Comment on Acemoglu "Labor- and Capital-augmenting technical change"

Li, Defu and Bental, Benjamin and Tang, Xuemei

School of Economics and Management, Tongji University, Department of Economics, University of Haifa, Shengxiang Business School of Shanghai Sanda University

18 December 2024

Comment on Acemoglu "Labor- and Capital-augmenting technical change"*

Defu Li[1]
School of Economics and Management, Tongji University
Benjamin Bental[2]
Department of Economics, University of Haifa.
Xuemei Tang[3]
Shengxiang Business School of Shanghai Sanda University

Abstract: Acemoglu's (2003) paper "Labor- and Capital-augmenting Technical Change" is a pioneering work that introduces a growth model with an endogenous direction of technical progress including microfoundations. At the steady-state equilibrium, the model indicates that there is only net labor-augmenting technical change, despite firms being able to pursue both labor- and capital-augmenting technological improvements. While this paper is a classic and original contribution to the field, it presents several significant shortcomings: (1) substantial mathematical errors in the proof of the main propositions; (2) the absence of a dynamic adjustment function for scientists across different innovation sectors, which is critical for the model; (3) neglect of the crucial condition required for the propositions to hold; (4) omission of important policy implications that diverge from existing literature; and (5) insufficient explanation of the intuition behind the model's core conclusions. This comment identifies and addresses these shortcomings.

Key Words：Acemoglu，Endogenous technical change，Direction of technical change，Balanced Growth Path；Relative income share of factors，Dynamic system

JEL：O33, O14, O31, E25

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* Li gratefully acknowledges support from the National Natural Science Foundation of China (NSFC:71773083; 72273096). All errors are our own.
[1] School of Economics and Management, Tongji University, 1500 Siping Road, Shanghai, P.R. China, tjldf@tongji.edu.cn.
[2] Department of Economics, University of Haifa, 199 Aba Khoushy Ave., Mount Carmel, Haifa, Israel, bbental@econ.haifa.ac.il.
[3] Shengxiang Business School of Shanghai Sanda University, Room 8314, Building 8, No. 2727 Jinhai Road, Pudong, Shanghai, 1815505000@qq.com.

I Introduction

To align the equilibrium of growth models with Kaldor's (1961) stylized facts and comply with Uzawa's (1961) steady-state theorem, both neoclassical growth models (Solow, 1956; Cass, 1965; Koopmans, 1965) and endogenous technological change models (Romer, 1990; Aghion and Howitt, 1992) assume that all technical change is purely labor-augmenting (Harrod-neutral). However, these models do not explain why profit-maximizing firms only pursue labor-augmenting innovations, even though other types of technological progress, such as Solow-neutral (capital-augmenting) and Hicks-neutral, are at least conceptually possible.

Acemoglu's (2003) paper addresses this gap by developing a growth model where the direction of technological progress is endogenously determined. In this model, firms can undertake both labor- and capital-augmenting innovations; however, driven by profit motives, the steady-state equilibrium ultimately results in only net labor-augmenting technological change. This indicates that purely labor-augmenting technological progress may be a rational choice for firms under specific constraints, thereby providing a microfoundation for the exogenous assumption regarding the direction of technological change in existing literature.

Indeed, as the inaugural paper in Volume 1, Issue 1 of the JEEA, it has attracted significant attention since its publication, garnering over 1,000 citations on Google Scholar. Additionally, it is an essential component of Acemoglu's classic textbook (2009, ch.15.6) and is prominently featured in Acemoglu's (2024) slides for the "Introduction to Economic Growth" course at MIT. Jones (2024) specifically discusses this paper in his teaching slides titled "The Direction of Technical Change," referring to it as a "Great Idea for a paper"!

However, this paper presents several serious issues that have not been publicly addressed in the literature or rectified by the author. **First**, there are substantial mathematical errors in the proof of the main propositions. Specifically, the premise Lemma A1 used to prove Propositions 1, 2, and 6, as well as the dynamic equation of scientists in the proof of Proposition 5, is mathematically incorrect. **Second**, the model is inadequately specified due to the absence of a behavioral function for the dynamic adjustments of scientists. These adjustments between different innovation sectors are critical determinants of the dynamics of technological progress and the outcomes of the steady-state equilibrium; however, the model does not explicitly provide such a dynamic adjustment function. **Third**, the paper presents results only for scenarios where investment is greater than zero, omitting analysis for the case when investment is zero. Even in the absence of investment, the model maintains a steady-state equilibrium but lacks a balanced growth path (BGP) equilibrium. This distinction between the two types of equilibria is overlooked in this paper and existing literature. **Fourth**, the paper neglects important policy implications of the model. It fails to indicate that the model suggests taxes or subsidies on innovation do not affect the direction or rate of technological progress in the steady state, nor do they influence economic growth; rather, they only impact income distribution. This contradicts established models of endogenous technological progress (Romer, 1990; Aghion and Howitt, 1992) and conflicts with the authors' assertion that tax policies have no effect on the long-run distribution of income. **Fifth**, the model does not provide the correct intuition behind its core conclusion. The authors claim that the asymmetry between capital and labor accumulation—specifically, that capital can be accumulated while labor cannot—leads to the conclusion that technological progress in the steady state is purely labor-augmenting. However, even the paper acknowledges the logical difficulties inherent in this argument. While the asymmetry may explain a bias toward labor-augmenting technological progress, it does not clarify why this progress is exclusively labor-augmenting. Therefore, the authors' objective of explaining why long-run technical change is labor-augmenting has not been achieved.

While these issues do not undermine the original value of the paper, identifying and addressing its errors could significantly enhance its contribution to the theory of economic growth. By highlighting the paper's failure to resolve its central issues, we can encourage further inquiry into what key factors were overlooked, ultimately leading to the development of a general framework for analyzing the direction of technological progress.

The structure of this comment is organized as follows: Section 2 identifies two significant mathematical errors in the proof of the proposition; Section 3 supplements the model with a behavioral function for scientists' dynamic adjustments that is currently missing; Section 4 presents the dynamic equations of the model, along with modifications to the propositions and the reproves; Section 5 highlights important policy implications of the model that have been overlooked in the paper; Section 6 discusses the intuition behind the core conclusions of the model; and Section 7 provides a summary of the comment.

II Two Significant Mathematical Errors in the Proof of the Propositions

This paper presents six propositions and one lemma, focusing on the existence, uniqueness and convergence of the steady-state equilibrium of the model, as well as specific results related to it. Propositions 1, 2, 3, and 4 delve into the existence and uniqueness, while Propositions 5 and 6, along with Lemma 1, address the convergence of the model's balanced growth path (BGP) equilibrium. To prove the main conclusions of the paper—specifically, Propositions 1, 2, and Lemma 1—the author first establishes Lemma A1. However, a significant mathematical error in Lemma A1 undermines the validity of the proofs for these propositions. Additionally, the proof of Proposition 5 contains another serious mathematical error, resulting in the proposition being only partially valid.

1. The mathematical error in Lemma A1

In Lemma A1, Acemoglu first defines $\Delta(t) \equiv \frac{m(t)V_k(t)}{n(t)V_l(t)}$, and then derives the following equation (1) from the equation (20) in the paper:

$$\Delta(t) \equiv \frac{m(t)V_k(t)}{n(t)V_l(t)} = \int_t^{\infty} \frac{r(v)K(v)}{w(v)L(v)} dv = \int_t^{\infty} k(v)^{(\varepsilon-1)/\varepsilon} dv \tag{1}$$

However, equation (1) is evidently mathematically incorrect. By substituting the equation (21) from Acemoglu (2003) into the equation (20) from the same paper, we arrive at the following equation (2):

$$\begin{cases} V_k(t) = \dfrac{1-\beta}{\beta} \displaystyle\int_t^{\infty} exp\left[-\int_t^{v} (r(\omega)+\delta) d\omega\right] \dfrac{r(v)K(v)}{m(v)} dv \\ V_l(t) = \dfrac{1-\beta}{\beta} \displaystyle\int_t^{\infty} exp\left[-\int_t^{v} (r(\omega)+\delta) d\omega\right] \dfrac{w(v)L(v)}{n(v)} dv \end{cases} \tag{2}$$

Substituting equation (2) into $\frac{m(t)V_k(t)}{n(t)V_l(t)}$ yields the following equation (3):

$$\Delta(t) \equiv \frac{m(t)V_k(t)}{n(t)V_l(t)} = \frac{m(t)\int_t^{\infty} exp\left[-\int_t^{v}(r(\omega)+\delta)d\omega\right]\frac{r(v)K(v)}{m(v)}dv}{n(t)\int_t^{\infty} exp\left[-\int_t^{v}(r(\omega)+\delta)d\omega\right]\frac{w(v)L(v)}{n(v)}dv} \tag{3}$$

The statement in (1) cannot be derived from (3). We suspect that the author made the following mathematical error, leading to the incorrect equation (1) from (3).

$$\Delta(t) \equiv \frac{m(t)V_k(t)}{n(t)V_l(t)} = \frac{\int_t^{\infty} exp\left[-\int_t^{v}(r(\omega)+\delta)d\omega\right]m(t)\frac{r(v)K(v)}{m(v)}dv}{\int_t^{\infty} exp\left[-\int_t^{v}(r(\omega)+\delta)d\omega\right]n(t)\frac{w(v)L(v)}{n(v)}dv}$$

$$= \int_t^{\infty} \frac{exp\left[-\int_t^{v}(r(\omega)+\delta)d\omega\right]r(v)K(v)}{exp\left[-\int_t^{v}(r(\omega)+\delta)d\omega\right]w(v)L(v)} dv = \int_t^{\infty} \frac{r(v)K(v)}{w(v)L(v)} dv$$

However, mathematically, the integral of a quotient does not equal the quotient of the integrands. Additionally, based on the model's production function, we have $\frac{r(v)K(v)}{w(v)L(v)} = \frac{(1-\gamma)}{\gamma} k(v)^{(\varepsilon-1)/\varepsilon}$. Therefore, equation (1), which represents the first expression in Lemma A1, is mathematically incorrect, specifically:

$$\Delta(t) \equiv \frac{m(t)V_k(t)}{n(t)V_l(t)} \neq \int_t^{\infty} \frac{r(v)K(v)}{w(v)L(v)} dv \neq \int_t^{\infty} k(v)^{(\varepsilon-1)/\varepsilon} dv \tag{4}$$

In fact, a simple example can demonstrate that equation (1) is not valid:

$$\frac{\int_t^\infty x dv}{\int_t^\infty x dv} = 1 \neq \int_t^\infty \frac{x}{x} dv = \infty$$

The relative wages of scientists engaged in two types of innovations are given by $\frac{b_k \phi(S_k(t))}{b_l \phi(S - S_k(t))}$ and $\frac{m(t)V_k(t)}{n(t)V_l(t)}$, where $\Delta(t) \equiv \frac{m(t)V_k(t)}{n(t)V_l(t)}$ is a key influencing factor. Therefore, the error in equation (1) not only affects the proof of the main propositions in the paper but also impacts the dynamic analysis of scientists' migration between different innovation sectors. It should be corrected. The correct form of $\Delta(t) = \frac{m(t)V_k(t)}{n(t)V_l(t)}$ should involve integrating the numerator and denominator separately, resulting in equation (5):[4]

$$\Delta(t) \equiv \frac{m(t)V_k(t)}{n(t)V_l(t)} = \frac{r(t)K(t)}{w(t)L(t)} \nabla(t) = \frac{(1-\gamma)}{\gamma} k(t)^{\frac{\varepsilon - 1}{\varepsilon}} \nabla(t) \tag{5}$$

where $\nabla(t) \equiv \frac{\int_t^\infty exp\left[\int_t^v [g_r(\omega) + g_K(\omega) - g_m(\omega)] d\omega - \int_t^v (r(t) exp\left[\int_t^\omega g_r(u) du\right] + \delta) d\omega\right] dv}{\int_t^\infty exp\left[\int_t^v [g_w(\omega) + g_L(\omega) - g_n(\omega)] d\omega - \int_t^v (r(t) exp\left[\int_t^\omega g_r(u) du\right] + \delta) d\omega\right] dv}$.

