Weak-Identification-Robust Bootstrap Tests after Pretesting for Exogeneity

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ABSTRACT

Pretesting for exogeneity has become a routine in many empirical applications involving instrumental variables (IVs) to decide whether the ordinary least squares (OLS) or the IV-based method is appropriate. Guggenberger (2010) shows that the second-stage t-test – based on the outcome of a Durbin-Wu-Hausman type pretest for exogeneity in the first stage – has extreme size distortion with asymptotic size equal to 1, even when the IVs are strong. In this paper, we propose a novel two-stage test procedure that switches between the OLS-based statistic and the weak-IV-robust statistic. Furthermore, we develop a size-corrected wild bootstrap approach, which combines certain wild bootstrap critical values along with an appropriate size-correction method. We establish uniform validity of this procedure under conditional heteroskedasticity in the sense that the resulting tests achieve correct asymptotic size no matter the identification is strong or weak. Monte Carlo simulations confirm our theoretical findings. In particular, our proposed method has remarkable power gains over the standard weak-identification-robust test.

Key words: DWH Pretest; Shrinkage; Weak Instruments; Asymptotic Size; Wild Bootstrap; Bonferroni-based Size-correction.

JEL classification: C12; C13; C26.

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1. Introduction

Inference after data-driven model selection is widely studied in both statistical and econometric literature. For instance, see Hansen (2005), Leeb and Pötscher (2005), who provide an overview of the importance and difficulty of conducting valid inference after model selection. In particular, it is now well known that widely used model-selection practices such as pretesting may have large impact on the size properties of two-stage procedures and thus invalidate inference on parameter of interest in the second stage. For the classical linear regression model with exogenous covariates, Kabaila (1995) and Leeb and Pötscher (2005) show that confidence intervals (CIs) based on consistent model selection have serious problem of under-coverage, while Andrews and Guggenberger (2009b) show that such CIs have asymptotic confidence size equal to 0. Furthermore, Andrews and Guggenberger (2009a) find extreme size distortion for the two-stage test after "conservative" model selection and propose various least favourable critical values (CVs).

In comparison, the literature on models that contain endogenous covariates, such as widely used instrumental variable (IV) regression models, remains relatively sparse. The uniform validity of post-selection inference for structural parameters in linear IV models with homoskedastic errors was studied by Guggenberger (2010a), who advised not to use Hausman-type pretesting to select between ordinary least squares (OLS) and two-stage least squares (2SLS)-based t-tests because such two-stage procedure can be extremely over-sized with asymptotic CVs.[1] Instead, Guggenberger (2010a) recommended to use the standard 2SLS-based t-test. However, it is well known that the 2SLS-based t-test itself may have undesirable finite-sample size properties when IVs are not strong enough. As such, in the quest for statistical power, many empirical practitioners still use pretesting in IV applications despite the important concern raised by Guggenberger (2010a).[2]

[1] Similar concerns were also raised by Guggenberger and Kumar (2012) about pretesting the instrument exogeneity using a test of overidentifying restrictions, and by Guggenberger (2010b) about pretesting for the presence of random effects before inference on the parameters of interest in panel data models.

[2] Their motivation of implementing the pretesting procedure also lies in the fact that valid IVs (i.e., exogenous IVs) found in practice are often rather uninformative, while strong IVs are typically more or less invalid and such deviation from IV exogeneity may also lead to serious size distortion in the 2SLS-based t-test; e.g., see Conley, Hansen and Rossi (2012), Guggenberger (2012), Andrews, Gentzkow and Shapiro (2017).

Recently, Young (2022) analyzes a sample of 1359 empirical applications involving IV regressions in 31 papers published in the American Economic Association (AEA): 16 in AER, 6 in AEJ: A.Econ., 4 in AEJ: E.Policy, and 5 in AEJ: Macro. He highlights that the IVs often do not appear to be strong in these papers, so that inference methods based-on standard normal CVs can be unreliable, especially in the case with heteroskedastic or clustered errors, and he advocates for the usage of bootstrap methods to improve the quality of inference. Furthermore, he argues that in these papers IV confidence intervals almost always include OLS point estimates and there is little statistical evidence of endogeneity and evidence that OLS is seriously biased, based on the low rejection rates of Hausman-type tests in his data. In his simulations based upon the published regressions (Table 14), the rejection frequencies can be as low as 0.232 and 0.382 for 1% and 5% significance levels, respectively, for asymptotic Hausman tests, and even as low as 0.098 and 0.200, respectively, for bootstrap Hausman tests. Similarly, Keane and Neal (2024) argue that a rather strong IV is necessary to give high confidence that 2SLS will outperform OLS (e.g., with a first-stage F higher than 50, which is well above the industry standard of 10).

However, Young (2022)'s finding from the AEA data that OLS estimates seem to be not very different from 2SLS estimates may be attributed to the fact that the used IVs are relatively weak so that 2SLS may be biased towards OLS, and Hausman-type tests also have low power in this case [e.g., see Doko Tchatoka and Dufour (2018, 2024)]. In particular, as shown by Guggenberger (2010a), the Hausman test is not able to reject the null hypothesis of exogeneity in situations where there is only a small degree of endogeneity, i.e., local endogeneity. Then, OLS-based inference is selected in the second stage with high probability. However, the OLS-based t-statistic often takes on very large values even under such local endogeneity, causing extreme size distortions in the two-stage test. Such issue with pretesting for exogeneity is highly relevant to empirical practice as endogeneity is mild in many IV applications. For example, Hansen, Hausman and Newey (2008) report that the median, 75th quantile, and 90th quantile of estimated endogeneity parameters are only 0.279, 0.466, and 0.555, respectively, in their investigated AER, JPE, and QJE papers. Angrist and Kolesár (2023) investigate three influential just-identified IV applicaitions: Angrist and Krueger (1991), Angrist and Evans (1998), Angrist and Lavy (1999), and find that the estimated endogeneity is no more than 0.175, 0.075, and 0.460 for different specifications and samples in these papers, respectively [see Section 3.1 and Table 1 in Angrist and Kolesár (2023)].

Motivated by these issues, we study in this paper the possibility of proposing a uniformly valid method for the above two-stage testing procedure and a closely related Stein-type shrinkage procedure proposed by Hansen (2017). First, we consider an asymptotic framework that allows for weak identification and conditional heteroskedasticity, which are paramount for the methodology to be useful in practice. Second, we propose a novel two-stage test procedure that switches between the OLS-based Wald statistic and the weak-IV-robust statistic such as the Anderson-Rubin statistic. Specifically, the switching is implemented by using a null-imposed Hausman-type Wald statisitc for testing exogeneity. We need to impose the null to ensure the validity of this test statistic even under weak identification. Third, we propose a novel size-corrected wild bootstrap procedure, which combines certain standard wild bootstrap CVs with an appropriate Bonferroni-based size-correction method, following the lead of McCloskey (2017). We show that the resulting CVs are uniformly valid with heteroskedastic errors in the sense that they yield two-stage and shrinkage tests with correct asymptotic size, including the case with weak IVs. In particular, as standard wild bootstrap procedures cannot mimic well the key localized endogeneity parameter, particular attention is taken on this parameter when designing bootstrap DGP, and a Bonferroni-based size-correction technique is implemented to deal with the presence of this localization parameter in the limiting distributions of interest. Different from the conventional Bonferroni bound, which may lead to conservative test with asymptotic size strictly less than the nominal level, the size-correction procedure always leads to desirable asymptotic size.

In terms of practical usage of our method, following the aforementioned studies by Hansen et al. (2008), Young (2022), and Angrist and Kolesár (2023), we are particularly interested in the IV applications where the values of endogeneity parameters are relatively small. These are the cases where the Hausman-type pretest would not reject exogeneity and the naive two-stage procedure would lead to extreme size distortion. On the other hand, as the problem of size distortion is circumvented by our method, we can take advantage of the power superiority of the OLS-based test over its IV counterpart. In addition, Hansen (2017) shows that his shrinkage estimator has substantially reduced median squared error relative to 2SLS, and Doko Tchatoka and Dufour (2024) show that their pretest estimators based on DWH tests can outperform both OLS and 2SLS estimators in terms of mean squared error, even with moderate endogeneity. As such, our proposed method is also attractive from the viewpoint of providing a valid inference method for such shrinkage or pretesting estimator. Monte Carlo experiments confirm that our size-corrected bootstrap procedure achieves reliable size correction and remarkable power gains over the standard weak-identification-robust method. We also note that the size-corrected bootstrap Hansen-type shrinkage procedure has superior finite-sample power performance than its Hausman-type counterpart.

The motivation of using bootstrap in the current testing problem originates from a growing literature illustrating that when applied to IV regressions, well designed bootstrap procedures typically have superior finite-sample performance than asymptotic approximations; see, e.g., Davidson and MacKinnon (2008, 2010), Wang and Kaffo (2016), Kaffo and Wang (2017), Wang and Doko Tchatoka (2018), Finlay and Magnusson (2019), Young (2022), MacKinnon (2023), and Wang and Zhang (2024). Furthermore, we are motivated by the growing literature showing the excellent performance of wild bootstrap methods with heteroskedastic or clustered errors, among them Davidson and Flachaire (2008), Cameron, Gelbach and Miller (2008), MacKinnon and Webb (2017), Djogbenou, MacKinnon and Nielsen (2019), and Mackinnon, Nielsen and Webb (2021, 2023). Our size-correction procedure follows closely the seminal study by McCloskey (2017), who proposed Bonferroni-based size-correction procedures for general nonstandard testing problems, and McCloskey (2020) applied this method to inference in linear regression model after consistent model selection. Additionally, Han and McCloskey (2019) applied it to inference in moment condition models where the estimating function may exhibit mixed identification strength and a nearly singular Jacobian, and Wang and Doko Tchatoka (2018) applied it to weak-identification-robust subvector inference in linear IV models. Different from our bootstrap procedures, these procedures are based on simulations from null limiting distributions.

The remainder of this paper is organized as follows. Section 2 presents the setting, test statistics, and parameter space of interest. Section 3 presents the main results of our size-corrected wild bootstrap methods. Section 4 investigates the finite sample power performance of our methods using simulations. The proofs are provided in the Supplementary Material.

Throughout the paper, for any positive integers n and m, I_n and $0_{n\times m}$ stand for the $n\times n$ identity matrix and $n\times m$ zero matrix, respectively. For any full-column rank $n\times m$ matrix A, $P_A = A(A'A)^{-1}A'$ is the projection matrix on the space spanned by the columns of A, and $M_A = I_n - P_A$. The notation $vec(A)$ is the $nm\times 1$ dimensional column vectorization of A. $\lambda_{min}(A)$ denote the minimum eigenvalue of a square matrix A. $\|U\|$ denotes the usual Euclidean or Frobenius norm for a matrix U. The usual orders of magnitude are denoted by $O_P(.)$ and $o_P(.)$, $\to^P$ stands for convergence in probability, while $\to^d$ stands for convergence in distribution. We write P^* to denote the probability measure induced by a bootstrap procedure conditional on the data, and E^* and Var^* to denote the expected value and variance with respect to P^*. For any bootstrap statistic T^* we write $T^* \to^{P^*} 0$ in probability P if for any $\delta > 0$, $\varepsilon > 0$, $lim_{n\to\infty}P[P^*(|T^*| > \delta) > \varepsilon] = 0$, i.e., $P^*(|T^*| > \delta) = o_P(1)$; e.g., see Gonçalves and White (2004). Also, we write $T^* = O_{P^*}(n^{\varphi})$ in probability P if and only if for any $\delta > 0$ there exists a $M_\delta < \infty$ such that $lim_{n\to\infty}P[P^*(|n^{-\varphi}T^*| > M_\delta) > \delta] = 0$, i.e., for any $\delta > 0$ there exists a $M_\delta < \infty$ such that $P^*(|n^{-\varphi}T^*| > M_\delta) = o_P(1)$. Finally, we write $T^* \to^{d^*} T$ in probability P if, conditional on the data, T^* weakly converges to T under P^*, for all samples contained in a set with probability approaching one.

2. Framework

2.1. Model and test statistics

We consider the following linear IV model

$$y = X\theta + u, \quad X = Z\pi + v, \tag{2.1}$$

where $y \in \mathbb{R}^n$ and $X \in \mathbb{R}^n$ are vectors of dependent and endogenous variables, respectively, $Z \in \mathbb{R}^{n\times k}$ is a matrix of instruments ($k \geq 1$), $(\theta, \pi')' \in \mathbb{R}^{k+1}$ are unknown parameters, and n is the sample size. Denote by u_i, v_i, y_i, X_i, and Z_i the i-th rows of u, v, y, X, and Z respectively, written as column vectors or scalars. For notational simplicity, we assume that the other exogenous variables have already been partialled out from the model.

The object of inferential interest is the structural parameter θ and we consider the problem of testing the null hypothesis $H_0 : \theta = \theta_0$. We study the two-stage testing procedure for assessing H_0, where an exogeneity test is undertaken in the first stage to decide whether an OLS or IV-based method is appropriate for testing H_0 in the second stage. Assume that the instruments Z are exogenous, i.e., $E_F[u_i Z_i] = 0$, where E_F denotes expectation under the distribution F. Under this orthogonality condition of the instruments, X is endogenous in (2.1) if and only if v and u are correlated. Consider the following linear projection of u on v:

$$u = va + e, \; a = (E_F[v_i^2])^{-1} E_F[v_i u_i], \tag{2.2}$$

where e is uncorrelated with v. Notice that the exogeneity of X in (2.1) can be assessed by testing the null hypothesis $H_a : \; a = 0$ in (2.2). Substituting (2.2) into (2.1), we obtain

$$y = X\theta + va + e, \tag{2.3}$$

where X and v are uncorrelated with e. Therefore, the null hypothesis of exogeneity $H_a:\ a=0$ can be assessed using a standard Wald statistic in the extended regression (2.3) [e.g., see Doko Tchatoka and Dufour (2014)]. To account for possible conditional heteroskedasticity, we propose the following control function-based Wald statistic imposing $H_0 : \theta = \theta_0$:[3]

$$H_n(\theta_0) = \hat{a}^2(\theta_0)/\hat{V}_a(\theta_0), \tag{2.4}$$

where $\hat{a} = (\hat{v}'\hat{v})^{-1}\hat{v}'\tilde{y}(\theta_0)$, $\tilde{y}(\theta_0) = y - X\theta_0$, $\hat{V}_a(\theta_0) = (n^{-1}\hat{v}'\hat{v})^{-1}\left(n^{-2}\sum_{i=1}^{n}\hat{v}_i^2\tilde{e}_i^2(\theta_0)\right)(n^{-1}\tilde{v}'\tilde{v})^{-1}$ is the (null-imposed) Eicker-White heteroskedasticity-robust estimator of the variance of $\hat{a}(\theta_0)$, $\hat{v} = M_Z X$, and $\tilde{e}(\theta_0) = M_{\hat{v}}\tilde{y}(\theta_0)$. Note that $\tilde{e}(\theta_0)$ is the residual vector from the OLS regression of $\tilde{y}(\theta_0)$ on $\hat{v}$. When H_0 is true and X is exogenous, $H_n(\theta_0)$ follows a χ_1^2 distribution asymptotically, no matter the IVs are strong or weak. The pretest rejects the null hypothesis that X is exogenous in (2.1) if $H_n(\theta_0) > \chi^2_{1,1-\beta}$, where $\chi^2_{1,1-\beta}$ is the $(1-\beta)$-th quantile of χ_1^2-distributed random variable for some $\beta \in (0,1)$.

