008a,b) can be viewed as generalizing the infinite horizon recursive dynamical systems approach popular in economics (e.g. Stokey and Lucas with Prescott (1989)) to continuous time infinite horizon continuous space systems, where Turing type instabilities can only appear if the future is discounted heavily enough.

If one expands the right hand side of (1) in a Taylor series, after setting $z - z' = \zeta$, one obtains:

$$X(t,z) \cong x(t,z) \int_{\zeta \in \mathcal{Z}} w(\zeta)\, d\zeta + x_z(t,z) \int_{\zeta \in \mathcal{Z}} \zeta w(\zeta)\, d\zeta + x_{zz}(t,z) \int_{\zeta \in \mathcal{Z}} \zeta^2 w(\zeta)\, d\zeta + \ldots \tag{2}$$

Here higher order terms have been dropped and subscripts denote partial differentiation with respect to z. In the context of the expansion (2), the BX papers can be viewed as the study of (2) where the first moment of the kernel

[1] There are a large number of papers that study forward looking new economic geography (NEG) models with a finite number of locations. Examples are Baldwin (2001), Ottaviano (2001), and Baldwin and Martin (2004). We focus on the continuous space case here.

$\int_{a\in\mathcal{Z}} aw\left(a\right)da = 0$. While the zero first moment and zero third moment is common in the literature,[2] truncation of the series at the second moment as in (2) is highly restrictive and fails to capture the tension between local centripetal forces and more distant centrifugal forces associated with the market potential of a location, which was stressed by the early writers, e.g., Krugman (1996) and Fujita, Krugman, and Venables (1999). In studying the emergence of economic agglomerations and clusters, later writers besides Quah, such as Lucas (2001), Lucas and Rossi-Hansberg (2002), Ioannides and Overman (2007), and Desmet and Rossi-Hansberg (2007), are heavy users of kernels of the type (1) in an attempt to incorporate into economic models spatial or geographical spillovers reflecting, for example, the impact of employment at neighboring sites on productivity at a given site, or the impact on accumulated knowledge at a given site of accumulated knowledge at neighboring sites.

As far as we know, our paper is the first relatively complete treatment of infinite horizon recursive dynamical systems (which include recursive infinite horizon optimal control systems) that includes kernel expressions in the law of motion and/or the payoff function. We present the technical aspects of our approach in an extensive and detailed Appendix. In the main body of the paper, we provide a summary of our theoretical results and illustrate our approach by applying it to four examples that are of interest in economic applications. We give a preview of the examples here; the details are developed in the main body of the paper.

Example 1 is a macro growth model along the lines of the standard textbook Solow (1956) model, but with spatial spillover externalities in the production function. It is given by

$$\frac{\partial x\left(t,z\right)}{\partial t} = sf\left(x\left(t,z\right),X\left(t,z\right),L\right)-\eta x\left(t,z\right), \text{for all } z\in\mathcal{Z} \tag{3}$$

$$X\left(t,z\right) = \int_{-Z}^{Z} w\left(z-z'\right)x\left(z',t\right)dz' \tag{4}$$

Here $x(t,z)$ denotes capital stock at site z at date t, L denotes labor, and $X(t,z)$ denotes an external effect on the production function $f(x,X,L)$ at site z at date t. We may think of this model as a spatial version of Solow (1956), Romer (1986) and Lucas (1988), with geographical spillovers given by a Krugman (1996), Chincarini and Asherie (2008) specification. Although we restrict z to be one-dimensional (e.g. Krugman's (1996) and Chincarini and Asherie's (2008)

[2]With symmetric kernels, $w\left(\zeta\right)=w\left(-\zeta\right)$, the odd moments of the kernel, that is, those with odd powers of ζ, are zero.

circle where $\mathcal{Z}$ is finite, or Krugman's (1996) line where $\mathcal{Z}$ is infinite), our methods of analysis should generalize to two-dimensional spatial settings by using an appropriate set of basis elements as in Chincarini and Asherie (2008).

In order to set the stage we linearize (1) at a flat steady state (FSS)[3] and locate sufficient conditions for the FSS to be destabilized by the spatially heterogeneous perturbations induced by kernel $w(.)$ in (4). We show below that analysis can be completely described by the dispersion relation presented in Murray (2003). We give closed form expressions for the dispersion relation for the case where $\mathcal{Z} = (-\infty, \infty)$ and for the case where $\mathcal{Z} = [-Z, Z]$. We locate sufficient conditions for existence of a heterogeneous steady state (HSS)[4] and compute an example below.

We shall call the above model the "spatial Solow" "descriptive" model with Romer/Lucas spatial externalities. Since this model is so close to the well-analyzed model in natural science which is treated in Murray (2003, Chapter 12), one might say this is in the received literature, although some of our results regarding the possibilities of spatial spillover induced instability and spatially heterogeneous steady state could provide further insights into the Solow model and regional convergence issues. Note that capital $x(t,z)$ as well as labor L is assumed to be immobile in model (3)-(4). We use this model here to set the stage for treatment of examples that we think are new.

Example 2 is the socially- optimized version of (3)-(4), i.e. consider the problem

$$\max_{\{c(t,z)\}} \int_{t=0}^{\infty} e^{-\rho t} \left[\int_{-Z}^{Z} U\left(c\left(t,z\right)\right) dz \right] dt \tag{5}$$

subject to

$$c\left(t,z\right) + \frac{\partial x\left(t,z\right)}{\partial t} = f\left(x\left(t,z\right), X\left(t,z\right), L\right) - \eta x\left(t,z\right) \ \text{, for all } z \tag{6}$$

Notice that each site has L units of labor and capital $x(t,z)$ can not be moved across sites. We call problem (5)-(6) the social optimization management problem (SOMP). However, model (5)-(6) has an extreme assumption that capital and labor are completely immobile across locations. If capital and labor are completely mobile, and consumption goods are completely mobile as well, then it can be shown that it is easy to reduce the problem one that is equivalent to a one-dimensional Ramsey type problem. Of course the cases of complete immobility

[3]An FSS is a spatially homogenous or "flat earth" steady state.

[4]An HSS is steady state where spatial patterns, agglomerations or clusters are present.

of capital and labor and complete mobility of capital and labor are polar cases. We use these polar cases to give insight into the more realistic case where there are frictional costs to the movement of capital and labor. We also study the concept of rational expectations equilibrium which we call the private optimization management problem (POMP).

Example 3 is a specialized version of the spatial agglomeration dynamics model developed by Quah (2002). Because we work on the circle and the line whereas Quah works on the sphere, we work on a simpler space here, i.e. the circle $[-Z, Z]$ of length $2Z$. Quah's (2002) equilibrium problem for studying spatial agglomerations or clusters in technology (or accumulated knowledge) is the following in our notation. A producer at location z solves:

$$\max_{\{u(t,z)\}} \int_{t=0}^{\infty} e^{-\rho t} \left[x(t,z) X^e(t,z) - \frac{\gamma}{2} u(t,z)^2 \right] dt \tag{7}$$

subject to

$$\frac{\partial x(t,z)}{\partial t} = u(t,z), \text{ for all } z \tag{8}$$

Here output is produced by a linear function of accumulated knowledge $x(t,z)$ and the productivity factor $X^e(t,z)$. The representative producer at site z takes $X^e(t,z)$ as given and chooses $\{u(t,z)\}$ which is costly investment in knowledge accumulation to maximize (7)-(8). We close the system with rational expectations by each producer located at $z \in [-Z, Z]$ where $X^e(t,z) = X(t,z)$.

Example 4 is an R&D based growth model (Jones (1995)) but with spatial spillovers along the lines of Quah (2002), so we call it a JQ model. We develop this example in quite a bit of detail, and we also compute heterogeneous optimal steady states (HOSSs) when they exist.

The rest of the paper is organized as follows. Section 2 contains our main results about necessary and sufficient conditions for the SOMP in a general setting as well as some basic analytics for the POMP, along with the results about instability of the flat optimal steady state (FOSS) and the rational equilibrium steady state (RESS). Section 3 contains the solutions to the examples and a discussion of the economics involved. All the proofs and mathematical details are contained in the Appendix.

2. Spatial Spillover Dynamics and Optimization

In this section we present the main results regarding necessary and sufficient conditions of optimal control under spatial spillovers and instability of the FOSS. Assume that the temporal growth of a state variable such as accumulated knowledge or technology at location z can be described by a function $g\left(x\left(t,z\right),u\left(t,z\right)\right)$ where $u\left(t,z\right)$ is a control variable. We assume that the state and the control variables are absolutely continuous square integrable functions and that the admissible control functions belong to a compact subset of a Hilbert space. Long range spatial effects describing the effects that the concentration of the state variable $x\left(t,z'\right)$ in locations z', has on $x\left(t,z\right)$, can be modelled using the kernel formulation as:

$$X\left(t,z\right)=\mathbf{K}x\left(t,z\right):=\int_{z'\in\mathcal{Z}}w\left(z-z'\right)x\left(z',t\right)dz' \tag{9}$$

where $\mathbf{K}=\int_{z'\in\mathcal{Z}}w\left(z-z'\right)dz'$ is a linear integral operator acting on a function $x\left(t,z\right):\mathcal{Z}\times[0,\infty)\rightarrow\Re^{n}$. For simplicity we use $\mathbf{K}x$ instead of $\mathbf{K}x\left(t,z\right)$. For the kernel function $w\left(\zeta\right)$, $\zeta=z'-z$ we assume square integrability along with symmetry, or $w\left(\left|z-z'\right|\right)=w\left(z-z'\right)=w\left(z'-z\right)$.[5] On the infinite domain $w\left(\zeta\right)$ is a continuous symmetric function such that geographical spillovers tend to zero for large $\left|z-z'\right|$, or $w\rightarrow0$ as $\left|z-z'\right|\rightarrow\infty$. The kernel function quantifies the impact of site z' on site z. When geographical spillovers are combined with the temporal growth function g, the rate of change of the state variable x at time t and location z depends on the values of the state variable at locations $z'\in\mathcal{Z}$ and can be written as

$$\frac{\partial x\left(z,t\right)}{\partial t}=g\left(x\left(z,t\right),u\left(z,t\right),X\left(t,z\right)\right)+\mu X\left(t,z\right)\ ,\ x\left(0,z\right)=x_{0}\left(z\right)\text{ given} \tag{10}$$

where $\mu\in\Re$ and $X\left(t,z\right)=\mathbf{K}x$. The integrodifferential equation (10) describes the spatiotemporal effects of geographical spillovers, since the temporal evolution of the state variable's spatial distribution depends on the control u and the spatial spillovers. The parameter μ reflects the intensity of the direct impact of the spillover variable X on the rate of change of the state variable. Thus in our formulation spatial effects along with temporal growth determine the evolution of the state variable in time and space.

[5] Thus $\mathbf{K}$ is a linear compact operator. Linearity means that $\mathbf{K}\left(\lambda_{1}x_{1}+\lambda_{2}x_{2}\right)=\lambda_{1}\mathbf{K}x_{1}+\lambda_{2}\mathbf{K}x_{2}$ for square integrable functions x and scalars λ. The operator notation will be used for certain derivations.

The integrodifferential equation (10) can be used as a dynamic constraint in an optimal control problem where the objective is to choose a spatiotemporal path for the control variable $u(t,z)$ which will maximize discounted benefits over the spatial domain $\mathcal{Z}$ associated with a payoff function. The payoff function can also be affected, in a general set up, by geographical spillovers and can be written as: $f(x(t,z), u(t,z), X(t,z))$. Thus our modelling approach provides tools for solving forward looking dynamic optimization problems which are at the core of dynamic economics under spatial spillovers.

In the rest of this section we develop an extension of Pontryagin's maximum principle which provides necessary conditions for the optimization problem, along with the corresponding sufficiency conditions.[6]

The infinite horizon optimal control problem with spatial spillovers can be stated as:

$$\max_{\{u(t,z)\}} \int_{z\in\mathcal{Z}} \int_0^\infty e^{-\rho t} f(x(t,z), u(t,z), X(t,z))\, dt dz \tag{11}$$

subject to (10).

As will be shown in the following sections, well-known growth models, when extended to include geographical spillovers, can be derived as special cases of (11).

Proposition 1 (Maximum principle under spatial spillovers). *Let $u^* = u^*(t,z)$ be a choice of instrument that solves problem (11) and let $x^* = x^*(t,z)$ be the associated path for the state variable. Then there exists a function $p(t,z)$ such that for each t and z, $u^* = u^*(t,z)$ maximizes the current value Hamiltonian function*

$$H(x,u,p,X) = f(x,u,X) + p(t,z)\left[g(x,u,X) + \mu X\right] \tag{12}$$

or for interior solutions:

$$\frac{\partial f}{\partial u} + p\frac{\partial g}{\partial u} = 0 \Rightarrow u^* = u^*(x(t,z), p(t,z), X(t,z)),\ \ X = \mathbf{K}x \tag{13}$$

Furthermore $x(t,z)$ and $p(t,z)$ satisfy the system of integrodifferential equations

$$\frac{\partial x}{\partial t} = g(x, u^*, \mathbf{K}x) + \mu\mathbf{K}x = H_p(x,p,X) \tag{14}$$

$$\frac{\partial p}{\partial t} = \rho p - (f_x + pg_x) - (\mathbf{K}f_X + \mathbf{K}pg_X + \mu\mathbf{K}p) = \tag{15}$$

$$= \rho p - H_x(x,p,X) - \mathbf{K}H_X(x,p,X) \tag{16}$$

[6]To make the presentation clearer, we use a one-state, one-control variable set up. Generalizations are provided in the Appendix.

where all functions in (14)-(16) are evaluated at $u^* = u^*(x, p, X)$. *The following limiting intertemporal transversality condition holds*

$$\lim_{T \to \infty} e^{-\rho T} \int_{z_0}^{z_1} p(T, z)\, x(T, z)\, dz = 0 \text{ for all } z \tag{17}$$

For a finite spatial domain with circle boundary conditions $x(t, -Z) = x(t, Z) = \bar{x}(t)$, *the following spatial transversality condition holds for all dates* t*:*

$$p(t, -Z) = p(t, Z) \tag{18}$$

For proof see Appendix 1. In the same Appendix the necessary conditions for the vector state and control variable problem are also presented.

Proposition 2 (Sufficient Conditions). *Assume that functions* $f(x, u, X)$ *and* $g(x, u, X)$ *are concave differentiable functions for problem (11) and suppose that functions* $x^*(t, z)$, $u^*(t, z)$ *and* $p(t, z)$ *satisfy necessary conditions (13)-(15) for all* $t \in [0, \infty)$, $z \in \mathcal{Z}$ *and that* $x(t, z)$ *and* $p(t, z)$ *are continuous with* $p(t, z) \geq 0$ *for all* t *and* z. *Then the functions* $x^*(t, z)$, $u^*(t, z)$ *solve the problem (11). That is, the necessary conditions (13) are also sufficient.*

For proof see Appendix 2.

2.1. Spillover Induced Spatial Instability and Emergence of Agglomerations

A question arising in the study of problems described by (11) is whether its solution exhibits spatial homogeneity or spatial heterogeneity. Spatial homogeneity implies that the state, costate and control variables which are solutions of (11) have a spatially uniform distribution along the optimal spatiotemporal path. Heterogeneity on the other hand, means that spatial distributions are not uniform so that geographical patterns are formed. This implies that clusters or economic agglomerations emerge and may become persistent at a spatially heterogeneous steady state.

