A note on Brandl and Brandt's axiomatic characterization of Nash equilibrium

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ABSTRACT: The axiomatic requirements in Brandl and Brandt (2024) make it possible to define solution concepts that do not select the set of all Nash equilibria as claimed. More precisely, it is possible to construct solution concepts that fulfill the axiomatic requirements, but in certain games no equilibrium is selected at all, as a simple example shows.

In their study „An axiomatic characterization of Nash equilibrium", Brandl and Brandt (2024) state in Theorem 1, that a solution concept f for a (non-cooperative) game G with complete information selects only and all Nash equilibria of G, if f is total and f satisfies axioms of "Consequentialism", "Consistency" and "Rationality". But this is in general not the case. Thus, a simple counterexample f^* will be constructed, that is total and satisfies consequentialism, consistency and rationality, but there are cases of G in which $f^* \neq NASH$.

In line with authors' formalism[1], finite sets of pure actions $A_i = S_i := \{s_{i1}, \dots, s_{im_i}\}$ and sets of probability distributions $\Delta A_i = P_i = \{p_{i1}, \dots, p_{im_i}\}, m_i \geq 2 \in \mathbb{N}$, for all player $i = 1, \dots, n$ will be considered. The set of strictly positive probability distributions P_i^+ of P_i is given by $P_i^+ = \{p_i \in P_i \mid p_{ij} > 0, j = 1, .., m_i\}$ for all player $i = 1, \dots, n$. The set of strategy profiles (mixed strategies) is given by $\Delta A = P = P_1 \times \dots \times P_n$, and the set of strictly positive strategy profiles by $P^+ = P_1^+ \times \dots \times P_n^+$. The set of action profiles (pure strategies) is given by $A = S = S_1 \times \dots \times S_n$. For simplicity[2], a (non-cooperative) game G will be specified by a vector $E = (E_1, \dots, E_n)$ of expected payoffs $E_i : P \longrightarrow \mathbb{R}$, which are based on a vector valued payoff function $U : A \longrightarrow \mathbb{R}^n$, with $U = (U_1, \dots, U_n)$ and $U_i : A \longrightarrow \mathbb{R}$ for all player $i = 1, \dots, n$. In the following a game G will be identified by $[S, U, P, E]$. Strategy profile $\hat{p} \in P$ will – as usually – be called *Nash equilibrium* of E, if for all $p_i \in P_i$ and for all $i = 1, \dots, n$: $E_i(\hat{p}_i, \hat{p}_{-i}) \geq E_i(p_i, \hat{p}_{-i})$. A solution concept for G, which returns all Nash equilibria will be denoted by $NASH$. In general, solution concepts f^* map games P to a set of strategy profiles $f^*(P)$, whereby specific requirements for elements $p \in f^*(P) \subseteq P$ will be

[1] A simple definition of pure and mixed strategies is used here, which is common in game theory, also to avoid misleading and ambiguous definitions ($A, \Delta A, A_i, \Delta A_i, p, p_i$ etc.) in the original publication.

[2] Note, that pure strategy profiles are also contained in the set of mixed strategy profiles: For $s_i^k \in S_i$ it will be defined $s_i^k = p_i^k = (p_{i1}, \dots, p_{ik}, \dots, p_{im_i})$ with $p_{ik} = 1$ and $p_{ij} = 0$, if $k \neq j \in \{1, \dots, m_i\}$. So, it holds: $s_i^k = p_i^k \in P_i$.

satisfied. Thus, a solution concept is a set valued self-mapping function (correspondence) $f^*: P \longrightarrow P$.

Instead of requiring Nash equilibria as a solution of G the following definition for game G is given: A strategy profile $p^* \in P$ is called a *complete mixture*, if $p^* \in P^+$ and $E_i(p_i^*, p_{-i}^*) \in \mathbb{R}$ for all $p_{-i}^* \in P_{-i}^+$ and $i = 1, \ldots, n$. The *set of all complete mixtures* will be denoted by P^*.

Obviously, each $p^* \in P^+$ satisfies $E_i(p_i^*, p_{-i}^*) \in \mathbb{R}$ by definition of E_i for $i = 1, \ldots, n$, i.e. $P^* \equiv P^+$. Thus, a solution concept $f^*: P \longrightarrow P$ can simply be derived by: $f^*(P) = P^+$ for all (non-cooperative) games G.

THEOREM. *Function $f^*(P) = P^+$ is a total solution concept, satisfying consequentialism, consistency and rationality, but there exist games for which $f^* \neq NASH$.*

PROOF:

By definition, $f^*(P) = P^+ \neq \emptyset$, for all games G. Thus, f^* is a total solution concept. If it would hold $f^* = Nash$, then the set of Nash equilibria for each (non-cooperative) game E must coincide with the whole set of complete mixed strategy profiles, which is typically not the case. If, for instance, game G has only one pure strategy equilibrium, it follows $f^* \neq NASH$.