Let $\hat{\theta}_{ols} = (X'X)^{-1}X'y$ be the OLS estimator of θ in (2.1). Also, define its corresponding variance estimator and Wald statistic as

$$\begin{aligned} \hat{V}_{ols} &= \left(n^{-1}X'X\right)^{-1}\left(n^{-2}\sum_{i=1}^{n}X_i^2\hat{u}_i^2(\hat{\theta}_{ols})\right)\left(n^{-1}X'X\right)^{-1}, \\ T_{ols}(\theta_0) &= (\hat{\theta}_{ols}-\theta_0)^2/\hat{V}_{ols}, \end{aligned} \tag{2.5}$$

where $\hat{u}_i(\hat{\theta}_{ols}) = y_i - X_i\hat{\theta}_{ols}$, and $\hat{\pi} = (Z'Z)^{-1}Z'X$. In addition, let us define the weak-identification-robust Anderson-Rubin (AR) statistic as:

$$T_{ar}(\theta_0) = \left(n^{-1/2}\hat{u}(\theta_0)'Z\right)\left(\hat{\Omega}(\theta_0)\right)^{-1}\left(n^{-1/2}Z'\hat{u}(\theta_0)\right), \tag{2.6}$$

[3] Alternative formulations of this exogeneity statistic (without imposing $H_0 : \theta = \theta_0$) are given in Hahn, Ham and Moon (2010), Doko Tchatoka and Dufour (2018, 2024) but the Wald version considered in (2.4) easily accommodates conditional heteroskedasticity or clustering, so we shall use this formulation.

where $\hat{u}_i(\theta_0) = y_i - X_i\theta_0$ and $\hat{\Omega}(\theta_0) = n^{-1}\sum_{i=1}^n Z_i Z_i' \hat{u}_i^2(\theta_0)$.

Then, the two-stage test statistic associated with the $H_n(\theta_0)$-based pretest of exogeneity in the first stage is given by

$$T_{1,n}(\theta_0) = T_{ols}(\theta_0)\mathbb{1}(H_n(\theta_0) \leq \chi^2_{1,1-\beta}) + T_{ar}(\theta_0)\mathbb{1}(H_n(\theta_0) > \chi^2_{1,1-\beta}), \tag{2.7}$$

Related to the two-stage procedure, Hansen (2017) proposed a Stein-like shrinkage approach in the context of IV regressions. His estimator follows Maasoumi (1978) in taking a weighted average of the 2SLS and OLS estimators, with the weight depending inversely on the test statistic for exogeneity, and the proposed shrinkage estimator is found to have substantially reduced finite-sample median squared error relative to the 2SLS estimator. Following Hansen (2017)'s approach, we define a Stein-like shrinkage test statistic as follows:

$$T_{2,n}(\theta_0) = T_{ols}(\theta_0)w(H_n(\theta_0)) + T_{ar}(\theta_0)(1 - w(H_n(\theta_0))), \tag{2.8}$$

where the weight function takes the form $w(H_n(\theta_0)) = \begin{cases} \tau/H_n(\theta_0) & \text{if} \quad H_n(\theta_0) \geq \tau \\ 1 \quad \text{if} \quad H_n(\theta_0) < \tau \end{cases}$, and τ is a shrinkage parameter chosen by the researcher. Compared with the Hausman-type procedure, the shrinkage statistic has a relatively smooth transition between the OLS-Wald and AR statistics. In Section 4, we evaluate the finite sample performance of the shrinkage procedure with different choices of τ.

2.2. Parameter space and asymptotic size

Assume that $\{(u_i, v_i, Z_i) : i \leq n\}$ in (2.1) are i.i.d. with distribution F. To characterize the asymptotic size of the two-stage and shrinkage tests, we define the parameter space Γ of the nuisance parameter vector γ following the seminal studies by Andrews and Guggenberger (2009, 2010a, 2010b), Guggenberger (2012), and Guggenberger and Kumar (2012). For the current testing problem, define the vector of nuisance parameters $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ by

$$\gamma_1 = a, \;\; \gamma_2 = (\gamma_{21}, \gamma_{22}, \gamma_{23}, \gamma_{24}, \gamma_{25}), \;\; \gamma_3 = F, \tag{2.9}$$

where a is defined in (2.2), $\gamma_{21} = \pi$, $\gamma_{22} = E_F e_i^2 Z_i Z_i'$, $\gamma_{23} = E_F e_i^2 v_i^2$, $\gamma_{24} = E_F Z_i Z_i'$, and $\gamma_{25} = E_F v_i^2$. Here, γ_1 measures the degree of endogeneity of X and is the key parameter in the current testing problem as it determines the point of discontinuity of the null limiting distributions of the two-stage and shrinkage test statistics. For the parameter space, let

$$\begin{aligned} \mathit{\Gamma}_1 = \mathbb{R}, \;\; \mathit{\Gamma}_2 = \Big\{ & (\gamma_{21}, \gamma_{22}, \gamma_{23}, \gamma_{24}, \gamma_{25}) : \gamma_{21} = \pi \in \mathbb{R}^k, \gamma_{22} = E_F e_i^2 Z_i Z_i' \in \mathbb{R}^{k \times k}, \\ & \gamma_{23} = E_F e_i^2 v_i^2 \in \mathbb{R}, \gamma_{24} = E_F Z_i Z_i' \in \mathbb{R}^{k \times k}, \gamma_{25} = E_F v_i^2 \in \mathbb{R}, \\ & s.t. \;\; \|\gamma_{21}\| \geq 0, \lambda_{min}(\gamma_{22}) \geq \underline{\kappa}, \gamma_{23} > 0, \lambda_{min}(\gamma_{24}) \geq \underline{\kappa}, \text{ and } \gamma_{25} > 0 \Big\}. \end{aligned} \tag{2.10}$$

In addition, $\mathit{\Gamma}_3(\gamma_1, \gamma_2)$ is defined as follows:

$$\begin{aligned} \mathit{\Gamma}_3(\gamma_1, \gamma_2) = \Big\{ & F : \; E_F e_i v_i = E_F e_i Z_i = E_F v_i Z_i = 0, \; E_F e_i^2 v_i Z_i = E_F e_i v_i^2 Z_i = E_F e_i v_i Z_i Z_i' = 0, \\ & E_F v_i^2 Z_i Z_i' \in \mathbb{R}^{k \times k} \;\text{ with }\; \lambda_{min}(E_F v_i^2 Z_i Z_i') \geq M^{-1}, \\ & \left\| E_F \left(||Z_i e_i||^{2+\xi}, ||Z_i v_i||^{2+\xi}, |v_i e_i|^{2+\xi}, ||Z_i Z_i'||^{2+\xi}, |X_i|^{2(2+\xi)} \right)' \right\| \leq M \Big\}, \end{aligned} \tag{2.11}$$

for some constant $\xi > 0$ and $M < \infty$. We then define the whole nuisance parameter space $\mathit{\Gamma}$ of γ as

$$\mathit{\Gamma} = \{\gamma = (\gamma_1, \gamma_2, \gamma_3) : \gamma_1 \in \mathit{\Gamma}_1, \gamma_2 \in \mathit{\Gamma}_2, \gamma_3 \in \mathit{\Gamma}_3(\gamma_1, \gamma_2)\}, \tag{2.12}$$

where $\mathit{\Gamma}_j, \; j = 1,2,3$ are given in (2.10) and (2.11). This nuisance parameter space extends the one defined in Guggenberger (2010a) to allows for conditional heteroskedasticity and is similar to those defined in Guggenberger (2012) and Guggenberger and Kumar (2012), which also allow for heteroskedastic errors. The condition that $E_F e_i^2 v_i Z_i = E_F e_i v_i^2 Z_i = E_F e_i v_i Z_i Z_i' = 0$ in (2.11) is similar to that imposed for $\Gamma_3(\gamma_1,\gamma_2)$ in Guggenberger (2010a) [see (A.2) in the Appendix of his paper for related discussions]. This condition simplifies the limiting distributions and its sufficient condition is, for example, independence between (v_i, e_i) and Z_i.

Now we define the asymptotic size. Let c_n denote a (possibly data-dependent) CV being used for the two-stage testing or shrinkage procedure. The finite sample null rejection probability (NRP) of the test statistic of interest evaluated at $\gamma \in \Gamma$ is given by $P_{\theta_0,\gamma}\left[T_{l,n}(\theta_0) > c_n\right]$ for $l \in \{1,2\}$, where $P_{\theta_0,\gamma}[E_n]$ denotes the probability of event E_n when θ_0 and γ are the true values of the parameters. Then, the asymptotic NRP of the test evaluated at $\gamma \in \Gamma$ is given by $\underset{n\to\infty}{limsup} P_{\theta_0,\gamma}\left[T_{l,n}(\theta_0) > c_n\right]$, while the asymptotic size is given by

$$AsySz[c_n] = \underset{n\to\infty}{limsup}\,\underset{\gamma\in\Gamma}{sup} P_{\theta_0,\gamma}\left[T_{l,n}(\theta_0) > c_n\right]. \tag{2.13}$$

In general, asymptotic NRP evaluated at a given $\gamma \in \Gamma$ is not equal to the asymptotic size of the test. To control the asymptotic size, one needs to control the null limiting behaviour of $T_{l,n}(\theta_0)$ under drifting parameter sequences $\{\gamma_n : n \geq 1\}$ indexed by the sample size; e.g., see Andrews and Guggenberger (2009, 2010a, 2010b), Guggenberger (2012), and Guggenberger and Kumar (2012).

Following the arguments used in these papers, to derive $AsySz[c_n]$ we can study the asymptotic NRP along certain parameter sequences of the type $\{\gamma_{n,h}\}$ (defined below) for some $h \in \mathscr{H}$, as the highest asymptotic NRP is materialized under such sequence, where

$$\begin{aligned}\mathscr{H} \;=\; & \Big\{ h = (h_1, h_{21}', vec(h_{22})', h_{23}, vec(h_{24})', h_{25})' \in \mathbb{R}_\infty^{2k^2+k+3} : \exists \{\gamma_n = (\gamma_{n,1},\gamma_{n,2},\gamma_{n,3}) \in \Gamma : n \geq 1\} \\ & s.t.\ n^{1/2}\gamma_{n,1} \to h_1 \in \mathbb{R}_\infty,\ \gamma_{n,2} \to h_2 = (h_{21},h_{22},h_{23},h_{24},h_{25}),\ \|h_{21}\| \geq 0,\ \lambda_{min}(A) \geq \underline{\kappa}, \\ & \text{for } A \in \{h_{22},h_{24}\},\ h_{23} > 0,\ h_{25} > 0 \Big\} \equiv \mathscr{H}_1 \times \mathscr{H}_{21} \times \mathscr{H}_{22} \times \mathscr{H}_{23} \times \mathscr{H}_{24} \times \mathscr{H}_{25},\end{aligned} \tag{2.14}$$

for some $\underline{\kappa} > 0$ and $\mathbb{R}_\infty = \mathbb{R} \cup \{\pm\infty\}$. Then, for $h \in \mathscr{H}$, the relevant sequence of parameters

$\{\gamma_{n,h}\} \subset \Gamma$ is defined following Guggenberger (2010a) as $\gamma_{n,h} = (\gamma_{n,h,1}, \gamma_{n,h,2}, \gamma_{n,h,3})$ where

$$\gamma_{n,h,1} = (E_{F_n}[v_i^2])^{-1} E_{F_n}[v_i u_i], \;\; \gamma_{n,h,2} = (\gamma_{n,h,21}, \gamma_{n,h,22}, \gamma_{n,h,23}, \gamma_{n,h,24}, \gamma_{n,h,25}), \tag{2.15}$$

with $\gamma_{n,h,21} = \pi_n$, $\gamma_{n,h,22} = E_{F_n} e_i^2 Z_i Z_i'$, $\gamma_{n,h,23} = E_{F_n} e_i^2 v_i^2$, $\gamma_{n,h,24} = E_{F_n} Z_i Z_i'$, $\gamma_{n,h,25} = E_{F_n} v_i^2$, s.t.

$$n^{1/2}\gamma_{n,h,1} \to h_1, \; \gamma_{n,h,2} \to h_2, \text{ and } \gamma_{n,h,3} = F_n \in \Gamma_3(\gamma_{n,h,1}, \gamma_{n,h,2}). \tag{2.16}$$

More specifically, under $H_0 : \theta = \theta_0$ and $\{\gamma_{n,h}\}$ satisfying (2.16) with $|h_1| = \infty$ (i.e., strong endogeneity), $H_n(\theta_0) \to^P \infty$, and the two-stage and shrinkage test statistics are asymptotically equivalent to the AR statistic. On the other hand, under $\{\gamma_{n,h}\}$ satisfying (2.16) with $|h_1| < \infty$ (i.e., local endogeneity), the following joint convergence results hold for $T_{ar}(\theta_0)$, $T_{ols}(\theta_0)$, $H_n(\theta_0)$, and the two-stage and shrinkage statistics $T_{l,n}(\theta_0)$ for $l \in \{1, 2\}$:

$$\frac{1}{\sqrt{n}} \begin{pmatrix} Z'u \\ (v'u - E_{F_n} v'u) \end{pmatrix} \to^d \begin{pmatrix} \psi_{Ze} \\ \psi_{ve} \end{pmatrix} \sim N\left(0, \begin{pmatrix} h_{22} & 0 \\ 0' & h_{23} \end{pmatrix}\right),$$

$$\begin{aligned} \begin{pmatrix} T_{ar}(\theta_0) \\ T_{ols}(\theta_0) \\ H_n(\theta_0) \end{pmatrix} &\to^d \begin{pmatrix} \eta_{1,h} \\ \eta_{2,h} \\ \eta_{3,h} \end{pmatrix} = \begin{pmatrix} \psi_{Ze}' h_{22}^{-1} \psi_{Ze} \\ (h_{21}' h_{22} h_{21} + h_{23})^{-1} (h_{21}' \psi_{Ze} + \psi_{ve} + h_{25} h_1)^2 \\ h_{23}^{-1} (\psi_{ve} + h_{25} h_1)^2 \end{pmatrix}, \\ T_{1,n}(\theta_0) &\to^d \tilde{T}_{1,h} = \eta_{2,h} \mathbb{1}(\eta_{3,h} \leq \chi^2_{1,1-\beta}) + \eta_{1,h} \mathbb{1}(\eta_{3,h} > \chi^2_{1,1-\beta}), \\ T_{2,n}(\theta_0) &\to^d \tilde{T}_{2,h} = \eta_{2,h} w(\eta_{3,h}) + \eta_{1,h}(1 - w(\eta_{3,h})), \end{aligned} \tag{2.17}$$

where $\eta_{1,h} \sim \chi^2_k$, $\eta_{2,h} \sim \chi^2_1\left((h_{21}' h_{22} h_{21} + h_{23})^{-1} h_{25}^2 h_1^2\right)$, $\eta_{3,h} \sim \chi^2_1\left(h_{23}^{-1} h_{25}^2 h_1^2\right)$, $w(\eta_{3,h}) = \tau/\eta_{3,h}$ if $\eta_{3,h} \geq \tau$, and $w(\eta_{3,h}) = 1$ if $\eta_{3,h} < \tau$.

3. Main Results

3.1. Standard wild bootstrap

In this section, we present the standard wild bootstrap procedure and further explain why in general it cannot achieve a correct size control for the two-stage testing and shrinkage procedures.

Wild Bootstrap Algorithm:

1. Compute the (null-restricted) residuals from the first-stage and structural equations: $\hat{v} = X - Z\hat{\pi}$, $\hat{u}(\theta_0) = y - X\theta_0$, where $\hat{\pi} = (Z'Z)^{-1}Z'X$ denotes the least squares estimator of π.

2. Generate the bootstrap pseudo-data following $X^* = Z\hat{\pi} + v^*$, $y^* = X^*\theta_0 + u^*$, where there are two options to generate the bootstrap disturbances:

   (a) v^* and u^* are generated independently from each other. Specifically, in the current case with heteroskedastic data, we set for each observation i: $v_i^* = \hat{v}_i\omega_{1i}^*$, and $u_i^* = \hat{u}_i(\theta_0)\omega_{2i}^*$, where ω_{1i}^* and ω_{2i}^* are two random variables with mean 0 and variance 1, i.e., $E^*\left[\omega_{1i}^*\right] = E^*\left[\omega_{2i}^*\right] = 0$ and $Var^*\left[\omega_{1i}^*\right] = Var^*\left[\omega_{2i}^*\right] = 1$, and they are independent from the data and independent from each other.