To study the emergence of agglomerations and the formation of spatial clusters in economics we follow the approach introduced by Turing (1952) which examines the stability of a stable FSS of reaction-diffusion systems to spatially heterogenous perturbation.[7] We extend this approach to deal with the system of integrodifferential equations (14), (16) which constitute the modified Hamiltonian dynamic system (MHDS) for problem (11). Assume that a FOSS, which is a special case of (14), (16) when spatially uniform spillovers are present, has the local saddle point property. As is well known, this implies that the Jacobian matrix $J^F(x^*, p^*)$ of the linearization of the MHDS evaluated at the FOSS (x^*, p^*) has two real eigenvalues, one positive and one negative, which characterize temporal growth. Furthermore, there is a one-dimensional stable manifold, which is tangent to the linear subspace spanned by the negative eigenvalue and which is tangent to the stable manifold at the FOSS, such that for any initial value of the state there is an initial value for the costate variable such that the dynamical system converges to the FOSS along the stable manifold. Thus along the stable manifold the FOSS is stable to spatially homogeneous perturbations. To check for the possible emergence of spatial clusters when spatial spillovers are introduced using Turing's approach, we examine whether the FOSS is stable to spatially heterogeneous perturbations. The linearization of the MHDS of (14), (16), which is the full system with spatial spillovers evaluated at the FOSS, can be written, using $X = \mathbf{K}x$ and a slight abuse of notation so that (x, p) denotes deviations from (x^*, p^*), as:

$$\frac{\partial x}{\partial t} = \left(H^*_{px} + H^*_{pX}\mathbf{K}\right) x + H^*_{pp} p \tag{19}$$

$$\frac{\partial p}{\partial t} = \left(-H^*_{xx} - 2H^*_{Xx}\mathbf{K} - H^*_{XX}\mathbf{K}^2\right) x + \left(\rho - H^*_{xp} - H^*_{Xp}\mathbf{K}\right) p \tag{20}$$

where the superscript $(^*)$ indicates that the derivatives are evaluated at the FOSS. Furthermore $X^* = \mathbf{K}x^*$ and $\mathbf{K}^2 = \int_{z' \in Z} w(z - z')\, w(z - z')\, dz'$. To study the stability of the FOSS when the spatial spillovers are present, we need to analyze the eigenvalues of the Jacobian matrix of the linearized system (19)-(20). Let J^{*S} denote the Jacobian matrix of (19)-(20) at (x^*, p^*, X^*) and let

$$W^m(k) = \int_{\zeta \in Z} [w(\zeta)]^m \cos(k\zeta)\, d\zeta, \zeta = z - z', m = 1, 2 \tag{21}$$

where $k = 2n\pi/L, n = \pm 1,\ \pm 2, \ldots$. The quantity k is called the *wave number,* while $1/k = L/2n\pi$ is a measure of the wave-like pattern in the spatial domain.

[7] Turing's approach has been used in new economic geography (e.g. Krugman (1996), Fujita Krugman and Venables (1999), Chincarini and Asherie (2008)), in biology (e.g. Okubo and Levin (2001), Murray (2002, 2003)) and in ecosystem management (Brock and Xepapadeas (2008a, 2008b)).

Thus, $1/k$ is proportional to the wavelength $l : l = 2\pi/k = L/n$ with $L = 2Z$ being the length of the spatial domain. As is shown in Appendix 3, treating spatial spillovers as a spatially heterogeneous perturbation implies that the FOSS will be unstable to such perturbation if there is a wave number k such that both eigenvalues of J^{*S} which characterize temporal growth have positive real parts. This means that the mode n corresponding to this wave number will keep growing in the spatial domain with the passage of time and eventually an agglomeration or spatial cluster might emerge, provided that the spatial domain is large enough to accommodate the pattern. To further analyze this potential instability, note that the trace of J^{*S} is $\rho > 0$, which means that at least one positive eigenvalue exists. This is consistent with the general result of optimal control in the temporal dimension only, which suggests, that eigenvalues at optimal steady states are either saddle point stable or completely unstable (Kurz (1968)). In a sense our result can be regarded as a generalization of Kurz's result to optimal control in both spatial and temporal dimensions. To have an unstable mode, which is equivalent to having both eigenvalues with positive real parts, the determinant of J^{*S} should be positive. Then the following results about instability induced by the presence of spatial spillovers can be stated:

Proposition 3 (Spillover induced instability). *Assume that a symmetric square integrable kernel function $w(\zeta)$ and wave numbers $k \in (k_1, k_2)$ exist such that*

$$\begin{aligned}\psi\left(W\left(k\right)\right) = &\left[H^*_{XX}H^*_{pp} - \left[H^*_{pX}\right]^2\right] W^2\left(k\right) + \\ &\left[H^*_{pX}\left(\rho - 2H^*_{px}\right) + 2H^*_{Xx}H^*_{pp}\right] W\left(k\right) + \\ &\left[\rho H^*_{px} - \left[H^*_{pX}\right]^2 + H^*_{pp}H^*_{xx}\right] > 0\end{aligned} \tag{22}$$

Then both eigenvalues of the Jacobian matrix J^{*S} of system (19)-(20) which characterize temporal growth are positive, and the FOSS for problem (11) is not stable to spatially heterogeneous perturbations which are induced by spatial spillovers.

For proof see Appendix 3.[8]

Condition (22) is a dispersion relationship. To obtain an idea of the way that spatial clusters are emerging we consider a solution for a specific mode which, as

[8] It should be noted that the approach used in the proof of this proposition can also be used to study, in addition to the MHDS which results from optimal control, the stability of the FSS of an arbitrary dynamical system to spatially heterogeneous perturbations induced by spillovers modelled through kernels.

explained in Appendix 3, will be of the form

$$v_k(t,z) = \sum_{i=1,2} e^{\lambda_i(k)t}[\alpha_k^v \cos(kz) + \beta_k^v \sin(kz)] \ , \ v = x, p \tag{23}$$

Assume that this solution corresponds to an unstable mode so that $(\lambda_1(k), \lambda_2(k)) > 0, k \in (k_1, k_2)$. Since there are no initial conditions on the costate variable $p(t,z)$, we can describe the spatiotemporal movement along the unstable path associated with only one of the two positive eigenvalues. This path is specified by setting in (23) the constants associated with one eigenvalue equal to zero and use existing initial conditions $x(0,z)$ to specify the constants associated with the other positive eigenvalue. If we set the constants associated with λ_1 equal to zero, then the path associated with λ_2 will be dominated by the single mode that corresponds to a $k_M \in (k_1, k_2) : \lambda_2(k_M) > \lambda_2(k)$ for all $k \in (k_1, k_2)$. In this case the spatiotemporal evolution near the FOSS, for a spatial domain of length 2π with $n = 1$, can be approximated by

$$\begin{pmatrix} x(t,z) \\ p(t,z) \end{pmatrix} \approx e^{\lambda_2(k_M)t}[\boldsymbol{\alpha}_1^v \cos(k_M z) + \boldsymbol{\beta}_2^v \sin(k_M z)] + \begin{pmatrix} x^* \\ p^* \end{pmatrix}$$

where the constants $(\boldsymbol{\alpha}_1^v, \boldsymbol{\beta}_2^v)$ can be determined by the eigenvector corresponding to $\lambda_2(k_M)$ and initial conditions on x. Since $\lambda_2(k_M) > 0$ the deviation for the FOSS (x^*, p^*) grows with the passage of time and a wave like pattern in the spatial domain emerges.

Proposition 3 shows that if we choose the control variable optimally in the spatiotemporal domain according to (13), then the flat steady state which was optimal for the model without spatial spillovers is no longer saddle point stable for the model with spatial spillovers, but is completely unstable. We call this result *spillover induced spatial instability* of the optimal control. This is a sign that clusters or economic agglomerations could emerge in the optimal control of a system with spatial spillovers. The clustering pattern at which the system might eventually settle in the long run will be determined by the solution, if it exists, of the system

$$0 = H_p(x, p, \mathbf{K}x; \boldsymbol{\omega}) \tag{24}$$

$$0 = \rho p - H_x(x, p, \mathbf{K}x; \boldsymbol{\omega}) - \mathbf{K}H_X(x, p, \mathbf{K}x; \boldsymbol{\omega}) \tag{25}$$

This is a system of nonlinear integral equations in the unknown functions $x(z)$, $p(z)$, $z \in \mathcal{Z}$, where $\boldsymbol{\omega}$ is a vector of parameters or known functions of

z. Conditions for the existence of a solution for the system of (24)-(25) may be formulated in terms of general conditions for the existence of solutions of nonlinear operator equations,[9] and could be approximated by numerical methods. If such a solution exists, then $x^*(z)$, $p^*(z)$ will provide the optimal long-

run equilibrium spatial distribution, or optimal equilibrium agglomeration, for the state and the costate variables, while $u^*(x^*(z), p^*(z))$ will provide the corresponding optimal agglomeration for the control variable.

2.2. Spillover Induced Spatial Instability and Rational Expectations Equilibrium

Problem (11) can be regarded as an optimization problem solved by a social planner who seeks to maximize discounted benefits over the whole spatial domain by taking into account both the temporal and the spatial constraints of the problem. Thus the social planner internalizes both the temporal and the spatial externalities. We call this the social optimization management problem. A related problem associated with market equilibrium is the problem where an economic agent considers certain external effects as outside her/his control and treats them as exogenous. In our case this can be interpreted as having a planner at each site z that maximizes discounted benefits on the site and considers the spatial spillover $X(t,z)$ affecting her/his site as an exogenous parameter $X^e(t,z)$. This is the private optimization management problem, which can be written as:

$$\max_{\{u(t,z)\}} \int_0^\infty e^{-\rho t} f\left(x(t,z), u(t,z), X^e(t,z)\right) dt, \ \forall \ z \in \mathcal{Z} \tag{26}$$

$$\text{s.t. } \frac{\partial x}{\partial t} = g\left(x(z,t), u(z,t), X^e(t,z)\right) + \mu X^e(t,z), \ x(0,z) = x_0(z) \tag{27}$$

This is a standard optimal control problem with current value Hamiltonian function $h = f(x,u,X^e) + p\left[g(x,u,X^e) + \mu X^e\right]$. Setting $X(t,z) = X^e$ in the optimality conditions of problem (26)-(27), a rational expectation equilibrium (REE)

[9]These conditions are based on generalizations of the implicit function theorem to Banach spaces (e.g. Dieudonne (1969, Vol I, Chapter X). The development however of more theory to guide the searching for a locally stable HOSS (if it exists), when the FOSS is unstable, is something we must allocate to future research.

is characterized by the MHDS system of integrodifferential equations

$$\frac{\partial x(z,t)}{\partial t} = g(x, u^*, \mathbf{K}x) + \mu \mathbf{K}x = h_p(x, p, X) \tag{28}$$
$$\frac{\partial p(z,t)}{\partial t} = \rho p - (f_x + p g_x) = \rho p - h_x(x, p, X) \tag{29}$$

where u^* maximizes the current value Hamiltonian h. The discussion in the previous section suggests that we can study spillover induced instability of the RESS. Following the theory developed in the previous section, the linearization of the MHDS of (28)-(29) at the RESS is:

$$\frac{\partial x}{\partial t} = \left(h^*_{px} + h^*_{pX}\mathbf{K}\right) x + h^*_{pp} p \tag{30}$$
$$\frac{\partial p}{\partial t} = \left(-h^*_{xx} - h^*_{xX}\mathbf{K}\right) x + \left(\rho - h^*_{xp}\right) p \tag{31}$$

By comparing the MHDSs for the SOMP and the POMP it is clear that the conditions for the destabilization of the FOSS or the RESS due to spatial spillovers are not the same. The following proposition can be stated:

Proposition 4 (Spillover induced instability for the RESS). *Assume that a symmetric square integrable kernel function $w(\zeta)$ and wave numbers $k \in (k_1, k_2)$ exist such that*

$$\varphi(W(k)) = \rho + h^*_{pX} W(k) > 0 \tag{32}$$
$$\xi(W(k)) = \left[h^*_{pX}\left(\rho - h^*_{px}\right) + h^*_{pp} h^*_{xX}\right] W(k) + h^*_{px}\rho - \left[h^*_{px}\right]^2 + h^*_{pp} h^*_{xx} > 0 \tag{33}$$

Then both eigenvalues of the Jacobian matrix of system (30)-(31), which characterize temporal growth, are positive and the RESS associated with problem (26)-(27) is not stable to spatially heterogeneous perturbations which are induced by spatial spillovers.

The proof can be obtained by following the proof of Proposition 3. $\varphi(W(k))$ is the trace and $\xi(W(k))$ is the determinant of the Jacobian matrix of system (30)-(31) for a mode k. By comparing the dispersion relationship for the social planner's problem (22) with the dispersion relationship (33) which relates to the

REE, it is clear that the potential emergence of agglomerations follows different routes. It is possible that spillover induced spatial instability is emerging for one problem but not the other, or that the emergence of clusters corresponds to different sets of parameters. Furthermore, the clustering pattern at which the system could eventually settle in the long run will be determined by the solution of the system

$$0 = H_p\left(x, p, \mathbf{K}x; \boldsymbol{\omega}\right) \tag{34}$$

$$0 = \rho p - H_x\left(x, p, \mathbf{K}x; \boldsymbol{\omega}\right) \tag{35}$$

The steady state spatial distribution resulting from (34)-(35), provided it exists, will be in general different from the distribution associated with the social planner's problem. Thus the use of the methods developed here might be useful not only in studying the emergence of agglomerations but also the deviations in the spatial patterns between socially optimal and market equilibrium outcomes, as well as the structure of spatially dependent regulation. We use the above theoretical framework to study some problems from growth theory.

3. Geographical Spillovers, Growth and Pattern Formation

3.1. A Spatial Solow Model

For the spatial Solow model (3), assume $(f_x, f_X) > 0$, $(f_{xx}, f_{XX}) < 0$, $f_{xX} > 0$. Since both $x(t,z), X(t,z)$ are treated as inputs, the quantity $X(t,z)$ will have different interpretations in different contexts. On the one hand, if $X(t,z)$ represents a type of "knowledge" which is produced proportionately to capital usage, then it is natural to assume for the kernel $w(\zeta)$ that it is single peaked with a maximum at $\zeta = 0$, like kernels $w_1(\zeta)$ in Appendix 4 (Figure A4.1). Indeed since knowledge is most likely to diffuse to production at (t,z) more strongly the closer is (t,z') to (t,z), then it seems natural to assume that the maximum of $w(\zeta)$, i.e. ζ^*, is taken at $\zeta^* = 0$. On the other hand, if $X(t,z)$ reflects aggregate benefits of knowledge produced at (t,z') for producers at (t,z) and damages to production at (t,z) by usage of capital at (t,z'), then nonmonotonic shapes of $w(\zeta)$ in ζ, like kernels $w_2(\zeta)$ in Appendix 4 (Figure 1, or Figure A4.3), are plausible.

Let $W(k) = \int_{\zeta\in\mathcal{Z}} w(\zeta)\cos(k\zeta)\,d\zeta$ as shown in Appendix 3. At an FSS $\bar{x}$, the spillovers externality is given by $X(t,z) = \bar{x}\mathbf{K} = \bar{x}\int_{\zeta\in\mathcal{Z}} w(\zeta)\,d\zeta = \bar{x}W(0)$. We assume fixed labor input normalized to unity and zero exogenous technical change so $x(t,z)$ denotes total and per capita capital. In this model the FSS solves

$0 = sf(\bar{x}, \bar{x}W(0)) - \eta\bar{x}$. Stability of the FSS with spatially uniform spillovers requires $s\bar{f}_x - \eta < 0$ where $(^-)$ indicates evaluation of the partial derivative at the FSS.[10] Linearizing (3) around the FSS and, following Appendix 3, using as trial solution for $x(t,z)$ either (126) or (128), results in

$$\dot{x}_k = s\left[\bar{f}_x + \bar{f}_X W(k)\right] x_k - \eta x_k \ , \ k = \frac{2n\pi}{L}, L = 2Z, n = 0, \pm 1, \pm 2, ... \tag{36}$$

This is a sequence of linear ordinary differential equations indexed by k which corresponds to mode n. Mode $n = 0$, $k = 0$ and $W(0)$ correspond to the flat Solow model. Mode n is stable if for some k

$$\lambda(k) = s\left[\bar{f}_x + \bar{f}_X W(k)\right] - \eta < 0 \tag{37}$$

and unstable if $\lambda(k) > 0$. In (37) λ is the eigenvalue for (36), which reflects the temporal growth factor and k is the *wave number* which is associated with the emergence of wave like spatial patterns. Thus, destabilization of a stable FSS due to spatial spillovers requires a positive temporal growth factor. Relationship (37) is the basic dispersion relation, which determines whether spatial patterns might emerge. The formula for $W(k)$ for alternative plausible kernel functions in infinite and finite spatial domains is presented in Appendix 4. To obtain more insights into the possibility of pattern formation in the Solow model we specify the production function to the usual Cobb-Douglas form $Y(t,z) = Ax(t,z)^{\alpha_1} X(t,z)^{\alpha_2}$, where $\alpha_1 + \alpha_2$ can be interpreted as social returns. For decreasing social returns $\alpha_1 + \alpha_2 < 1$, a positive FSS exists and is given by $\bar{x} = \left[\frac{\eta}{sAW(0)^{\alpha_2}}\right]^{\frac{1}{\alpha_1+\alpha_2-1}}$. Then the eigenvalue defined in (37) becomes:

$$\lambda(k) = \eta\left(\alpha_1 + \alpha_2 \frac{W(k)}{W(0)} - 1\right) \tag{38}$$

The FSS, at mode $k = 0$, will be stable if and only if the production function exhibits decreasing social returns or $\alpha_1 + \alpha_2 < 1$. To destabilize a stable FSS the ratio $W(k)/W(0)$ should be positive and greater than one. This ratio depends on the kernel function. For example for the A-2 kernel of Appendix 4 (Figures A4.3 and A4.4), the ratio $\frac{W(1)}{W(0)} = 1.08$. Therefore for (α_1, α_2) satisfying $\alpha_1 + \alpha_2 < 1$ and $\alpha_1 + 1.08\alpha_2 > 1$, the FSS is destabilized at mode $k = 1$.