2. Mathematical Error in the Proof of Proposition 5

The growth model should not only address the existence of a steady-state equilibrium but also examine the stability of that equilibrium. Proposition 5 of the paper specifically discusses the stability of the balanced growth path of the model. When the economy is on the balanced growth path, the wage rates of scientists across different innovation sectors are equal, as expressed in equation (6):

$$b_l \phi(S - S_k(t)) n(t) V_l(t) = b_k \phi(S_k(t)) m(t) V_k(t) \tag{6}$$

Equation (6) indicates that the allocation of scientists across different sectors has reached equilibrium. However, what happens to the movement of scientists when the economy deviates from the balanced growth path? Acemoglu does not provide explicit assumptions regarding this in the paper, meaning that he does not specify a clear flow function for scientists, $\dot{S}_k(t)/S_k(t)$. Nonetheless, when the economy is not on the balanced growth path, scientists may need to shift across different innovation sectors, and this movement will affect whether the model's BGP equilibrium converges. Therefore, in order to analyze the convergence characteristics of the model's balanced growth path, a flow equation for scientists is typically required. To this end, the paper derives an equation that includes $\dot{S}_k(t)/S_k(t)$ by taking the growth rate of both sides of equation (6), resulting in equation (7):

[4] The derivation process is detailed in Appendix A. Fortunately, the equation (29) of the paper is derived by solving for V_k and V_l separately, as shown in the equation (27) of the paper. Therefore, the specific results concerning the BGP equilibrium in the paper are unaffected.

$$-\frac{\dot{S_k}(t)}{S_k(t)}e_k+\frac{\dot{n}(t)}{n(t)}+\frac{\dot{V_l}(t)}{V_l(t)}=\frac{\dot{S_k}(t)}{S_k(t)}e_l+\frac{\dot{m}(t)}{m(t)}+\frac{\dot{V_k}(t)}{V_k(t)} \tag{7}$$

Although equation (7) can be derived from equation (6), the function $\dot{S}_k(t)/S_k(t)$ obtained from equation (7) is inadequate for analyzing the convergence of the model's balanced growth path. This is because when $\dot{S}_k(t)/S_k(t)=0$ in equation (7), it does not guarantee that $S_k(t)$ converges to the steady state $S_k{}^*$ defined by equation (6). This limitation arises from fundamental principles of differentiation and integration: while equation (6) can lead to equation (7), it is generally not possible to derive equation (6) from equation (7). For any constant $\boldsymbol{A}\neq\boldsymbol{0}$, if the following equation (8) holds:

$$b_l\phi\big(S-S_k(t)\big)n(t)V_l(t)=\boldsymbol{A}.\,b_k\phi\big(S_k(t)\big)m(t)V_k(t) \tag{8}$$

Therefore, not only can equation (6) lead to equation (7), but equation (8) can also yield equation (7). However, when $\boldsymbol{A}\neq\boldsymbol{1}$, equation (8) indicates that the economy is not on the balanced growth path. Consequently, the function $\dot{S}_k(t)/S_k(t)$ derived from equation (7) cannot be used to discuss the dynamic properties of the balanced growth path, rendering both the proof and the conclusions of Proposition 5 incorrect.

III Absence of Dynamic Behavior Function for Scientists in the Model

The core content of this paper is that firms can undertake both labor- and capital-augmenting technological improvements. However, at steady-state equilibrium, only net labor-augmenting innovations will occur, while net capital-augmenting innovation will be zero. The allocation of scientists between the two sectors ultimately determines the direction of technological progress at steady state. Thus, how scientists move between sectors is crucial for the dynamic adjustments that shape technological progress and its outcomes. However, the paper lacks a clear specification of scientists' dynamic adjustment behaviors and merely suggests that the free entry of innovations leads to an equilibrium where scientists' wages equal those in the higher-return sector, as indicated in equation (9):

$$\omega_s=max\{b_l\phi(S_l)nV_l,b_k\phi(S_k)mV_k\} \tag{9}$$

Although equation (9) provides the results at the final equilibrium, it does not offer sufficient information for dynamic adjustments during non-equilibrium states. Without a function describing the movement of scientists between sectors during non-equilibrium, the paper cannot analyze the convergence characteristics of the model's steady-state equilibrium. When the paper examines the convergence properties of the Balanced Growth Path (BGP) equilibrium in the appendix, it derives the function $\dot{S}_k(t)/S_k(t)$ using equation (6), further demonstrating that without $\dot{S}_k(t)/S_k(t)$, the description of the model's dynamic system is incomplete. However, this derived function is both economically unreasonable and mathematically inadequate. Thus, the lack of a behavioral function for scientists' movement between innovation sectors is a significant flaw in the model's specification that needs to be addressed.

Although scientists are homogeneous and their wages must be identical at the long-term equilibrium, wage rates for scientists in different innovation sectors may not be equal during non-steady-state equilibria. Therefore, we denote the wage rates for scientists in the two sectors as ω_{sl} and ω_{sk}, respectively.

$$\begin{cases} \omega_{sl}(t) = b_l \phi\big(S_l(t)\big) n(t) V_l(t) \\ \omega_{sk}(t) = b_k \phi\big(S_k(t)\big) m(t) V_k(t) \end{cases} \tag{10}$$

Since scientists are homogeneous, any wage differentials will incentivize them to migrate from lower-wage sectors to higher-wage sectors until equilibrium is achieved and all scientists receive the same wage. However, the transfer process takes time, and it is reasonable to assume that the greater the wage differential, the faster the rate of scientist flow, leading to a more rapid reduction in the wage disparity. Therefore, we supplement the model by introducing a dynamic adjustment function for scientists as shown in equation (11):

$$\frac{\dot{S_k}(t)}{S_k(t)} = G\left[\frac{\omega_{sk}}{\omega_{sl}}\right] = G\left[\frac{b_k \phi\big(S_k(t)\big) m(t) V_k(t)}{b_l \phi\big(S_l(t)\big) n(t) V_l(t)}\right], \tag{11}$$

where G(.) is assumed to satisfy G(1)=0, $G'(.) > 0$, $G''(.) < 0$.

Equation (11) indicates that when $\frac{\omega_{sk}}{\omega_{sl}} > 1$, $\frac{\dot{S_k}(t)}{S_k(t)} > 0$, resulting in a continuous increase in $S_k(t)$ and a corresponding decrease in $S_l(t)$. Conversely, when $\frac{\omega_{sk}}{\omega_{sl}} < 1$, $\frac{\dot{S_k}(t)}{S_k(t)} < 0$, leading to a continuous decrease in $S_k(t)$ and an increase in $S_l(t)$. Given that the total number of scientists S is fixed, three possible states can exist, at which point the flow of scientists will cease, indicated by $\frac{\dot{S_k}(t)}{S_k(t)} = 0$, achieving equilibrium in the distribution of scientists.

First, when $\frac{\omega_{sk}}{\omega_{sl}} = 1$, there are $0 < S_k(t) < S$ and $0 < S_l(t) < S$;

Second, when $\frac{\omega_{sk}}{\omega_{sl}} > 1$, there are $S_k(t) = S$ and $S_l(t) = 0$;

Third, when $\frac{\omega_{sk}}{\omega_{sl}} < 1$, $S_k(t) = 0$ and $S_l(t) = S$.

The first scenario represents an equilibrium where scientists coexist in both innovation sectors with identical wage rates, resulting in no further movement among scientists. [5] In contrast, the second and third scenarios indicate that all scientists are concentrated in a single

[5] If $\phi(0) < \infty$, it is possible for the case to exist where $\frac{\omega_{sk}}{\omega_{sl}} = 1$ and either $S_k(t) = 0$ or $S_l(t) = S$.

sector, where the wage in that sector exceeds that of the other, leading to a cessation of scientist flow as well.

Substituting $\Delta(t) \equiv \frac{m(t)V_k(t)}{n(t)V_l(t)}$ determined by equation (5) into equation (11) yields:

$$\frac{\dot{S_k}(t)}{S_k(t)} = G\left[\frac{b_k \phi\big(S_k(t)\big)}{b_l \phi\big(S_l(t)\big)} \frac{(1-\gamma)}{\gamma} k(t)^{\frac{\varepsilon-1}{\varepsilon}} \nabla(\mathrm{t})\right] \tag{12}$$

Since $\nabla(t)$ depends on the discounted integral of the growth rates of $r(v)$, $K(v)$, $w(v)$, $L(v)$, and $n(v)$ from time t to infinity, it can be assumed that the impact of short-term shocks on $\nabla(t)$ is negligible. Therefore, equation (12) indicates that the primary factors influencing the wage differentials for scientists across different sectors are: the relative crowding effect of innovation $\frac{\phi(S_k(t))}{\phi(S-S_k(t))}$ and the relative income share of factors $\frac{(1-\gamma)}{\gamma}k(t)^{\frac{\varepsilon-1}{\varepsilon}}$. Holding other factors constant, $\frac{\dot{S_k}(t)}{S_k(t)}$ is positively correlated with $\frac{\phi(S_k(t))}{\phi(S-S_k(t))}$. According to the crowding effect hypothesis, the crowding effect of innovation reduces the wage rate gap of scientists between the two sectors. The relationship between $\frac{\dot{S_k}(t)}{S_k(t)}$ and $k(t)$ depends on whether ε is greater than 1. If $\varepsilon < 1$, then $\frac{\dot{S_k}(t)}{S_k(t)}$ is negatively correlated with $k(t)$, which also contributes to narrowing the wage rate gap among scientists. Conversely, if $\varepsilon > 1$, then $\frac{\dot{S_k}(t)}{S_k(t)}$ is positively correlated with $k(t)$, which serves to widen the wage rate differentials among scientists.

From equation (12), all the core conclusions required for this paper can be derived, thereby enhancing the clarity of the analytical process. Thus, it aligns with the ideas presented in Acemoglu (2003).

IV Dynamic system of equations, revision and reproof of propositions

The mathematical errors in the proof process of the propositions in this paper stem not only from oversight by the authors and reviewers but also from a common shortcoming in the existing economic growth literature. Growth models are inherently dynamic systems described by a set of dynamic equations. However, prior to solving for the steady-state equilibrium, the literature often fails to provide a complete set of these equations, typically introducing them only when discussing convergence. This approach could create inconsistencies between the dynamic equations analyzed for steady-state convergence and those used in the solution process, making such errors difficult to detect.

Conversely, if a complete set of dynamic equations describing the model is provided before solving for the steady state, and if the steady state and its convergence are analyzed using these equations, the mathematical errors in this paper are less likely to occur. Specifically, substituting $\Delta(t) = \int_t^{\infty} k(v)^{(\varepsilon-1)/\varepsilon} dv$ from Lemma A1 into the scientist's dynamic adjustment function (equation 11) would prevent the derivation of Propositions 1, 2, 3, and 4 when $\dot{S_k}/S_k = 0$. In this scenario, for any $0 < k(v) < \infty$, we must have $\Delta(t) = \infty$, implying that scientists would focus exclusively on capital-augmenting technological innovation. This situation makes the existence of a balanced growth equilibrium impossible and directly indicates that $\Delta(t) = \int_t^{\infty} k(v)^{(\varepsilon-1)/\varepsilon} dv$ is incorrect. Similarly, using the dynamic adjustment function for scientists presented in equation (7) which used in the appendix of the paper also fails to derive Propositions 1, 2, 3, and 4. This indicates that the formulation of $\dot{S_k}/S_k$ cannot accurately describe the dynamic behavior of scientists, thus precluding its use for analyzing the convergence characteristics of the steady state.

Therefore, this comment aims to address this deficiency. First, we will provide a complete set of dynamic equations for the model using the function of $\dot{S_k}/S_k$ in the equation (12) introduced in the previous section. Next, we will use these dynamic equations to derive the equilibrium of the model and prove Propositions 1, 2, 3, and 4 as outlined in the paper. Finally, we will linearize the equation system around the balanced growth path (BGP) to analyze the local convergence characteristics of the BGP equilibrium, thereby revising Propositions 5 and 6 and Lemma 1.

1. Dynamic Equations of the Model

As in the original paper, we define $c(t) \equiv C(t)/K(t)$. By substituting the dynamic function for scientists $\dot{S_k}/S_k$ from equation (12) of the previous section, we obtain the dynamic equations of the model as stated in equation (13):

$$\begin{cases} \dfrac{\dot{S_k}}{S_k} = G\left[\dfrac{b_k \phi(S_k)}{b_l \phi(S - S_k)} \dfrac{(1-\gamma)\nabla}{\gamma} k^{(\varepsilon-1)/\varepsilon}\right] \\ \dfrac{\dot{M}}{M} = \dfrac{1-\beta}{\beta}[b_k \phi(S_k) S_k - \delta] \\ \dfrac{\dot{c}}{c} = \left(\dfrac{M\beta f'(k)}{\theta} - \dfrac{\rho}{\theta}\right) - \left(\dfrac{Mf(k)}{k} - c\right) \\ \dfrac{\dot{k}}{k} = \left(\dfrac{Mf(k)}{k} - c\right) + \dfrac{1-\beta}{\beta}[b_k \phi(S_k) S_k - b_l \phi(S - S_k)(S - S_k)] \end{cases} \tag{13}$$

Equation system (13) consists of four variables: S_k, M, c, and k, along with four dynamic equations that collectively describe a dynamic system. It encompasses all behavioral functions of the model. Apart from the dynamic adjustment function for scientists $\dot{S_k}/S_k$, the Euler equation for consumption and the capital accumulation function are included within the $\dot{c}/c$ function, while the $\dot{k}/k$ function incorporates both the innovation function and the capital accumulation function.