   (b) v^* and u^* are drawn dependently from each other. We set for each observation i: $v_i^* = \hat{v}_i\omega_{1i}^*$, and $u_i^* = \hat{u}_i(\theta_0)\omega_{1i}^*$.

   Following Young (2022), we refer to (a) as *independent transformation* of disturbances and (b) as *dependent transformation* of disturbances.[4]

3. Compute the wild bootstrap analogues of the two-stage and shrinkage test statistics:

$$T_{1,n}^*(\theta_0) = T_{ols}^*(\theta_0)\mathbb{1}(H_n^*(\theta_0) \leq \chi_{1,1-\beta}^2) + T_{ar}^*(\theta_0)\mathbb{1}(H_n^*(\theta_0) > \chi_{1,1-\beta}^2),$$

[4]For the purpose of better size control, it is often recommended that for bootstrap exogeneity tests, (u^*, v^*) should be generated using the independent transformation scheme, so that the bootstrap samples are obtained under the null hypothesis of exogeneity. However, this is not necessarily the case for the bootstrap two-stage or shrinkage test statistic.

$$T^*_{2,n}(\theta_0) \;=\; T^*_{ols}(\theta_0)w(H^*_n(\theta_0)) + T^*_{ar}(\theta_0)(1-w(H^*_n(\theta_0))), \tag{3.1}$$

where $w(H^*_n(\theta_0)) = \begin{cases} \tau/H^*_n(\theta_0) & \text{if} \quad H^*_n(\theta_0) \geq \tau \\ 1 & \text{if} \quad H^*_n(\theta_0) < \tau \end{cases}$, $T^*_{ols}(\theta_0)$, $T^*_{ar}(\theta_0)$ and $H^*_n(\theta_0)$ are the bootstrap analogues of $T_{ols}(\theta_0)$, $T_{ar}(\theta_0)$ and $H_n(\theta_0)$, respectively, which are obtained from the bootstrap samples generated in Step 2.

4. For $l \in \{1,2\}$, repeat Steps 2-3 B times and obtain $\{T^{*(b)}_{l,n}(\theta_0), b = 1,...,B\}$. The bootstrap test with the test statistic $T_l(\theta_0)$ rejects H_0 if the corresponding bootstrap p-value $\frac{1}{B}\sum_{b=1}^{B} \mathbb{1}\left[T^{*(b)}_{l,n}(\theta_0) > T_{l,n}(\theta_0)\right]$ is less than the nominal level α.

Following the standard arguments for bootstrap validity, to check whether (conditional on the data) the bootstrap is able to consistently estimate the distribution of the two-stage or shrinkage test statistic, one needs to check whether under H_0 and both cases of strong endogeneity ($|h_1| = \infty$) and local endogeneity ($|h_1| < \infty$), $\sup_{x \in R}\left|P^*\left(T^*_{l,n}(\theta_0) \leq x\right) - P\left(T_{l,n}(\theta_0) \leq x\right)\right| \to^P 0$, for $l \in \{1,2\}$. However, we notice below that neither bootstrap procedure is able to consistently estimate the distribution of interest under local endogeneity.

More specifically, it holds for the bootstrap statistics with dependent or independent transformation (for the dependent transformation, we further require $E^*\left[\omega^{*3}_{1i}\right] = 0$ and $E^*\left[\omega^{*4}_{1i}\right] = 1$) that

$$n^{-1/2}\begin{pmatrix} Z' u^* \\ \left(u^{*'} v^* - E^*[u^{*'} v^*]\right) \end{pmatrix} \to^{d^*} \begin{pmatrix} \psi^*_{Ze} \\ \psi^*_{ve} \end{pmatrix}, \tag{3.2}$$

in probability P (i.e., with probability approaching one according to P), where the bootstrap (conditional) weak limit $(\psi^{*'}_{Ze}, \psi^*_{ve})'$ is the same as $(\psi'_{Ze}, \psi_{ve})'$, i.e., the weak limit of $n^{-1/2}\left((Z'u)', (u'v - E_F[u'v])\right)'$. Therefore, the bootstrap procedures do replicate well the randomness in the original sample.

On the other hand, under local endogeneity the standard wild bootstraps are not able to mimic well the key localization parameter h_1, thus resulting in the discrepancy between the original and bootstrap samples. In particular, let h_1^b denote the localization parameter of endogeneity in the bootstrap world, then $h_1^b = 0$ for the bootstrap with independent transformation, while $h_1^b = h_1 + h_{25}^{-1}\psi_{ve}$ for the one with dependent transformation, where $\psi_{ve} \sim N(0, h_{23})$. That is, while the bootstrap with dependent transformation is able to mimic the situation of local endogeneity in the original sample (h_1^b is finite with probability approaching one when h_1 is finite), the approximation is imprecise and results in an extra error term $h_{25}^{-1}\psi_{ve}$, whose value depends on the actual realization of the sample. Therefore, in general, neither bootstrap procedure is able to achieve a correct size control for the two-stage testing and shrinkage procedures. We show formally the bootstrap failure in Theorem S.5 of the Supplementary Material.

3.2. Size-corrected wild bootstrap

As the standard wild bootstrap procedures are not able to provide uniform size control, in this section we propose Bonferroni-based size-correction methods for the two-stage testing and shrinkage procedures, following the seminal study by McCloskey (2017). As explained in McCloskey (2017), the idea behind such size-correction is to construct CVs that use the data to determine how far the key nuisance parameter (i.e., the endogeneity parameter in the current testing problem) is from the point that causes the discontinuity in the limiting distributions of the test statistics. Although the key nuisance parameter cannot be consistently estimated under the drifting sequences in (2.16), it is still possible to construct an asymptotically valid confidence set for it and then construct adaptive CVs that control the asymptotic size.

First, we will construct a size-corrected wild bootstrap CV by using the wild bootstrap CVs with the independent transformation and Bonferroni bounds. Note that although the localization parameter h_1 cannot be consistently estimated, we may still construct an asymptotically valid confidence set for h_1 by defining $\hat{h}_{n,1}(\theta_0) = n^{1/2}\hat{a}(\theta_0)$, where $\hat{a}(\theta_0) = (\hat{v}'\hat{v})^{-1}\hat{v}'\tilde{y}(\theta_0)$. A confidence set of h_1 can be constructed by using the fact that under the drifting parameter sequences and H_0,

$$\hat{h}_{n,1}(\theta_0) \to^d \tilde{h}_1 \sim N\Big(h_1, h_{25}^{-2} h_{23}\Big). \tag{3.3}$$

Then, uniformly valid size-corrected bootstrap CVs for testing $H_0 : \theta = \theta_0$ under the two-stage or shrinkage procedure can be constructed by using Bonferroni bounds: we may construct a $1-(\alpha-\delta)$ level first-stage confidence set for h_1, and then take the maximal $(1-\delta)$-th quantile of appropriately generated bootstrap statistics over the first-stage confidence set. Specifically, let $\hat{h}_{n,2} = \Big(\hat{h}'_{n,21}, vec(\hat{h}_{n,22})', \hat{h}_{n,23}, vec(\hat{h}_{n,24})', \hat{h}_{n,25}\Big)'$ be the estimators of $h_2 = (h'_{21}, vec(h_{22})', h_{23}, vec(h_{24})', h_{25})'$, and define the $1-(\alpha-\delta)$ level confidence set of h_1 for some $0 < \delta \leq \alpha < 1$ as

$$CI_{\alpha-\delta}(\hat{h}_{n,1}(\theta_0)) = \Big[\hat{h}_{n,1}(\theta_0) - z_{1-(\alpha-\delta)/2} \cdot \big(n\hat{V}_a(\theta_0)\big)^{1/2},\ \hat{h}_{n,1}(\theta_0) + z_{1-(\alpha-\delta)/2} \cdot \big(n\hat{V}_a(\theta_0)\big)^{1/2}\Big],$$

where $\hat{V}_a(\theta_0)$ is defined in (2.4). The wild bootstrap-based simple Bonferroni critical value (SBCV) is defined as

$$c_l^{B-S}(\alpha, \alpha-\delta, \hat{h}_{n,1}(\theta_0), \hat{h}_{n,2}) = \sup_{h_1 \in CI_{\alpha-\delta}(\hat{h}_{n,1}(\theta_0))} c^*_{l,(h_1,\hat{h}_{n,2})}(1-\delta), \tag{3.4}$$

for $l \in \{1,2\}$, where $c^*_{l,(h_1,\hat{h}_2)}(1-\delta)$ is the $(1-\delta)$-th quantile of the distribution of $T^*_{l,n,(h_1,\hat{h}_{n,2})}(\theta_0)$, i.e., the distribution of the bootstrap analogue of $T_{l,n}(\theta_0)$ generated under the value of localization parameter equal to h_1.

As we have seen in the previous section, the standard wild bootstrap procedures cannot mimic well the localization parameter h_1, no matter with independent or dependent transformation. Therefore, attention has to be taken when considering the bootstrap DGP. In particular, we propose to generate the bootstrap statistics under the localization parameter h_1 as follows:

$$T^*_{1,n,(h_1,\hat{h}_{n,2})}(\theta_0) = T^*_{ols,(h_1,\hat{h}_{n,2})}(\theta_0)\mathbb{1}\left(H^*_{n,(h_1,\hat{h}_{n,2})}(\theta_0) \le \chi^2_{1,1-\beta}\right) + T^*_{ar}(\theta_0)\mathbb{1}\left(H^*_{n,(h_1,\hat{h}_{n,2})}(\theta_0) > \chi^2_{1,1-\beta}\right),$$
$$T^*_{2,n,(h_1,\hat{h}_{n,2})}(\theta_0) = T^*_{ols,(h_1,\hat{h}_{n,2})}(\theta_0)w\left(H^*_{n,(h_1,\hat{h}_{n,2})}(\theta_0)\right) + T^*_{ar}(\theta_0)\left(1 - w\left(H^*_{n,(h_1,\hat{h}_{n,2})}(\theta_0)\right)\right), \quad (3.5)$$

where $T^*_{ols,(h_1,\hat{h}_{n,2})}(\theta_0)$ and $H^*_{n,(h_1,\hat{h}_{n,2})}(\theta_0)$ are the bootstrap analogues of $T_{ols}(\theta_0)$ and $H_n(\theta_0)$, respectively, evaluated at the value of localization parameter equal to h_1. More precisely, to obtain these bootstrap analogues, we first generate the bootstrap counterparts of the OLS and regression endogeneity parameter estimators under h_1:

$$\hat{\theta}^*_{ols,(h_1,\hat{h}_{n,2})} = \hat{\theta}^*_{ols} + \left(\hat{h}'_{n,21}\hat{h}_{n,24}\hat{h}_{n,21} + \hat{h}_{n,25}\right)^{-1}\hat{h}_{n,25}\left(n^{-1/2}h_1\right), \quad \hat{a}^*_{(h_1,\hat{h}_{n,2})} = \hat{a}^* + n^{-1/2}h_1, \quad (3.6)$$

where $\hat{\theta}^*_{ols}$ and $\hat{a}^*$ are generated by the standard wild bootstrap procedure in Section 3.1 with *independent transformation* of disturbances, so that $\hat{\theta}^*_{ols}$ and $\hat{a}^*$ have localization parameter equal to zero in the bootstrap world. By doing so, $\sqrt{n}\left(\hat{\theta}^*_{ols,(h_1,\hat{h}_{n,2})} - \theta_0\right)$ and $\sqrt{n}\hat{a}^*_{(h_1,\hat{h}_{n,2})}$ have appropriate null limiting distribution conditional on the data. Then, we obtain

$$T^*_{ols,(h_1,\hat{h}_{n,2})}(\theta_0) = (\hat{\theta}^*_{ols,(h_1,\hat{h}_{n,2})} - \theta_0)/\hat{V}^{*1/2}_{ols}, \quad H^*_{n,(h_1,\hat{h}_{n,2})}(\theta_0) = \hat{a}^{*2}_{(h_1,\hat{h}_{n,2})}/\hat{V}^*_a, \quad (3.7)$$

and we can show that the following (conditional) convergence in distribution holds:

$$\begin{pmatrix} T^*_{ols,(h_1,\hat{h}_{n,2})}(\theta_0) \\ H^*_{n,(h_1,\hat{h}_{n,2})}(\theta_0) \end{pmatrix} \to^{d^*} \begin{pmatrix} (h'_{21}h_{22}h_{21} + h_{23})^{-1}(h'_{21}\psi^*_{Ze} + \psi^*_{ve} + h_{25}h_1)^2 \\ h_{23}^{-1}(\psi^*_{ve} + h_{25}h_1)^2 \end{pmatrix},$$

in probability P, where ψ^*_{Ze} and ψ^*_{ve} are the bootstrap analogues of ψ_{Ze} and ψ_{ve}, respectively. This implies that $T^*_{1,n,(h_1,\hat{h}_{n,2})}(\theta_0)$ and $T^*_{2,n,(h_1,\hat{h}_{n,2})}(\theta_0)$, the resulting bootstrap counterparts of the two-stage and shrinkage test statistics, have the desired null limiting distributions evaluated at the value of localization parameter equal to h_1.

As seen from (3.4), the bootstrap SBCV equals the maximal quantile $c^*_{l,(h_1,\hat{h}_{n,2})}(1-\delta)$ over the values of the localization parameter h_1 in the set $CI_{\alpha-\delta}(\hat{h}_{n,1}(\theta_0))$. We can now state the following asymptotic size result for $c_l^{B-S}(\alpha,\alpha-\delta,\hat{h}_{n,1},\hat{h}_{n,2})$, where $l\in\{1,2\}$.

Theorem 3.1 *Suppose that H_0 holds, then we have for any $0<\delta\leq\alpha<1$ and for $l\in\{1,2\}$, $AsySz\left[c_l^{B-S}(\alpha,\alpha-\delta,\hat{h}_{n,1}(\theta_0),\hat{h}_{n,2})\right]\leq\alpha$.*

Theorem **3.1** states that tests based on $c_l^{B-S}(\alpha,\alpha-\delta,\hat{h}_{n,1}(\theta_0),\hat{h}_{n,2})$ control the asymptotic size. In practice, $c_l^{B-S}(\alpha,\alpha-\delta,\hat{h}_{n,1}(\theta_0),\hat{h}_{n,2})$ can be obtained by using the following algorithm.

Wild Bootstrap Algorithm for $\boldsymbol{c_l^{\text{B-S}}(\alpha,\alpha-\delta,\hat{h}_{\text{n,1}}(\theta_0),\hat{h}_{\text{n,2}})}$:

1. Generate the bootstrap statistics $\left\{\hat{\theta}^{*(b)}_{ols},\hat{a}^{*(b)},\hat{V}^{*(b)}_{ols},\hat{V}^{*(b)}_{a},T^{*(b)}_{ar}(\theta_0)\right\},b=1,...,B$, using the standard wild bootstrap procedure with independent transformation of disturbances.

2. Choose α, δ, and compute $CI_{\alpha-\delta}(\hat{h}_{n,1}(\theta_0))$. Create a fine grid for $CI_{\alpha-\delta}(\hat{h}_{n,1}(\theta_0))$ and call it $\mathscr{C}^{grid}_{\alpha-\delta}$.

3. For $l\in\{1,2\}$ and for $h_1\in\mathscr{C}^{grid}_{\alpha-\delta}$, generate $T^{*(b)}_{l,n,(h_1,\hat{h}_{n,2})}(\theta_0)$, $b=1,...,B$, using the bootstrap statistics generated in Step 1. The same set of bootstrap statistics can be used repeatedly for each h_1.