[10] The well-known Inada conditions guarantee the existence and stability of the FSS.

3.1.1. Steady state agglomerations in the Solow model

Destabilization of an FSS by spatial spillovers is a sign that spatial clusters start emerging. The question is whether this emergence will eventually induce persistent steady state agglomerations or clusters. To answer this question we study conditions for the existence of an HSS. We keep the Cobb-Douglas formulation and we specify $X(t,z) = \exp\left(\int_{-Z}^{Z} w(z-z')\, x(t,z')\, dz'\right)$. Then the HSS $\bar{x}(z)$ must solve

$$0 = sAx(t,z)^{\alpha_1} X(t,z)^{\alpha_2} - \eta x(t,z) \text{ for all } z \in [-Z, Z] \tag{39}$$

Define $(1-\alpha_1)\ln \bar{x}(z) := \phi(z)$, then $\phi(z)$ must solve the linear Fredholm equation of the second kind:

$$\phi(z) - \sigma \int_{-Z}^{Z} w(z-z')\, \phi(z')\, dz' = b,\ \sigma = \frac{\alpha_2}{(1-\alpha_1)}\ ,\ b = \frac{1}{1-\alpha_1} \ln\left(\frac{\eta}{sA}\right) \tag{40}$$

Using the operator notation this equation is written as $(1-\sigma\mathbf{K})\,\phi = b$. The kernel $w(\cdot)$ is symmetric and square integrable so that the operator $\mathbf{K}$ is compact. Then the solution of (40), which can be obtained in the form of an infinite Neumann series (Porter and Stirling (1990)), will be a unique square integrable function $\phi(z)$, if $\sigma \|\mathbf{K}\|_2 < 1$, where $\|\mathbf{K}\|_2 \le \left[\int_{-Z}^{Z}\int_{-Z}^{Z} |w(z,z')|^2\, dzdz'\right]^{1/2} = \bar{N}$, and $\bar{N}$ is the upper bound of the norm $\|\mathbf{K}\|_2$ of the kernel $w(\cdot)$.[11] The following proposition can then be stated.

Proposition 5. *For the Solow growth model specified by (39), assume that* $\|\mathbf{K}\|_2 < 1$*, or* $\sigma < 1/\bar{N}$*, then under decreasing social returns spatial spillovers will not result in a steady state agglomeration. The steady state will be flat and no HSS exists.*

Proof. *The assumptions imply that the solution to* $(1-\sigma\mathbf{K})\,\phi = b$ *is unique. Since both the FSS,* $\bar{x}$*, as well as the HSS* $\bar{x}(z)$*, solve (39), and the FSS exists under decreasing social returns, both solutions should coincide. Thus a persistent economic agglomeration does not emerge.* ■

This result raises the question of whether an HSS exists for this specification of the Solow model. To study this problem we move away from decreasing social returns and study a model which at the flat state takes the AK form

[11] Note that if $\sigma < 1/\bar{N}$ the uniqueness condition is satisfied.

and generates endogenous growth (e.g. Barro and Sala-i-Martin (2004, p. 63)). At a flat earth state the production function used in (39) can be written as $Ax^{\alpha_1}(\exp(W(0)\ln x))^{\alpha_2}$ or $\ln A + \alpha_1 \ln x + \alpha_2 W(0)\ln x$. Assume that $\alpha_1 + \alpha_2 W(0) = 1$ then the production function is of the AK form and the corresponding flat Solow model is $\dot{x} = sAx - \eta x$. We know that if $sA > \eta$ this model does not have an FSS but implies positive long-run per capita growth. To look for an HSS we use the fact that the HSS must solve $(1 - \sigma \mathbf{K})\phi = b$. Then the following proposition follows.

Proposition 6. *Assume that the Solow growth model specified by (39) is characterized at a 'flat-earth state' by an AK production function which generates endogenous growth. If $\sigma \|\mathbf{K}\|_2 < 1$, then this Solow growth model has a unique HSS.*

This result implies that while the growth of capital per capita will be positive, the stock of capital will not be the same across space but that there will be geographical clusters with different rates of growth. This spatial growth pattern will be persistent with the passage of time. Since $\alpha_1 = 1 - \alpha_2 W(0)$, $\sigma = \frac{\alpha_2}{1-\alpha_1} = \frac{1}{W(0)}$. Thus the condition for the existence of an HSS becomes $\|\mathbf{K}\|_2 / W(0) < 1$, $\|\mathbf{K}\|_2 \leq \bar{N}$. To have meaningful production elasticities $W(0) < 1$, which implies that the social returns can be defined as $\alpha_1 + \alpha_2 = 1 + \alpha_2(1 - W(0))$. Thus the existence of an HSS requires increasing social returns. Using the Hilbert-Schmidt theorem the unique HSS can be expressed (e.g. Moiseiwitsch (2005, p. 145)) as:

$$\phi(z) = b_1 + \sigma \int_{-Z}^{Z} w(z, z') b_1 dz' + \sigma^2 \sum_{\nu=1}^{\infty} \frac{(b_1, \phi_v)\phi_\nu(z)}{\sigma_\nu(\sigma_\nu - 1)} \tag{41}$$

where $\phi_\nu(z), \sigma_\nu$ $(\nu = 1, 2, ...)$ are the characteristic functions and values of $w(z, z')$, respectively and (b_1, ϕ_ν) defines the inner product.

It should be noted that the above results regarding steady state agglomerations were derived under the assumption that $X = \exp(\mathbf{K}\ln x)$ which allowed us to express the steady state of the Solow model as a linear Fredholm equation of the second kind in logarithms. In the more general case where $X = \mathbf{K}x$ the HSS must solve the nonlinear integral equation $Ax^{\alpha_1 - 1}(\mathbf{K}x)^{\alpha_2} = \eta$, $x = x(z)$. As mentioned in Section 2.1, sufficient conditions for local existence of solutions can be explored in terms of the implicit function theorem in Banach spaces.

3.2. A Spatial Ramsey Model

Keeping the same interpretation for (x, X) we study the Ramsey growth model in the presence of spatial spillovers. We start with the analysis of the rational expectations equilibrium at the POMP.

3.2.1. Rational Expectations Equilibrium

Consider a planner at each site z that takes $X(z,t)$ as parametric and solves the Ramsey problem:

$$\max_{\{c(t,z)\}} \int_0^\infty e^{-\rho t} U(c(z,t))\, dt \tag{42}$$

subject to

$$c(t,z) + \frac{\partial x(t,z)}{\partial t} = f(x(t,z), X^e(t,z)) - \eta x(t,z) \ , \ x(0,z) = x_0(z) \text{ given}$$

This type of planner's problem at each z generates a competitive equilibrium where representative consumers at z rent out their capital at rate $r(t,z)$, and profits $\pi(t,z) = f(x(t,z), X^e(t,z)) - r(t,z)\, x(t,z)$ are distributed lump sum. Consumers maximize discounted sum of utilities, while representative firms hire capital to maximize profits by facing rental rates on capital parametrically. We assume capital is completely immobile. If it were completely mobile, rent $r(t,z)$ would be equated to a common value $r(t)$ across all sites z. Hence we must interpret the "capital" as a type of capital embodied in humans, knowledge or technology which does not move across "countries" z. A richer model would allow mobility of capital but impose some type of "haste makes waste" adjustment costs. Using the results of Section 2.2, the Ramsey type optimality conditions for an REE defined for $X^e(t,z) = X(t,z)$ are:

$$\frac{\partial x(t,z)}{\partial t} = f(x, \mathbf{K}x) - c(p) - \eta x, \ c(p) : U'(c) = p(t,z) \tag{43}$$

$$\frac{\partial p(t,z)}{\partial t} = p\left[\rho + \eta - f_x(x, \mathbf{K}x)\right], \ X = \mathbf{K}x \tag{44}$$

The equilibrium steady state (ESS) $(\bar{x}, \bar{p})$, determined by the REE equilibrium of the POMP, solves

$$c(\bar{p}) = f(\bar{x}, W(0)\bar{x}) - \eta \bar{x} \tag{45}$$

$$\rho + \eta = f_x(\bar{x}, W(0)\bar{x}) \tag{46}$$

The ESS depends on $W(0)$ and, by differentiating (55), we obtain $d\bar{x}/W(0) = -f_{xX}\bar{x}/(f_{xx} + f_{xX}W(0))$. Thus $W(0)$ can be thought of as a bifurcation parameter in the analysis of the ESS. Furthermore multiple ESS may exist. Assume that an ESS exists. Its stability properties depend on the eigenvalues of the linearization matrix of system (43)-(44) at the ESS. To identify conditions under which geographical spillovers reflected in the kernel function $w(\zeta)$ might destabilize the ESS so that spatial agglomerations might emerge at the REE, we apply Proposition 4 to obtain:

Proposition 7. *In the POMP model with spatial spillovers we have instability of the REE determined by a particular ESS if there is a mode k such that*

$$\begin{aligned} \varphi(k) &= \rho + \bar{f}_X W(k) > 0 && (47) \\ \xi(k) &= -\bar{p}c'(\bar{p})\left[\bar{f}_{xx} + \bar{f}_{xX}W(k)\right] > 0, \text{ or} && (48) \\ \xi_1(k) &= \bar{f}_{xx} + \bar{f}_{xX}W(k) > 0 && (49) \end{aligned}$$

From the linearization it follows that $\varphi(k)$ is the trace and $\xi(k)$ is the determinant of the linearization matrix of system (43)-(44) at the ESS. If spatial spillovers are positive as in kernels $w_1(\zeta)$ of Appendix 4, then agglomerations emerge at the REE if $\xi(k) > 0$.[12] On the other hand, if nearby spatial externalities are positive but farther away externalities are negative as in kernels $w_2(\zeta)$ of Appendix 4, then $W(k)$ could be negative for some k, which suggests a wide range of possible results for the simple POMP model. A detailed analysis of the possible bifurcations, possible ESSs and local stability (instability) is outside the purpose of this paper, but it can be obtained by straightforward application of our theoretical framework.

To study a potential spatially heterogeneous RESS (HRESS), it is clear from (44) and the definition of the RESS that an HRESS for the capital stock must solve the nonlinear integral equation $\rho + \eta = f_x(x, \mathbf{K}x)$. Using the Cobb-Douglas specification with $X = \exp(\mathbf{K}x)$ as in the spatial Solow model, the HRESS is the solution of $(1 - \sigma\mathbf{K})\phi = b$, $b = \frac{1}{1-\alpha_1}\ln\left(\frac{\rho+\eta}{\alpha_1}\right)$. Results similar to the Solow model hold. With decreasing social returns and $\|\mathbf{K}\|_2 < 1$, only a flat RESS exists. With an AK production function at the flat Ramsey model, increasing social returns, and $\sigma\|\mathbf{K}\|_2 < 1$, a unique HRESS, $\bar{x}(z)$, exists and persistent agglomerations for capital emerge at the REE. The corresponding consumption clusters can be determined by (43) at the HRESS. For more general specifications

[12] In this case $\varphi(k)$ is always positive.

of the production function, numerical approximations are required for the study of the HRESS.

3.2.2. The Social Optimum

The social planner, assuming that capital $x(t,z)$ is immobile in the sense described above and that consumption goods $c(t,z)$ are produced on site, solves:

$$\max_{\{c(t,z)\}} \int_0^\infty e^{-\rho t} \left(\int_{z\in\mathcal{Z}} U(c(z,t))\, dz \right) dt \tag{50}$$

subject to

$$c(t,z) + \frac{\partial x(t,z)}{\partial t} = f(x(t,z), X(t,z)) - \eta x(t,z) \ \text{, for all } z$$

Using Proposition 1, the Ramsey type optimality conditions for the social optimum (the SOMP) are:

$$\begin{aligned}
\frac{\partial x(t,z)}{\partial t} &= f(x, \mathbf{K}x) - c(p) - \eta x,\ c(p) : U'(c) = p(t,z) && (51)\\
\frac{\partial p(t,z)}{\partial t} &= p[\rho + \eta - f_x(x, \mathbf{K}x)] - \mathbf{K}p f_X \ , && (52)\\
\mathbf{K}p f_X &= \int_z w(z'-z)\, p(t,z')\, f_X(x(t,z'), X(t,z'))\, dz' && (53)
\end{aligned}$$

The flat optimal steady state (FOSS) $(\bar{x}, \bar{p})$ will solve:

$$\begin{aligned}
c(\bar{p}) &= f(\bar{x}, W(0)\bar{x}) - \eta\bar{x}, && (54)\\
\rho + \eta &= f_x(\bar{x}, W(0)\bar{x}) + W(0) f_X(\bar{x}, W(0)\bar{x}) && (55)
\end{aligned}$$

Assume that a FOSS defined by (54)-(55) exists, with the saddle point property, as explained in Appendix 3 in terms of (130)-(131). Using Proposition 3 we obtain sufficient conditions for geographical spillovers to destabilize the FOSS so that spatial agglomerations might emerge at the social optimum.

Proposition 8. *In the SOMP model with spatial spillovers we have instability of the social optimum determined by a FOSS with the local saddle point property, if there is a mode k such that*

$$\begin{aligned}
&\psi(k) = \left(\rho + \eta - \bar{f}_x - \bar{f}_X W(k)\right)\left(\bar{f}_x + \bar{f}_X W(k) - \eta\right) - \\
&\bar{p}c'(\bar{p}) \left[\bar{f}_{xx} + 2\bar{f}_{xX} W(k) + \bar{f}_{XX} W(k)^2\right] > 0
\end{aligned} \tag{56}$$

It follows from Appendix 3 that $\psi(k)$ is the determinant of the linearization, at the FOSS, of the system (51)-(52). The instability means that both eigenvalues of this matrix are positive under the spatial spillovers and clusters might appear at the SOMP.

It might be interesting to compare the ESS and the FOSS, with respect to their size and likelihood of becoming unstable due to spatial spillovers.

Let $(\bar{x}_E, \bar{p}_E), (\bar{x}_S, \bar{p}_S)$ denote the ESS and the FOSS respectively, and assume that the production function is Cobb-Douglas with decreasing social returns and $X = \mathbf{K}x$. Then it can be easily shown using (44) and (52) that

$$\bar{x}_E = \left(\frac{\rho+\eta}{\alpha_1 AW(0)^{\alpha_2}}\right)^{\beta} < \bar{x}_S = \left(\frac{\rho+\eta}{(\alpha_1+\alpha_2)AW(0)^{\alpha_2}}\right)^{\beta}, \beta = \frac{1}{\alpha_1+\alpha_2-1} \quad (57)$$

To compare instability tendencies we compare (47)-(48) with (56). Write (56) as $\psi(k) = T_1(k) - T_2(k)$ and assume that the function $g(x,k) := f(x, xW(k))$ is concave in x for each k. Define $T_3(k) = \bar{f}_{xx} + 2\bar{f}_{xX}W(k) + \bar{f}_{XX}W(k)^2 < 0$ then, $T_2(k) = \bar{p}c'(\bar{p})T_3(k) > 0$. At a flat steady state $\rho + \eta = \bar{f}_x - \bar{f}_X W(0)$, thus $T_1(k) = \bar{f}_X(W(0) - W(k))\left(\bar{f}_x + \bar{f}_X W(k) - \eta\right)$. For the emergence of clusters at the FOSS we need $T_1(k) > 0$ and $T_1(k) > |T_2(k)|$. On the other hand the emergence of clusters at the ESS requires that there be a mode k such that $W(k) > 0$ and $\bar{f}_{xx} + \bar{f}_{xX}W(k) > 0$. Possible $W(k)$ functions are presented in Appendix 4. Numerical simulations presented below suggest that the RESS is more likely to become unstable under spatial spillovers relative to the FOSS.