This comment will employ the dynamic equations presented in equation (13) to derive the steady-state equilibrium of the model and analyze the stability of the balanced growth path (BGP) equilibrium, thus proving all the propositions outlined in the original paper. Since the Euler equation $\frac{\dot{C}}{C} = \frac{1}{\theta}(r(t) - \rho)$ holds only when *I*>0, equation (13) is also valid only when *I*>0. As this paper analyzes the equilibrium results of the model solely under the condition *I*>0, we will supplement the results for the case when *I*=0.

2. Steady-State Equilibrium of the Model

Acemoglu defines two concepts of equilibrium in this paper: asymptotic path (AP) equilibrium and balanced growth path (BGP) equilibrium. However, he does not provide a definition for steady-state (SS) equilibrium, nor does he clarify the distinction between SS and BGP equilibria. Based on the dynamic equations in equation (13), and drawing on Acemoglu's definition of steady state (Acemoglu, 2009, pp.60), this comment asserts that there exists a finite time $T < \infty$ such that AP equilibrium refers to the condition where at least one dynamic equation equals zero for $t \geq \mathrm{T}$. In contrast, SS equilibrium is characterized by all dynamic equations of the model's system equaling zero for $t \geq \mathrm{T}$. To date, the existing literature, including Acemoglu's work, has not clearly differentiated between BGP equilibrium and SS equilibrium, often treating them as synonymous. This confusion arises primarily because current growth literature has yet to propose a SS equilibrium that is distinct from BGP equilibrium. However, according to Acemoglu's definition, BGP equilibrium is a specific subset of SS equilibrium, requiring not only the conditions of SS equilibrium but also specific relationships among the growth rates of certain variables for $t \geq \mathrm{T}$, such as $\dot{Y}(t)/Y(t) = \dot{C}(t)/C(t) = \dot{K}(t)/K(t)$, and $\dot{M}(t)/M(t) = 0$. This primarily pertains to the SS equilibrium of the neoclassical growth model, where the capital-output ratio remains constant and the rate of net capital-augmenting technological progress equals zero.

(1) Refinements and Reproving of Propositions 1, 2, 3, and 4

Propositions 1, 3, and 4 focus on the existence of BGP equilibria and the specific outcomes of various variables at equilibrium, while Proposition 2 explores the existence of equilibria beyond the BGP. Due to a mathematical error in Lemma A1, the proofs of these propositions contain inaccuracies that require re-evaluation. To enhance the clarity of the proof's logic, we will make slight modifications to Propositions 1, 2, 3, and 4, reorganizing them into Propositions A1, A2, and A3, which will address the three cases of $\varepsilon < 1$, $\varepsilon > 1$ and $\varepsilon = 1$, respectively.[6]

[6] Since we prove Propositions 1 and 2 by solving the equilibrium solutions of the dynamic system represented by equation (13), the results for Propositions 3 and 4 will be provided concurrently with the proofs of Propositions 1 and 2. For the cases of $\varepsilon > 1$ and $\varepsilon = 1$, not only do the equilibrium results differ, but the dynamic equations also vary. Therefore, we will separate Proposition 2 into two distinct cases for $\varepsilon > 1$ and $\varepsilon = 1$ and prove them individually.

Proposition A1: For $\varepsilon < 1$, there exists a finite $T < \infty$ such that for $t \geq T$, the solution $\underset{t\geq T}{S_k(t)} = {S_k}^* = \frac{\delta}{b_k\phi({S_k}^*)}$ represents the model's asymptotic path (AP) equilibrium, which is also a balanced growth path (BGP) equilibrium. When $\delta > 0$, there exists a unique set of values (k^*, M^*, c^*); when $\delta = 0$, there are infinite sets of values (k^*, M^*, c^*) and ${S_k}^* = 0$ such that $\frac{\dot{S_k}}{S_k} = \frac{\dot{M}}{M} = \frac{\dot{c}}{c} = \frac{\dot{k}}{k} = 0$. However, for different (k^*, M^*, c^*), the growth rates of all variables of the model remain the same.

The detailed proof of the proposition can be found in Appendix B.

Proposition A2: For $\varepsilon > 1$, there exist three asymptotic path (AP) equilibria:

(1) When ≥ 0, ${S_k}^* = \frac{\delta}{b_k\phi({S_k}^*)}$, there exists a BGP equilibrium that is also an AP equilibrium;

(2) When $\delta \geq 0$, ${S_k}^* = S$, there exists a non-steady-state AP equilibrium;

(3) When $\delta > 0$, ${S_k}^* = 0$, there is also a non-steady-state AP equilibrium.

The detailed proof of the proposition can be found in Appendix C.

Proposition A3: For $\varepsilon = 1$, the model has a unique asymptotic path (AP) equilibrium, which is also a balanced growth path (BGP) equilibrium.

The detailed proof of the proposition can be found in Appendix D.

Although Propositions 1, 2, 3, and 4 in the paper remain valid under the condition I>0, the proof process in this comment serves two main purposes: first, to correct mathematical errors in the original proof in the paper; second, to show that solving the model's dynamic equations (13) allows us to obtain all the results originally intended in the paper.

(2) Supplementary Proposition B

Propositions 1, 2, 3, and 4 hold only when *I*>0; however, the paper not only fails to point this out but also does not analyze the equilibrium of the model when *I*=0. Nonetheless, the equilibrium results when *I*=0 still hold economic significance and therefore warrant further analysis. If I=0, then $C(t) = Y(t)$, $\dot{K}(t)/K(t) = 0$. In this case, the Euler equation $\frac{\dot{C}}{C} = \frac{1}{\theta}(r(t) - \rho)$ no longer applies, and the model's dynamic equations are described solely by $\dot{S_k}/S_k$ and $\dot{k}/k$ (where $k \equiv \frac{MK}{NL}$), as follows:

$$
\begin{cases}
\frac{\dot{S}_k}{S_k} = G\left[\frac{b_k\phi(S_k)}{b_l\phi(S-S_k)}\frac{(1-\gamma)\nabla}{\gamma}k^{(\varepsilon-1)/\varepsilon}\right] \\
\frac{\dot{k}}{k} = \frac{1-\beta}{\beta}[b_k\phi(S_k)S_k - b_l\phi(S-S_k)(S-S_k)]
\end{cases}
\tag{14}
$$

At this point, Propositions 1, 2, 3, and 4 proposed in the paper do not hold, and the equilibrium situation of the model can be summarized by the following Proposition B:

Proposition B: If $I = 0$, the model has asymptotic path (APs) equilibrium and a steady-state (SS) equilibrium, but no balanced growth path (BGP) equilibrium. When $\varepsilon < 1$, the model has a unique SS equilibrium; when $\varepsilon > 1$, the model has three AP equilibria, one of which is an SS equilibrium and the other two are non-stationary AP equilibria. When $\varepsilon = 1$, the model typically has only an SS equilibrium.

The detailed proof of the proposition can be found in Appendix E.

Proposition B not only demonstrates that Propositions 1, 2, 3, and 4 proposed in the paper hold only when I>0, [7] but also indicates that the SS equilibrium is distinct from the BGP equilibrium. The model achieves both an AP equilibrium and an SS equilibrium; however, it does not satisfy the BGP equilibrium as defined in the paper, since $\dot{K}/K \neq \dot{Y}/Y = \dot{C}/C$, and $\dot{M}/M > 0$.

Another important role of Proposition B is that the form of the capital accumulation function is crucial for determining whether capital-augmenting technological progress is included in the steady state. However, this point has not been adequately recognized in the existing literature. Acemoglu not only fails to highlight the importance of I>0 for the core conclusions in this paper, but he also explicitly assumes a capital accumulation function $\dot{K} = s_K$K in his simplified model (Acemoglu, 2009, ch.15.6; 2024), where s_K is an exogenous parameter, implicitly implying I = 0, while still hoping to derive results for the case of I>0, which leads to erroneous conclusions.

3. Convergence of BGP Equilibrium

(1) Revised Proposition 5: When **I>0**, the model's BGP equilibrium is saddle-stable regardless of whether $\varepsilon < 1$ or $\varepsilon > 1$.

The detailed proof of the proposition can be found in Appendix F.

The proof indicate that the BGP is saddle-stable not only when the elasticity of substitution $\varepsilon < 1$ but also when $\varepsilon > 1$. This is because, when $\varepsilon > 1$, the neoclassical properties of the

[7] Jones and Scrimgeour (2008) also note that I>0 is a key condition for the validity of Uzawa's (1961) steady-state theorem; however, in the case of exogenous technological progress, a growth model with I>0 is not meaningful.

production function ensure that, for given M^*, c^* and ${S_k}^*$, $\dot{k}/k$ converges, implying that the BGP is not completely divergent but is instead saddle-stable.

(2) Revised Proposition 6 and Lemma 1

The paper analyzes the convergence properties of the model's steady-state equilibrium more clearly by specifically examining the case where the utility function parameter $\theta = 0$. When $\theta = 0$, the consumption Euler equation degenerates to equation (15) as follow:

$$M\beta f'(k) = \rho \tag{15}$$

From equation (15), we obtain $k = k(M)$ with $\mathrm{d}k/\mathrm{d}M > 0$, that is, k is an increasing function of M. Using this condition, equations (13) simplify to the following pair of dynamic equations:

$$\begin{cases} \dfrac{\dot{S_k}}{S_k} = G\left[\dfrac{b_k \phi(S_k)}{b_l \phi(S - S_k)} \dfrac{(1-\gamma)\nabla}{\gamma} k(M)^{(\varepsilon-1)/\varepsilon}\right] \\ \dfrac{\dot{M}}{M} = \dfrac{1-\beta}{\beta} b_k \phi(S_k) S_k - \delta \end{cases} \tag{16}$$

Equation (16) indicates that even when θ=0, the dynamics of the model must be described by $\dot{M}/M$ and $\dot{S_k}/S_k$, rather than solely by $\dot{M}/M = \psi(M)$ as suggested in Lemma 1 of the paper. Therefore, Proposition 6 should be revised as follows:

Revised Proposition 6: When $\varepsilon < 1$, the steady state is locally stable, and when $\varepsilon > 1$, it is locally saddle-path stable.

The detailed proof of the proposition can be found in Appendix G.

From equation (16), it can be seen that Lemma 1 and Proposition 6 in the paper hold only under a specific condition. We will now present this condition. If the scientists' wages are the same overtime (even when the economy is not in a steady state), then the following equation (17) must hold at all times:

$$\frac{b_k \phi(S_k)}{b_l \phi(S - S_k)} \frac{(1-\gamma)\nabla}{\gamma} k(M)^{(\varepsilon-1)/\varepsilon} = 1 \tag{17}$$

At this point, from equation (17), it can be deduced that S_k is a function of k(M), that is, $S_k = S_k(k(M))$. Substituting this into $\dot{M}/M$ yields:

$$\frac{\dot{M}}{M} = \frac{1-\beta}{\beta} b_k \phi\left(S_k(k(M))\right) S_k(k(M)) - \delta \tag{18}$$

Let $\psi(M) \equiv \frac{1-\beta}{\beta} b_k \phi\left(S_k(k(M))\right) S_k(k(M)) - \delta$, equation (18) corresponds to the equation (32) in the paper, $\dot{M}/M = \psi(M)$.

From equation (18) we can obtain the equation (19) as follow:

$$\frac{\partial\psi(M)}{\partial M} = \frac{1-\varepsilon}{\varepsilon}k^{-\frac{1}{\varepsilon}}\frac{\frac{\partial\psi(M)}{\partial S_k}\frac{\partial k}{\partial M}b_k[\phi(S_k)]^2(1-\gamma)\nabla}{b_l\gamma[\phi'(S-S_k)\phi(S_k)+\phi(S-S_k)\phi'(S_k)]} \tag{19}$$

Since $\frac{\partial\psi(M)}{\partial S_k}\frac{\partial k}{\partial M} > 0$, and due to the crowding-out effect, $\phi'() < 0$, it follows that when $\varepsilon < 1$, $\frac{\partial\psi(M)}{\partial M} < 0$, indicating that the model's steady-state equilibrium is globally stable. Conversely, when $\varepsilon > 1$, $\frac{\partial\psi(M)}{\partial M} > 0$, which means the model's steady-state equilibrium is unstable. In other words, Lemma 1 and Proposition 6 are only correct when equation (17) holds in all time; however, the paper also asserts that the scientists' wages are considered equal only when the economy is in a BGP equilibrium.