4. Compute $c^*_{l,(h_1,\hat{h}_{n,2})}(1-\delta)$, the $(1-\delta)$-th quantile of the distribution of $T^*_{l,n,(h_1,\hat{h}_2)}(\theta_0)$ from these B draws of bootstrap samples.

5. Find $c_l^{B-S}(\alpha,\alpha-\delta,\hat{h}_{n,1}(\theta_0),\hat{h}_{n,2})=\sup_{h_1\in\mathscr{C}^{grid}_{\alpha-\delta}}c^*_{l,(h_1,\hat{h}_{n,2})}(1-\delta)$.

Note that as shown in Theorem **3.1**, although controlling the asymptotic size, the bootstrap SBCV may yield a conservative test whose asymptotic size does not reach its nominal level. For further refinement on the Bonferroni bounds, we propose a size-adjustment method to adjust the bootstrap SBCV so that the resulting test is not conservative with asymptotic size exactly equal to

α. Specifically, for $l \in \{1,2\}$, the size-adjustment factor for the bootstrap SBCV is defined as:

$$\hat{\eta}_{l,n} = \inf\left\{\eta : \sup_{h_1 \in \mathscr{H}_1} P^*\left[T^*_{l,n,(h_1,\hat{h}_{n,2})}(\theta_0) > c_l^{B-S}(\alpha, \alpha-\delta, \hat{h}^*_{n,1}(h_1), \hat{h}_{n,2}) + \eta\right] \leq \alpha\right\}, \quad (3.8)$$

where $\hat{h}^*_{n,1}(h_1)$ denotes the bootstrap analogue of $\hat{h}_{n,1}$ with localization parameter equal to h_1 and is generated by the same bootstrap samples as those for $T^*_{n,(h_1,\hat{h}_{n,2})}(\theta_0)$. More precisely, we define

$$\hat{h}^*_{n,1}(h_1) = \hat{h}^*_{n,1} + h_1, \quad (3.9)$$

where $\hat{h}^*_{n,1} = n^{1/2}\hat{a}^* = \left(\hat{v}^{*\prime}\hat{v}^*\right)^{-1}\hat{v}^{*\prime}u^*$, $\hat{v}^* = M_Z X^*$, is generated by the standard wild bootstrap procedure with independent transformation so that the localization parameter equals zero in the bootstrap world. Notice that we have the following convergence in distribution (jointly with the other bootstrap statistics), $\hat{h}^*_{n,1}(h_1) \to^{d^*} N\left(h_1, h_{25}^{-2}h_{23}\right)$, in probability P, i.e., the same limiting distribution as that of $\hat{h}_{n,1}(\theta_0)$ in (3.3).

The goal of the size-adjustment method is to decrease the bootstrap SBCV as much as possible by using the factor η while not violating the inequality in (3.8), so that the asymptotic size of the resulting tests can be controlled. Then, the bootstrap size-adjusted CV (BACV) can be defined as

$$\begin{aligned} & c_l^{B-A}(\alpha, \alpha-\delta, \hat{h}_{n,1}(\theta_0), \hat{h}_{n,2}) \\ = \; & c_l^{B-S}(\alpha, \alpha-\delta, \hat{h}_{n,1}(\theta_0), \hat{h}_{n,2}) + \hat{\eta}_{l,n} \text{ for } l \in \{1,2\}, \end{aligned} \quad (3.10)$$

and one can expect that relatively small $\hat{\eta}_{l,n}$ results in relatively less conservative (and more powerful) test. Under a proper algorithm for the size-adjustment method, and given some fixed $\alpha \in (0,1)$ and $\delta \in (0,\alpha]$, the size-adjustment factor $\hat{\eta}_{l,n}(\cdot)$ is continuous as a function of $\hat{h}^*_{n,1}(h_1)$. Furthermore, we notice that the bootstrap-based size-adjustment method in (3.10) is in the same spirit as the adjusted Bonferroni CV proposed in McCloskey (2017, Section 3.2), which is based on adjusting the quantile level of the underlying localized quantile in the simple Bonferroni CV.

Below we state the theorem on the uniform size control of the wild bootstrap CVs based on the size-adjustment method, and we assume a continuity condition on the NRP function, following similar continuity assumptions in Andrews and Cheng [2012, p.2195, Assumption Rob2(i)] and Han and McCloskey [2019, p.1052, Assumption DF2(ii)]. Define $c_l^{B-S}(\alpha, \alpha-\delta, \tilde{h}_1, h_2) = \sup_{h_1 \in CI_{\alpha-\delta}(\tilde{h}_1)} c_{l,h}(1-\delta)$, where $c_{l,h}(1-\delta)$ is the $(1-\delta)$-th quantile of $\tilde{T}_{l,h}$ and $\tilde{T}_{l,h}$ is the weak limit of $T_{l,n}(\theta_0)$ under the sequence $\{\gamma_{n,h}\} \subset \Gamma$ satisfying (2.16) for $l \in \{1,2\}$.

Assumption 3.2 *$P\left[\tilde{T}_{l,h} = c_l^{B-S}(\alpha, \alpha-\delta, \tilde{h}_1, h_2) + \eta\right] = 0$, $\forall h_1 \in H_1$ and $\eta \in [-c_l^{B-S}(\alpha, \alpha-\delta, \tilde{h}_1, h_2), 0]$, where $l \in \{1,2\}$.*

Theorem 3.3 *Suppose that H_0 and Assumption* **3.2** *hold, then we have for any $0 < \delta \leq \alpha < 1$ and for $l \in \{1,2\}$: $AsySz\left[c_l^{B-A}(\alpha, \alpha-\delta, \hat{h}_{n,1}(\theta_0), \hat{h}_{n,2})\right] = \alpha$.*

Furthermore, let $CS_{l,n}(1-\alpha)$ denote the nominal level $1-\alpha$ confidence set for θ constructed by collecting all the values of θ that cannot be rejected by the corresponding size-adjusted two-stage or shrinkage test at nominal level α.

Corollary 3.4 *Suppose that Assumption* **3.2** *holds, then we have for any $0 < \delta \leq \alpha < 1$ and for $l \in \{1,2\}$: $\liminf_{n\to\infty} \inf_{\gamma \in \Gamma} P_{\theta,\gamma}\left[\theta \in CS_{l,n}(1-\alpha)\right] = 1-\alpha$.*

Theorem **3.3** shows that $c_l^{B-A}(\alpha, \alpha-\delta, \hat{h}_{n,1}(\theta_0), \hat{h}_{n,2})$ yield two-stage and shrinkage tests with the correct asymptotic size, and Corollary **3.4** states that the confidence sets constructed from inverting these tests have correct asymptotic coverage probability.[5] To implement such size-adjusted tests in practice, we must compute $c_l^{B-S}(\alpha, \alpha-\delta, \hat{h}_{n,1}(\theta_0), \hat{h}_{n,2})$ and $\hat{\eta}_{l,n}$. These values can be computed sequentially starting with $c_l^{B-S}(\alpha, \alpha-\delta, \hat{h}_{n,1}(\theta_0), \hat{h}_{n,2})$. Then the size-adjustment factor $\hat{\eta}_{l,n}$ can be computed by evaluating (3.8) over a fine grid of $\mathscr{H}_1$ as follows.

Wild Bootstrap Algorithm for $c_l^{\text{B-A}}(\alpha, \alpha-\delta, \hat{h}_{n,1}(\theta_0), \hat{h}_{n,2})$:

[5] Also see, e.g., Section 6 in Davidson and MacKinnon (2010) and Section 3.5 in Roodman, Nielsen, MacKinnon and Webb (2019) for detailed guidance on constructing confidence set from inverting a wild bootstrap test.

1. Generate the bootstrap statistics $\left\{\hat{\theta}^{*(b)}_{ols}, \hat{a}^{*(b)}, \hat{V}^{*(b)}_{ols}, \hat{V}^{*(b)}_{a}, T^{*(b)}_{ar}(\theta_0)\right\}, b=1,...,B$, using the standard wild bootstrap procedure with independent transformation.

2. For $l \in \{1,2\}$, let $c^{B-S}_{l}(\alpha, \alpha-\delta, \hat{h}_{n,1}(\theta_0), \hat{h}_{n,2})$ be the obtained SBCV.

3. Create a fine grid of the set $\mathscr{H}_1$ in (3.8) and call it $\mathscr{H}^{grid}_1$. For $l \in \{1,2\}$ and for each $h_1 \in \mathscr{H}^{grid}_1$, obtain $T^{*(b)}_{l,n,(h_1,\hat{h}_{n,2})}(\theta_0)$ and $c^{B-S}_{l}(\alpha, \alpha-\delta, \hat{h}^{*(b)}_{n,1}(h_1), \hat{h}_{n,2})$, $b=1,...,B$, using the bootstrap statistics generated in Step 1. Note that the same set of bootstrap statistics can be used for each h_1.

4. Create a fine grid of $[-c^{B-S}_{l}(\alpha, \alpha-\delta, \hat{h}_{n,1}(\theta_0), \hat{h}_{n,2}), 0]$ and call it $\mathbb{S}^{grid}$.

5. Find all $\eta \in \mathbb{S}^{grid}$ s.t. $\sup_{h_1 \in \mathscr{H}^{grid}_1} \frac{1}{B}\sum_{b=1}^{B} \mathbb{1}\left[T^{*(b)}_{l,n,(h_1,\hat{h}_{n,2})}(\theta_0) > c^{B-S}_{l}(\alpha, \alpha-\delta, \hat{h}^{*(b)}_{n,1}(h_1), \hat{h}_{n,2}) + \eta\right] \leq \alpha$, and set $\hat{\eta}_{l,n}$ equal to the smallest η.

6. The BACV is given by $c^{B-A}_{l}(\alpha, \alpha-\delta, \hat{h}_{n,1}(\theta_0), \hat{h}_{n,2}) = c^{B-S}_{l}(\alpha, \alpha-\delta, \hat{h}_{n,1}(\theta_0), \hat{h}_{n,2}) + \hat{\eta}_{l,n}$.

We emphasize that $\hat{h}^{*}_{n,1}(h_1)$ needs to be generated simultaneously with $T^{*}_{l,n,(h_1,\hat{h}_{n,2})}(\theta_0)$ using the same bootstrap samples, so that the dependence structure between the statistics $T_{l,n}(\theta_0)$ and $\hat{h}_{n,1}(\theta_0)$ is well mimicked by the bootstrap statistics. This is important for the size-adjustment procedure to correct the conservativeness of the Bonferroni bound. Similarly, for the implementation of the size-adjustment, one cannot replace $c^{B-S}_{l}(\alpha, \alpha-\delta, \hat{h}^{*}_{n,1}(h_1), \hat{h}_{n,2})$ in (3.8) with $c^{B-S}_{l}(\alpha, \alpha-\delta, \hat{h}_{n,1}(\theta_0), \hat{h}_{n,2})$, as it also breaks down the dependence structure.

4. Finite sample power performance

In this section, we study the finite-sample power performance of the size-corrected wild bootstrap procedure by conducting simulations for the linear IV model under conditional heteroskedasticity. For all simulations, the number of Monte Carlo replications is set at $5,000$, and the number of bootstrap replications is set at $B = 399$. We compare the performance of the AR-based wild bootstrap test (without pretest or shrinkage), our two-stage test based on the size-adjusted wild bootstrap CVs, and our test that is based on Hansen (2017)'s shrinkage apporach and its corresponding size-adjusted wild bootstrap CVs. We set $\alpha = .05$ for the CVs of the three tests. In addition, we set $\beta = .05$ for the nominal level of the pretest. The algorithms for the size-adjusted wild bootstrap CVs are executed with $\delta = \alpha - \alpha/10 = .045$, following the recommendation in McCloskey (2017). As explained by McCloskey (2017, Section 3.5), this choice of δ tends to have good power performance in both regions of the parameter space in which the key nuisance parameter (i.e., γ_1 or γ_1^c in the current context) is far from zero and those in which it is close to zero. The shrinkage parameter τ in Hansen (2017)'s procedure is set to equal $1/2, 1/3$, or $1/4$. The random weights for the wild bootstrap are generated from the standard normal distribution throughout the simulations.

The simulation model follows the IV model in (2.1), and the DGP is specified as

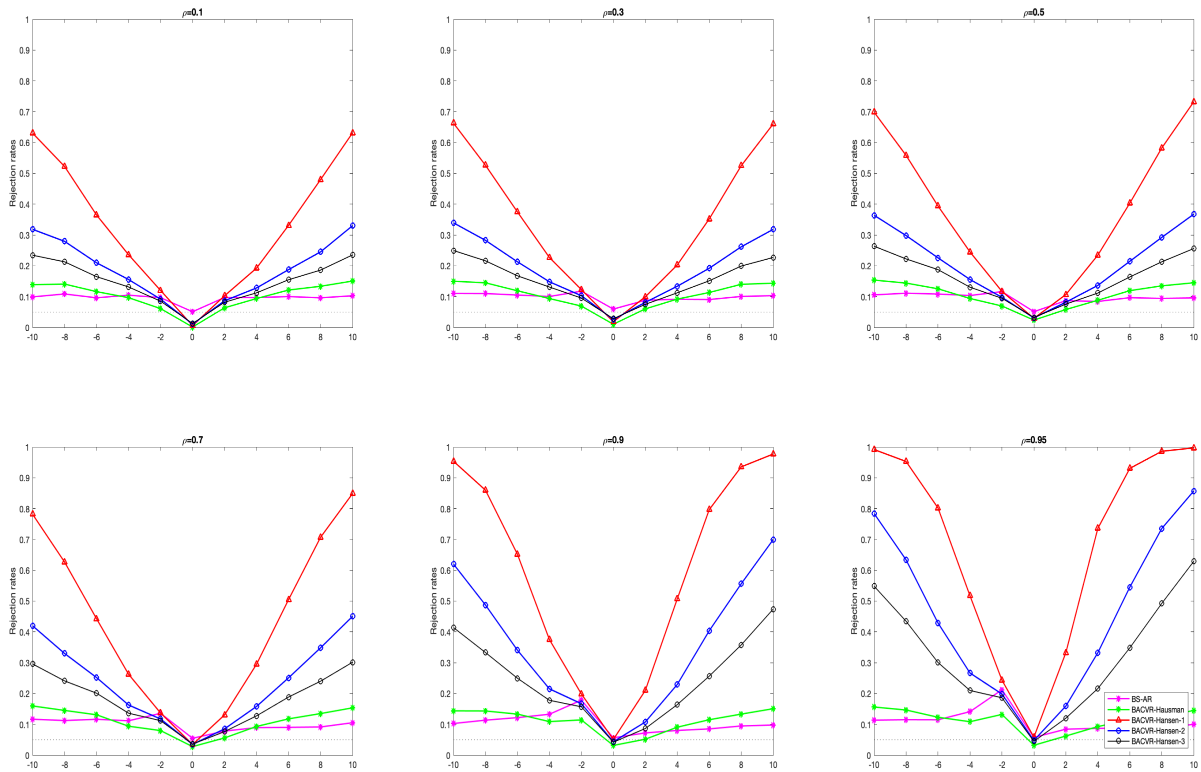
$$\begin{aligned}
&(\tilde{u}_i, \tilde{\varepsilon}_i)' \sim i.i.d.\, N(0, I_2),\ Z_i \sim i.i.d.\, N(0,1) \text{ and is independent from } (\tilde{u}_i, \tilde{\varepsilon}_i)', \\
&\tilde{v}_i = \rho \tilde{u}_i + (1-\rho^2)^{1/2} \tilde{\varepsilon}_i,\ u_i = f(Z_i)\tilde{u}_i, \text{ and } v_i = f(Z_i)\tilde{v}_i,
\end{aligned} \tag{4.11}$$

where $i = 1, ..., n$ and $f(x) = |x|$. The sample size is set at $n = 100$. The value of the null hypothesis θ_0 is fixed at zero throughout the simulations. Following the IV literature, we capture the instrument strength by the concentration parameter $\phi = \pi^2 \cdot Z'Z$ and let $\phi \in \{1, 10, 25, 50\}$. In addition, the true values of the endogeneity parameter are set at $\rho \in \{0.1, 0.3, 0.5, 0.7, 0.9, 0.95\}$.