To show that consequentialism (see Definition 1 in Brandl and Brandt 2024) will be satisfied, one has to show: $f^*(P) = \phi_*^{-1}(f^*(P'))$ for surjection $\phi = (\phi_1, \ldots, \phi_n)$. Considering blow-up $A = S_1 \times \ldots \times S_n$ of action profile $A' = S_1' \times \ldots \times S_n'$. Action set S_i differs from action set $S_i' = (s_{i1}', \ldots, s_{im_i}')$ by the fact, that in S_i' some elements are replaced by its clones. Thus, S_i contains instead of s_{ik}' clones of number $|\phi^{-1}(s_{ik}')|$ for some or all $k \in \{1, \ldots, m_i\}$. Now, considering for an action set $S_i' = (s_{i1}', \ldots, s_{im_i}')$ its respective probability distribution $p_i' = (p_{i1}', \ldots, p_{im_i}')$ in P_i', and denote $\phi_*^{-1} = (\phi_{1*}^{-1}, \ldots, \phi_{n*}^{-1})$. Then ϕ_{i*}^{-1} applied on p_i' extends[3] p_i' to the respective $p_i \in P_i$ with regard to the difference between S_i' and S_i. Therefore, it applies for any P and P': $P = \phi_*^{-1}(P')$. Now, let P' be specified with P'^+, i.e. P'^+ denotes the set of strictly positive strategy profiles with respect to $A' = S_1' \times \ldots \times S_n'$, and let P^+ represent the set of strictly positive strategy profiles with respect to the blow-up $A = S_1 \times \ldots \times S_n$. Then, $f^*(P) = P^+$, $f^*(P') = P'^+$ and $P^+ = \phi_*^{-1}(P'^+)$. Together, $f^*(P) = P^+ = \phi_*^{-1}(P'^+) = \phi_*^{-1}(f^*(P'))$, i.e. consequentialism is satisfied.

[3] Note that here – as in the original text of Brandl and Brandt (2024) – it must be assumed, that ϕ_{i*}^{-1} applied on $p_i' \in P_i'$ "generates" a probability distribution $p_i \in P_i$ again, and that ϕ_{i*} is invertible.

Showing that f^* satisfies consistency (see Definition 2 in Brandl and Brandt 2024) is simple: If two different expected payoff functions E and E' mapping from the set of strategy profiles P into $\mathbb{R}^n$, based on the same set of action profiles $A = S = S_1 \times \ldots \times S_n$, then for $t \in [0,1]$ convex combination $t \cdot E + (1-t) \cdot E'$ maps also from P into $\mathbb{R}^n$. Thus, $f^*(P) = P^+ = f^*(P')$, i.e. $f^*(P) \cap f^*(P') = P^+$, and $f^*(t \cdot E + (1-t) \cdot E') = P^+$, which confirms consistency.

Rationality is defined in Definition 3 (Brandl and Brandt 2024). For a dominating strategy $a_i = s_{ij}$ with $j \in \{1, \ldots, m_i\}$ for players $i \in \{1, \ldots, n\}$ it is requiered $p_i(a_i) = p_{ij} > 0$. Because $f^*(P) = P^+ = P_1^+ \times \ldots \times P_n^+$ and by definition of $P_i^+ = \{p_i \in P_i \mid p_{ij} > 0, j = 1, .., m_i\}$ for all player $i = 1, \ldots, n$, all – and thus also such probabilities for dominating strategies – are strictly positive.

□

Finally, it follows a very brief discussion. The construction of axiom systems to characterize solution concepts for Nash equilibria in (non-cooperative) games in normal form poses a challenge, as also underlined by the small number of references in Brand and Brandt (2024). From the point of view of the author of this note, important assumptions about the behavior of real human players should be reflected in corresponding axiom systems, similar to the axiomatization of Peleg and Tijs (1996), for instance. An important axiomatic requirement should focus on the aspect of maximizing the expected value of player i, given that all other players $-i$ also follow this optimization principle and this is common knowledge. A maximization criterion is neither explicitly nor implicitly reflected in any of the axioms in Brand and Brandt (2024). In particular, the axiom of rationality allows positive probabilities p_{il}^- to exist for dominated strategies s_{il}^- for some $l \in \{1, \ldots, m_i\}$, so that related to a positive probability p_{ik} of a dominating strategy s_{ik} with $k \neq l \in \{1, \ldots, m_i\}$, it could hold $p_{il}^- > p_{ik}$, what seems to contradict maximization.

Another aspect that can be included in the axiomatic requirements is – if players' behavior deviates from the purely (mathematical and) rational *Homo Economicus*, e.g. if risk attitudes play a role as in expected utility theory – how the players' beliefs about their opponents are theoretically motivated[4].

REFERENCES

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[4] An answer to this question may be given by so-called *social projection* as in Schade et a. (2010).

Schade, Christian, Andreas Schroeder and Kai Oliver Krause (2010) "Coordination after gains and losses: Is prospect theory's value function predictive for games?" Journal of Mathematical Psychology 55, 426-445.