V. Overlooked the Important Policy Implications

The paper overlooks the important policy implications that significantly differ from those in traditional endogenous technological progress literature (Romer, 1990; Aghion and Howitt, 1992). Contrary to the paper's assertion that tax policy does not influence income distribution, this comment finds that, within the model, tax and subsidy policies have a substantial impact on income distribution, while not affecting technological progress or economic growth.

1. Tax Policy and Steady-State Income Shares of Factors

If we consider that scientists may have $\omega_{sk} \neq \omega_{sl}$ during non-steady states, and that taxes are applied to scientists' wages or the monopoly profits of innovative firms, then the budget constraint should be modified to:[8]

$$C + I \leq wL + rK + (1-\tau_k)\omega_{sk}S_k + (1-\tau_l)\omega_{sl}S_l + \Pi + T \tag{20}$$

where $T = \tau_k\omega_{sk}S_k + \tau_l\omega_{sl}S_l$.

At this point, the flow equation for scientists becomes as follows:

$$\frac{\dot{S_k}}{S_k} = G\left[\frac{(1-\tau_k)b_k\phi(S_k)}{(1-\tau_l)b_l\phi(S-S_k)}\frac{(1-\gamma)\nabla}{\gamma}k^{(\varepsilon-1)/\varepsilon}\right] \tag{21}$$

Equation (21) indicates that when a uniform tax rate is imposed on all scientists (i.e., $\tau_k = \tau_l$), taxation has no effect on $\dot{S_k}/S_k$, and consequently no impact on the equilibrium results of the model. However, when $\tau_k \neq \tau_l$, it follows from (21) that the relative income shares in the steady state after taxation are as follows:

$$\sigma_K{}^\tau = \frac{rK}{wL} = \frac{(1-\tau_l)b_l\phi(S-S_k{}^*)}{(1-\tau_k)b_k\phi(S_k{}^*)\nabla^*} = \frac{1-\tau_l}{1-\tau_k}\sigma_K \tag{22}$$

Equation (22) indicates that taxation affects the relative income share $\sigma_K{}^\tau$ in the BGP

[8] Alternatively, providing subsidies for the inputs used in the production of machines as intermediate goods would reduce the production costs for patent-holding firms to $(1-\tau_k)rK$ and $(1-\tau_l)wL$, which would have the same effect as taxing monopoly profits or scientists' wages.

steady-state equilibrium, and is negatively correlated with τ_l and positively correlated with τ_k. In other words, taxing innovations in capital-intensive machinery benefits the relative income of capital, while taxing innovations in labor-intensive machinery increases the relative income share of labor. Therefore, the paper's assertion that taxation does not affect long-term relative income shares, based solely on the premise that direct taxation of factor incomes does not influence income distribution, is inaccurate.

2. The Impact of Tax Policy on Technical Change and Economic Growth

The traditional literature on endogenous technological progress (Romer, 1990; Aghion and Howitt, 1992) posits that innovation has positive externalities, and that subsidies or taxes on innovation can influence innovation and promote economic growth. However, the model presented in this paper challenges this conventional view. Although the paper's innovation maintains economies of scale similar to the Romer model, in the BGP equilibrium, taxation does not affect the direction of technological progress; that is, it does not influence the relative magnitude of $\frac{\dot{M}/M}{\dot{N}/N}$, nor does it affect the magnitudes of $\dot{M}/M$ and $\dot{N}/N$. Since the economic growth rate equals $\dot{N}/N$, taxation also does not impact the economic growth rate in the BGP equilibrium, nor the interest rate $r^* = \rho + \theta g^*$. This result may be surprising for the model, as a key feature is that firms can undertake both labor- and capital-augmenting technological improvement, which would allow the direction of technological progress to be endogenously determined in the steady state. However, the outcome shows that, in the steady state, technological progress must be purely labor-augmenting, and no policy can influence this.

It is unfortunate that the implicit policy implications of taxation or subsidies are overlooked in this paper. Since the introduction of endogenous technological progress theories, government subsidies for innovation have been viewed as essential policies for promoting economic growth. However, the analysis presented above suggests that unless uniform subsidies are applied to all innovations, the outcomes of such taxes or subsidies may be entirely unforeseen. In practice, taxation and subsidy policies for innovation tend to be highly selective; certain sectors receive substantial innovation subsidies, while much innovation remains outside the government's purview. From the perspective of this model, such subsidy policies appear to provide minimal benefits for technological progress and economic growth, primarily affecting income distribution instead.

VI Intuition Behind the Core Conclusions

Both the exogenous and endogenous technological progress growth models assume that technological progress is purely labor-augmenting (Uzawa, 1961). However, this intuition of the assumption has long been a challenge for growth theory. The significance of Acemoglu's paper lies in the assertion that firms can pursue both labor- and capital-augmenting technological improvements; however, under profit incentives, firms will ultimately realize only net labor-augmenting technological progress in the steady state. This suggests that the exogenous assumption of purely labor-augmenting technological progress may align with the rational choices of firms, which is a much-anticipated outcome in existing growth theories. Nevertheless, why do these rational firms ultimately choose only labor-augmenting technological improvements? In other words, why can profit maximization be achieved solely through the selection of labor-augmenting technologies, while firms opting for capital-augmenting improvements fail to maximize profits? Acemoglu still needs to provide an economic intuition for the steady-state equilibrium results of the model.

The paper argues that the asymmetry between capital and labor accumulation—specifically, that capital, K, can be accumulated, while labor, L, cannot—is the key reason for this outcome. The author contends that the steady state necessitates balanced growth between *MK* and *NL*. Since capital can be accumulated, both *M* and *K* can grow, while in *NL*, only *N* can increase. Consequently, the author asserts that "capital accumulation, therefore, implies that technical change has to be, on average, more labor-augmenting than capital-augmenting." However, this does not fully explain the model's result regarding why "all technical change will be labor-augmenting". Indeed, Acemoglu (2009, ch.15.6; 2024) demonstrates the fallacy of this logic in his simplified model. In the simplified model, he assumes that capital is accumulable but is no longer a function of investment, expressed as $\dot{K}(t) = s_K K(t)$, while labor is not accumulable. Acemoglu confidently asserts that, in the steady state, technological progress remains purely labor-augmenting. However, in reality, the opposite is true; it is generally impossible for technological progress to be purely labor-augmenting (Li, 2016).

What, then, causes firms to opt for labor-augmenting innovations rather than capital-augmenting ones, even when they have the option to choose capital-augmenting technology? Recently, Li and Bental (2023) extended Acemoglu's model to develop a general framework for examining the determinants of the direction of technological progress in the steady state. They argue that the absence of net capital-augmenting technological progress in this context is attributable to the infinite elasticity of capital accumulation, rather than the asymmetry between capital and labor accumulation.

This intuition was actually proposed by Hicks (1932), who argued that an increase in factor prices would incentivize firms to innovate in order to economize those factors. However, Hicks overlooked the other side of price incentives: rising factor prices also motivate factor suppliers to increase the supply of those factors. If the supply of factors is infinitely elastic in response to price changes, then, in the long run, there is no potential for factor prices to rise.

Consequently, firms would lack the incentive to invest resources in technological innovations aimed at conserving these factors. The function $\dot{K}(t) = I(t) > 0$ implies that capital accumulation possesses infinite elasticity. This further explains why the core proposition of this paper holds only when $I(t) > 0$. When $I(t) = 0$ or $\dot{K}(t) = s_K K(t)$with s_K being exogenous, capital accumulation does not exhibit infinite elasticity, and therefore, the model in steady-state equilibrium includes capital-augmenting technological progress.

VII Summary

Acemoglu (2003) represents a significant contribution to growth theory by developing a model in which the direction of technological progress is determined endogenously. The model allows firms to pursue both labor-augmenting and capital-augmenting technologies; however, it ultimately converges to a steady-state equilibrium characterized solely by net labor-augmenting progress, with no advancements in capital-augmenting technologies. This framework provides a microeconomic foundation for earlier models that assume labor-augmenting technical change as exogenous, while also establishing a basis for analyzing the determinants of technological direction.

Despite its significant impact and enduring relevance in the field, the original paper has notable shortcomings that limit its utility for understanding economic growth and the dynamics of technological progress. This commentary identifies and rectifies mathematical errors in the original proofs, providing new proofs for all propositions. Additionally, it introduces a behavioral function for scientists' dynamic adjustments across different innovation sectors and revises several propositions from Acemoglu's work. While Acemoglu only analyzed the equilibrium results of the model under the condition I>0, this commentary supplements the model by examining the equilibrium results when I=0. The model exhibits a steady-state equilibrium under I=0 but does not possess a balanced growth path equilibrium, clearly delineating the distinctions between steady-state equilibrium and balanced growth path equilibrium. Furthermore, it elucidates that capital-augmenting technological progress does not affect the existence of a steady-state equilibrium but is inherently incompatible with the balanced growth path equilibrium. In addition, the model's equilibrium results indicate that taxes or subsidies for innovative firms do not affect the direction of technological progress, the rate of technological advancement, or overall economic growth; rather, they primarily influence income distribution. This important policy implication was overlooked in the original paper and contradicts the author's own positions.

Although the model posits that the direction of technological progress is endogenous, it fails to propose any policy instruments capable of altering this equilibrium direction. Firms may choose various technological paths; however, implicit constraints lead them to adopt only labor-augmenting technologies. The factors limiting firms' choices regarding the direction of technological progress in steady state remain unexplored, leaving the question of why long-run technical change is labor-augmenting inadequately addressed. The explanation provided—rooted in the asymmetry between capital and labor accumulation—does not satisfactorily clarify this issue.

Building on Acemoglu's work, Li and Bental (2023) delve deeper into the determinants of technological direction in the steady state. They extend the production and factor accumulation functions of the original model, proposing a general framework for analyzing this direction. Their findings suggest that the relative size of elasticity of factor supply is the key determinant of the direction of technological progress, establishing that infinite elasticity in capital accumulation leads to purely labor-augmenting technological advancement.

Additionally, the paper highlights that the innovation function for the lab-equipment model (Rivera-Batiz and Romer,1991) does not yield a steady-state equilibrium, indicating a sensitivity of the model's equilibrium to the form of the innovation function. This observation introduces a broader concern in growth theory: the apparent knife-edge nature of steady-state equilibria in growth models, a longstanding issue that merits further investigation in a dedicated study.

Appendix A: The derivation process of $\Delta(t)=\frac{m(t)V_k(t)}{n(t)V_l(t)}=\frac{(1-\gamma)}{\gamma}k(t)^{\frac{\varepsilon-1}{\varepsilon}}.\nabla(t)$