Figures 1-4 show the finite-sample power curves of the tests. We highlight some findings below. First, it is clear that our size-adjusted bootstrap tests have remarkable power gain over the AR-based bootstrap test, especially when the IV is rather weak (e.g., $\phi \in \{1, 10\}$). Such power gain originates from the inclusion of the OLS-based Wald test in the two-stage and shrinkage test statistics. Second, we notice that the shrinkage bootstrap tests (in red, blue, and black) have power advantage over the two-stage bootstrap test (in green), especially for distant alternative hypotheses.

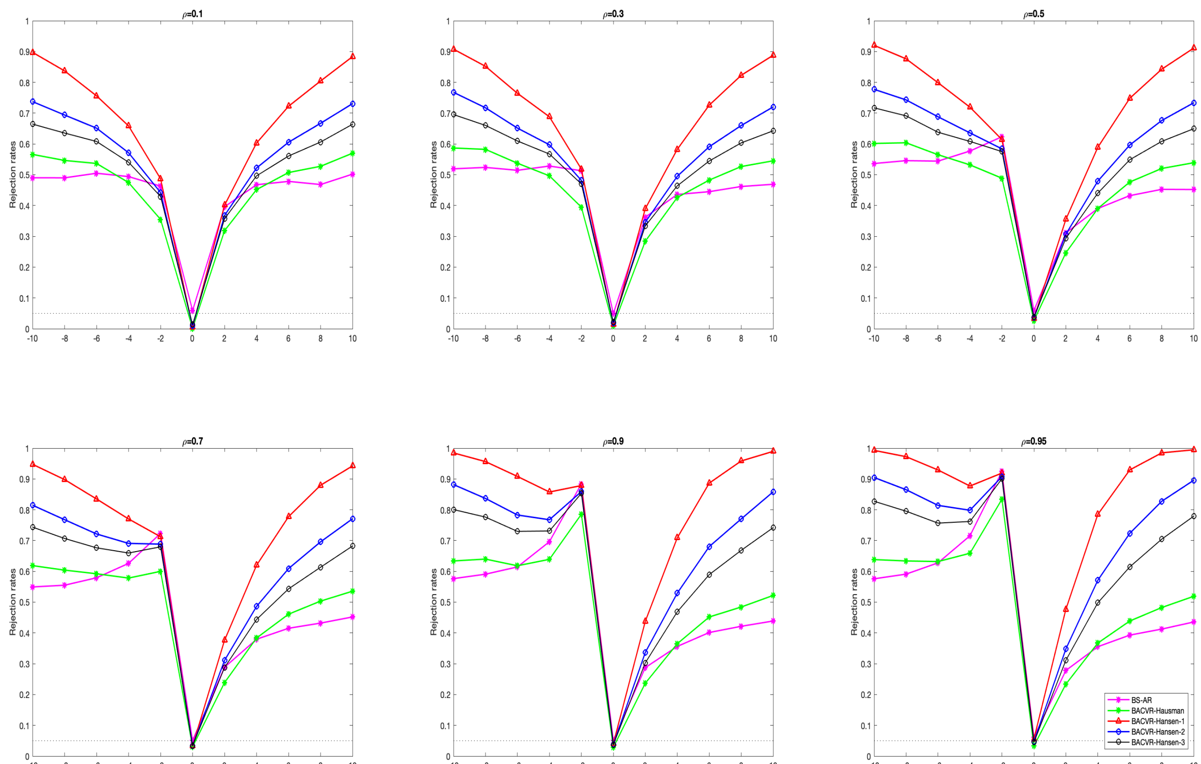
Third, the shrinkage bootstrap test with $\tau = 1/2$ typically has the best power performance among the size-adjusted bootstrap tests. Furthermore, the standard bootstrap AR test has good size control across different settings, as it is weak-IV-robust. However, although its size is well controlled, its power can also be rather low when the identification is weak. On the other hand, our proposed bootstrap tests have the advantage of providing power improvement by incorporating the OLS-based Wald-test.

Figure 1: Power of wild bootstrap tests with $\phi = 1$



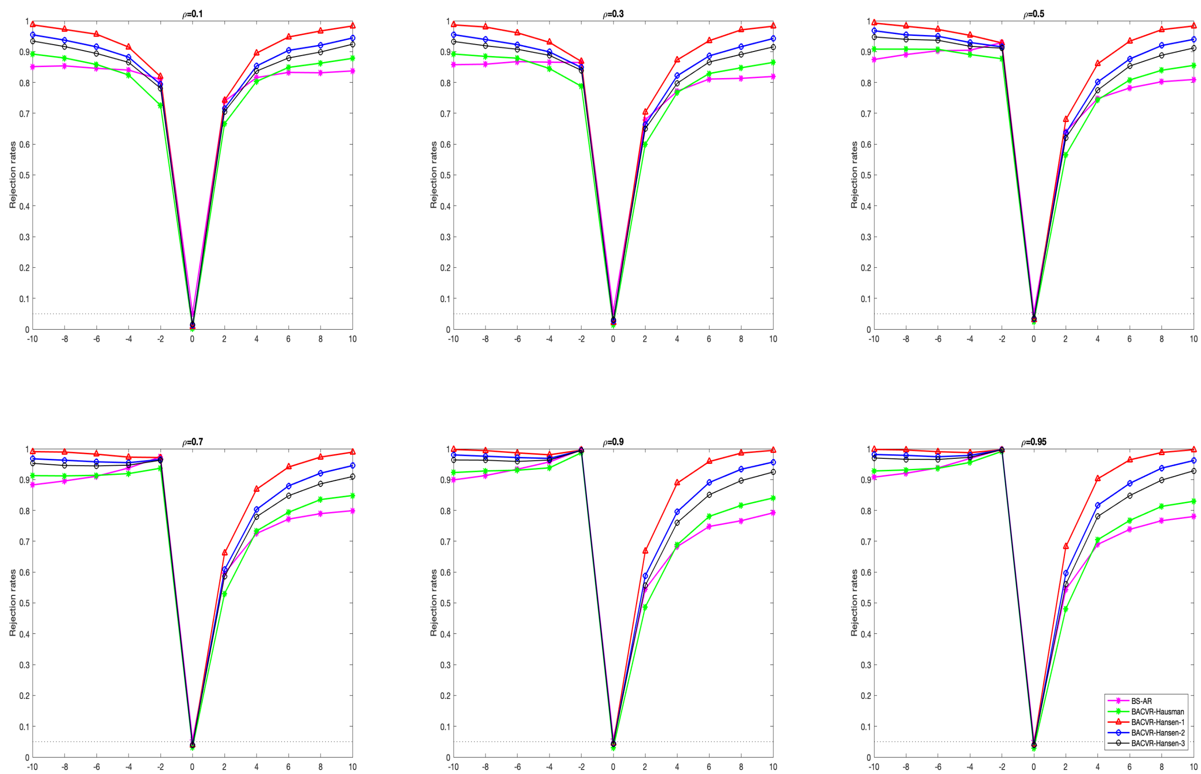
Notes: The power curves for the bootstrap AR test, the two-stage test with hybrid-BACVs, and the shrinkage test with BACVs with $\tau = 1/2, 1/3, 1/4$ are illustrated by the curves in pink, green, red, blue, and black, respectively.

Figure 2: Power of wild bootstrap tests with $\phi = 10$



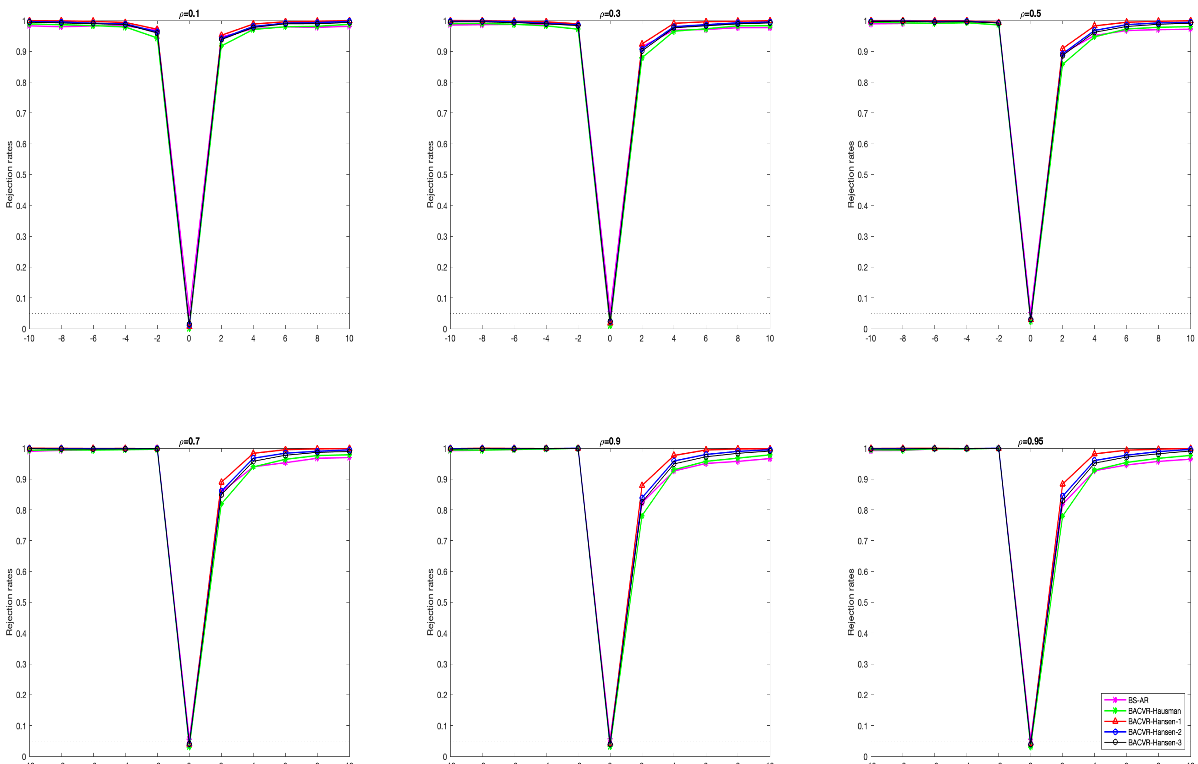
Notes: The power curves for the bootstrap AR test, the two-stage test with hybrid-BACVs, and the shrinkage test with BACVs with $\tau = 1/2, 1/3, 1/4$ are illustrated by the curves in pink, green, red, blue, and black, respectively.

Figure 3: Power of wild bootstrap tests with $\phi = 25$



Notes: The power curves for the bootstrap AR test, the two-stage test with hybrid-BACVs, and the shrinkage test with BACVs with $\tau = 1/2, 1/3, 1/4$ are illustrated by the curves in pink, green, red, blue, and black, respectively.

Figure 4: Power of wild bootstrap tests with $\phi = 50$



Notes: The power curves for the bootstrap AR test, the two-stage test with hybrid-BACVs, and the shrinkage test with BACVs with $\tau = 1/2, 1/3, 1/4$ are illustrated by the curves in pink, green, red, blue, and black, respectively.

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Supplementary Material for

"Weak-Identification-Robust Bootstrap Tests after Pretesting for Exogeneity "

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In this Supplementary Material, Section S.1 contains several technical lemmas. Section S.2 contains the proofs of the theorems in the main text. Section S.3 presents the details of the bootstrap inconsistency under local endogeneity.

S.1. Technical Lemmas

The following lemma gives the limiting distributions of the estimators and test statistics under the sequences of drifting endogeneity parameter $n^{1/2}\gamma_{n,h,1} \to h_1 \in R$.

Lemma S.1 *Under H_0 and the drift sequences of parameters $\{\gamma_{n,h}\}$ in (2.16) with $|h_1| < \infty$, the following results hold*:

(a) *Asymptotic distributions of the estimators:*

$$\begin{pmatrix} n^{1/2}\hat{a}(\theta_0) \\ n^{1/2}(\hat{\theta}_{ols} - \theta) \end{pmatrix} \to^d \begin{pmatrix} \psi_a \\ \psi_{ols} \end{pmatrix} = \begin{pmatrix} h_{25}^{-1}\psi_{ve} + h_1 \\ (h_{21}' h_{24} h_{21} + h_{25})^{-1}\,(h_{21}'\psi_{Ze} + \psi_{ve} + h_{25}h_1) \end{pmatrix},$$

where $\psi_a \sim N\Big(h_1, h_{25}^{-2}h_{23}\Big)$, and $\psi_{ols} \sim N\Big(h_{25}h_1/\big(h_{21}'h_{24}h_{21} + h_{25}\big), (h_{21}'h_{22}h_{21} + h_{23})/\big(h_{21}'h_{24}h_{21} + h_{25}\big)^2\Big)$.

(b) *Asymptotic distributions of the test statistics:*

$$\begin{pmatrix} T_{ar}(\theta_0) \\ T_{ols}(\theta_0) \\ H_n(\theta_0) \end{pmatrix} \to^d \eta_h = \begin{pmatrix} \eta_{1,h} \\ \eta_{2,h} \\ \eta_{3,h} \end{pmatrix}$$

$$= \begin{pmatrix} \psi_{Ze}' h_{22} \psi_{Ze} \\ (h_{21}' h_{22} h_{21} + h_{23})^{-1} (h_{21}' \psi_{Ze} + \psi_{ve} + h_{25} h_1)^2 \\ h_{23}^{-1} (\psi_{ve} + h_{25} h_1)^2 \end{pmatrix}$$

$$T_{1,n}(\theta_0) \to^d \tilde{T}_{1,h} = \eta_{2,h} \mathbb{1}(\eta_{3,h} \leq \chi^2_{1,1-\beta}) + \eta_{1,h} \mathbb{1}(\eta_{3,h} > \chi^2_{1,1-\beta}),$$

$$T_{2,n}(\theta_0) \to^d \tilde{T}_{2,h} = \eta_{2,h} w(\eta_{3,h}) + \eta_{1,h}(1 - w(\eta_{3,h})),$$

where $\eta_{1,h} \sim \chi^2_k$, $\eta_{2,h} \sim \chi^2_1 \Big((h_{21}' h_{22} h_{21} + h_{23})^{-1} h_{25}^2 h_1^2 \Big)$, *and* $\eta_{3,h} \sim \chi^2_1 \Big(h_{23}^{-1} h_{25}^2 h_1^2 \Big)$.

PROOF OF LEMMA **S.1** (a) It is sufficient to characterize the asymptotic distributions of estimators separately: (a1) $n^{1/2}\hat{a}(\theta_0)$, and (a2) $n^{1/2}(\hat{\theta}_{ols} - \theta)$.