The deviations between ESS and FOSS, both in terms of size and stability properties, raises the issue of regulation so that the regulated REE will replicate the SOMP solution. Since the deviation is caused by the spatial externality, regulation should take the form of a subsidy on the cost of capital to reflect the unaccounted, at the REE, social returns due to geographical spillovers. Let $s(z,t)$ the subsidy per unit of capital. The firm's problem becomes $\pi(t,z) = f(x(t,z), X^e(t,z)) - r(t,z)x(t,z) + s(t,z)x(t,z)$, with $f_x = r - s$ in profit maximizing equilibrium. Subsidies are paid by consumers in a lump-sum form. Then the national income identity in each site becomes $\partial x/\partial t + \eta x + c = rx + w - sx$. Competition in each site and constant returns to scale imply that $w = f(x, X^e) - f_x x = f - (r-s)x$. Substituting w in the national income identity we obtain under REE where, $X = X^e$,

$$\frac{\partial x(t,z)}{\partial t} = f(x(t,z), X(t,z)) - c(t,z) - \eta x(t,z) \quad (58)$$

The representative consumer lifetime utility maximization in each site implies $\partial p/\partial t = (\rho + \eta - r)\, p$, or $\partial p/\partial t = (\rho + \eta - f_x(x, X) - s)\, p$, where $p = p(t, z) = U'(c(t, z))$, which implies $c = c(p(t, z))$. If we set the subsidy at a given site equal to the value of the marginal spatial externality in terms of marginal utility at this site, or

$$s(t, z) = \frac{1}{p(t, z)} \int_{z \in \mathcal{Z}} w(z - z')\, p(t, z')\, f_X(x(t, z'), X(t, z'))\, dz' = \frac{1}{p(t, z)} \mathbf{K} p f_X \tag{59}$$

then

$$\frac{\partial p(t, z)}{\partial t} = (\rho + \eta - f_x(x(t, z), X(t, z))) - \mathbf{K} p f_X \tag{60}$$

However, with $c = c(p(t, z))$, the dynamical system (58), (60) is the same as the dynamical system (51)-(52) which determines the SOMP. Thus the optimal spatial subsidy is $s(t, z) = \frac{1}{p(t,z)} \mathbf{K} p f_X$. If the SOMP corresponds to a spatially uniform steady state $(\bar{x}_S, \bar{p}_S)$, then the steady state subsidy will be $\bar{s}(z) = W(0)\, f_X(\bar{x}_S, \bar{x}_S W(0))$ for all z.

To obtain a clearer picture of the above results we present a numerical example. We use a Cobb-Douglas production function and assume $\alpha_1 = 0.4$, $\alpha_2 = 0.2$, $\rho = 0.03$, $\eta = 0.04$. We assume that the kernel is of the form KQE-2 in Appendix 4, or $w(\zeta) = b_1 \exp\left[-(\zeta/d_1)^2\right] - b_2 \exp\left[-(\zeta/d_2)^2\right]$, with $b_1 = 1, d_1 = 0.75, b_2 = 0.7, d_2 = 1$. The functions $w(\zeta)$ and $W(k)$ are shown in Figures 1 and 2 below.

[Figures 1 and 2]

From (57) we obtain $\bar{x}_E = 34.16$, $\bar{x}_S = 94.14$. To study the stability of these steady states we use (48) and (56). The graph of $\xi_1(k)$ is shown in Figure 3.

[Figure 3]

It is clear that for modes in the neighborhood of $k = 2$, the RESS will became unstable under the influence of spatial spillovers and economic agglomerations will start emerging. To study the stability properties of the FOSS of the SOMP, we present the function $\psi(k)$ in Figure 4.

[Figure 4]

Since this function is always negative, the FOSS of the SOMP is *not* destabilized by spatial spillovers. The optimal steady state of the social planner's problem is spatially homogeneous. For this problem the optimal steady state subsidy per unit of capital is spatially homogeneous with $\bar{s} = 0.02$. Under this subsidy the REE will reproduce the FOSS of the SOMP.

3.3. Spatial Agglomeration Dynamics of Knowledge Accumulation

We study the REE of problem (7)-(8). The optimality conditions for each producer, where $p(t,z)$ is the costate variable, are given by

$$\frac{\partial p(t,z)}{\partial t} = \rho p(t,z) - X(t,z) \tag{61}$$

$$\frac{\partial x(t,z)}{\partial t} = \frac{p(t,z)}{\gamma} \tag{62}$$

As we show in Appendix 3, if the trial solution $(x,p)\exp(\lambda t + ikz)$ is inserted into (61) and (62) we obtain equations (63) and (64) for the eigenvalue characterizing temporal growth and its corresponding eigenvector.

$$\lambda p = \rho p - W(k)\,x\ , \tag{63}$$

$$\lambda x = \frac{p}{\gamma} \tag{64}$$

where $W(k) = \int_{-Z}^{Z} e^{ik\zeta} w(\zeta)\,d\zeta = \int_{-Z}^{Z} \cos(k\zeta) w(\zeta)\,d\zeta$ and possible functional forms for $W(k)$ are shown in Appendix 4. The requirement that the determinant of the relevant matrix be zero in order for a nontrivial eigenvector to exist yields the eigenvalue equation

$$\lambda^2 - \lambda\rho - \frac{W(k)}{\gamma} = 0 \tag{65}$$

Note that if one graphs the parabola (65) with $W(0) < 0$, it immediately evident that there is one negative root and one positive root for $k = 0$, which is the usual saddle point result of the FSS of optimal control. It is also evident that as soon as $W(k)$ becomes positive as k increases, then there are two positive roots. I.e. the system has lost stability. The system of (61) and (62) using the trial solutions can be written as the sequence of ordinary differential equations:

$$\dot{p}_k = p_k - W(k)\,x_k \tag{66}$$

$$\dot{x}_k = \frac{1}{\gamma} p_k \tag{67}$$

Assume that for a given kernel, $\lambda^M(k_M)$ is the maximum eigenvalue corresponding to mode k_M. The spatiotemporal evolution of (x,p) will be dominated by this eigenvalue. At a balanced growth path $\frac{\dot{p}_k}{p_k} = \frac{\dot{x}_k}{x_k} = g_k$, and the ratio $\frac{p_k}{x_k}$ is constant through time. Then (66)-(67) imply that g_k is the solution of $g_k^2 - \rho g_k - \frac{W(k)}{\gamma} =$

0, which is the eigenvalue equation (65). Let $g_k^M(k_M)$ be the maximum positive solution for g_k at mode k_M which is less than $\rho/2$ so that the integral (7) is less than infinity at the REE equilibrium. Since this growth rate is also the positive eigenvalue $\lambda^M(k_M)$, the temporal and spatial evolution of (x, p) will be determined by this eigenvalue. Then the agglomeration dynamics of knowledge accumulation along the balanced growth path will be proportional to $\exp\left(\lambda^M(k_M)\,t\right)[\alpha\cos(k_M z) + \beta\sin(k_M z)]$, where α, β are constants to be determined by the eigenspace of $\lambda^M(k_M)$. If we use the same kernel as in the previous section, $\rho = 0.03$ and $\gamma = 1$, the maximum feasible balanced growth rate (and eigenvalue) is $g_k = 0.012$ for $k = 7.9$. The approximate agglomeration dynamics of knowledge accumulation are shown in Figure 5. Knowledge grows along the balanced growth path as expected by the AK structure of the production function.

[Figure 5]

3.4. Agglomeration Dynamics and R&D Based Growth

Building on the previous section we consider a model where knowledge accumulation and overall productivity in a location (or country) depends on the resources devoted to the development of new knowledge locally and the knowledge accumulated in neighboring countries. This approach allows us to bring together knowledge or R&D based growth models,[13] which allow for knowledge generation by using scarce resources, with spatial models which incorporate geographical spillovers.

Consider a one-dimensional spatial domain as described above and assume that new knowledge at time t and location $z \in Z$ is produced by scarce labor $L_x(t, z)$, which is used in knowledge generation, and the existing stock of knowledge $x(t, z)$. Assume that the influence of neighboring locations on knowledge accumulated in location z is given by the kernel formulation $\mu \int_{z \in Z} w(z - z')\, x(t, z')\, dz' = \mu \mathbf{K} x = \mu X$, where $w(z - z')$ is a symmetric kernel function which characterizes the influence of knowledge accumulated in neighboring locations on local knowledge accumulation, and μ reflects the overall effectiveness of geographical knowledge spillovers.[14] Following Jones (1995) we assume that new knowledge generated at

[13]See for example Romer (1990), Grossman and Helpman (1991a, 1991b), Aghion and Howitt (1992), and Jones (1995).

[14]$\mu > 0$ is the usual case of positive knowledge spillovers. $\mu < 0$ can be regarded as describing 'knowledge drainage' or 'knowledge absorbtion' by neighbors.

time t and location z is given by $\delta L_x(t,z)(x(t,z))^{\phi}$, $\delta > 0, .0 < \phi < 1$. Then the accumulation of knowledge can be described by

$$\frac{\partial x(t,z)}{\partial t} = \delta L_x x^{\phi} - mx + \mu \mathbf{K} x \ , \ m > 0, \ x(0,z) \text{ given} \tag{68}$$

where $m > 0$ is a depreciation term reflecting knowledge or technologies that become obsolete. Thus equation (68) describes knowledge (or technology) accumulation by combining the dynamic law of knowledge accumulation developed in R&D based growth models, with geographical knowledge spillovers.

To specify the growth model assume that aggregate labor input at each spatial point is fixed $L(z)$ through time t and immobile. labor is allocated to output production L_Y and knowledge generation L_x as: $L(z) = L_Y(t,z) + L_x(t,z)$, for all t. Let output $Y(t,z)$ at location z be produced by labor $L_Y(t,z)$ allocated to output production and knowledge $x(t,z)$. In order to keep the formulation relatively simple so that our main points become clear, we do not introduce physical capital at this stage, thus the production function can be written as $Y = xL_Y^{\alpha}$, $0 < \alpha < 1$.

Consider the problem of a social planner seeking to allocate at each point in time t the fixed amount of labor existing at a given location z between output production and creation of new knowledge, in order to maximize discounted utility over the given spatial domain Z. Assuming a logarithmic utility function, the social planner's problem is:

$$\max_{\{L_x(t,z)\}} \int_{z \in Z} \int_0^{\infty} e^{-\rho t} \ln(c(t,z))\, dt dz \ , \ c(t,z) = x(t,z)(L(z) - L_x(t,z))^{\alpha}$$
$$\text{subject to (68)}$$

Assuming that a solution to this problem exists, Proposition 1 implies the following optimality conditions:

$$L_x = L - \frac{\alpha}{\delta p x^{\phi}} \tag{69}$$

$$\frac{\partial x(t,z)}{\partial t} = \delta L x^{\phi} - mx - \frac{\alpha}{p} + \mu \mathbf{K} x \tag{70}$$

$$\frac{\partial p(t,z)}{\partial t} = \left(\rho + m - \phi \delta L x^{\phi - 1}\right) p - \frac{(1 - \phi\alpha)}{x} - \mu \mathbf{K} p \tag{71}$$

The FOSS (x^*, p^*)[15] is characterized by $x^*p^* = 1/\left[\rho + (1-\phi)(m - \mu W(0))\right]$, while the optimal labor allocation is $L_x^* = L - \frac{\alpha[\rho-(\phi-1)m]}{\delta(x^*)^{\phi-1}}$. Assume that the FOSS has the saddle point property. Applying Proposition 3, destabilization of the FOSS due to spatial spillovers occurs if a kernel function $w(\zeta)$ and wave numbers $k \in (k_1, k_2)$ exist such that the dispersion relationship becomes positive, or $\psi(k) > 0$, $\psi(k) = \det J$.

$$J = \begin{pmatrix} \delta\phi L x^{\phi-1} - m + \mu W(k) & \frac{\alpha}{p^2} \\ \frac{(1-\phi\alpha)}{x^2} & \rho - \delta\phi L x^{\phi-1} + m - \mu W(k) \end{pmatrix} \tag{72}$$

where J is evaluated at (x^*, p^*).[16] At an REE the agent at each site considers $X = \mu \mathbf{K} x$ as parametric and solves at each z

$$\max_{\{L_x(t)\}} \int_0^\infty e^{-\rho t} \ln(c(t))\, dt \ , \ c(t) = x(t)(L - L_x(t))^\alpha \tag{73}$$

$$\text{s. t. } \dot{x} = \delta L_x x^\phi - mx + X^e \tag{74}$$

Assuming $X^e = X = \mu \mathbf{K} x$ for an REE, the optimality conditions include (69) along with

$$\frac{\partial x(t,z)}{\partial t} = \delta L x^\phi - mx - \frac{\alpha}{p} + \mu \mathbf{K} x \tag{75}$$

$$\frac{\partial p(t,z)}{\partial t} = \left(\rho + m - \phi\delta L x^{\phi-1}\right) p - \frac{(1-\phi\alpha)}{x} \tag{76}$$

Then for an RESS $(\bar{x}, \bar{p})$:

$$0 = \delta L \bar{x}^\phi - m\bar{x} - \frac{\alpha}{\bar{p}} + \mu \bar{x} W(0) \tag{77}$$

$$0 = \left(\rho + m - \phi\delta L \bar{x}^{\phi-1}\right) \bar{p} - \frac{(1-\phi\alpha)}{\bar{x}} \tag{78}$$

[15] The FOSS solves

$$\begin{aligned} 0 &= \delta L x^{*\phi} - mx^* - \alpha/p^* + \mu W(0)\, x^* \\ 0 &= \left(\rho + m - \phi\delta L x^{*\phi-1}\right) p^* - (1-\phi\alpha)/x^* - \mu W(0)\, p^* \end{aligned}$$

[16] Saddle point stability of the FOSS requires that $\psi(0) < 0$.

Using Proposition 4, instability of the RESS due to spatial spillovers requires that both the trace and the determinant $\psi^R(k) = \det J^R$ of the linearization matrix of (75)-(76) evaluated at the ESS $(\bar{x}, \bar{p})$ be positive. The J^R matrix is:

$$J^R = \begin{pmatrix} \delta\phi L\bar{x}^{\phi-1} - m + \mu W(k) & \frac{\alpha}{\bar{p}^2} \\ \frac{(1-\phi\alpha)}{\bar{x}^2} & \rho + m - \phi\delta L\bar{x}^{\phi-1} \end{pmatrix} \tag{79}$$

To obtain more insight into the instability result we construct a numerical example. Using the kernel of the previous examples (see Figure 1), we set $\phi = 0.5$, $a = 0.7$, $m = 0.01$, $\delta = 1$, $\rho = 0.03$, $\mu = 0.1$, $L = 1$, and spatial length 2π. The FOSS is $(x^*, p^*) = (1969.27, 0.0166)$ with optimal labor allocation to R&D, $L_x^* = 0.05$, while the RESS is $(\bar{x}, \bar{p}) = (1210.77, 0.021)$ with $\bar{L}_x = 0.04$. The FOSS and the RESS both have the saddle point property with eigenvalues $(0.0389, -0.0089)$ and $(0.0502, -0.0113)$ respectively. Figure 6 shows the dispersion relationship $\psi(k)$ for the SOMP.

[Figure 6]

Since the dispersion relationship remains negative, the FOSS is stable to spatial spillovers. Figure 7 shows the dispersion relationship $\psi^R(k)$ for the POMP.