Assuming that the variables *r*(*v*), *K*(*v*), *m*(*v*), *w*(*v*), *L*(*v*), and *n*(*v*) exhibit growth rates as follows:

$$g_x(\omega)\equiv\frac{\dot{x}(\omega)}{x(\omega)},x=r,w,K,L,m,n \tag{A1}$$

Using (A1), the variables *r*(*v*), *K*(*v*), *m*(*v*), *w*(*v*), *L*(*v*), and *n*(*v*) can be expressed as follows:

$$x(v)=x(t)exp\left[\int_t^v g_x(\omega)\,d\omega\right] \tag{A2}$$

(A2) indicates that *x*(*v*) at time *v* is the value of *x*(*t*) at the initial time *t* accumulated to time *v* at the growth rate $g_x(\omega)$. Therefore, *x*(*t*) is merely the initial value of *x*(*v*) and is not equal to *x*(*v*) at other times. Substituting (A2) into equation (3) of this comment yields:

$$\Delta(t)\equiv\frac{m(t)V_k(t)}{n(t)V_l(t)}$$

$$=\frac{m(t)\int_t^\infty exp\left[-\int_t^v\left(r(t)exp\left[\int_t^\omega g_r(u)\,du\right]+\delta\right)d\omega\right]\frac{r(t)exp\left[\int_t^v g_r(\omega)\,d\omega\right]K(t)exp\left[\int_t^v g_K(\omega)\,d\omega\right]}{m(t)exp\left[\int_t^v g_m(\omega)\,d\omega\right]}dv}{n(t)\int_t^\infty exp\left[-\int_t^v\left(r(t)exp\left[\int_t^\omega g_r(u)\,du\right]+\delta\right)d\omega\right]\frac{w(t)exp\left[\int_t^v g_w(\omega)\,d\omega\right]L(t)exp\left[\int_t^v g_L(\omega)\,d\omega\right]}{n(t)exp\left[\int_t^v g_n(\omega)\,d\omega\right]}dv}$$

$$=\frac{r(t)K(t)\int_t^\infty exp\left[-\int_t^v\left(r(t)exp\left[\int_t^\omega g_r(u)\,du\right]+\delta\right)d\omega\right]\frac{exp\left[\int_t^v g_r(\omega)\,d\omega\right]exp\left[\int_t^v g_K(\omega)\,d\omega\right]}{exp\left[\int_t^v g_m(\omega)\,d\omega\right]}dv}{w(t)L(t)\int_t^\infty exp\left[-\int_t^v\left(r(t)exp\left[\int_t^\omega g_r(u)\,du\right]+\delta\right)d\omega\right]\frac{exp\left[\int_t^v g_w(\omega)\,d\omega\right]exp\left[\int_t^v g_L(\omega)\,d\omega\right]}{exp\left[\int_t^v g_n(\omega)\,d\omega\right]}dv}$$

$$=\frac{r(t)K(t)\int_t^\infty exp\left[\int_t^v[g_r(\omega)+g_K(\omega)-g_m(\omega)]d\omega-\int_t^v\left(r(t)exp\left[\int_t^\omega g_r(u)\,du\right]+\delta\right)d\omega\right]dv}{w(t)L(t)\int_t^\infty exp\left[\int_t^v[g_w(\omega)+g_L(\omega)-g_n(\omega)]d\omega-\int_t^v\left(r(t)exp\left[\int_t^\omega g_r(u)\,du\right]+\delta\right)d\omega\right]dv} \tag{A3}$$

Let $\nabla(t)\equiv\frac{\int_t^\infty exp\left[\int_t^v[g_r(\omega)+g_K(\omega)-g_m(\omega)]d\omega-\int_t^v\left(r(t)exp\left[\int_t^\omega g_r(u)du\right]+\delta\right)d\omega\right]dv}{\int_t^\infty exp\left[\int_t^v[g_w(\omega)+g_L(\omega)-g_n(\omega)]d\omega-\int_t^v\left(r(t)exp\left[\int_t^\omega g_r(u)du\right]+\delta\right)d\omega\right]dv}$, substituting it into equation (A3) gives

$$\Delta(t)=\frac{r(t)K(t)}{w(t)L(t)}.\nabla(t) \tag{A4}$$

Substituting the equation (26) in the paper, that is, $\frac{r(t)K(t)}{w(t)L(t)}=\frac{(1-\gamma)}{\gamma}k(t)^{\frac{\varepsilon-1}{\varepsilon}}$, into equation (A4) gives

$$\Delta(t)=\frac{(1-\gamma)}{\gamma}k(t)^{\frac{\varepsilon-1}{\varepsilon}}.\nabla(t) \tag{A5}$$

Equation (A5) is equation (5) in the comment.

Appendix B: Proof of Proposition A1

Proposition A1: For $\varepsilon < 1$, there exists a finite $T < \infty$ such that for $t \geq T$, the solution $\underset{t\geq T}{S_k(t)} = {S_k}^* = \frac{\delta}{b_k \phi({S_k}^*)}$ represents the model's asymptotic path (AP) equilibrium, which is also a balanced growth path (BGP) equilibrium. When $\delta > 0$, there exists a unique set of values (k^*, M^*, c^*); when $\delta = 0$, there are infinite sets of values (k^*, M^*, c^*) and ${S_k}^* = 0$ such that $\frac{\dot{S_k}}{S_k} = \frac{\dot{M}}{M} = \frac{\dot{c}}{c} = \frac{\dot{k}}{k} = 0$. However, for different (k^*, M^*, c^*), the growth rates of all variables of the model remain the same.

Proof.

Step 1: To prove that *there exists* a finite $T < \infty$ such that for all $t \geq T$, if $\frac{\dot{S_k}}{S_k} = \frac{\dot{M}}{M} = \frac{\dot{c}}{c} = \frac{\dot{k}}{k} = 0$ holds in the equation (13), then a Balanced Growth Path (BGP) equilibrium exists;

From the production function in equation (23) of the paper and the equation $k(t) \equiv \frac{M(t)K(t)}{N(t)L(t)}$, we obtain:

$$\begin{cases} \frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{M}(t)}{M(t)} + \frac{\dot{K}(t)}{K(t)} - \frac{\gamma}{\gamma + (1-\gamma)(k)^{(\varepsilon-1)/\varepsilon}} \frac{\dot{k}(t)}{k(t)} \\ \frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{N}(t)}{N(t)} + \frac{\dot{L}(t)}{L(t)} + \frac{(1-\gamma)(k)^{(\varepsilon-1)/\varepsilon}}{\gamma + (1-\gamma)(k)^{(\varepsilon-1)/\varepsilon}} \frac{\dot{k}(t)}{k(t)} \end{cases} \quad (B1)$$

Substituting $\frac{\dot{M}}{M} = \frac{\dot{k}}{k} = 0$ and $\frac{\dot{L}(t)}{L(t)} = 0$ into equation (B1) and using $\frac{\dot{c}}{c} = 0$ yields

$$g^* = \frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = \frac{\dot{C}}{C} = \frac{\dot{N}}{N} = \frac{1-\beta}{\beta}[b_l \phi(S - {S_k}^*)(S - {S_k}^*) - \delta] \quad (B2)$$

Substituting equation (B2) in Euler equation $\frac{\dot{C}}{C} = \frac{1}{\theta}(r(t) - \rho)$ yields

$$r^* = \theta g^* + \rho \quad (B3)$$

(B2) and (B3) indicate that when $t \geq T$, $\underset{t\geq T}{\dot{Y}(t)/Y(t)} = \underset{t\geq T}{\dot{K}(t)/K(t)} = \underset{t\geq T}{\dot{C}(t)/C(t)} = g^*$, and $\underset{t\geq T}{\dot{M}(t)/M(t)} = 0$. Therefore, the model is in a Balanced Growth Path (BGP) equilibrium.

Next, we will derive the specific results for ${S_k}^*$ and (k^*, M^*, c^*) when $\frac{\dot{S_k}}{S_k} = \frac{\dot{M}}{M} = \frac{\dot{c}}{c} = \frac{\dot{k}}{k} = 0$.

Substituting $r = \beta(1-\gamma)M\left[\gamma(k^*)^{-(\varepsilon-1)/\varepsilon} + (1-\gamma)\right]^{1/(\varepsilon-1)}$ into equation (B3) yields

$$M^* = \frac{\theta g^* + \rho}{\beta(1-\gamma)}\left[\gamma(k^*)^{(1-\varepsilon)/\varepsilon} + (1-\gamma)\right]^{1/(1-\varepsilon)} \quad (B4)$$

Substituting (B4) and $f(k) = \left[\gamma + (1-\gamma)k^{(\varepsilon-1)/\varepsilon}\right]^{\varepsilon/(\varepsilon-1)}$ into $\frac{\dot{k}}{k} = \frac{Mf(k)}{k} - c - g^* = 0$ yields

$$c^* = \frac{\theta g^* + \rho}{\beta(1-\gamma)}\left[\gamma(k^*)^{-(\varepsilon-1)/\varepsilon} + (1-\gamma)\right] - g^* \tag{B5}$$

From $\frac{\dot{M}}{M} = 0$ we can obtain

$${S_k}^* = \frac{\delta}{b_k \phi({S_k}^*)} \tag{B6}$$

When $\delta > 0$, (B6) indicates that ${S_k}^* > 0$. Given that the paper assumes $S > {S_k}^* = \frac{\delta}{b_k \phi({S_k}^*)}$, it follows that ${S_l}^* = S - {S_k}^* > 0$. Since the scientists are homogeneous, when ${S_l}^* = S - {S_k}^* > 0$ and both ${S_k}^*$ and ${S_l}^*$ are positive, from $\frac{\dot{S_k}}{S_k} = 0$ the following equation (B7) must hold:

$$\frac{b_k \phi(S_k)}{b_l \phi(S - S_k)} \frac{(1-\gamma)\nabla}{\gamma} k^{(\varepsilon-1)/\varepsilon} = 1 \tag{B7}$$

From (B7), we obtain the following (B8):

$$(k^*)^{\frac{\varepsilon-1}{\varepsilon}} = \frac{b_l \phi(S - {S_k}^*)\gamma}{b_k \phi({S_k}^*)(1-\gamma)\nabla^*} \tag{B8}$$

From the equation (18) in the paper, we have $w = \beta\gamma N\left[\gamma + (1-\gamma)k^{(\varepsilon-1)/\varepsilon}\right]^{1/(\varepsilon-1)}$. Taking the growth rate of both sides of this equation and using $\frac{\dot{k}}{k} = 0$, we obtain $g_w(\omega) = g^*$. From $M \equiv m^{(1-\beta)/\beta}$, $\frac{\dot{M}}{M} = 0$, and (B3), we have $g_r(\omega) = g_m(\omega) = 0$. Additionally, from $N \equiv n^{(1-\beta)/\beta}$, it follows that $g_n(\omega) = \frac{\beta}{1-\beta} g^*$.

Substituting $g_w(\omega)$, $g_r(\omega)$, $g_m(\omega)$, $g_n(\omega)$, $g_K(\omega)$, $g_L(\omega)$ and $r^* = \theta g^* + \rho$ into $\nabla(t)$ yields

$$\nabla^* = \frac{[\theta + (2\beta - 1)/(1-\beta)]g^* + \rho + \delta}{(\theta - 1)g^* + \rho + \delta} \tag{B9}$$

Equations (B4), (B5), (B6), and (B8) provide the specific values of ${S_k}^*$ and (k^*, M^*, c^*) when $\frac{\dot{S_k}}{S_k} = \frac{\dot{M}}{M} = \frac{\dot{c}}{c} = \frac{\dot{k}}{k} = 0$, summarized as follows:

$$
\begin{cases}
\underset{t\geq T}{S_k(t)} = {S_k}^* = \dfrac{\delta}{b_k \phi({S_k}^*)} & (B6) \\
\underset{t\geq T}{k(t)}^{\frac{\varepsilon-1}{\varepsilon}} = (k^*)^{\frac{\varepsilon-1}{\varepsilon}} = \dfrac{b_l \phi(S - {S_k}^*)\gamma}{b_k \phi({S_k}^*)(1-\gamma)\nabla^*} & (B8) \\
\underset{t\geq T}{M(t)} = M^* = \dfrac{\theta g^* + \rho}{\beta(1-\gamma)}\left[\gamma (k^*)^{(1-\varepsilon)/\varepsilon} + (1-\gamma)\right]^{1/(1-\varepsilon)} & (B4) \\
\underset{t\geq T}{c(t)} = c^* = \dfrac{\theta g^* + \rho}{\beta(1-\gamma)}\left[\gamma (k^*)^{-(\varepsilon-1)/\varepsilon} + (1-\gamma)\right] - g^* & (B5)
\end{cases}
$$

When $\delta = 0$, since ${S_k}^* = \frac{\delta}{b_k \phi({S_k}^*)} = 0$, there will be no movement of scientists even if $\frac{\omega_{sk}}{\omega_{sl}} < 1$. Therefore, from $\frac{\dot{S_k}}{S_k} = 0$, we can only derive the following inequality (B10):

$$
\frac{b_k \phi(S_k)}{b_l \phi(S - S_k)} \frac{(1-\gamma)\nabla}{\gamma} k^{(\varepsilon-1)/\varepsilon} \leq 1 \qquad (B10)
$$

The following inequality (B11) can be obtained from inequality (B10):

$$
(k^*)^{\frac{\varepsilon-1}{\varepsilon}} \geq \frac{b_l \phi(S - {S_k}^*)\gamma}{b_k \phi({S_k}^*)(1-\gamma)\nabla^*} \qquad (B11)
$$

All k^* that satisfy (B11) ensure that $\frac{\dot{S_k}}{S_k} = 0$, allowing the model to achieve BGP equilibrium. Since M^* and c^* are functions of k^*, there are also numerous M^* and c^* that can lead the model to BGP equilibrium.