(a1) Asymptotic distribution of $n^{1/2}\hat{a}(\theta_0)$. First, note that for the denominator,

$$n^{-1}\hat{v}'\hat{v} = n^{-1}X' M_Z X \to^P h_{25}. \tag{S.1}$$

Second, for the numerator, we have

$$n^{-1/2}\hat{v}' e = n^{-1/2} v' M_Z e = n^{-1/2} v' e - n^{-1/2} v' P_Z e = n^{-1/2} v' e + o_P(1) \to^d \psi_{ve}, \tag{S.2}$$

by applying Lyapunov Central Limit Theorem (CLT), where $\psi_{ve} \sim N(0, h_{23})$. Therefore, we obtain

$$n^{1/2}(\hat{a}(\theta_0) - \gamma_{n,h,1}) \to^d h_{25}^{-1} \psi_{ve} \sim N\left(0, h_{25}^{-2} h_{23}\right). \tag{S.3}$$

Since $n^{1/2}\hat{a}(\theta_0) = n^{1/2}(\hat{a}(\theta_0) - \gamma_{n,h,1}) + n^{1/2}\gamma_{n,h,1}$, it follows that

$$n^{1/2}\hat{a}(\theta_0) \to^d \psi_a = h_{25}^{-1} \psi_{ve} + h_1 \sim N\Big(h_1, h_{25}^{-2} h_{23}\Big). \tag{S.4}$$

(a2) Asymptotic distribution of $n^{1/2}(\hat{\theta}_{OLS} - \theta_0)$. First, we have

$$n^{1/2}(\hat{\theta}_{OLS} - \theta) = \left(n^{-1} X' X\right)^{-1} \left(n^{-1/2} X' u\right), \tag{S.5}$$

where $n^{-1}X'X \to^P h_{21}'h_{24}h_{21}+h_{25}$, and

$$\begin{aligned} n^{-1/2}X'u &= n^{-1/2}(\gamma_{n,h,21}'Z'+v')(v\gamma_{n,h,1}+e) \\ &= \gamma_{n,h,21}'\left(n^{-1/2}Z'e\right)+\gamma_{n,h,21}'\left(n^{-1/2}Z'v\right)\gamma_{n,h,1}+n^{-1/2}v'e+\left(n^{-1}v'v\right)n^{1/2}\gamma_{n,h,1} \\ &\to^d h_{21}'\psi_{Ze}+\psi_{ve}+h_{25}h_1, \end{aligned} \quad \text{(S.6)}$$

since $\gamma_{n,h,21}'(n^{-1/2}Z'v)\gamma_{n,h,1}=o_P(1)$, $n^{-1}(v'v)=h_{25}+o_P(1)$, and $n^{1/2}\gamma_{n,h,1}\to h_1$ as $n\to\infty$.

Therefore, we obtain

$$\begin{aligned} n^{1/2}(\hat{\theta}_{ols}-\theta) &\to^d \psi_{ols}=(h_{21}'h_{24}h_{21}+h_{25})^{-1}(h_{21}'\psi_{Ze}+\psi_{ve}+h_{25}h_1) \\ &\sim N\Big(\frac{h_{25}h_1}{h_{21}'h_{24}h_{21}+h_{25}},\frac{h_{21}'h_{22}h_{21}+h_{23}}{(h_{21}'h_{24}h_{21}+h_{25})^2}\Big). \end{aligned} \quad \text{(S.7)}$$

(b) It also suffices to characterize the asymptotic distributions of each statistic separately. Below we show that $n\hat{V}_{ols}\to^P \frac{h_{21}'h_{22}h_{21}+h_{23}}{\left(h_{21}'h_{24}h_{21}+h_{25}\right)^2}$. The argument for $\hat{V}_a(\theta_0)$ is similar and thus omitted.

For $\hat{V}_{ols}$ we use the decomposition

$$\frac{\hat{V}_{ols}}{V_{ols}}-1=V_{ols}^{-1}\left(\hat{V}_{ols}-V_{ols}\right)=V_{ols}^{-1}\left(A_{ols,1}-2A_{ols,2}+A_{ols,3}\right)+o_P(1), \quad \text{(S.8)}$$

where $V_{ols}=n^{-2}Q_{ols}^{-1}\sum_{i=1}^n E_F[X_i^2u_i^2]Q_{ols}^{-1}$, $A_{ols,1}=n^{-2}Q_{ols}^{-1}\sum_{i=1}^n X_i^2u_i^2Q_{ols}^{-1}-n^{-2}Q_{ols}^{-1}\sum_{i=1}^n E_F[X_i^2u_i^2]Q_{ols}^{-1}$, $A_{ols,2}=n^{-2}Q_{ols}^{-1}\sum_{i=1}^n X_i^3u_i(\hat{\theta}_{ols}-\theta)Q_{ols}^{-1}$, $A_{ols,3}=n^{-2}Q_{ols}^{-1}\sum_{i=1}^n X_i^4(\hat{\theta}_{ols}-\theta)^2Q_{ols}^{-1}$, and $Q_{ols}=plim_{n\to\infty}n^{-1}X'X$. Thus, we need to show that $V_{ols}^{-1}A_{ols,m}=o_P(1)$, for $m=1,2,3$.

For $m=1$, we let $r_i=n^{-1}V_{ols}^{-1/2}Q_{ols}^{-1}X_iu_i$, and we have $E_F\left[\sum_{i=1}^n r_i^2-1\right]=E_F\left[V_{ols}^{-1}A_{ols,1}\right]=0$. Also define the truncated variable $q_i=r_i\mathbb{1}(|r_i|\le\varepsilon)$ such that $r_i^2=q_i^2+r_i^2\mathbb{1}(|r_i|>\varepsilon)$. Then,

$$E_F\left|\sum_{i=1}^n r_i^2-1\right|\le E_F\left|\sum_{i=1}^n\left(q_i^2-E_F[q_i^2]\right)\right|+E_F\left|\sum_{i=1}^n\left(r_i^2\mathbb{1}(|r_i|>\varepsilon)-E_F[r_i^2\mathbb{1}(|r_i|>\varepsilon)]\right)\right|. \quad \text{(S.9)}$$

by the triangle inequality. The first term is $o(1)$ because

$$Var_F\left[\sum_{i=1}^{n} q_i^2\right] = \sum_{i=1}^{n} Var_F\left[q_i^2\right] \leq \varepsilon^2 \sum_{i=1}^{n} Var_F\left[|q_i|\right] \leq \varepsilon^2 \sum_{i=1}^{n} E_F\left[q_i^2\right] \leq \varepsilon^2 \sum_{i=1}^{n} E_F\left[r_i^2\right] = \varepsilon^2, \quad \text{(S.10)}$$

where ε is arbitrary. For the second term, we have

$$\begin{aligned} & E_F\left|\sum_{i=1}^{n}\left(r_i^2 \mathbb{1}(|r_i| > \varepsilon) - E_F(r_i^2 \mathbb{1}(|r_i| > \varepsilon))\right)\right| \leq 2\sum_{i=1}^{n} E_F\left[|r_i|^{2+\xi}|r_i|^{-\xi}\mathbb{1}(|r_i| > \varepsilon)\right] \\ & \leq 2\varepsilon^{-\xi}\sum_{i=1}^{n} E_F|r_i|^{2+\xi} \to 0, \end{aligned} \quad \text{(S.11)}$$

where the result of convergence to zero holds by the moment restriction on $E_F[||Z_i e_i||^{2+\xi}]$, $E_F[|v_i e_i|^{2+\xi}]$, $E_F[||Z_i Z_i'||^{2+\xi}]$ and $E_F[|X_i|^{2(2+\xi)}]$, and by $V_{ols} = O(n^{-1})$. For $m = 3$, we have

$$|nA_{ols,3}| = n^{-1} Q_{ols}^{-2}(\hat{\theta}_{ols} - \theta)^2 \sum_{i=1}^{n} X_i^4 = o_P(1), \quad \text{(S.12)}$$

where the second equality follows from the moment restriction on $E_F[|X_i|^{2(2+\xi)}]$. Therefore, we obtain that $V_{ols}^{-1} A_{ols,3} = o_P(1)$. For $m = 2$, by the Cauchy-Schwarz inequality,

$$\begin{aligned} \left|V_{ols}^{-1} A_{ols,2}\right| & \leq \left(V_{ols}^{-1} n^{-2} Q_{ols}^{-1} \sum_{i=1}^{n} X_i^2 u_i^2 Q_{ols}^{-1}\right)^{1/2} \left(V_{ols}^{-1} A_{ols,3}\right)^{1/2} \\ & = \left(1 + V_{ols}^{-1} A_{ols,1}\right)^{1/2} \left(V_{ols}^{-1} A_{ols,3}\right)^{1/2} = o_P(1), \end{aligned} \quad \text{(S.13)}$$

so that the results follows from those for $m = 1$ and $m = 3$.

Finally, the proof for the asymptotic distribution of $T_{ar}(\theta_0)$ is straightforward and thus also omitted.

□

Lemmas **S.2-S.3** are needed for the arguments with regard to the limiting distributions of the bootstrap analogues of the estimators and test statistics.

Lemma S.2 *For the independent bootstrap, suppose that* $E^*\left[|\omega_{1i}^*|^{2+\xi}\right] \leq C$ *and* $E^*\left[|\omega_{2i}^*|^{2+\xi}\right] \leq C$*; for the dependent bootstrap, suppose that* $E^*\left[|\omega_{1i}^*|^{2(2+\xi)}\right] \leq C$*, for some* $\xi > 0$ *and some large enough constant* C*. If further* $E_F\left[w_i^{2+\xi}\right] < \infty$ *for all* $w_i \in \left\{||Z_iu_i||, ||Z_iv_i||, ||Z_iZ_i'||, |u_iv_i|\right\}$ *and some* $\xi > 0$*, then under* H_0*,* $n^{-1}\sum_{i=1}^n E^*\left[||Z_iu_i^*||^{2+\xi}\right]$*,* $n^{-1}\sum_{i=1}^n E^*\left[||Z_iv_i^*||^{2+\xi}\right]$ *and* $n^{-1}\sum_{i=1}^n E^*\left[|u_i^*v_i^*|^{2+\xi}\right]$ *are bounded in probability.*

PROOF OF LEMMA S.2

The proof is straightforward for $n^{-1}\sum_{i=1}^n E^*\left[||Z_iu_i^*||^{2+\xi}\right]$. Indeed, we have

$$\begin{aligned}
& n^{-1}\sum_{i=1}^n E^*\left[||Z_iu_i^*||^{2+\xi}\right] = n^{-1}\sum_{i=1}^n E^*\left[||Z_iu_i(\theta_0)\omega_{1i}^*||^{2+\xi}\right] = n^{-1}\sum_{i=1}^n E^*\left[||Z_iu_i(\theta_0)||^{2+\xi}|\omega_{1i}^*|^{2+\xi}\right] \\
= \; & n^{-1}\sum_{i=1}^n ||Z_iu_i(\theta_0)||^{2+\xi} E^*\left[|\omega_{1i}^*|^{2+\xi}\right] \leq Cn^{-1}\sum_{i=1}^n ||Z_iu_i(\theta_0)||^{2+\xi} = O_P(1), \qquad \text{(S.14)}
\end{aligned}$$

where the last equality follows from $\theta = \theta_0$ under the null hypothesis, $E_F[||Z_iu_i||^{2+\xi}] < \infty$, and $n^{-1}\sum_{i=1}^n ||Z_iu_i||^{2+\xi} - E_F[||Z_iu_i||^{2+\xi}] \to^P 0$ by Law of Large Numbers (LLN). Now, consider $n^{-1}\sum_{i=1}^n E^*\left[||Z_iv_i^*||^{2+\xi}\right]$. As in (S.14) we have for $j = 1$ or 2,

$$n^{-1}\sum_{i=1}^n E^*\left[||Z_iv_i^*||^{2+\xi}\right] = n^{-1}\sum_{i=1}^n ||Z_i\hat{v}_i||^{2+\xi} E^*\left[|\omega_{ji}^*|^{2+\xi}\right] \leq Cn^{-1}\sum_{i=1}^n ||Z_i\hat{v}_i||^{2+\xi}. \qquad \text{(S.15)}$$

By using Minkowski and Cauchy-Schwartz inequalities, along with $\hat{v}_i = v_i - Z_i'(\hat{\pi} - \pi)$, we obtain

$$\begin{aligned}
& n^{-1}\sum_{i=1}^n ||Z_i\hat{v}_i||^{2+\xi} = n^{-1}\sum_{i=1}^n ||Z_iv_i - Z_iZ_i'(\hat{\pi} - \pi)||^{2+\xi} \\
& \leq C_1\left\{n^{-1}\sum_{i=1}^n ||Z_iv_i||^{2+\xi} + ||\hat{\pi} - \pi||^{2+\xi} n^{-1}\sum_{i=1}^n ||Z_iZ_i'||^{2+\xi}\right\} = O_P(1), \qquad \text{(S.16)}
\end{aligned}$$

where C_1 denotes some large enough constant, and (S.16) holds because $\hat{\pi} - \pi \to^P 0$, $E_F\left[||Z_iv_i||^{2+\xi}\right] < \infty$, $E_F\left[||Z_iZ_i'||^{2+\xi}\right] < \infty$, $n^{-1}\sum_{i=1}^n ||Z_iv_i||^{2+\xi} - E_F\left[||Z_iv_i||^{2+\xi}\right] \to^P 0$ and $n^{-1}\sum_{i=1}^n ||Z_iZ_i'||^{2+\xi} - E_F\left[||Z_iZ_i'||^{2+\xi}\right] \to^P 0$ by LLN. Therefore, $n^{-1}\sum_{i=1}^n E^*\left[||Z_iv_i^*||^{2+\xi}\right]$ is bounded in probability from (S.15)-(S.16).

We now show that $n^{-1}\sum_{i=1}^{n} E^{*}\left[|u_i^{*}v_i^{*}|^{2+\xi}\right]$ is bounded in probability. For $j=1$ or 2, we have

$$\begin{aligned} n^{-1}\sum_{i=1}^{n} E^{*}\left[|u_i^{*}v_i^{*}|^{2+\xi}\right] &= n^{-1}\sum_{i=1}^{n} E^{*}\left[|u_i(\theta_0)\hat{v}_i|^{2+\xi}|\omega_{1i}^{*}\omega_{ji}^{*}|^{2+\xi}\right] \\ &= n^{-1}\sum_{i=1}^{n} |u_i(\theta_0)\hat{v}_i|^{2+\xi} E^{*}\left[|\omega_{1i}^{*}\omega_{ji}^{*}|^{2+\xi}\right]. \end{aligned} \tag{S.17}$$

Note that $j=2$ for the wild bootstrap scheme with independent transformation, so that $E^{*}\left[|\omega_{1i}^{*}\omega_{ji}^{*}|^{2+\xi}\right] = E^{*}\left[|\omega_{1i}^{*}\omega_{2i}^{*}|^{2+\xi}\right] = E^{*}\left[|\omega_{1i}^{*}|^{2+\xi}\right]E^{*}\left[|\omega_{2i}^{*}|^{2+\xi}\right] \leq C_2$ for some large enough constant C_2. For the wild bootstrap scheme with dependent transformation, $j=1$, and we have $E^{*}\left[|\omega_{1i}^{*}\omega_{ji}^{*}|^{2+\xi}\right] = E^{*}\left[|\omega_{1i}^{*}|^{2(2+\xi)}\right] \leq C$. Combining both cases into (S.17) along with the fact that $u_i(\theta_0)\hat{v}_i = u_i(\theta_0)v_i - u_i(\theta_0)Z_i'(\hat{\pi}-\pi)$, $\theta=\theta_0$ under the null hypothesis, $E_F||Z_iu_i||^{2+\xi}<\infty$, $E_F|u_iv_i|^{2+\xi}<\infty$, and by using the arguments with Minkowski and Cauchy-Schwartz inequalities, we have

$$n^{-1}\sum_{i=1}^{n} E^{*}\left[|u_i^{*}v_i^{*}|^{2+\xi}\right] \leq C_3\left\{n^{-1}\sum_{i=1}^{n}|u_i(\theta_0)v_i|^{2+\xi} + ||\hat{\pi}-\pi||^{2+\xi}n^{-1}\sum_{i=1}^{n}||Z_iu_i(\theta_0)||^{2+\xi}\right\} = O_P(1),$$

for some large enough constants C_3. □

Lemma S.3 *Suppose that H_0 holds, the conditions of Lemma* **S.2** *are satisfied, $E^{*}[\omega_{1i}^{*}] = E^{*}[\omega_{2i}^{*}] = 0$, and $Var^{*}[\omega_{1i}^{*}] = Var^{*}[\omega_{2i}^{*}] = 1$. For the dependent bootstrap, further suppose that $E^{*}[\omega_{1i}^{*3}] = 0$ and $E^{*}[\omega_{1i}^{*4}] = 1$. Then, under the sequence $\{\gamma_{n,h}\}$ defined in (2.16) with $|h_1|<\infty$ we have*:

$$\begin{pmatrix} n^{-1/2}Z'u^{*} \\ n^{-1/2}\left(u^{*\prime}v^{*} - E^{*}\left[u^{*\prime}v^{*}\right]\right) \end{pmatrix} \to^{d^{*}} \begin{pmatrix} \psi_{ze}^{*} \\ \psi_{ve}^{*} \end{pmatrix} \sim N\left(0, \begin{pmatrix} h_{22} & 0 \\ 0' & h_{23} \end{pmatrix}\right), \tag{S.18}$$

in probability P.