[Figure 7]

This relationship becomes positive for a finite range of modes, which means that for these modes the ESS of the REE is destabilized by spatial spillovers. For $k = 2$, the eigenvalues of the linearization matrix (79) are $(0.0659, 0.0031)$, which implies that the spatial perturbation for the mode corresponding to $k = 2$ grows with the passage of time and knowledge clusters are emerging. To study the steady state REE knowledge agglomeration, we need to solve the nonlinear integral equation system

$$0 = \delta Lx(z)^{\phi} - mx(z) - \frac{\alpha}{p(z)} + \mu\mathbf{K}x(z) \tag{80}$$

$$0 = \left(\rho + m - \phi\delta Lx(z)^{\phi-1}\right)p(z) - \frac{(1-\phi\alpha)}{x(z)} \tag{81}$$

A search for a local numerical approximation can be conducted by choosing a net of n equal sub-intervals with length $\delta_n = 2Z/n$ given by $-Z = z_0 < z_1 < \dots <$

$z_r < ... < z_n = Z$ with $z_r = -Z + r\delta_n$. Approximating the Riemann integral in $\mathbf{K}x$ by a finite sum as[17]

$$\int_{-Z}^{Z} w(z-z')\,x(z')\,dz' \simeq \delta_n \sum_{l=1}^{n} w(z_r - z_l')\,x(z_l') \ \ , z = z_r$$

the nonlinear system of integral equation can be replaced by a system of $2n$ nonlinear algebraic equations

$$\delta Lx(z_r)^{\phi} - mx(z_r) - \frac{\alpha}{p(z_t)}\mu\delta_n \sum_{l=1}^{n} w(z_r - z_l')\,x(z_l') = 0 \tag{82}$$

$$\left(\rho + m - \phi\delta Lx(z_r)^{\phi-1}\right)p(z_r) - \frac{(1-\phi\alpha)}{x(z_r)} = 0 \ , r = 1, ..., n \tag{83}$$

The system is solved in the neighborhood of the RESS. Figure 8 shows the REE steady state knowledge agglomeration, along with the spatially uniform ESS, while Figure 9 shows the spatial pattern of the shadow price for knowledge.

[Figures 8, 9]

The numerical solution suggests that three clusters occur at the steady state, with quantities x_r and prices p_r following, as expected, mirror patterns. Knowledge clustering implies that consumption distribution in not spatially uniform at the steady state of the REE.

4. Conclusions

This paper developed a fairly general approach to the study of infinite dimensional infinite horizon intertemporal recursive dynamical systems models in continuous spatial settings as well as analytical techniques for local stability analysis of spatially flat optimal steady states and computational techniques for spatially heterogeneous optimal steady states. Our work is related to the stability analysis of infinite dimensional, infinite horizon optimal control problems in Hilbert space settings (Carlson, Haurie, and Leizarowitz (1991, Chapter 9) and Leizarowitz (2008), but we formulated and analyzed models with spillovers represented by

[17] This is based on the method introduced by Fredholm where the integral equation is treated as a limiting form of a finite system of linear algebraic equations.

kernels as in the new economic geography literature, technology spillover models, and elsewhere. We also exploited Fourier basis techniques to organize the local stability analysis around an analytically tractable dispersion relation.

Section 2 developed a quite general approach to the social optimization management problem by posing it as an infinite horizon optimal control problem on a continuous space (e.g. a circle). This section developed the concept of flat optimal steady state where all variables (e.g. capital stock and output) have the same value at all spatial sites but are optimal given the same initial conditions as in the FOSS. Analytical techniques were developed to locate sufficient conditions for the local stability (and local instability) of a FOSS. Techniques were developed to compute a heterogeneous optimal steady state when it exists. A parallel concept of private optimization management problem and a concept of rational expectations equilibrium for the POMP were developed and REEs were compared to solutions to the SOMP. As one would expect, REE solutions to the POMP are the same as solutions to the SOMP when there are no spillovers (i.e. no externalities). But the presence of externalities and their potential for causing agglomerations was the main economic interest. The analysis proceeded by using Fourier analysis to develop an analytically tractable quantity called the dispersion relation which is a function of modes. In the optimization case we showed that local instability occurs if a FOSS occurs when the dispersion relation has two positive roots for some mode whose frequency fits inside the space (if it is a circle with finite circumference). This corresponds to the usual condition for instability of an n-dimensional optimal control system, i.e. there is at least one more unstable root of the linearized modified Hamiltonian dynamical system than there are stable roots (Brock and Malliaris (1989, Chapter 5, page 149)).

In Section 3 of the paper we conducted a detailed investigation of local stability (or instability) of FOSSs and computation of HOSSs for four examples of economic interest. The four examples were: (i) a descriptive Solow type model with spatial spillovers, (ii) a Ramsey type optimization model with spatial spillovers, (iii) a technology clustering model studied by Quah, and (iv) an R&D based optimal growth model with potential clustering.

We report here in more detail on the four examples studied in Section 3. The first example was a spatial differential equation Solow type model with immobile capital and immobile labor at each site but with kernel-type spillover externalities across sites. Hence the capital stock at each site represented a type of knowledge which is produced at each site and spills over to assist production at nearby sites. Our interest was in locating sufficient conditions for agglomerations in this model.

It was used mainly as a warm-up exercise because it had no optimization. But it was rich enough to allow the illustration of the concept of dispersion relation which was extended to optimization for the second, third, and fourth examples.

The second example was an infinite horizon version of the spatial Solow model with kernel-type spillover externalities across sites. It was much more difficult to analyze. The theory developed in Section 2 was used to analyze it. However, when the system was linearized, the Fourier series technique allowed us to construct a countable basis of modes and the equivalent of approximation linear quadratic optimal control problems, one for each mode. While this was an oversimplification of what we actually did, it provides some insight into the procedure that we used. We computed the REE for the POMP as well as the SOMP and exhibited a range of parameter values where the REE was locally unstable and the SOMP was locally stable. We computed a heterogeneous steady state for the REE for the POMP and exhibited a capital subsidy program that implemented the SOMP solution. This exercise illustrated the economic point that in a world of low enough discounting, the SOMP would be stable due to the usual logic behind turnpike theorems (Cass and Shell (1976), McKenzie (1983), Scheinkman (1976)), but the REE for the POMP can easily be unstable. I.e. it is socially optimal not to have agglomerations form, yet the REE for the POMP produced agglomerations.

The third example was a technology clustering model developed by Quah. It has a linear quadratic structure, so the procedure of using a Fourier type basis to decompose the infinite dimensional optimal control problem into a countable sequence of tractable finite dimensional optimal control problems was exact for the SOMP. The POMP was an REE system so care had to be taken in the analysis of local stability of rational expectations equilibria around a flat rational expectations equilibrium steady state. But a Fourier basis type approach was still available to decompose the infinite dimensional equilibrium problem into a countable number of finite dimensional equilibrium problems, one for each mode.

The fourth example was an R&D based growth model. We computed the SOMP and POMP and located a parameter set for which the REE for the POMP was locally unstable but the SOMP was locally stable. We computed the dispersion relation for both the SOMP and POMP. Moreover we computed the heterogeneous REE steady state and showed how its shape was governed by economic parameters.

A basic theme runs through the general theory and the examples. Asymptotic stability analysis in infinite horizon recursive optimal control theory in finite dimensional spaces prompts us to expect that a FOSS is locally asymptotically stable provided that the analog of the underlying systems matrix is stable or the discount rate is smaller than the product of local Hamiltonian curvatures in the costate, and the state is smaller than the square of the discount rate divided by four.[18] In the case of diffusion driven dynamics, Brock and Xepapadeas (2008a,b) used a Fourier basis technique to decompose the analysis of local stability of a FOSS for the original infinite dimensional optimal control problem into an analysis of local stability of a countable set of easily analyzed finite dimensional linear quadratic optimal control problems, one for each mode. If any of these finite dimensional modal problems has an unstable optimal solution, then the FOSS is locally unstable. One might view the current paper as an extension of this work to the case of kernel driven dynamics.

What about future research? We think the top priority for future research is to extend the general forward looking infinite dimensional, infinite horizon optimization approach developed here to new economic geography (NEG) models, to structural change models, and to the general study of symmetry breaking in economics (Matsuyama (2008a,b)). I.e. we need to enrich the models studied here to include endogenous product variety at each site, increasing returns to production of each variety at each site, imperfect competition amongst varieties, backward/forward linkages, costly movement of resources, and other ingredients that expose the role of depth of increasing returns, elasticity of substitution amongst varieties, costliness of moving resources, etc. Work on equilibrium analysis in infinite horizon forward looking NEG models has already been done for NEG models with a finite number of sites (e.g. two sites as in Baldwin (2001), Ottaviano (2001)). Excellent reviews of this work and related work, including history versus expectations in NEG models, can be found in Matsuyama (2008a,b). We view our paper as a contribution to the set of analytical techniques useful for analyzing models in this area. A serious policy analysis must deal with not only analytical issues and theory but also measurement issues (see, for example, Martin and Sunley (2003)).

Appendix 1

Proof of Proposition 1

The problem is

[18] See Brock and Malliaris (1989, Chapter 5), Carlson, Haurie, and Leizarowitz (1991) for a review and discussion of the literature.

$$\max_{\{u(t,z)\}} \int_{z\in Z}\int_0^\infty e^{-\rho t} f\left(x,u,X\right) dtdz \ , \ \ X = \mathbf{K}x \tag{84}$$

$$\text{subject to } \frac{\partial x\left(z,t\right)}{\partial t} = g\left(x,u,X\right) + \mu X \tag{85}$$

To develop a version of the maximum principle for this problem we use a variational argument along the lines of Kamien and Schwartz (1981, pp. 115-116). Problem (84) - (85) can be written as:

$$J = \int_{z\in \mathcal{Z}}\int_0^\infty e^{-\rho t} f\left(x\left(t,z\right), u\left(t,z\right), X\left(t,z\right)\right) dtdz = \tag{86}$$

$$\int_{z\in \mathcal{Z}}\int_0^\infty e^{-\rho t} \left\{ f\left(x\left(t,z\right), u\left(t,z\right), X\left(t,z\right)\right) + \right.$$

$$\left. p\left(t,z\right)\left[g\left(x\left(t,z\right), u\left(t,z\right), X\left(t,z\right)\right) + \mu X\left(t,z\right) - \frac{\partial x}{\partial t}\right]\right\} dtdz \tag{87}$$

Integrating by parts the term $e^{-\rho t} p\left(t,z\right) \frac{\partial x}{\partial t}$ of (87) we obtain:

$$(-1)\int_{z\in \mathcal{Z}}\int_0^\infty e^{-\rho t} p\left(t,z\right) \frac{\partial x}{\partial t} dt =$$

$$-\int_{z\in \mathcal{Z}}\left[\left[e^{-\rho t} p\left(t,z\right) x\left(t,z\right)\right]_0^T + \int_0^\infty x\left(t,z\right)\frac{\partial\left(e^{-\rho t}p\right)}{\partial t} dt\right] dz = \tag{88}$$

$$-\int_{z\in \mathcal{Z}}\left[\left[e^{-\rho t} p\left(t,z\right) x\left(t,z\right)\right]_0^T + \int_0^\infty e^{-\rho t} x\left(t,z\right)\left(-\rho p + \frac{\partial p}{\partial t}\right)\right] dz \tag{89}$$

where $t = 0, \ t = T$ and $T \to \infty$ in the second line of the right hand side.

Assuming the following limiting intertemporal transversality condition holds

$$\lim_{T\to\infty} e^{-\rho T}\int_{z\in \mathcal{Z}} p\left(T,z\right) x\left(T,z\right) dz = 0 \tag{90}$$

then in (89) the first term in T goes to zero as $T \to \infty$ by the intertemporal transversality condition. The initial term at $t = 0$ that is left does not impact the expression where control appears. Thus (87) becomes

$$\begin{aligned}
&\int_{z\in\mathcal{Z}}\int_0^\infty e^{-\rho t} f\left(x\left(t,z\right),u\left(t,z\right),X\left(t,z\right)\right)dtdz = \\
&\int_{z\in\mathcal{Z}}\int_0^\infty e^{-\rho t}\left[f\left(x\left(t,z\right),u\left(t,z\right),X\left(t,z\right)\right)+p\left(t,z\right)g\left(x\left(t,z\right),u\left(t,z\right),X\left(t,z\right)\right)\right. \\
&\left.+x\left(t,z\right)\left(-\rho p+\frac{\partial p}{\partial t}\right)+p\left(t,z\right)\mu X\left(t,z\right)\right]dtdz \\
&-\int_{z\in\mathcal{Z}}\left[e^{-\rho t}p\left(t,z\right)x\left(t,z\right)\right]_0^T dz \qquad (91)
\end{aligned}$$

where $t=0,\ t=T$ and $T\rightarrow\infty$ in the third line of the right hand side.

We use the following variational argument. Consider a one parameter family of comparison controls, $u^*\left(t,z\right)+\epsilon\eta\left(t,z\right)$, where $u^*\left(t,z\right)$ is the optimal control, $\eta\left(t,z\right)$ is a fixed function and ϵ is a small parameter. Let $y\left(t,z,\epsilon\right)$, $t\in[0,\infty)$, $z\in[-Z,Z]$ be the state variable generated by (85) and circle spatial boundary conditions with control $u^*\left(t,z\right)+\epsilon\eta\left(t,z\right)$, $t\in[0,\infty)$, $z\in[-Z,Z]$. We assume that $y\left(t,z,\epsilon\right)$ is a smooth function of all its arguments and that ϵ enters parametrically. For $\epsilon=0$, we have the optimal path $x^*\left(t,z\right)$. Furthermore all comparison paths must satisfy initial conditions. Thus,

$$\begin{aligned}
y\left(t,z,0\right) &= x^*\left(t,z\right),\ y\left(0,z,\epsilon\right)=x\left(0,z\right)\text{ fixed} \\
y\left(t,-Z,0\right) &= y\left(t,Z,0\right)
\end{aligned}$$

Let

$$Y\left(t,z,\epsilon\right)=\int_{z\in\mathcal{Z}} w\left(z-z'\right)y\left(z',t,\epsilon\right)dz'=\mathbf{K}y\left(\epsilon\right)$$

When the functions u^*, x^* and η are held fixed, the value of (86) evaluated along the control function $u^*\left(t,z\right)+\epsilon\eta\left(t,z\right)$ and the corresponding state function $y\left(t,z,\epsilon\right)$ depends only on the single parameter ϵ. Therefore,

$$J\left(\epsilon\right)=\int_{z\in\mathcal{Z}}\int_0^\infty e^{-\rho t}\left[f\left(y\left(t,z,\epsilon\right),u^*\left(t,z\right)+\epsilon\eta\left(t,z\right),Y\left(t,z,\epsilon\right)\right)\right]dtdz$$

or using (91)

$$
\begin{aligned}
&J(\epsilon) = \\
&\int_{z\in\mathcal{Z}}\int_0^\infty e^{-\rho t}\left[f\left(y(t,z,\epsilon), u^*(t,z)+\epsilon\eta(t,z), \mathbf{K}y(\epsilon)\right)\right. \\
&+p(t,z)\, g\left(y(t,z,\epsilon), u^*(t,z)+\epsilon\eta(t,z), \mathbf{K}y(\epsilon)\right)+p(t,z)\,\mu\mathbf{K}y(\epsilon) \\
&\left.+y(t,z,\epsilon)\left(-\rho p(t,z)+\frac{\partial p(t,z)}{\partial t}\right)+\right] dtdz \\
&-\int_{z\in\mathcal{Z}} e^{-\rho t}\left[p(t,z)\, y(t,z,\epsilon)\right]_0^T dz
\end{aligned}
\tag{92}
$$

where $t=0,\ t=T$ and $T\to\infty$ in the fourth line of the right hand side.

Since u^* is a maximizing control, the function $J(\epsilon)$ assumes a maximum when $\epsilon=0$. Thus $\left.\frac{dJ(\epsilon)}{d\epsilon}\right|_{\epsilon=0}=0$ or

$$
\begin{aligned}
&\left.\frac{dJ(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \\
&\int_{z\in\mathcal{Z}}\int_0^\infty e^{-\rho t}\left[\left[\left(f_x+pg_x+\frac{\partial p}{\partial t}-\rho p(t,z)+\right) y_\epsilon(t,z,\epsilon)+\right.\right. \\
&\left.\left. f_X\mathbf{K}y_\epsilon+p(t,z)\, g_X\mathbf{K}y_\epsilon+p(t,z)\,\mu\mathbf{K}y_\epsilon+(f_u+pg_u)\,\eta(t,z)\right]\right] dtdz \\
&-\int_{z\in\mathcal{Z}}\left[e^{-\rho t}p(t,z)\, y_\epsilon(t,z,0)\right]_0^T dz
\end{aligned}
\tag{93}
$$

where $t=0, t=T$ and $T\to\infty$ in the third line of the right hand side.