Substituting (B8) into the relative income shares of capital and labor $\sigma_K \equiv \frac{r(t)K(t)}{w(t)L(t)} = \frac{(1-\gamma)}{\gamma} k^{\frac{\varepsilon-1}{\varepsilon}}$, if $\delta > 0$ then yields

$$
{\sigma_K}^* = \frac{(1-\gamma)}{\gamma}(k^*)^{\frac{\varepsilon-1}{\varepsilon}} = \frac{b_l \phi(S - {S_k}^*)}{b_k \phi({S_k}^*)\nabla^*} \qquad (B12a);
$$

if $\delta = 0$ then yields

$$
{\sigma_K}^* = \frac{(1-\gamma)}{\gamma}(k^*)^{\frac{\varepsilon-1}{\varepsilon}} \geq \frac{b_l \phi(S - {S_k}^*)}{b_k \phi({S_k}^*)\nabla^*} \qquad (B12b).
$$

Thus, when $\delta > 0$, the BGP equilibrium corresponds to a unique relative income share, while for $\delta = 0$, there are infinitely many relative income shares.

Step two: Prove that there is only one equilibrium ${S_k}^* = \frac{\delta}{b_k \phi({S_k}^*)}$, or in other words, that no equilibrium exists with $\frac{\dot{M}}{M} \neq 0$.

First, we prove by contradiction that there is no equilibrium $S_k = {S_k}^{**} > {S_k}^* = \frac{\delta}{b_k \phi({S_k}^*)}$ in the model.

Suppose there exists another equilibrium $S_k = {S_k}^{**} > {S_k}^* = \frac{\delta}{b_k \phi({S_k}^*)}$, we have $\frac{\dot{S_k}}{S_k} = 0$. In this case, it must hold that $\frac{b_k \phi(S_k{}^{**})}{b_l \phi(S-S_k{}^{**})} \frac{(1-\gamma)\nabla}{\gamma} k^{(\varepsilon-1)/\varepsilon} \geq 1$; otherwise, $\frac{\dot{S_k}}{S_k} < 0$, which contradicts $\frac{\dot{S_k}}{S_k} = 0$. Substituting $S_k = S_k{}^{**}$ into $\frac{\dot{k}}{k}$ yields:

$$\frac{\dot{k}}{k} = \left(\frac{Mf(k)}{k} - c\right) + \frac{1-\beta}{\beta}[b_k \phi(S_k{}^{**}) S_k{}^{**} - b_l \phi(S - S_k{}^{**})(S - S_k{}^{**})] \quad (B13)$$

Since $S_k{}^{**} > S_k{}^{*}$, it follows that $\frac{\dot{M}}{M} > 0$. Consequently, *M* will continue to rise, leading to an ongoing increase in $\frac{\dot{k}}{k}$. As long as $\frac{\dot{M}}{M} > 0$, there will eventually be $\frac{\dot{k}}{k} > 0$, resulting in an increase in k. Given that $\varepsilon < 1$, $k^{(\varepsilon-1)/\varepsilon}$ will decline. Therefore, as long as $\frac{\dot{M}}{M} > 0$, $k^{(\varepsilon-1)/\varepsilon}$ will keep decreasing, ultimately leading to $\frac{b_k \phi(S_k{}^{**})}{b_l \phi(S-S_k{}^{**})} \frac{(1-\gamma)\nabla}{\gamma} k^{(\varepsilon-1)/\varepsilon} < 1$, which implies $\frac{\dot{S_k}}{S_k} < 0$. This proves that any $S_k = S_k{}^{**} > S_k{}^{*}$ cannot be a steady-state equilibrium.

Similarly, it can be proved that there is no other equilibrium $S_k = S_k{}^{**} < S_k{}^{*} = \frac{\delta}{b_k \phi(S_k{}^{*})}$ in the model.

The above proof demonstrates that the dynamic system described by equation (13) has not only one balanced growth path (BGP) equilibrium at $\frac{\dot{S_k}}{S_k} = \frac{\dot{M}}{M} = \frac{\dot{c}}{c} = \frac{\dot{k}}{k} = 0$, but also that this equilibrium is unique at $S_k{}^{*} = \frac{\delta}{b_k \phi(S_k{}^{*})}$, providing specific values of $S_k{}^{*}$ and (k^*, M^*, c^*). Thus, this proves Proposition A1, as well as Propositions 1, 3, and 4.

Appendix C: Proof of Proposition A2

Proposition A2: For $\varepsilon > 1$, there exist three asymptotic path (AP) equilibria:

(1) When ≥ 0, $S_k{}^{*} = \frac{\delta}{b_k \phi(S_k{}^{*})}$, there exists a BGP equilibrium that is also an AP equilibrium;

(2) When $\delta \geq 0$, $S_k{}^{*} = S$, there exists a non-steady-state AP equilibrium;

(3) When $\delta > 0$, $S_k{}^{*} = 0$, there is also a non-steady-state AP equilibrium.

Proof.

In the first step, we can similarly prove that when $\varepsilon > 1$, the conditions $\frac{\dot{S_k}}{S_k} = \frac{\dot{M}}{M} = \frac{\dot{c}}{c} = \frac{\dot{k}}{k} = 0$ yield the same balanced growth path (BGP) equilibrium as in the case of $\varepsilon < 1$. Here, $S_k{}^{*} = \frac{\delta}{b_k \phi(S_k{}^{*})}$ is the unique value that satisfies $\frac{\dot{S_k}}{S_k} = \frac{\dot{M}}{M} = \frac{\dot{c}}{c} = \frac{\dot{k}}{k} = 0$ for $\delta > 0$, resulting in a unique set of values (k^*, M^*, c^*). When $\delta = 0$, there are infinitely many sets of (k^*, M^*, c^*) and a unique $S_k^* = \frac{\delta}{b_k \phi(S_k^*)}$ that satisfy the same conditions. The growth rates of each variable at equilibrium are also the same as in the case of $\varepsilon < 1$, regardless of whether $\delta > 0$ or $\delta = 0$.

In the second step, we prove that when $S_k^* = S$, the model has an asymptotic path (AP) equilibrium.

When $S_k = S$, substituting it into the dynamic system represented by equation (13) yields $\dot{M}/M = \frac{1-\beta}{\beta}[b_k \phi(S)S - \delta] > 0$. Substituting $= M(0)\exp\left(\frac{1-\beta}{\beta}[b_k \phi(S)S - \delta]t\right)$ and $S_k = S$ into the $\dot{k}/k$ function in equation (13) results in the following equation:

$$\dot{k}/k = \left(\frac{M(0)\exp\left(\frac{1-\beta}{\beta}[b_k \phi(S)S - \delta]t\right)}{f(k)/k} - c \right) + \frac{1-\beta}{\beta} b_k \phi(S)S \qquad (C1)$$

Since $\dot{M}/M > 0$, we have $\lim_{t\to\infty} \dot{k}/k = \infty$, which implies $\lim_{t\to\infty} k = \infty$. Given that $\varepsilon > 1$, it follows that $\lim_{t\to\infty} k^{(\varepsilon-1)/\varepsilon} = \infty$. As long as $\phi(0) < \infty$, it must be that $\lim_{t\to\infty} \frac{b_k \phi(S)}{b_l \phi(0)} \frac{(1-\gamma)\nabla}{\gamma} k^{(\varepsilon-1)/\varepsilon} > 1$. Since we already have $S_k = S$, it follows that $\frac{\dot{S_k}}{S_k} = 0$. Therefore, $S_k = S$ is an asymptotic path (AP) equilibrium. However, at this point, $\dot{M}(t)/M(t) > 0$ and $\dot{N}(t)/N(t) = \frac{1-\beta}{\beta}[b_l \phi(0)(0) - \delta] = -\frac{1-\beta}{\beta}\delta < 0$.

Substituting $\lim_{t\to\infty} k^{(\varepsilon-1)/\varepsilon} = \infty$ into equation (B1) yields

$$\lim_{t\to\infty} \frac{\dot{Y}(t)}{Y(t)} = \lim_{t\to\infty} \left(\frac{\dot{M}(t)}{M(t)} + \frac{\dot{K}(t)}{K(t)} \right) = \lim_{t\to\infty} \left(\frac{\dot{N}(t)}{N(t)} + \frac{\dot{k}(t)}{k(t)} \right) = \infty \qquad (C2)$$

Since $\frac{\dot{M}(t)}{M(t)} < \infty$ and $\frac{\dot{N}(t)}{N(t)} < \infty$, we can derive the equation (C3) from equation (C2) as follows:

$$\lim_{t\to\infty} \frac{\dot{Y}(t)}{Y(t)} = \lim_{t\to\infty} \left(\frac{\dot{K}(t)}{K(t)} \right) = \lim_{t\to\infty} \left(\frac{\dot{k}(t)}{k(t)} \right) = \infty \qquad (C3)$$

Since $\lim_{t\to\infty} M(t) = \infty$ and $\lim_{t\to\infty} k(t)^{-\frac{\varepsilon-1}{\varepsilon}} \to 0$, we can derive the equation (C4) as follows:

$$\lim_{t\to\infty} r(t) = \beta(1-\gamma)M(t)\left[\gamma k^{-\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma)\right]^{\frac{1}{\varepsilon-1}} = \infty \qquad (C4)$$

Substituting (C4) into Euler equation yields

$$\lim_{t\to\infty}\frac{C(\dot{t})}{C(t)}=\lim_{t\to\infty}\frac{1}{\theta}(r(t)-\rho)=\infty \quad (C5)$$

From equations (C3) and (C5) we can obtain the following equations:

$$\lim_{t\to\infty}\frac{\dot{Y}(t)}{Y(t)}=\lim_{t\to\infty}\left(\frac{\dot{K}(t)}{K(t)}\right)=\lim_{t\to\infty}\left(\frac{C(\dot{t})}{C(t)}\right)=\infty \quad (C6)$$

At this point, in the dynamic system represented by equation (13), there is $\frac{\dot{S_k}}{S_k}=0$. However, $\lim_{t\to\infty}\left(\frac{\dot{k}(t)}{k(t)}\right)=\infty$ and $\frac{\dot{M}(t)}{M(t)}=\frac{1-\beta}{\beta}[b_k\phi(S)S-\delta]>0$. Therefore, the model represents an asymptotic path (AP) equilibrium, rather than a steady-state (SS) equilibrium or a balanced growth path (BGP) equilibrium.

In the third step, when $\delta>0$ and ${S_k}^*=0$, the model has another asymptotic path (AP) equilibrium.

When $S_k=0$, substituting this into the dynamic system represented by equation (13) yields $\dot{M}/M=-\frac{1-\beta}{\beta}\delta<0$. By substituting $M=M(0)\exp\left(-\frac{1-\beta}{\beta}\delta t\right)$ and $S_k=0$ into the $\dot{k}/k$ function from equation (13), we obtain the following equation (C7):

$$\dot{k}/k=\left(\frac{M(0)\exp\left(-\frac{1-\beta}{\beta}\delta t\right)}{f(k)/k}-c\right)-\frac{1-\beta}{\beta}b_l\phi(S)S \quad (C7)$$

Since $\dot{M}/M<0$, it follows that $\lim_{t\to\infty}\dot{k}/k<0$, which implies $\lim_{t\to\infty}k=0$. Given that $\varepsilon>1$, we have $\lim_{t\to\infty}k^{(\varepsilon-1)/\varepsilon}=0$. As long as $\phi(0)<\infty$, it follows that $\lim_{t\to\infty}\frac{b_k\phi(0)}{b_l\phi(S)}\frac{(1-\gamma)\nabla}{\gamma}k^{(\varepsilon-1)/\varepsilon}<1$. Since we already have $S_k=0$, it follows that $\frac{\dot{S_k}}{S_k}=0$. Therefore, $S_k=0$ is an asymptotic path (AP) equilibrium.