PROOF OF LEMMA **S.3**

Let c_1 denote k-dimensional nonzero vectors, and c_2 denote a nonzero scalar. Define

$$\begin{aligned} U^*_{n,i} &= \left\{c_1' u_i^* Z_i + c_2 \left(u_i^* v_i^* - E^*[u_i^* v_i^*]\right)\right\} / \sqrt{n} \\ &= \left\{c_1' \omega^*_{1i} \hat{u}_i(\theta_0) Z_i + c_2 \left(\hat{u}_i(\theta_0)\hat{v}_i \omega^*_{1i}\omega^*_{ji} - E^*[\hat{u}_i(\theta_0)\hat{v}_i \omega^*_{1i}\omega^*_{ji}]\right)\right\} / \sqrt{n}, \qquad \text{(S.19)} \end{aligned}$$

where $j = 1$ for the dependent bootstrap scheme and $j = 2$ for the independent bootstrap scheme. It suffices to verify that the conditions of the Liapounov CLT hold for $U^*_{n,i}$. For brevity, we shall focus on the proof for the case with independent transformation (i.e., $j = 2$). Note that the proof for the case with dependent transformation ($j = 1$) follows similar steps.

(a) We have $E^*[U^*_{n,i}] = 0$ as $E^*[\omega^*_{1i}\hat{u}_i(\theta_0)Z_i] = \hat{u}_i(\theta_0)Z_i E^*[\omega^*_{1i}] = 0$, and $E^*[\hat{u}_i(\theta_0)\hat{v}_i\omega^*_{1i}\omega^*_{2i} - E^*[\hat{u}_i(\theta_0)\hat{v}_i\omega^*_{1i}\omega^*_{2i}]] = \hat{u}_i(\theta_0)\hat{v}_i E^*[\omega^*_{1i}\omega^*_{2i}] - \hat{u}_i(\theta_0)\hat{v}_i E^*[\omega^*_{1i}\omega^*_{2i}] = 0$.

(b) Note that

$$\begin{aligned} E^*[u_i^{*2} Z_i Z_i'] &= E^*[\hat{u}_i^2(\theta_0)\omega^{*2}_{1i} Z_i Z_i'] = \hat{u}_i^2(\theta_0) Z_i Z_i' E^*[\omega^{*2}_{1i}] = \hat{u}_i^2(\theta_0) Z_i Z_i', \\ E^*[u_i^{*2} v_i^{*2}] &= E^*[\hat{u}_i^2(\theta_0)\hat{v}_i^2 \omega^{*2}_{1i}\omega^{*2}_{2i}] = \hat{u}_i^2(\theta_0)\hat{v}_i^2 E^*[\omega^{*2}_{1i}\omega^{*2}_{2i}] = \hat{u}_i^2(\theta_0)\hat{v}_i^2 E^*[\omega^{*2}_{1i}] E^*[\omega^{*2}_{2i}] = \hat{u}_i^2(\theta_0)\hat{v}_i^2, \\ E^*[u_i^{*2} v_i^* Z_i] &= E^*[\hat{u}_i^2(\theta_0)\hat{v}_i Z_i \omega^{*2}_{1i}\omega^*_{2i}] = \hat{u}_i^2(\theta_0)\hat{v}_i Z_i E^*[\omega^{*2}_{1i}\omega^*_{2i}] = \hat{u}_i^2(\theta_0)\hat{v}_i Z_i E^*[\omega^{*2}_{1i}] E^*[\omega^*_{2i}] = 0, \end{aligned}$$

which implies that under H_0,

$$\sum_{i=1}^n E^*[U^{*2}_{n,i}] = c_1' \left(n^{-1}\sum_{i=1}^n \hat{u}_i^2(\theta_0) Z_i Z_i'\right) c_1 + c_2^2 \left(n^{-1}\sum_{i=1}^n \hat{u}_i^2(\theta_0)\hat{v}_i^2\right) = c_1' h_{22} c_1 + c_2^2 h_{23} + o_P(1) = O_P(1). \qquad \text{(S.20)}$$

(c) We note that by Minkowski inequality, for some $\xi > 0$ and some large enough constant C_4,

$$\sum_{i=1}^n E^*[|U^*_{n,i}|^{2+\xi}] \le C_4 n^{-\frac{\xi}{2}} n^{-1} \sum_{i=1}^n E^*\left[|c_1' Z_i^* u_i^*|^{2+\xi} + |c_2 u_i^* v_i^*|^{2+\xi}\right] \to^P 0, \qquad \text{(S.21)}$$

where the convergence in probability is obtained by using Lemma **S.2**.

From (a)-(c) above, $U_{n,i}^*$ satisfies the Lyapunov CLT conditions, and the result of Lemma **S.3** follows for the independent bootstrap. For the dependent bootstrap, notice that for (b),

$$E^*[u_i^{*2} v_i^{*2}] = \hat{u}_i^2(\theta_0)\hat{v}_i^2 E^*[\omega_{1i}^{*4}] = \hat{u}_i^2(\theta_0)\hat{v}_i^2, \text{ and } E^*[u_i^{*2} v_i^* Z_i] = \hat{u}_i^2(\theta_0)\hat{v}_i Z_i E^*[\omega_{1i}^{*3}] = 0, \quad \text{(S.22)}$$

and the desired result follows.

□

S.2. Proofs of Theorems in the Main Text

PROOF OF THEOREM **3.1**

First, note that by following similar arguments as those in the proofs of Theorem **S.4**, we can obtain that the following (conditional) convergence in distribution holds:

$$\begin{pmatrix} T^*_{OLS,(h_1,\hat{h}_{n,2})}(\theta_0) \\ H^*_{n,(h_1,\hat{h}_{n,2})}(\theta_0) \end{pmatrix} \to^{d^*} \begin{pmatrix} (h_{21}' h_{22} h_{21} + h_{23})^{-1/2} (h_{21}' \psi^*_{Ze} + \psi^*_{ve} + h_{25} h_1) \\ h_{23}^{-1} (\psi^*_{ve} + h_{25} h_1)^2 \end{pmatrix} \quad \text{(S.23)}$$

in probability P. Then, based on the formula of $T^*_{l,n,(h_1,\hat{h}_{n,2})}(\theta_0)$ for $l \in \{1,2\}$, we conclude that the (conditional) null limiting distribution of $T^*_{l,n,(h_1,\hat{h}_{n,2})}(\theta_0)$ is the same as the null limiting distribution of $T_{l,n}(\theta_0)$ with the value of localization parameter equal to h_1, and this implies that $c^*_{l,(h_1,\hat{h}_{n,2})}(1-\delta) \to^P c_{l,(h_1,h_2)}(1-\delta)$, where $c_{l,(h_1,h_2)}(1-\delta)$ denotes the $(1-\delta)$-th quantile of $\tilde{T}_{l,h}$ with $h = (h_1, h_2)$.

Then, the arguments for the proof is similar to those in McCloskey (2017). We note that there exists a "worst case sequence" $\gamma_n \in \Gamma$ such that $AsySz\left[c_l^{B-S}(\alpha, \alpha-\delta, \hat{h}_{n,1}(\theta_0), \hat{h}_{n,2})\right]$ equals:

$$\limsup_{n\to\infty} \sup_{\gamma\in\Gamma} P_{\theta_0,\gamma}\left[T_{l,n}(\theta_0) > c_l^{B-S}(\alpha, \alpha-\delta, \hat{h}_{n,1}(\theta_0), \hat{h}_{n,2})\right]$$

$$
\begin{aligned}
&= \limsup_{n\to\infty} P_{\theta_0,\gamma_n}\left[T_{l,n}(\theta_0) > c_l^{B-S}(\alpha,\alpha-\delta,\hat{h}_{n,1}(\theta_0),\hat{h}_{n,2})\right] \\
&= \lim_{n\to\infty} P_{\theta_0,\gamma_{m_n}}\left[T_{l,m_n}(\theta_0) > c_l^{B-S}(\alpha,\alpha-\delta,\hat{h}_{m_n,1}(\theta_0),\hat{h}_{m_n,2})\right] \qquad \text{(S.24)}
\end{aligned}
$$

where $\{m_n : n \geq 1\}$ is a subsequence of $\{n : n \geq 1\}$ and such a subsequence always exists. Furthermore, there exists a subsequence $\{\omega_n : n \geq 1\}$ of $\{m_n : n \geq 1\}$ such that:

$$
\begin{aligned}
&\lim_{n\to\infty} P_{\theta_0,\gamma_{m_n}}\left[T_{l,m_n}(\theta_0) > c_l^{B-S}(\alpha,\alpha-\delta,\hat{h}_{m_n,1}(\theta_0),\hat{h}_{m_n,2})\right] \\
&= \lim_{n\to\infty} P_{\theta_0,\gamma_{\omega_n,h}}\left[T_{l,\omega_n}(\theta_0) > c_l^{B-S}(\alpha,\alpha-\delta,\hat{h}_{\omega_n,1}(\theta_0),\hat{h}_{\omega_n,2})\right] \qquad \text{(S.25)}
\end{aligned}
$$

for some $h \in \mathscr{H}$. But, for any $h \in \mathscr{H}$, any subsequence $\{\omega_n : n \geq 1\}$ of $\{n : n \geq 1\}$, and any sequence $\{\gamma_{\omega_n,h} : n \geq 1\}$, we have $\left(T_{l,\omega_n}(\theta_0),\hat{h}_{\omega_n,1}(\theta_0)\right) \to^d \left(\tilde{T}_{l,h},\tilde{h}_1\right)$ jointly. In addition, $c_l^{B-S}(\alpha,\alpha-\delta,\hat{h}_{\omega_n,1}(\theta_0),\hat{h}_{\omega_n,2})$ is continuous in $\hat{h}_{\omega_n,1}$ by the definition of the SBCV and Maximum Theorem. Hence, the following convergence holds jointly by the Continuous Mapping Theorem:

$$
\left(T_{l,\omega_n}(\theta_0), c_l^{B-S}(\alpha,\alpha-\delta,\hat{h}_{\omega_n,1}(\theta_0),\hat{h}_{\omega_n,2})\right) \to^d \left(\tilde{T}_{l,h}, c_l^{B-S}(\alpha,\alpha-\delta,\tilde{h}_1,h_2)\right) \qquad \text{(S.26)}
$$

where $c_l^{B-S}(\alpha,\alpha-\delta,\tilde{h}_1,h_2) = \sup_{h_1\in CI_{\alpha-\delta}(\tilde{h}_1)} c_{l,(h_1,h_2)}(1-\delta)$. Then, (S.24)-(S.26) imply that

$$
\begin{aligned}
&AsySz\left[c_l^{B-S}(\alpha,\alpha-\delta,\hat{h}_{n,1}(\theta_0),\hat{h}_{n,2})\right] \\
&= \lim_{n\to\infty} P_{\theta_0,\gamma_{\omega_{n,h}}}\left[T_{l,\omega_n}(\theta_0) > c_l^{B-S}(\alpha,\alpha-\delta,\hat{h}_{\omega_{n,1}}(\theta_0),\hat{h}_{\omega_{n,2}})\right] \\
&= \sup_{h\in\mathscr{H}} P\left[\tilde{T}_{l,h} > c_l^{B-S}(\alpha,\alpha-\delta,\tilde{h}_1,h_2)\right], \qquad \text{(S.27)}
\end{aligned}
$$

Now, for any $h \in \mathscr{H}$, we have:

$$
\begin{aligned}
&P\left[\tilde{T}_{l,h} \geq c_l^{B-S}(\alpha,\alpha-\delta,\tilde{h}_1,h_2)\right] \\
&= P\left[\tilde{T}_{l,h} \geq c_l^{B-S}(\alpha,\alpha-\delta,\tilde{h}_1,h_2) \geq c_{l,h}(1-\delta)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \;\; P\left[\tilde{T}_{l,h} \geq c_{l,h}(1-\delta) \geq c_l^{B-S}(\alpha, \alpha-\delta, \tilde{h}_1, h_2)\right] \\
&+ \;\; P\left[c_{l,h}(1-\delta) \geq \tilde{T}_{l,h} \geq c_l^{B-S}(\alpha, \alpha-\delta, \tilde{h}_1, h_2)\right] \\
&\leq \;\; P\left[\tilde{T}_{l,h} \geq c_{l,h}(1-\delta)\right] + P\left[c_{l,h}(1-\delta) \geq c_l^{B-S}(\alpha, \alpha-\delta, \tilde{h}_1, h_2)\right] \\
&= \;\; P\left[\tilde{T}_{l,h} \geq c_{l,h}(1-\delta)\right] + P\left[h_1 \notin CI_{\alpha-\delta}(\tilde{h}_1)\right] \\
&= \;\; \delta + (\alpha - \delta) = \alpha, \qquad \text{(S.28)}
\end{aligned}
$$

where the inequality and the second equality follow from the form of $c_l^{B-S}(\alpha, \alpha-\delta, \tilde{h}_1, h_2)$, and the third equality follows from the definition of $CI_{\alpha-\delta}(\tilde{h}_1)$. As (S.28) holds for any $h \in \mathscr{H}$, it is clear from (S.27) that $AsySz[c_l^{B-S}(\alpha, \alpha-\delta, \hat{h}_{n,1}, \hat{h}_{n,2})] \leq \alpha$, as stated. □