In the right hand side of (93) the terms of the form

$$\int_{z\in\mathcal{Z}}\phi(t,z)\left[\int_{z'\in\mathcal{Z}} w(z-z')\, y_\epsilon(t,z',\epsilon)\, dz'\right] dz$$

can be written by changing the order of integration as

$$\int_{z'\in\mathcal{Z}}\left[\int_{z\in\mathcal{Z}}\phi(t,z)\, w(z-z')\, dz\right] y_\epsilon(t,z',\epsilon)\, dz'.$$

Since the integration area is the same by re-labeling z as z' and z' as z, we obtain finally that

$$\int_{z\in\mathcal{Z}} \phi(t,z)\left[\int_{z'\in\mathcal{Z}} w(z-z')\, y_\epsilon(t,z',\epsilon)\, dz'\right] dz = \tag{94}$$

$$\int_{z\in\mathcal{Z}} \left[\int_{z'\in\mathcal{Z}} \phi(t,z')\, w(z'-z)\, dz'\right] y_\epsilon(t,z,\epsilon)\, dz = \tag{95}$$

$$\int_{z\in\mathcal{Z}} (\mathbf{K}\phi)\, y_\epsilon(t,z,\epsilon)\, dz \tag{96}$$

Substituting (96) into (93) and using the linearity of the operator $\mathbf{K}$ we obtain

$$\left.\frac{dJ(\epsilon)}{d\epsilon}\right|_{\epsilon=0} =$$

$$\int_{z\in\mathcal{Z}}\int_0^\infty e^{-\rho t}\left[\left[\left(f_x + pg_x + \frac{\partial p}{\partial t} - \rho p(t,z) + \mathbf{K}\left(f_X + p\left(g_X+\mu\right)\right)\right) y_\epsilon(t,z,\epsilon) + \right.\right.$$

$$\left. + \left(f_u + pg_u\right)\eta(t,z)\right]\Big] dtdz - \int_{z\in\mathcal{Z}} \left[e^{-\rho t} p(t,z)\, y_\epsilon(t,z,0)\right]_0^T dz \tag{97}$$

where $t = 0, t = T$ and $T \to \infty$ in the second line of the right hand side.

In (97), $y_\epsilon(0,z,\epsilon) = 0$, since $y(0,z,\epsilon) = x(0,z)$ fixed by initial conditions. We show next, by using (90) when the state and costate variables are positive and the state variable is bounded away from zero, that the last term of (97) vanishes. Let

$$\int_{z\in\mathcal{Z}} \xi(T,z)\,\beta(T,z) = 0 \tag{98}$$

for all $\beta(T,z)$ piecewise continuous functions in z. It follows, using Athans and Falb's (1966, p. 260) fundamental lemma, that

$$\xi(T,z) = 0\ ,\ z\in\mathcal{Z} \tag{99}$$

By writing $\xi(T,z) = e^{-\rho T}p(T,z)$ and assuming the intertemporal transversality condition $\lim_{T\to\infty} e^{-\rho T}p(T,z) = 0$, we obtain

$$\lim_{T\to\infty} e^{-\rho T}\int_{z\in\mathcal{Z}} p(T,z)\, dz = 0 \tag{100}$$

$$\lim_{T\to\infty} e^{-\rho T}\int_{z\in\mathcal{Z}} p(T,z)\, x(T,z)\, dz = 0 \text{ or,} \tag{101}$$

$$\lim_{T\to\infty} e^{-\rho T}p(T,z) = 0\ ,\ \lim_{T\to\infty} e^{-\rho T}p(T,z)\, x(T,z) = 0 \tag{102}$$

Then since $y_\epsilon(t,z,\epsilon)$ is arbitrary

$$-\left[e^{-\rho t}p(t,z)\,y_\epsilon(t,z,\epsilon)\right]_0^T = 0,\ T\to\infty$$

Since y_ϵ and $\eta(t,z)$ are arbitrary, we obtain from (97) that the necessary condition for a local maximum is:

$$\frac{\partial p}{\partial t} = \rho p - (f_x + pg_x) - \mathbf{K}\left(f_X + p\left(g_X + \mu\right)\right) \tag{103}$$

$$f_u + pg_u = 0 \tag{104}$$

So if we define a current value Hamiltonian function

$$\tilde{H} = f(x,u,X) + p\left[g(x,u,X) + \mu X\right],\ X = \mathbf{K}x \tag{105}$$

then by (103) and (104) we obtain the necessary conditions of the maximum principle.

$$\frac{\partial \tilde{H}}{\partial u} = 0,\ or\ f_u + pg_u = 0 \Rightarrow u^* = u^*(x,p,X) \tag{106}$$

$$\frac{\partial x}{\partial t} = \left[g(x,u^*,X) + \mu X\right]\ ,\ X = \mathbf{K}x \tag{107}$$

$$\frac{\partial p}{\partial t} = \rho p - (f_x + pg_x) - (\mathbf{K}f_X + \mathbf{K}pg_X + \mu\mathbf{K}p) \tag{108}$$

where $u^*(x,p,X)$ is the control that maximizes the Hamiltonian function (105). With circle boundary conditions for the state variable $x(t,-Z) = x(t,Z) = \bar{x}(t)$, similar spatial transversality conditions $p(t,-Z) = p(t,Z)$ for all t, should be satisfied for the costate variable for the solution of the system of integrodifferential equations (107)-(108).■

The maximum principle for the vector problem

We consider a generalization of problem (84) - (85) defined as:

$$\max_{\{u(t,z)\}} \int_{z\in Z}\int_0^\infty e^{-\rho t} f\left(\mathbf{x}(t,z),\mathbf{u}(t,z),\mathbf{X}(t,z)\right) dtdz\ \mathbf{x}\in\Re^n, \mathbf{u}\in\Re^m \tag{109}$$

$$\mathbf{x} = (x_1,\ldots,x_n)\,,\ \mathbf{u} = (u_1,\ldots,u_m)\,, \mathbf{X}(t,z) = (\mathbf{K}_1 x_1,...,\mathbf{K}_n x_n) \tag{110}$$

$$\mathbf{K}_i x_i = \int_{z'\in Z} w_i(z-z')\,x_i(z',t)\,dz'\ ,\ \ i = 1,...,n. \tag{111}$$

subject to

$$\frac{\partial x_i(z,t)}{\partial t} = g_i\left(\mathbf{x}(z,t),\mathbf{u}(z,t),\mathbf{X}(t,z)\right) + \mu_i X_i(t,z)\,,\ ,\ \ i = 1,...,n. \tag{112}$$

With the current value Hamiltonian function defined as

$$\tilde{H} = f(\mathbf{x}, \mathbf{u}, \mathbf{X}) + \sum_{i=1}^{n} p_i \left[g_i (\mathbf{x}, \mathbf{u}, \mathbf{X}) + \mu_i \mathbf{K}_i x_i \right] \tag{113}$$

the necessary conditions of the maximum principle can be stated as:

$$\frac{\partial \tilde{H}}{\partial u_j} = 0, \; or \; \frac{\partial f}{\partial u_j} + \sum_{i=1}^{n} p_i \frac{\partial g_i}{\partial u_j} = 0 \Rightarrow u_j^* = u_j (\mathbf{x}, \mathbf{p}, \mathbf{X}) \tag{114}$$

$$\frac{\partial x_i}{\partial t} = g(\mathbf{x}, \mathbf{u}^*, \mathbf{X}) + \mu_i \mathbf{K}_i x_i \;,\; i = 1, ..., n \;, j = 1, ..., m \tag{115}$$

$$\frac{\partial p_i}{\partial t} = \tag{116}$$

$$\rho p_i - \left(\frac{\partial f}{\partial x_i} + \sum_{l=1}^{n} p_l \frac{\partial g_l}{\partial x_i} \right) - \left(\mathbf{K}_i \frac{\partial f}{\partial X_i} + \sum_{l=1}^{n} \mathbf{K}_l p_l \frac{\partial g_l}{\partial X_i} + \mu_i \mathbf{K}_i p_i \right)$$

Define

$$J = \int_{z \in Z} \int_0^{\infty} e^{-\rho t} f(\mathbf{x}, \mathbf{u}, \mathbf{X}) \, dt dz = \tag{117}$$

$$\int_{z \in Z} \int_0^{\infty} e^{-\rho t} \{ f(\mathbf{x}, \mathbf{u}, \mathbf{X}) +$$

$$\sum_{i=1}^{n} p_i (t, z) \left[g_i (\mathbf{x}, \mathbf{u}, \mathbf{X}) + \mu_i \mathbf{K}_i x_i - \frac{\partial x_i}{\partial t} \right] \right\} dt dz$$

and consider again a one parameter family of comparison controls, $\mathbf{u}^*(t, z) + \epsilon \boldsymbol{\eta}(t, z)$, where $\mathbf{u}^*(t, z)$ is the optimal control, $\boldsymbol{\eta}(t, z)$ is a fixed vector function, ϵ is a small parameter and $\mathbf{y}(t, z, \epsilon)$, $t \in [0, \infty)$, $z \in [-Z, Z]$ is the state variable vector generated by (112) with control $\mathbf{u}^*(t, z) + \epsilon \boldsymbol{\eta}(t, z)$, $t \in [0, \infty)$, $z \in [-Z, Z]$. The necessary conditions are derived by following the same steps as in the one variable case and using similar intertemporal and spatial transversality condition.

Appendix 2

Proof of Proposition 2

Suppose that $x^*(t, z)$, $u^*(t, z)$, $p(t, z)$ satisfy conditions (106)-(108) and let $x(t, z)$, $u(t, z)$ functions satisfy (85), initial and boundary conditions. Let f^*, g^* denote functions evaluated along $(x^*(t, z), u^*(t, z), X^*(t, z))$ and let f, g denote functions evaluated along the feasible path $(x(t,z), u(t,z), X(t,z))$. To prove sufficiency we need to show that

$$W \equiv \int_{z\in\mathcal{Z}} \int_0^\infty e^{-\rho t} (f^* - f)\, dt dz \geq 0$$

From the concavity of f it follows that

$$(f^* - f) \geq (x^* - x) f_x^* + (u^* - u) f_u^* + (X^* - X) f_X^* \tag{118}$$

Using $X = \mathbf{K}x$, and using from (96) $\int_z \phi(\mathbf{K}\psi)\, dz = \int_z \psi(\mathbf{K}\phi)\, dz$, we obtain

$$\int_{z\in\mathcal{Z}} (f^* - f)\, dz \geq \int_{z\in\mathcal{Z}} [(x^* - x)(f_x^* + \mathbf{K}f_X^*) + (u^* - u) f_u^*]\, dz$$

Then

$$W \geq \int_{z\in\mathcal{Z}} \int_0^\infty e^{-\rho t} [(x^* - x)(f_x^* + \mathbf{K}f_X^*) + (u^* - u) f_u^*]\, dt dz \tag{119}$$

$$= \int_{z\in\mathcal{Z}} \int_0^\infty e^{-\rho t} \left[(x^* - x)\left(\rho p - \frac{\partial p}{\partial t} - p g_x^* - \mathbf{K} p g_X^* - \mu \mathbf{K} p\right)\right. \tag{120}$$

$$\left. + (u^* - u)(-p g_u^*)\right] dt dz$$

$$= \int_{z\in\mathcal{Z}} \int_0^\infty e^{-\rho t} p [(g^* - g) - (x^* - x) g_x^* - (X^* - X) g_X^* - (u^* - u) g_u^*]\, dt dz \geq 0 \tag{121}$$

Condition (120) follows from (119) by using conditions (106) and (108) to substitute for f_u^* and $f_x^* + \mathbf{K}f_X^*$. Condition (121) is derived in the following way.

The term $\int_0^\infty e^{-\rho t} (x^* - x)\left(\rho p - \frac{\partial p}{\partial t}\right) dt$ is replaced using (89) by:

$$\int_0^\infty e^{-\rho t} p \left(\frac{\partial x^*}{\partial t} - \frac{\partial x}{\partial t}\right) dt \text{ with} \tag{122}$$

$$\frac{\partial x^*}{\partial t} = g^* + \mu \mathbf{K} x^*, \quad \frac{\partial x}{\partial t} = g + \mu \mathbf{K} x$$

Furthermore

$$\begin{aligned}
\int_{z\in\mathcal{Z}} (x^* - x) \mu \mathbf{K} p dz &= \int_{z\in\mathcal{Z}} \mu p \mathbf{K} (x^* - x)\, dz \\
\int_{z\in\mathcal{Z}} (x^* - x)(\mathbf{K} p g_X^*)\, dz &= \int_{z\in\mathcal{Z}} p (\mathbf{K}(x^* - x) g_X^*)\, dz = \\
&\int_{z\in\mathcal{Z}} p (X^* - X) g_X^* dz
\end{aligned}$$

Substituting into (120), the first term of (120) can be written as:

$$\int_{z\in\mathcal{Z}}\int_0^{\infty} e^{-\rho t}\left[p\left(g^*-g\right)-p\left(x^*-x\right)g_x^*-p\left(X^*-X\right)g_X^*\right]dtdz \tag{123}$$

By substituting (123) into (120) we obtain (121) which holds by the concavity assumption about g and the assumption that $p\geq 0.$■

Appendix 3

Proof of Proposition 3

The linearized MHDS system of (19)-(20) at the FOSS, can be written as:

$$\frac{\partial x}{\partial t} = H_{px}^*x+H_{pX}^*\mathbf{K}x+H_{pp}^*p \tag{124}$$

$$\frac{\partial p}{\partial t} = \left(-H_{xx}^*-2H_{Xx}^*\right)x-H_{XX}^*\mathbf{K}^2x+\left(\rho-H_{xp}^*\right)p-H_{Xp}^*\mathbf{K}p \tag{125}$$

To study the stability of the FOSS to spatially heterogeneous perturbations, we consider trial solutions for the state and costate variables which can be expressed as convergent Fourier series. If we consider solutions $x\left(t,z\right),p\left(t,z\right)$ for state and costate variables which are square integrable and periodic functions in z,[19] then the Fourier series expansions of the solutions converge to the value of the function in the mean square (Priestley (1981, p. 190)). The trial solutions can then be expressed as:

$$v\left(t,z\right) = e^{\lambda t}\sum_{k=0}^{\infty}\left[\alpha_k^v\cos\left(kz\right)+\beta_k^v\sin\left(kz\right)\right],\ z\in\left[-Z.Z\right] \tag{126}$$

$$v = \left\{x,p\right\},\ k=\frac{2n\pi}{L},L=2Z,n=0,\pm1,\pm2,... \tag{127}$$

where the constants $\left\{\alpha_k^v\right\},\left\{\beta_k^v\right\}$ are the Fourier coefficients.[20] It should be noted that the same results can be obtained by using trial solutions of the form

$$v\left(t,z\right)=c^ve^{\lambda t+ikz} \tag{128}$$

[19]This is the class of functions denoted by $L^2\left(-\pi,\pi\right)$. Square integrable functions of general periodicity not necessarily 2π can be transformed to functions with periodicity 2π by an appropriate transformation of the time scale (Priestley (1981, page 194)).