Substituting $\lim_{t\to\infty}k^{(\varepsilon-1)/\varepsilon}=0$ into (B1) yields

$$\lim_{t\to\infty}\frac{\dot{Y}(t)}{Y(t)}=\lim_{t\to\infty}\left(\frac{\dot{M}(t)}{M(t)}+\frac{\dot{K}(t)}{K(t)}-\frac{\dot{k}(t)}{k(t)}\right)=\lim_{t\to\infty}\left(\frac{\dot{N}(t)}{N(t)}\right)=\frac{1-\beta}{\beta}[b_l\phi(S)S-\delta] \quad (C8)$$

From $r(t)=\beta(1-\gamma)M(t)\left[\gamma k(t)^{-\frac{\varepsilon-1}{\varepsilon}}+(1-\gamma)\right]^{\frac{1}{\varepsilon-1}}$, we can obtain the following equation:

$$\frac{r(\dot{t})}{r(t)}=\frac{\dot{M}(t)}{M(t)}-\frac{1}{\varepsilon\left[1+\frac{(1-\gamma)}{\gamma}k(t)^{\frac{\varepsilon-1}{\varepsilon}}\right]}\frac{\dot{k}(t)}{k(t)} \quad (C9)$$

Substituting $\lim_{t\to\infty} k^{\frac{\varepsilon-1}{\varepsilon}} = 0$ into (C9) we can obtain the following equation

$$\lim_{t\to\infty}\frac{\dot{r(t)}}{r(t)} = \lim_{t\to\infty}\left(\frac{\dot{M}(t)}{M(t)} - \frac{1}{\varepsilon}\frac{\dot{k}(t)}{k(t)}\right) \qquad (C10)$$

Substituting (C10) into Euler equation yields

$$\lim_{t\to\infty}\frac{\dot{C(t)}}{C(t)} = \lim_{t\to\infty}\frac{1}{\theta}\left(r(0)exp\left[-\frac{1-\beta}{\beta}\delta - \frac{1}{\varepsilon}\frac{\dot{k}(t)}{k(t)}\right]t - \rho\right) \qquad (C11)$$

If $\lim_{t\to\infty}\left[-\frac{1-\beta}{\beta}\delta - \frac{1}{\varepsilon}\frac{\dot{k}(t)}{k(t)}\right] > 0$, then $\lim_{t\to\infty}\frac{\dot{C(t)}}{C(t)} = \infty$. Conversely, if $\lim_{t\to\infty}\left[-\frac{1-\beta}{\beta}\delta - \frac{1}{\varepsilon}\frac{\dot{k}(t)}{k(t)}\right] < 0$, then $\lim_{t\to\infty}\frac{\dot{C(t)}}{C(t)} = -\frac{1}{\theta}\rho < 0$, leading to $\lim_{t\to\infty} C(t) = 0$. Given that $0 < C(t) < Y(t)$ and $\lim_{t\to\infty}\frac{\dot{Y(t)}}{Y(t)} = \frac{1-\beta}{\beta}[b_l\phi(S)S - \delta] < \infty$, it follows that $0 < \lim_{t\to\infty}\frac{\dot{C(t)}}{C(t)} < \infty$. Therefore, we must have $\lim_{t\to\infty}\left[-\frac{1-\beta}{\beta}\delta - \frac{1}{\varepsilon}\frac{\dot{k}(t)}{k(t)}\right] = 0$. Substituting it into equation (C11) yields

$$\lim_{t\to\infty}\frac{\dot{C(t)}}{C(t)} = \frac{1}{\theta}(r(0) - \rho) = g_c < \infty \qquad (C12)$$

and

$$\lim_{t\to\infty}\frac{\dot{k}(t)}{k(t)} = -\frac{1-\beta}{\beta}\varepsilon\delta = g_k \qquad (C13)$$

Substituting (C13) into (C9) yields

$$\lim_{t\to\infty}\frac{\dot{K}(t)}{K(t)} = \frac{1-\beta}{\beta}[b_l\phi(S)S - \varepsilon\delta] < \lim_{t\to\infty}\frac{\dot{Y}(t)}{Y(t)} \qquad (C14)$$

From $\lim_{t\to\infty}\frac{\dot{K}(t)}{K(t)} = \lim_{t\to\infty}\frac{I(t)}{K(t)} = \frac{1-\beta}{\beta}[b_l\phi(S)S - \varepsilon\delta]$ we can obtain the following equation:

$$\lim_{t\to\infty}\frac{\dot{K}(t)}{K(t)} = \lim_{t\to\infty}\frac{\dot{I}(t)}{I(t)} \qquad (C15)$$

From $C(t) + I(t) = Y(t)$ and $\lim_{t\to\infty}\frac{\dot{K}(t)}{K(t)} < \lim_{t\to\infty}\frac{\dot{Y}(t)}{Y(t)}$, we can obtain $\lim_{t\to\infty}\frac{\dot{C(t)}}{C(t)} > \lim_{t\to\infty}\frac{\dot{Y}(t)}{Y(t)}$. Combining with (C14), we obtain the following chain of inequalities:

$$\lim_{t\to\infty}\frac{\dot{K}(t)}{K(t)} < \lim_{t\to\infty}\frac{\dot{Y}(t)}{Y(t)} < \lim_{t\to\infty}\frac{\dot{C(t)}}{C(t)} \qquad (C16)$$

Since $\frac{\dot{c}}{c} = \frac{\dot{C(t)}}{C(t)} - \frac{\dot{K}(t)}{K(t)}$, it follows that $\lim_{t\to\infty}\frac{\dot{c}}{c} > 0$. Therefore, in the dynamic system represented by equation (13), while $\frac{\dot{S_k}}{S_k} = 0$, the other three equations—$\frac{\dot{M}}{M}$, $\frac{\dot{c}}{c}$ and $\frac{\dot{k}}{k}$—are all nonzero. Thus, although the distribution of scientists has reached a steady state, the model itself has not, which is why Acemoglu refers to it as an asymptotic path (AP) equilibrium. It is certainly not a balanced growth path (BGP) equilibrium.

In summary, Proposition A2 is proven, which confirms the main part of Proposition 2 in the paper.

Appendix D: Proof of Proposition A3

Proposition A3: For $\varepsilon = 1$, the model has a unique asymptotic path (AP) equilibrium, which is also a balanced growth path (BGP) equilibrium.

Proof.

When $\varepsilon = 1$, the production function is no longer a CES function but a Cobb-Douglas (C-D) function, namely:

$$Y(t) = [M(t)K(t)]^{1-\gamma}[N(t)L(t)]^{\gamma} \quad (D1)$$

Since $\Delta(t) \equiv \frac{m(t)V_k(t)}{n(t)V_l(t)} = \frac{r(t)K(t)}{w(t)L(t)}\nabla(t)$, for a Cobb-Douglas function, the relative income shares of the factors are given by:

$$\sigma_K \equiv \frac{r(t)K(t)}{w(t)L(t)} = \frac{1-\gamma}{\gamma} \quad (D2)$$

At this point, the dynamic equation $\frac{\dot{S_k}}{S_k}$ becomes:

$$\frac{\dot{S_k}}{S_k} = G\left[\frac{b_k\phi(S_k)}{b_l\phi(S-S_k)}\frac{(1-\gamma)\nabla}{\gamma}\right] \quad (D3)$$

When $\frac{\dot{S_k}}{S_k} = 0$, it determines $\frac{\dot{M}}{M} = \frac{1-\beta}{\beta}[b_k\phi(S_k)S_k - \delta]$; thus, $\frac{\dot{M}}{M}$ is no longer an independent dynamic equation. On the other hand, we need to redefine $k(t) \equiv \frac{K(t)}{[M(t)]^{(1-\gamma)/\gamma}N(t)L(t)}$, resulting in the new $\frac{\dot{k}}{k}$ function as follows:

$$\frac{\dot{k}}{k} = k^{-\gamma} - c - \frac{1-\beta}{\beta}\left(\left[\frac{1-\gamma}{\gamma}b_k\phi(S_k)S_k + b_l\phi(S-S_k)(S-S_k) - \frac{\delta}{\gamma}\right]\right) \quad (D4)$$

Using $k(t) \equiv \frac{K(t)}{[M(t)]^{(1-\gamma)/\gamma}N(t)L(t)}$ to yield $\frac{\dot{c}}{c}$ function as follows:

$$\frac{\dot{c}}{c} = \left(\frac{\beta(1-\gamma)-\theta}{\theta}k^{-\gamma}\right) - \left(\frac{\rho}{\theta} - c\right) \quad (D5)$$

(D3), (D4), and (D5) form a new dynamic system involving the three variables S_k, k and c, which describes the dynamic behavior of the model.

If there exists a finite $T < \infty$ such that for $t \geq T$, $\frac{\dot{S_k}}{S_k} = \frac{\dot{k}}{k} = \frac{\dot{c}}{c} = 0$, we have:

$$\begin{cases} \frac{b_k \phi(S_k)}{b_l \phi(S - S_k)} \frac{(1-\gamma)\nabla}{\gamma} = 1 \\ \left(\frac{\beta(1-\gamma) - \theta}{\theta} k^{-\gamma}\right) - \left(\frac{\rho}{\theta} - c\right) = 0 \\ k^{-\gamma} - c - \frac{1-\beta}{\beta}\left(\frac{1-\gamma}{\gamma} g_m + g_n\right) = 0 \end{cases} \quad \text{(D6)}$$

It can be derived from (D6) that:

$$\begin{cases} \frac{\phi({S_k}^*)}{\phi(S - {S_k}^*)} = \frac{b_l \gamma}{b_k (1-\gamma)\nabla^*} \\ c^* = \frac{\rho}{\theta} - \frac{\beta(1-\gamma) - \theta}{\theta}\left(\frac{\rho}{\beta(1-\gamma)} + \frac{(1-\beta)\theta}{\beta\beta(1-\gamma)}\left(\frac{1-\gamma}{\gamma} {g_m}^* + {g_n}^*\right)\right) \\ (k^*)^{-\gamma} = \frac{\rho}{\beta(1-\gamma)} + \frac{(1-\beta)\theta}{\beta\beta(1-\gamma)}\left(\frac{1-\gamma}{\gamma} g_m + g_n\right) \end{cases} \quad \text{(D7)}$$

Substituting ${S_k}^*$ into the innovation function yields:

$$\begin{cases} {g_m}^* \equiv \frac{\dot{m}}{m} = b_k \phi({S_k}^*){S_k}^* - \delta \\ {g_n}^* \equiv \frac{\dot{n}}{n} = b_l \phi(S - {S_k}^*)(S - {S_k}^*) - \delta \end{cases} \quad (D8)$$

From $\frac{\dot{k}}{k} = 0$, the following results can be obtained:

$$g_K{}^* \equiv \frac{\dot{K}}{K} = \frac{1-\beta}{\beta}\left(\frac{1-\gamma}{\gamma} {g_m}^* + {g_n}^*\right) \quad (D9)$$

From production function (D1), we can derive the following results:

$$g^* \equiv \frac{\dot{Y}}{Y} = \frac{1-\beta}{\beta}\left(\frac{1-\gamma}{\gamma} {g_m}^* + {g_n}^*\right) \quad (D10)$$

From $\frac{\dot{c}}{c} = 0$, we can derive the following results:

$$g_C{}^* \equiv \frac{\dot{C}}{C} = \frac{\dot{K}}{K} = \frac{1-\beta}{\beta}\left(\frac{1-\gamma}{\gamma} {g_m}^* + {g_n}^*\right) \quad (D11)$$

Substituting (D11) into Euler equation yields

$$r^* = \theta g^* + \rho \quad (D12)$$

From equation (18) in the paper, we have $w = \gamma[M(t)K(t)]^{1-\gamma}[N(t)]^{\gamma}[L(t)]^{\gamma-1}$. This leads to $g_w(\omega) = g^*$. From (D12), we obtain $g_r(\omega) = 0$. Substituting $g_w(\omega)$, $g_r(\omega)$, $g_m(\omega)$, $g_n(\omega)$, $g_K(\omega)$, $g_L(\omega)$ and $r^* = \theta g^* + \rho$ into $\nabla(t)$ yields:

$$\nabla^*=\frac{\int_t^{\infty} exp\left[\int_t^{v}[(1-\theta)g^* - g_m - \rho - \delta]d\omega\right]dv}{\int_t^{\infty} exp\left[\int_t^{v}[(1-\theta)g^* - g_n - \rho - \delta)]d\omega\right]dv} = \frac{(1-\theta)g^* - g_n - \rho - \delta}{(1-\theta)g^* - g_m - \rho - \delta} \quad (D13)$$

Although if ${S_k}^* = S$ results in $\frac{b_k\phi(S_k)}{b_l\phi(S-S_k)}\frac{(1-\gamma)\nabla}{\gamma} > 1$, we still have $\frac{\dot{S_k}}{S_k} = 0$, there is only one equilibrium at ${S_k}^* = S$. Similarly, if ${S_k}^* = 0$ leads to $\frac{b_k\phi(S_k)}{b_l\phi(S-S_k)}\frac{(1-\gamma)\nabla}{\gamma} < 1$, we also have $\frac{\dot{S_k}}{S_k} = 0$, but again, there is only one equilibrium at ${S_k}^* = 0$. Furthermore, the preceding solution process is identical, and the model's results remain the same.

Although $\frac{\dot{M}}{M} = \frac{1-\beta}{\beta}{g_m}^* > 0$, since the production function is a Cobb-Douglas function and $r^* = \theta g^* + \rho$ is a constant, we have $\frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = \frac{\dot{C}}{C} = \frac{1-\beta}{\beta}\left(\frac{1-\gamma}{\gamma}{g_m}^* + {g_n}^*\right)$. Therefore, the unique steady state (SS) equilibrium of the model is still the balanced growth path (BGP) equilibrium.