PROOF OF THEOREM **3.3**

As in Theorem **3.1**, we can show that there exists a sequence $\gamma_n \in \Gamma$, a subsequence $\{m_n : n \geq 1\}$ of $\{n : n \geq 1\}$, and a subsubsequnce $\{\omega_n : n \geq 1\}$ of $\{m_n : n \geq 1\}$ such that the following result holds for $l \in \{1, 2\}$:

$$
\begin{aligned}
& AsySz\left[c_l^{B-A}(\alpha, \alpha-\delta, \hat{h}_{n,1}(\theta_0), \hat{h}_{n,2})\right] \\
= \;\; & \limsup_{n\to\infty} \sup_{\gamma\in\Gamma} P_{\theta_0,\gamma}\left[T_{l,n}(\theta_0) > c_l^{B-S}(\alpha, \alpha-\delta, \hat{h}_{n,1}(\theta_0), \hat{h}_{n,2}) + \hat{\eta}_{l,n}\right] \\
= \;\; & \limsup_{n\to\infty} P_{\theta_0,\gamma_n}\left[T_{l,n}(\theta_0) > c_l^{B-S}(\alpha, \alpha-\delta, \hat{h}_{n,1}(\theta_0), \hat{h}_{n,2}) + \hat{\eta}_{l,n}\right] \\
= \;\; & \lim_{n\to\infty} P_{\theta_0,\gamma_{m_n}}\left[T_{l,m_n}(\theta_0) > c_l^{B-S}(\alpha, \alpha-\delta, \hat{h}_{m_n,1}(\theta_0), \hat{h}_{m_n,2}) + \hat{\eta}_{l,m_n}\right] \\
= \;\; & \lim_{n\to\infty} P_{\theta_0,\gamma_{\omega_n,h}}\left[T_{l,\omega_n}(\theta_0) > c_l^{B-S}(\alpha, \alpha-\delta, \hat{h}_{\omega_n,1}(\theta_0), \hat{h}_{\omega_n,2}) + \hat{\eta}_{l,\omega_n}\right] \qquad \text{(S.29)}
\end{aligned}
$$

for some $h \in \mathscr{H}$. Furthermore, as in the proof of Theorem **3.1**, for any $h \in \mathscr{H}_h$, any subsequence $\{\omega_n : n \geq 1\}$ of $\{n : n \geq 1\}$, and any sequence $\{\gamma_{\omega_n,h} : n \geq 1\}$, we have $\left(T_{l,\omega_n}(\theta_0), \hat{h}_{\omega_n,1}\right) \to^d \left(\tilde{T}_{l,h}, \tilde{h}_1\right)$ jointly. Hence,

$$
\lim_{n\to\infty} P_{\theta_0,\gamma_{\omega_n,h}}\left[T_{l,\omega_n}(\theta_0) > c_l^{B-S}(\alpha, \alpha-\delta, \hat{h}_{\omega_n,1}(\theta_0), \hat{h}_{\omega_n,2}) + \hat{\eta}_{l,\omega_n}\right]
$$

$$= \sup_{h\in\mathscr{H}} P\left[\tilde{T}_{l,h} > c_l^{B-S}(\alpha,\alpha-\delta,\tilde{h}_1,h_2)+\bar{\eta}_l\right] \quad \text{(S.30)}$$

$$\equiv \sup_{h\in\mathscr{H}} P\left[\tilde{T}_{l,h} > c_l^{B-A}(\alpha,\alpha-\delta,\tilde{h}_1,h_2)\right], \quad \text{(S.31)}$$

where $\bar{\eta}_l = \inf\left\{\eta : \sup_{h_1\in\mathscr{H}_1} P\left[\tilde{T}_{l,h} > c_l^{B-S}(\alpha,\alpha-\delta,\tilde{h}_1,h_2)+\eta\right] \leq \alpha\right\}$. For the simplicity of exposition, define the following asymptotic rejection probability:

$$NRP_l[h,\eta] \equiv P[\tilde{T}_{l,h} > c_l^{B-S}(\alpha,\alpha-\delta,\tilde{h}_1,h_2)+\eta]. \quad \text{(S.32)}$$

It is clear from (S.29)-(S.32) that $AsySz[c_l^{B-A}(\alpha,\alpha-\delta,\hat{h}_{n,1},\hat{h}_{n,2})] = \sup_{h\in\mathscr{H}} NRP_l[h,\bar{\eta}_l]$. Hence, it suffices to show that $\sup_{h\in\mathscr{H}} NRP_l[h,\bar{\eta}_l] = \alpha$ to establish Theorem **3.3**.

First, from the result of Theorem **3.1** and the definition of the size-correction criterion, it is clear that $\sup_{h\in\mathscr{H}} NRP_l[h,\bar{\eta}_l] \leq \alpha$. We proceed to show that $\sup_{h\in\mathscr{H}} NRP_l[h,\bar{\eta}_l] < \alpha$ leads to contradiction. Assume that $\sup_{h\in\mathscr{H}} NRP_l[h,\bar{\eta}_l] < \alpha$ and define the function $K_l(\cdot): \mathbb{R}_- \to [-\alpha,\ 1-\alpha]$ such that

$$K_l(x) = \sup_{h\in\mathscr{H}} NRP_l[h,x] - \alpha. \quad \text{(S.33)}$$

Notice that given Assumption **3.2**, $NRP_l[h,\cdot]$ is continuous on $\mathbb{R}_-$. Therefore, the Maximum Theorem entails that $K_l(\cdot)$ is also continuous on $\mathbb{R}_-$. Moreover, we have[6]

$$K_l\left(-c_l^{B-S}(\alpha,\alpha-\delta,\tilde{h}_1,h_2)\right) = \sup_{h\in\mathscr{H}} NRP_l[h,-c_l^{B-S}(\alpha,\alpha-\delta,\tilde{h}_1,h_2)] - \alpha = 1-\alpha > 0$$

and $K_l\left(\bar{\eta}_l\right) = \sup_{h\in\mathscr{H}} NRP_l[h,\bar{\eta}_l] - \alpha < 0$ (by assumption).

[6]We notice that the proof is focused on the symmetric two-sided test and uses the fact that $NRP_l[h,-c_l^{B-S}(\alpha,\alpha-\delta,\tilde{h}_1,h_2)] = P[\tilde{T}_{l,h} > 0] = 1$ in this case. This proof can be adapted to the case of a lower/upper one-sided test by noting that for any $\varepsilon > 0$ small enough, there exists a large enough positive constant $c \equiv c(\varepsilon)$ such that $NRP_l[h,-c(\varepsilon)] = 1-\varepsilon$, for all $h \in \mathscr{H}$. Therefore, $K_l(-c(\varepsilon)) = \sup_{h\in\mathscr{H}} NRP_l[h,-c(\varepsilon)] = 1-\varepsilon-\alpha$. As this holds for any $\varepsilon > 0$ small enough, the result for the case with lower/upper one-sided test follows by choosing ε such that $\varepsilon \to 0$.

Then, we note that by the Intermediate Value Theorem, there exists $\dot{\eta}_l$ such that

i) $-c_l^{B-S}(\alpha,\alpha-\delta,\tilde{h}_1,h_2) < \dot{\eta}_l < \bar{\eta}_l$ almost surely,

ii) $K_l(\dot{\eta}_l) = 0$; *i.e.*, $\sup_{h\in\mathscr{H}} NRP_l[h,\dot{\eta}_l] = \alpha$.

However, this contradicts the size-correction procedure where

$$\bar{\eta}_l = \inf\left\{\eta : \sup_{h_1\in\mathscr{H}_1} P\left[\tilde{T}_{l,h} > c_l^{B-S}(\alpha,\alpha-\delta,\tilde{h}_1,h_2)+\eta\right] \leq \alpha\right\}.$$

It follows that $\sup_{h\in\mathscr{H}} NRP_l[h,\bar{\eta}_l] = \alpha$; i.e., $AsySz[c_l^{B-A}(\alpha,\alpha-\delta,\hat{h}_{n,1}(\theta_0),\hat{h}_{n,2})] = \alpha$. □

PROOF OF COROLLARY **3.4** We notice that for $l \in \{1,2\}$,

$$\begin{aligned} & \liminf_{n\to\infty} \inf_{\gamma\in\Gamma} P_{\theta,\gamma}\left[\theta \in CS_{l,n}(1-\alpha)\right] \\ = \; & \liminf_{n\to\infty} \inf_{\gamma\in\Gamma} P_{\theta,\gamma}\left[T_{l,n}(\theta) \leq c_l^{B-A}(\alpha,\alpha-\delta,\hat{h}_{n,1}(\theta_0),\hat{h}_{n,2})\right], \end{aligned} \tag{S.34}$$

where $c_l^{B-A}(\alpha,\alpha-\delta,\hat{h}_{n,1}(\theta_0),\hat{h}_{n,2})$ denotes the BACV corresponding to $T_{l,n}(\theta)$. Then, the result follows by Theorem **3.3** and by exploiting the duality between confidence set and inverting the test of each of the individual null hypothesis $H_0 : \theta = \theta_0$.

□

S.3. Asymptotic Results for the Bootstrap Inconsistency

This section contains the details of the bootstrap inconsistency under local endogeneity. In the following theorem, we give the results of bootstrap inconsistency for the two-stage and shrinkage tests under local endogeneity. For this purpose, we notice that there are two sources of randomness in the bootstrap: the randomness from the original data and the randomness from the bootstrap procedure (i.e., the random weights of the wild bootstrap). Specifically, take the original sample as from the probability space $(\Omega, \mathscr{F}, P)$. In addition, suppose the randomness from the bootstrap is defined on a probability space $(\Lambda, \mathscr{G}, P^*)$, which is independent of $(\Omega, \mathscr{F}, P)$. Then, in the following theorem we view the bootstrap statistics as being defined on the product probability space $(\Omega, \mathscr{F}, P) \times (\Lambda, \mathscr{G}, P^*) = (\Omega \times \Lambda, \mathscr{F} \times \mathscr{G}, \mathbb{P})$, where $\mathbb{P} = P \times P^*$. Theorem **S.4** gives the null limiting distributions of the bootstrap statistics under $\mathbb{P}$. In particular, this framework is needed to characterize the asymptotic behaviour of the bootstrap statistics generated under the dependent transformation of disturbances.

Theorem S.4 *Suppose that H_0 and the conditions of Lemmas* **S.2** *and* **S.3** *hold. Then, under the sequence $\{\gamma_{n,h}\}$ defined in* (2.16) *with $|h_1| < \infty$:*

$$\begin{pmatrix} T^*_{ar}(\theta_0) \\ T^*_{ols}(\theta_0) \\ H^*_n(\theta_0) \end{pmatrix} \rightsquigarrow \eta^*_h \equiv \begin{pmatrix} \eta^*_{1,h} \\ \eta^*_{2,h} \\ \eta^*_{3,h} \end{pmatrix} = \begin{pmatrix} \psi^{*'}_{Ze} h_{22} \psi^*_{Ze} \\ (h'_{21} h_{22} h_{21} + h_{23})^{-1} \left(h'_{21} \psi^*_{Ze} + \psi^*_{ve} + h_{25} h^b_1\right)^2 \\ h^{-1}_{23} \left(\psi^*_{ve} + h_{25} h^b_1\right)^2 \end{pmatrix},$$

$$T^*_{1,n}(\theta_0) \rightsquigarrow \tilde{T}^*_{1,h} = \eta^*_{2,h} \mathbb{1}(\eta^*_{3,h} \leq \chi^2_{1,1-\beta}) + \eta^*_{1,h} \mathbb{1}(\eta^*_{3,h} > \chi^2_{1,1-\beta}),$$

$$T^*_{2,n}(\theta_0) \rightsquigarrow \tilde{T}^*_{2,h} = \eta^*_{2,h} w(\eta^*_{3,h}) + \eta^*_{1,h}(1 - w(\eta^*_{3,h})),$$

where $h^b_1 = 0$ for the bootstrap based on independent transformation of disturbances, and $h^b_1 = h_1 + h^{-1}_{25} \psi_{ve}$ with $\psi_{ve} \sim N(0, h_{23})$, for the bootstrap based on dependent transformation of disturbances, and $\rightsquigarrow$ signifies the weak convergence under $\mathbb{P}$.

PROOF OF THEOREM **S.4**

First, we note that

$$n^{-1} X^{*'} P_Z X^* = n^{-1} (Z\hat{\pi} + v^*)' P_Z (Z\hat{\pi} + v^*) = n^{-1} \hat{\pi}' Z' Z \hat{\pi} + n^{-1} \hat{\pi}' Z' v^* + n^{-1} v^{*'} Z \hat{\pi} + n^{-1} v^{*'} P_Z v^*$$

$$= n^{-1} \hat{\pi}' Z' Z \hat{\pi} + o_{P^*}(1) \to^{P^*} h'_{21} h_{24} h_{21}, \text{ in probability } P, \tag{S.35}$$

which follows from $\hat{\pi} - h_{21} \to^P 0$, $n^{-1}Z'Z - h_{24} \to^P 0$, and $n^{-1}Z'v^* \to^{P^*} 0$ in probability P. Using similar arguments, we obtain

$$n^{-1}X^{*'}X^* \to^{P^*} h_{21}^{'}h_{24}h_{21} + h_{25}, \tag{S.36}$$

in probability P. Furthermore, using similar arguments as those for $\hat{V}_a$, $\hat{V}_{ols}$ and $\hat{V}_{2sls}$ in the proof of Lemma **S.1**, we obtain

$$\begin{aligned} &n\hat{V}_a^* \to^{P^*} (h_{21}^{'}h_{24}h_{21})^{-2}h_{21}^{'}h_{22}h_{21} + h_{25}^{-2}h_{23}, \;\; n\hat{V}_{ols}^* \to^{P^*} (h_{21}^{'}h_{24}h_{21} + h_{25})^{-2}\left(h_{21}^{'}h_{22}h_{21} + h_{23}\right), \\ &n\hat{V}_{2sls}^* \to^{P^*} \left(h_{21}^{'}h_{24}h_{21}\right)^{-2}h_{21}^{'}h_{22}h_{21}, \end{aligned} \tag{S.37}$$

in probability P.

Second, we note that

$$\begin{aligned} &n^{-1/2}X^{*'}P_Z u^* = n^{-1/2}\left(Z\hat{\pi} + v^*\right)' P_Z u^* = n^{-1/2}\hat{\pi}'Z^{'}u^* + \left(n^{-1}v^{*'}Z\right)\left(n^{-1}Z^{'}Z\right)^{-1}\left(n^{-1/2}Z^{'}u^*\right) \\ &= n^{-1/2}\hat{\pi}'Z^{'}u^* + o_{P^*}(1) \to^{d^*} h_{21}^{\prime}\psi_{Ze}^*, \end{aligned} \tag{S.38}$$

in probability P, where the last equality follows from: (a) by Lemma **S.3**, $n^{-1/2}Z^{'}u^* = O_{P^*}(1)$ in probability P; (b) $n^{-1}Z'v^* \to^{P^*} 0$ in probability P as $E^*[n^{-1}Z'v^*] = 0$; (c) $n^{-1}Z^{'}Z \to^P h_{24}$, which is positive definite, and therefore $\left(n^{-1}Z^{'}Z\right)^{-1} \to^P h_{24}^{-1}$. Then, the (conditional) convergence in distribution in (S.35) follows from Lemma **S.3**, along with the fact that $\hat{\pi} - h_{21} \to^P 0$.

Third, following the same arguments as above, we have $n^{-1/2}X^{*'}u^* = n^{-1/2}\hat{\pi}'Z^{'}u^* + n^{-1/2}\left(v^{*'}u^* - E^*[v^{*'}u^*]\right) + n^{-1/2}E^*[v^{*'}u^*]$, where

$$n^{-1/2}\hat{\pi}'Z^{'}u^* + n^{-1/2}\left(v^{*'}u^* - E^*[v^{*'}u^*]\right) \to^{d^*} h_{21}^{\prime}\psi_{Ze}^* + \psi_{ve}^*, \tag{S.39}$$

in probability P. Then, for $n^{-1/2}E^*[v^{*'}u^*]$, we notice that it is equal to zero under the independent transformation of disturbances. Under the dependent transformation, $n^{-1/2}E^*[v^{*'}u^*] = n^{1/2}\left(n^{-1}\sum_{i=1}^n \hat{v}_i\hat{u}_i(\theta_0)\right)$, where

$$\begin{aligned} n^{1/2}\left(n^{-1}\sum_{i=1}^{n}\hat{v}_i\hat{u}_i(\theta_0)\right) &= n^{1/2}\left(n^{-1}\sum_{i=1}^{n}(v_iu_i(\theta_0)-E_F[v_iu_i(\theta_0)])\right)+n^{1/2}E_F[v_iu_i(\theta_0)]+o_P(1) \\ &\to^d \psi_{ve}+h_{25}h_1. \end{aligned} \tag{S.40}$$

Finally, notice that the results in probability P in (S.35)-(S.39) are invariant to the original data, so they hold under $\mathbb{P}$ as well. Then, by (S.40) and the Continuous Mapping Theorem, we obtain that under H_0,

$$\begin{pmatrix} n^{1/2}\hat{a}^* \\ n^{1/2}(\hat{\theta}^*_{ols}-\theta_0) \\ T^*_{ar}(\theta_0) \end{pmatrix} \rightsquigarrow \begin{pmatrix} h_{25}^{-1}\psi^*_{ve}+h_1^b \\ (h_{21}^{'}h_{24}h_{21}+h_{25})^{-1}(h_{21}^{'}\psi^*_{Ze}+\psi^*_{ve}+h_{25}h_1^b) \\ \psi^{*'}_{Ze}h_{22}\psi^*_{Ze} \end{pmatrix}, \tag{S.41}$$

and the results in the statement of Theorem **S.4** follow. □