[20]The set of functions $\cos\left(2n\pi z/L\right),\sin\left(2n\pi z/L\right),n=0,\pm1,\pm2,...$ is a complete orthogonal basis over $\left[-Z,Z\right]$ which is used for the expansion of a function.

where c^v is a constant. The two approaches of constructing the trial solution are equivalent since for any given k (128) expresses the kth term of (126). By the symmetry of the kernel $w(z-z') = w(z'-z)$, setting $\zeta = z' - z$ we obtain $\int_{z' \in Z} w(z'-z)\,\omega(t,z')\,dz' = \int_{\zeta \in Z} w(\zeta)\,v(t,\zeta+z)\,d\zeta,\ v = x,p$. Substituting the trial solution under the integral we obtain, dropping t to simplify notation:

$$\begin{aligned}\mathbf{K}v &= \int_{\zeta \in Z} w(\zeta)\,v(\zeta+z)\,d\zeta = \\ & e^{\lambda t} \int_{\zeta \in Z} w(\zeta) \sum_{k=0}^{\infty} \left[\alpha_k^v \cos(k(\zeta+z)) + \beta_k^v \sin(k(\zeta+z))\right] d\zeta\end{aligned}$$

Using the formulas

$$\begin{aligned}\cos(A+B) &= \cos A \cos B - \sin A \sin B \\ \sin(A+B) &= \sin A \cos B + \sin B \cos A\end{aligned}$$

and noting that because of the symmetry of the kernel $w(\zeta) = w(-\zeta)$, it holds that $\int_{-Z}^{Z} w(\zeta)\sin(k\zeta) = 0$ for any constants Z, k, we obtain

$$\mathbf{K}v = e^{\lambda t} \sum_k \left[\alpha_k^v \cos(kz) + \beta_k^v \sin(kz)\right] \int_{\zeta \in \mathcal{Z}} w(\zeta)\cos(k\zeta)\,d\zeta = v(t,z)\,W(k)$$

If we use (128) as a trial solution for an infinite spatial domain $\mathcal{Z}$, then $W(k)$ is the Fourier transform of the kernel $w(\zeta)$, or

$$W(k) = \int_{-\infty}^{\infty} w(\zeta)\,e^{ik\zeta} d\zeta \tag{129}$$

Expressions for $W(k)$ are presented in Appendix 4. Furthermore

$$\mathbf{K}^2 x = v(t,z) \int_{\zeta \in Z} [w(\zeta)]^2 \cos(k\zeta)\,d\zeta = v(t,z)\,W^2(k)$$

Substituting the rest of the trial solutions into (19)-(20) and collecting terms we obtain

$$\frac{dx_k}{dt} = \left[H_{px}^* + H_{pX}^* W(k)\right] x_k + H_{pp}^* p_k \tag{130}$$

$$\frac{dp_k}{dt} = \left[-H_{xx}^* - 2H_{Xx}^* W(k) - H_{XX}^* W^2(k)\right] x_k + \tag{131}$$

$$\left[\rho - H_{xp}^* - H_{Xp}^* W(k)\right] p_k \tag{132}$$

This is a sequence of linear systems of ordinary differential equations indexed by k which corresponds to mode n. Mode $n = 0$, $k = 0$ and $W(0)$ correspond to the MHDS of a spatially homogenous system, the FOSS. From the Jacobian matrix of the sequence of the linear systems (130)-(131) it follows that mode n is saddle point stable if the pair of eigenvalues of (130)-(131) have opposite signs, and it is unstable if both eigenvalues have positive real parts. In (130)-(131), $\mathrm{tr}.J^{LR} = \rho > 0$, while the determinant defines a quadratic expression in terms of $W(k)$. This is the dispersion relationship for the optimal control problem with spatial spillovers, which can be written as:

$$\psi(W(k)) = \left[H_{XX}^* H_{pp}^* - \left[H_{pX}^*\right]^2\right] W^2(k) +$$
$$\left[H_{pX}^*\left(\rho - 2H_{px}^*\right) + 2H_{Xx}^* H_{pp}^*\right] W(k) + \left[\rho H_{px}^* - \left[H_{pX}^*\right]^2 + H_{pp}^* H_{xx}^*\right] \quad (133)$$

If there exist $k \in (k_1, k_2)$ such that $\psi(W(k)) > 0$ for $k \in (k_1, k_2)$, then both eigenvalues (λ_1, λ_2) of the MHDS which characterize temporal growth are positive and the FOSS is not stable to spatially heterogeneous perturbations. The eigenvalues are obtained as the solution of the characteristic equation

$$\lambda^2 - \rho\lambda + \psi(W(k)) = 0$$

with eigenvalues:

$$\lambda_{1,2}(k) = \frac{1}{2}\left(\rho \pm \sqrt{\rho^2 - 4\psi(W(k))}\right) \quad (134)$$

Spillovers induced spatial instability requires $\psi(W(k)) > 0$ for $k \in (k_1, k_2)$. The solution for (19)-(20) can be obtained by a linear combination of the solutions (126) for the two eigenvalues (134).■

Appendix 4

We consider two types of simple exponential kernels (SEK): (i) kernels with quadratic exponents (KQE), and (ii) kernels where the exponent is defined in terms of absolute values (KAVE). In the following we present the kernel $w(\zeta)$ and the corresponding $W(k)$ for one SEK and the sum of two SEKs. Generalizations to more complicated linear combinations of SEKs are straightforward.

1. KQE

Kernel	$w_1(\zeta) = b_1 \exp\left[-(\zeta/d_1)^2\right], b_1, d_1 > 0$	Q-1
$z \in (-\infty, \infty)$	$W(k) = \sqrt{\pi} b_1 d_1 \exp\left[-\frac{(d_1 k)^2}{4}\right]$	Q-11
$z \in [-\pi, \pi]$	$W(k) = \frac{i\sqrt{\pi}}{2}\left[b_1 d_1 \exp\left[-\frac{(d_1 k)^2}{4}\right]\right] \times$ $\times \left[\operatorname{erf} \mathrm{i}\left(\frac{d_1 k}{d_1} - \frac{i\pi}{d_1}\right) + \operatorname{erf} \mathrm{i}\left(\frac{d_1 k}{d_1} + \frac{i\pi}{d_1}\right)\right]$	QE-12
Kernel	$w_2(\zeta) = b_1 \exp\left[-(\zeta/d_1)^2\right] - b_2 \exp\left[-(\zeta/d_2)^2\right]$ $b_1 > b_2,\ d_1 < d_2$	Q-2
$z \in (-\infty, \infty)$	$W(k) = \sqrt{\pi}\left\{b_1 d_1 \exp\left[-\frac{(d_1 k)^2}{4}\right] - b_2 d_2 \exp\left[-\frac{(d_2 k)^2}{4}\right]\right\}$	Q-21
$z \in [-\pi, \pi]$	$W(k) = \frac{\sqrt{\pi}}{2}(A_1 - A_2),\ A_j = \left[b_j d_j \exp\left[-\frac{(d_j k)^2}{4}\right]\right] \times$ $\times \left[\operatorname{erf}\left(\frac{Z}{d_j} - \frac{i d_j k}{2}\right) + \operatorname{erf}\left(\frac{Z}{d_j} + \frac{i d_j k}{2}\right)\right], j = 1, 2$	QE-22
	$\operatorname{erfi}(z) = \operatorname{erf}(iz/i)$: imaginary error function $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2} du$: the error function	

2. KAVE

Kernel	$w_1(\zeta) = b_1 \exp\left[-d_1 \lvert\zeta\rvert\right], b_1, d_1 > 0$	A-1
$z \in (-\infty, +\infty)$	$W(k) = \frac{2 b_1 d_1}{d_1^2 + k^2}$	A-11
$z \in [-\pi, \pi]$	$W(k) = \frac{2 b_1 \exp(-d_1 \pi)[d_1 \exp(-d_1 \pi) - d_1 \cos(k\pi) + k \sin(k\pi)]}{d_1^2 + k^2}$	A-12
Kernel	$w_2(\zeta) = b_1 \exp\left[-d_1 \lvert\zeta\rvert\right] - b_2 \exp\left[-d_2 \lvert\zeta\rvert\right]$ $(b_1, d_1) > (b_2, d_2)$	A-21
$z \in (-\infty, +\infty)$	$W(k) = -\frac{2\left(b_2 d_1^2 d_2 - b_1 d_1 d_2^2 - b_1 d_1 k^2 + b_2 d_2 k^2\right)}{\left(d_1^2 + k^2\right)\left(d_2^2 + k^2\right)}$	
$z \in [-\pi, \pi]$	$W(k) = -\frac{2 \exp\left[-(d_1 + d_2)\pi\right]}{\left(d_1^2 + k^2\right)\left(d_2^2 + k^2\right)} \times$ $(\exp\left[(d_1 + d_2)\pi\right]\left[b_2 d_2 (d_1^2 + k^2) - b_1 d_1 (d_2^2 + k^2)\right]) +$ $b_1 \exp(d_2 \pi)(d_2^2 + k^2)(d_1 \cos(k\pi) - k \sin(k\pi)) +$ $b_2 \exp(d_1 \pi)(d_1^2 + k^2)(-d_2 \cos(k\pi) + k \sin(k\pi))$	A-22

[Figures A4.1-A4-4]

Figures A4.1-A4.2 present some typical shapes for $w(\zeta)$ and the corresponding $W(\zeta)$. Kernels of the type $w_1(\zeta)$ imply that the influence of neighboring state variables on a local state variable is a weighted average of the state variable at neighboring locations, with weights decaying exponentially, and with this influence being always nonnegative. This is similar for example to Lucas' (2001) assumption for the case of labor productivity. Kernels of the type $w_2(\zeta)$ imply similarly that the influence of neighboring state on local state is a weighted average of the state at neighboring locations, but that the influence from nearby locations is positive, while the influence is negative from relatively more distant locations. This is similar to Krugman's (1996) modelling of a market potential function.

References

AGHION, P., AND P. HOWITT (1992): "A Model of Growth Through Creative Destruction," *Econometrica*, 60, 323–351.

ATHANS, M., and P. FALB (1966): *Optimal Control*. New York: McGraw-Hill.

BALDWIN, R. (2001): "Core-Periphery Model with Forward-Looking Expectations," *Regional Science and Urban Economics*, 31, 21–49.

BALDWIN, R., AND P. MARTIN (2004): "Agglomeration and Regional Growth," in *Handbook of Regional and Urban Economics*, Part III, 60, ed. by V. Henderson, and J.F. Thisse. Amsterdam: North Holland.

BARRO, R. J., AND X. SALA-I-MARTIN (2004): *Economic Growth*, Second Edition. Cambridge, MA: MIT Press.

BOUCEKKINE, R., C. CAMACHO, AND B. ZOU (2006): "Bridging the Gap between Growth Theory and the New Economic Geography: The Spatial Ramsey Model," CORE Discussion Paper, 2006/72, Department of Economics, Catholic University of Louvain.

BROCK, W., AND A. MALLIARIS (1989): *Differential Equations, Stability and Chaos in Dynamic Economics*. Amsterdam: North-Holland.

BROCK, W., AND A. XEPAPADEAS (2008a): "Diffusion-Induced Instability and Pattern Formation in Infinite Horizon Recursive Optimal Control," *Journal of Economic Dynamics and Control*, 32, 2745–2787.

BROCK, W., AND A. XEPAPADEAS (2008b): "Pattern Formation, Spatial Externalities and Regulation in Coupled Economic-Ecological Systems," Working paper, Available at SSRN: http://ssrn.com/abstract=1144071.

CARLSON, D., A. HAURIE, AND A. LEIZAROWITZ (1991): *Infinite Horizon Optimal Control*, Second Edition. Berlin: Springer-Verlag.

CASS, D., AND K. SHELL (eds.) (1976): *The Hamiltonian Approach to Dynamic Economics*. New York: Academic Press.

CHINCARINI, L., AND N. ASHERIE (2008): "An Analytical Model for the Formation of Economic Clusters," *Regional Science and Urban Economics*, 38, 252–270.

DESMET, K., AND E. ROSSI-HANSBERG (2007): "Spatial Growth and Industry Age," CEPR Discussion Paper No. DP6421.

DIEUDONNE, J. (1969): *Foundations of Modern Analysis*. New York: Academic Press.

FUJITA, M., P. KRUGMAN, AND A. VENABLES (1999): *The Spatial Economy.* Cambridge, MA: The MIT Press.

GROSSMAN, G. M., AND E. HELPMAN (1991a): *Innovation and Growth in the Global Economy.* Cambridge, MA: MIT Press.

GROSSMAN, G. M., AND E. HELPMAN (1991b): "Quality Ladders and Product Cycles," *Quarterly Journal of Economics*, 106, 557-586.

IOANNIDES, Y. M., AND H. G. OVERMAN (2007): "Spatial Evolution and the U.S. Urban System," *Journal of Economic Geography*, 4, 131–156.

JONES, C. I. (1995): "R&D-Based Models of Economic Growth," *Journal of Political Economy*, 103, 759–784.

KAMIEN, M., AND N. SCHWARTZ (1981): *Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management*, Second Edition. New York: North-Holland.

KRUGMAN, P. (1996): *The Self-Organizing Economy.* Cambridge, MA: Blackwell Publishers.

KURZ, M. (1968): "The General Instability of a Class of Competitive Growth Processes," *Review of Economic Studies,* 35, 155–174.

LEIZAROWITZ, A. (2008): "Turnpike properties of a class of aquifer control problems," *Automatica*, 44, 1460–1470.

LUCAS, R. E. (1988): "On the Mechanics of Economic Development," *Journal of Monetary Economics*, 22 (1), 3–42.

LUCAS, R. E. (2001): "Externalities and Cities," *Review of Economic Dynamics*, 4, 245–274.

LUCAS, R. E., AND E. ROSSI-HANSBERG (2002): "On the Internal Structure of Cities," *Econometrica*, 70(4), 1445–1476.

MARTIN, R., AND P. SUNLEY (2003): "Deconstructing Clusters: Chaotic Concept or Policy Panacea?" *Journal of Economic Geography*, 3, 5–35.

MATSUYAMA, K. (2008a): "Structural Change," in *The New Palgrave,* ed. by L. Blume, and S. Durlauf. London: McMillan.

MATSUYAMA, K. (2008b): "Symmetry Breaking," in *The New Palgrave,* ed. by L. Blume, and S. Durlauf. London: McMillan.

MCKENZIE, L. (1983): "Optimal Economic Growth and Turnpike Theorems," in *The Handbook of Mathematical Economics*, Volume 3, ed. by K. Arrow, and M. Intriligator. Amsterdam: North Holland.

MOISEIWITSCH, B. L. (2005): *Integral Equations.* Dover Publications.

MOSSAY, P. (2006): "A Theory of Rational Spatial Agglomerations," Department of Economics, University of Alicante.

MURRAY, J. (2002): *Mathematical Biology,* Volume I, Third Edition. Berlin: Springer.

MURRAY, J. (2003): *Mathematical Biology,* Volume II, Third Edition. Berlin: Springer.

OKUBO, A., AND S. LEVIN (Eds) (2001): *Diffusion and Ecological Problems: Modern Perspectives*, 2nd Edition. Berlin: Springer.

OTTAVIANO, G. (2001): "Monopolistic Competition, Trade, and Endogenous Spatial Fluctuations," *Regional Science and Urban Economics*, 31, 51–77.

PORTER, D., AND D. S. G. STIRLING (1990): *Integral Equations.* Cambridge: Cambridge University Press.

PRIESTLEY, M. (1981): *Spectral Analysis and Time Series*, Vol. 1. London: Academic Press.

QUAH, D. (2002): "Spatial Agglomeration Dynamics," *AEA Papers and Proceedings*, 92, 247–252.

ROMER, P. M. (1986): "Increasing Returns and Long-Run Growth," *Journal of Political Economy*, 94 (5), 1002–10037.

ROMER, P. M. (1990): "Endogenous Technological Change," *Journal of Political Economy*, 98, S71–S102.

SCHEINKMAN, J. (1976): "On Optimal Steady States of N-Sector Growth Models When Utility Is Discounted," *Journal of Economic Theory*, 12, 11–30.

SOLOW, R. (1956): "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics*, 70, 65–94.

STOKEY, N. L., R. E. LUCAS WITH E. C. PRESCOTT (1989): *Recursive Methods in Economic Dynamics.* Cambridge, MA: Harvard University Press.

TURING, A. (1952): "The Chemical Basis of Morphogenesis," *Philosophical Transactions of the Royal Society of London*, 237, 37–72.

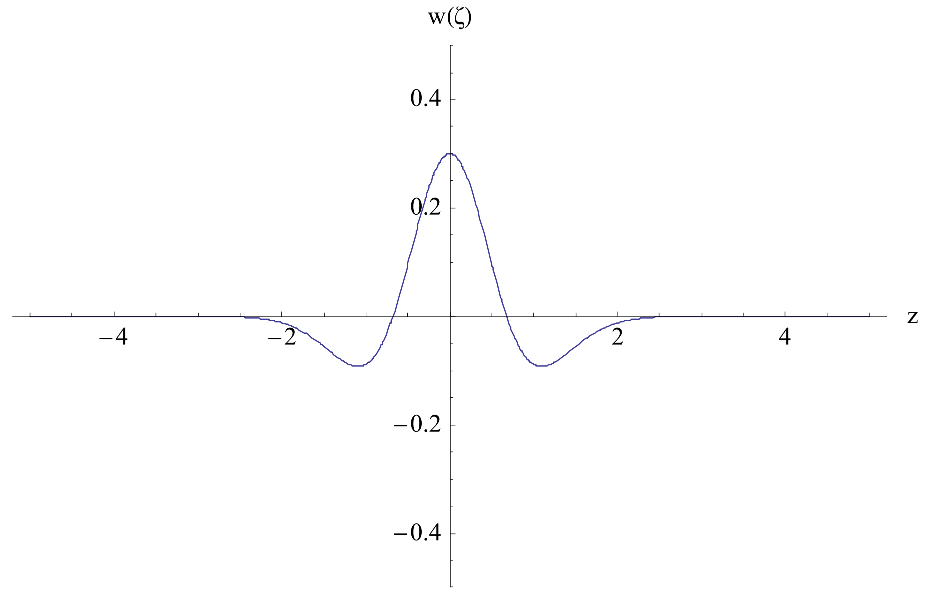

Figure 1: The kernel function $w\,(\zeta)$

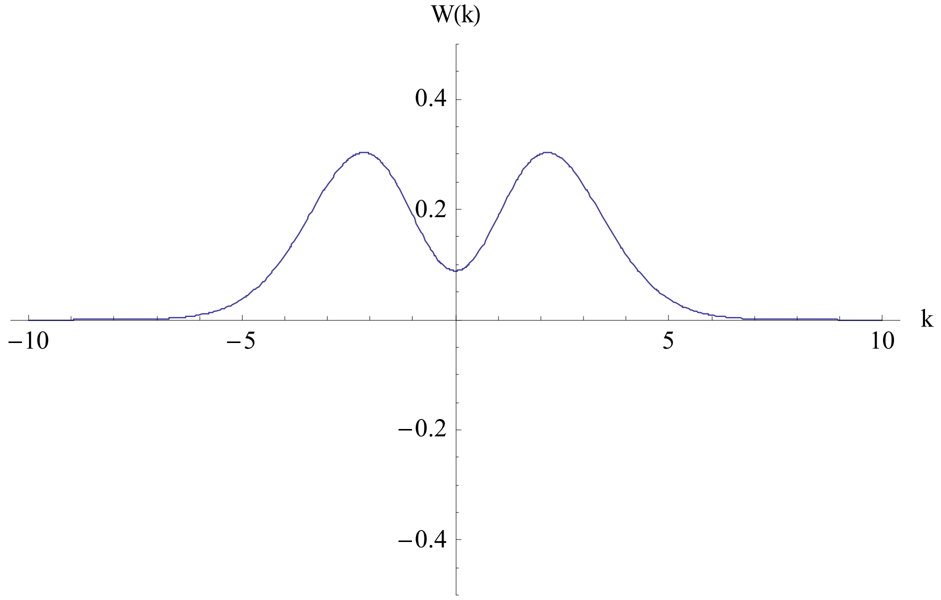

Figure 2: The $W\,(k)$ function

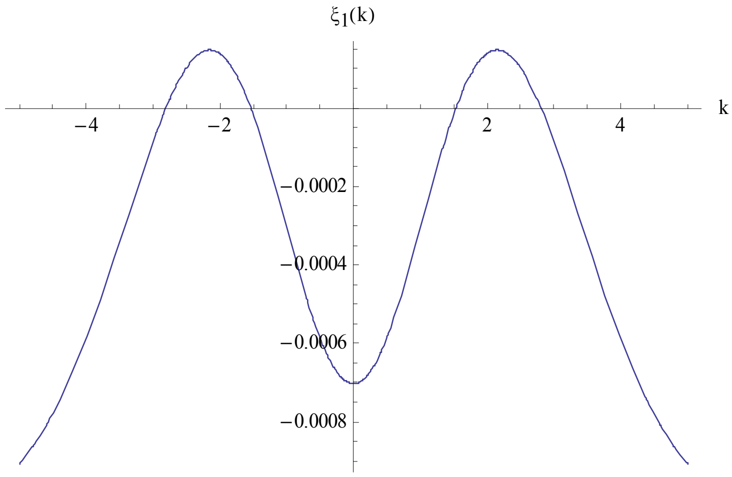

Figure 3: The $\xi_1(k)$ function

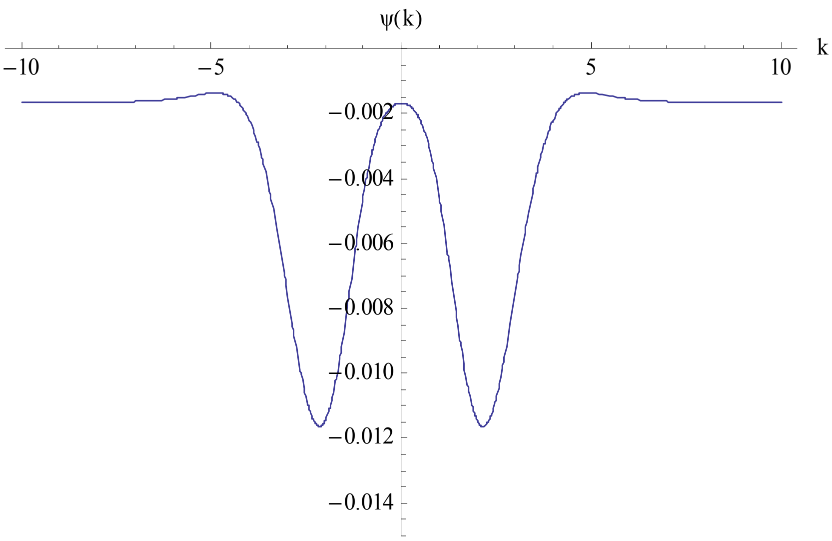

Figure 4: The $\psi(k)$ function

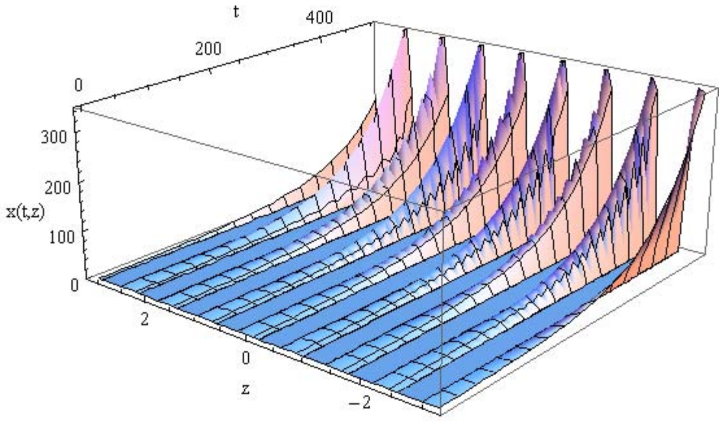

Figure 5: Knowledge agglomeration dynamics

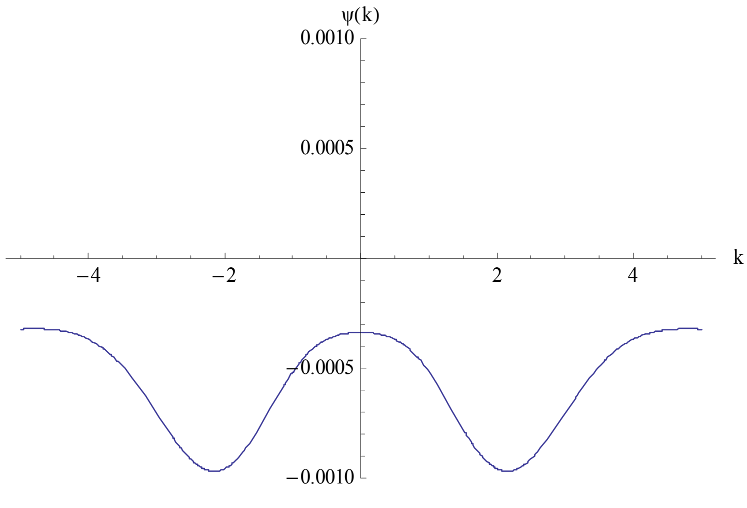

Figure 6: The dispersion relationship for the SOMP

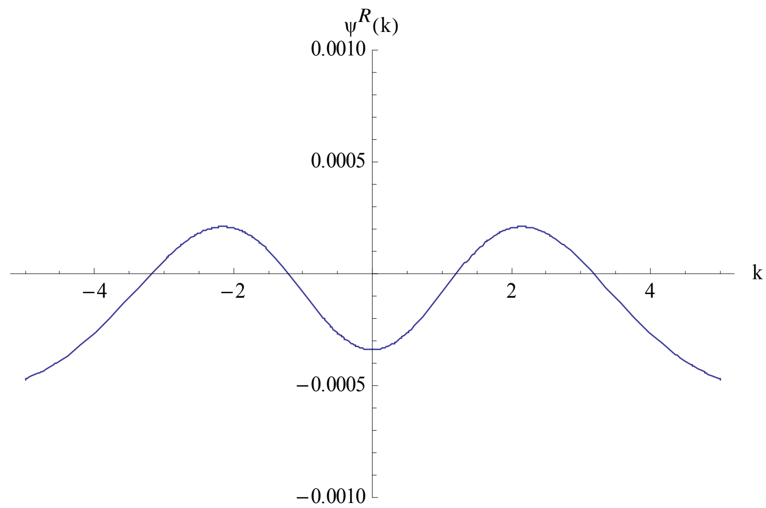

Figure 7: The dispersion relationship for the POMP

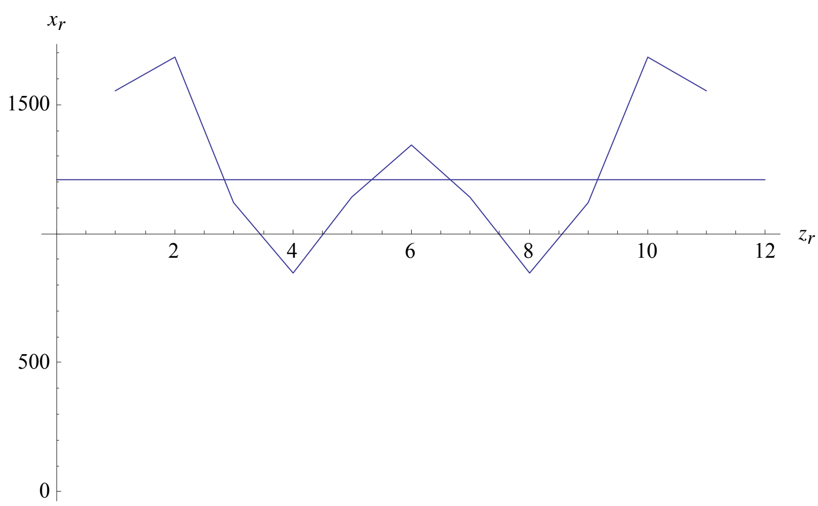

Figure 8: Steady state knowledge agglomeration

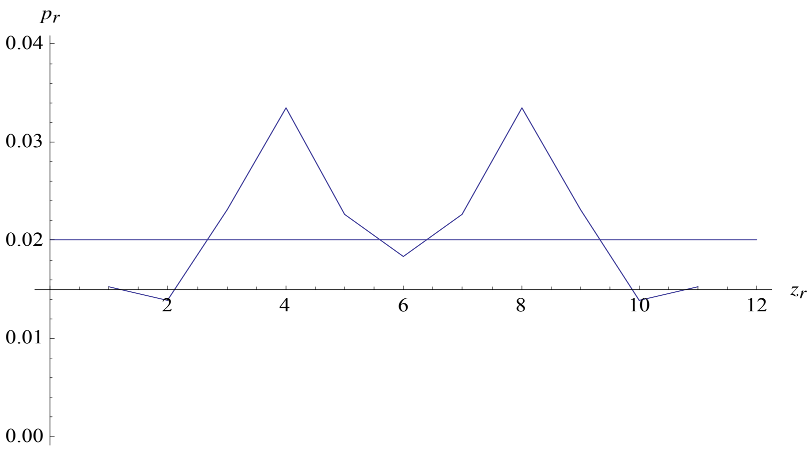

Figure 9: The shadow price of knowledge

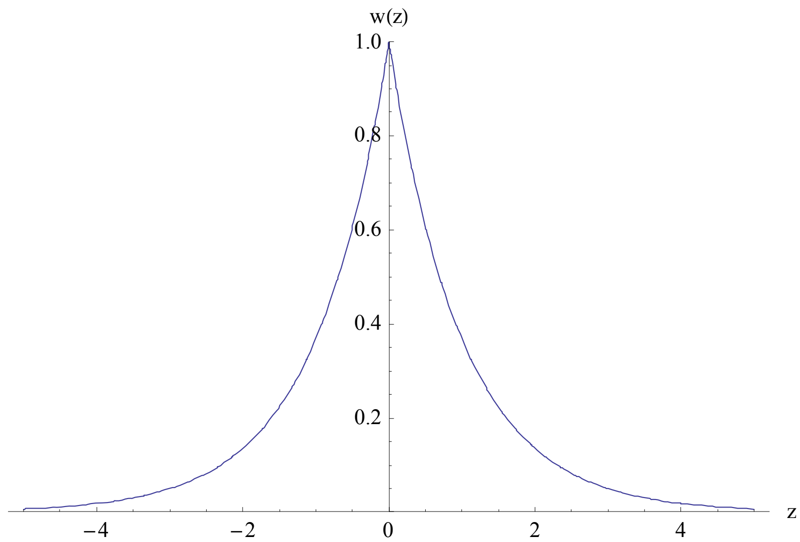

Figure A4.1: A-1

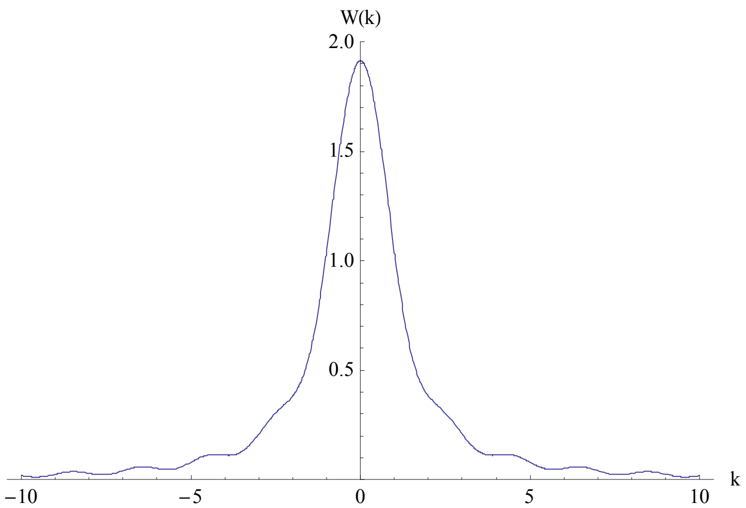

Figure A4.2: A-12

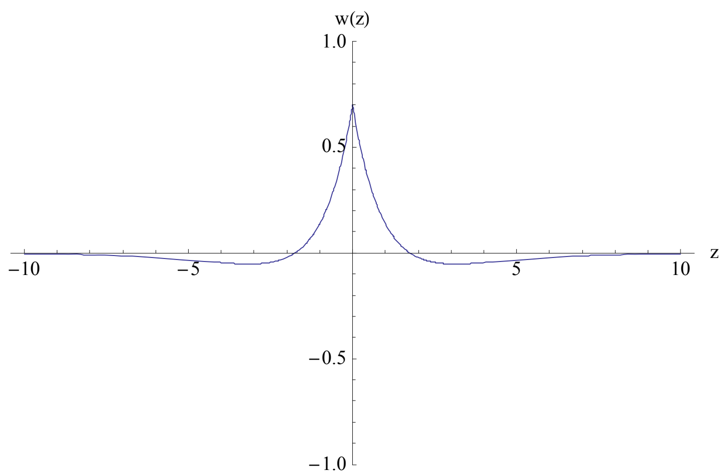

Figure A4.3: A-2

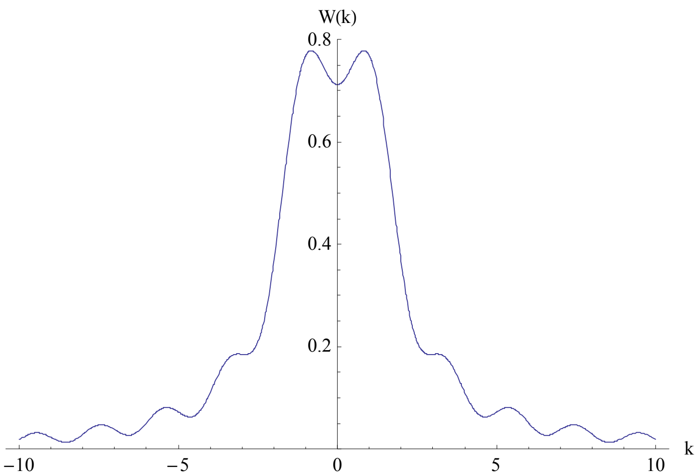

Figure A4.4: A-22