In summary, Proposition A3 is proved. Propositions A2 and A3 together indicate the validity of Proposition 2 in the paper.

Appendix E: Proof of Proposition B

Proposition B: If $I = 0$, the model has asymptotic path (APs) equilibrium and a steady-state (SS) equilibrium, but no balanced growth path (BGP) equilibrium. When $\varepsilon < 1$, the model has a unique SS equilibrium; when $\varepsilon > 1$, the model has three AP equilibria, one of which is an SS equilibrium and the other two are non-stationary AP equilibria. When $\varepsilon = 1$, the model typically has only an SS equilibrium.

Proof.

For equation (14), if there exists a finite $T < \infty$ such that for $t \geq T$, $\frac{\dot{S_k}}{S_k} = \frac{\dot{k}}{k} = 0$, we have:

$$\begin{cases} \frac{b_k\phi(S_k)}{b_l\phi(S-S_k)}\frac{(1-\gamma)\nabla}{\gamma}k^{(\varepsilon-1)/\varepsilon} = 0 \\ \frac{1-\beta}{\beta}[b_k\phi(S_k)S_k - b_l\phi(S-S_k)(S-S_k)] = 0 \end{cases} \quad (E1)$$

The unique solution can be derived as follows:

$$\begin{cases} {S_k}^* = \frac{b_l\phi(S-{S_k}^*)}{b_k\phi({S_k}^*) + b_l\phi(S-{S_k}^*)}S \\ (k^*)^{(\varepsilon-1)/\varepsilon} = \frac{b_l\phi(S-{S_k}^*)\gamma}{b_k\phi({S_k}^*)(1-\gamma)\nabla^*} \end{cases} \quad (E2)$$

Substituting ${S_k}^*$ into equation $\frac{\dot{k}}{k} = 0$ yields

$$g_m{}^* = g_n{}^* = b_k\phi(S_k{}^*)S_k{}^* - \delta \tag{E3}$$

From production function in equation (23) of the paper, the result in the following equation can be derived:

$$g^* \equiv \frac{\dot{Y}}{Y} = \frac{1-\beta}{\beta}[b_k\phi(S_k{}^*)S_k{}^* - \delta] \tag{E4}$$

Since I=0, we have C=Y, leading to the following result:

$$g_C{}^* \equiv \frac{\dot{C}}{C} = \frac{\dot{Y}}{Y} = \frac{1-\beta}{\beta}[b_k\phi(S_k{}^*)S_k{}^* - \delta] \tag{E5}$$

From the equilibrium prices of capital and labor at profit maximization, we can obtain the following results:

$$g_w{}^* = g_r{}^* = \frac{1-\beta}{\beta}[b_k\phi(S_k{}^*)S_k{}^* - \delta] \tag{E6}$$

Substituting $g_w(\omega)$, $g_r(\omega)$, $g_m(\omega)$, $g_n(\omega)$, $g_K(\omega)$, $g_L(\omega)$ and $r^* = \theta g^* + \rho$ into $\nabla(t)$ yields

$$\nabla^* = \frac{\int_t^\infty exp\left[\int_t^v [g_r - g_m]d\omega - \int_t^v (r(t)exp\left[\int_t^\omega g_r\, du\right] + \delta)d\omega\right]dv}{\int_t^\infty exp\left[\int_t^v [g_w - g_n]d\omega - \int_t^v (r(t)exp\left[\int_t^\omega g_r\, du\right] + \delta)d\omega\right]dv} = 1 \tag{E7}$$

Substituting (E7) into (E1) yields

$$(k^*)^{(\varepsilon-1)/\varepsilon} = \frac{b_l\phi(S - S_k{}^*)\gamma}{b_k\phi(S_k{}^*)(1-\gamma)} \tag{E8}$$

Substituting (E8) into the relative income share yields

$$\sigma_K{}^* = \frac{b_l\phi(S - S_k{}^*)}{b_k\phi(S_k{}^*)} = \frac{S_k{}^*}{S - S_k{}^*} \tag{E9}$$

Since for $t \geq T$, $\frac{\dot{S_k}}{S_k} = \frac{\dot{k}}{k} =$, the model achieves a steady-state (SS) equilibrium. However, because $\frac{\dot{C}}{C} = \frac{\dot{Y}}{Y} = \frac{\dot{M}}{M} = g^* > 0$ and $\frac{\dot{K}}{K} = 0$, it does not satisfy Acemoglu's definition of a balanced growth path (BGP) equilibrium.

When $\varepsilon < 1$, if $S_k(t) > S_k{}^*$, then $\frac{\dot{k}}{k} > 0$, causing k to increase. This leads to a decrease in $k^{(\varepsilon-1)/\varepsilon}$, which reduces $\frac{\dot{S_k}}{S_k}$, ultimately bringing $S_k(t)$ back to $S_k{}^*$. Conversely, if $S_k(t) < S_k{}^*$, a similar opposite process occurs, also resulting in $S_k(t)$ back to $S_k{}^*$. Thus, the model has a unique steady-state equilibrium.

When $\varepsilon > 1$, if $S_k(t) = S$, then $\frac{\dot{k}}{k} = \frac{1-\beta}{\beta}[b_k\phi(S)S] > 0$. Thus, $\lim_{t\to\infty} k^{(\varepsilon-1)/\varepsilon} = \infty$, leading to $\lim_{t\to\infty} \frac{b_k\phi(S)}{b_l\phi(0)}\frac{(1-\gamma)\nabla}{\gamma}k^{(\varepsilon-1)/\varepsilon} > 1$. However, since $S_k = S$, we have $\frac{\dot{S_k}}{S_k} = 0$. Therefore, $S_k(t) = S$ represents an AP equilibrium.

Substituting $\lim_{t\to\infty} k^{(\varepsilon-1)/\varepsilon} = \infty$ into (B1) yields

$$\lim_{t\to\infty}\frac{\dot{Y}(t)}{Y(t)} = \lim_{t\to\infty}\frac{\dot{M}(t)}{M(t)} = \lim_{t\to\infty}\left(\frac{\dot{N}(t)}{N(t)} + \frac{\dot{k}(t)}{k(t)}\right) = \frac{1-\beta}{\beta}[b_k\phi(S)S - \delta] \qquad (E10)$$

where $\lim_{t\to\infty}\frac{\dot{N}(t)}{N(t)} = -\frac{1-\beta}{\beta}\delta$, $\lim_{t\to\infty}\frac{\dot{k}(t)}{k(t)} = \frac{1-\beta}{\beta}[b_k\phi(S)S]$.

Although $\lim_{t\to\infty}\frac{\dot{Y}(t)}{Y(t)} = \lim_{t\to\infty}\frac{\dot{M}(t)}{M(t)} = \frac{1-\beta}{\beta}[b_k\phi(S)S - \delta]$ and $\lim_{t\to\infty}\frac{\dot{N}(t)}{N(t)} = -\frac{1-\beta}{\beta}\delta$, with $\frac{\dot{K}(t)}{K(t)} = \frac{\dot{L}(t)}{L(t)} = 0$, all variables are constants. However, since there is no $T < \infty$ such that for $t \geq T$, $\frac{\dot{S_k}}{S_k} = \frac{\dot{k}}{k} = 0$, and there is no $\frac{\dot{Y}(t)}{Y(t)}_{t\geq T} = \frac{\dot{M}(t)}{M(t)}_{t\geq T} = \frac{1-\beta}{\beta}[b_k\phi(S)S - \delta]$, this equilibrium is merely an AP equilibrium, not a SS equilibrium. This distinguishes SS equilibrium from AP equilibrium.

Similarly, when $S_k = 0$, it is also an AP equilibrium, not a SS equilibrium.

When $\varepsilon = 1$, the dynamic equation (14) of the model becomes

$$\begin{cases}\dfrac{\dot{S_k}}{S_k} = G\left[\dfrac{b_k\phi(S_k)}{b_l\phi(S-S_k)}\dfrac{(1-\gamma)\nabla}{\gamma}\right] \\ \dfrac{\dot{k}}{k} = \dfrac{1-\beta}{\beta}[b_k\phi(S_k)S_k - b_l\phi(S-S_k)(S-S_k)]\end{cases} \qquad (E11)$$

Since $\frac{\dot{S_k}}{S_k} = 0$ and $\frac{\dot{k}}{k} = 0$ are two independent equations regarding S_k, there is generally no solution, meaning a SS equilibrium typically does not exist. However, there must exist $S_k(t) = {S_k}^*$ such that $\frac{\dot{S_k}}{S_k} = 0$, indicating the presence of an AP equilibrium.

The Proposition B is proven.

Appendix F: Proof of the Revised Proposition 5

Revised Proposition 5: When **I>0**, the model's BGP equilibrium is saddle-stable regardless of whether $\varepsilon < 1$ or $\varepsilon > 1$.

Proof.

Linearizing equation (13) around the BGP yields the dynamic equation (F1):

$$\begin{cases}\dfrac{\dot{c}}{c} = a_{cc}(c-c^*) + a_{ck}(k-k^*) + a_{cm}(M-M^*) \\ \dfrac{\dot{k}}{k} = a_{kc}(c-c^*) + a_{kk}(k-k^*) + a_{km}(M-M^*) + a_{ks}(S_k - {S_k}^*) \\ \dfrac{\dot{M}}{M} = a_{ms}(S_k - {S_k}^*) \\ \dfrac{\dot{S_k}}{S_k} = a_{sk}(k-k^*) + a_{ss}(S_k - {S_k}^*)\end{cases} \qquad (F1)$$

where $a_{ss} = G'.\frac{b_k}{b_l}\frac{(1-\gamma)\nabla}{\gamma}k^{\frac{\varepsilon-1}{\varepsilon}}\frac{\phi'(S_k)\phi(S-S_k)+\phi(S_k)\phi'(S-S_k)}{[\phi(S-S_k)]^2} < 0$, $a_{sk} = G'.\frac{\varepsilon-1}{\varepsilon}\frac{b_k\phi(S_k)}{b_l\phi(S-S_k)}\frac{(1-\gamma)\nabla}{\gamma}k^{\frac{-1}{\varepsilon}}$ the sign of which depends on the value of substitution elasticity ε, $a_{sm} = a_{sc} = 0$; $a_{ms} = \frac{1-\beta}{\beta}b_k\phi(S_k) > 0$, $a_{mm} = a_{mc} = a_{mk} = 0$; $a_{cs} = 0$, $a_{cm} = \left(\frac{\beta}{\theta}\frac{kf'(k)}{k} - \frac{f(k)}{k}\right)$ the sign of which is unknown, $a_{cc} = 1$, $a_{ck} = M\left(\frac{\beta}{\theta}f''(k) - \frac{kf'(k)-f(k)}{k*k}\right)$ the sign of which is also is unknown; $a_{ks} = \frac{1-\beta}{\beta}\left(b_k\phi(S_k) + b_l\phi(S-S_k)\right) > 0$, $a_{km} = \frac{f(k)}{k} > 0$, $a_{kc} = -1 < 0$, $a_{kk} = M\frac{kf'(k)-f(k)}{k*k} < 0$.

The characteristic equation is as follows:

$$det\begin{vmatrix} a_{cc}-\lambda & a_{ck} & a_{cm} & 0 \\ a_{kc} & a_{kk}-\lambda & a_{km} & a_{ks} \\ 0 & 0 & -\lambda & a_{ms} \\ 0 & a_{sk} & 0 & a_{ss}-\lambda \end{vmatrix} = 0 \qquad (F2)$$

Expanding of the characteristic equation yields:

$$\lambda^4 - \lambda^3(a_{ss} + 1 + a_{kk}) + \lambda^2(a_{ss} + a_{ss}a_{kk} + a_{kk} + a_{ck} - a_{sk}a_{ks}) + \lambda(-a_{ss}a_{ck} - a_{ss}a_{kk} + a_{sk}a_{ks} - a_{sk}a_{ms}a_{km}) + a_{sk}a_{ms}(a_{km} + a_{cm}) = 0 \qquad (F3)$$

From the Vieta theorem:

$$\lambda_1\lambda_2\lambda_3\lambda_4 = a_{sk}a_{ms}(a_{km} + a_{cm}) \qquad (F4)$$

When $\varepsilon > 1$, $a_{sk} = \frac{\varepsilon-1}{\varepsilon}\frac{b_k\phi(S_k)}{b_l\phi(S-S_k)}\frac{(1-\gamma)}{\gamma}k^{\frac{-1}{\varepsilon}} > 0$, $a_{ms} > 0$, $(a_{km} + a_{cm}) = \frac{\beta}{\theta}\frac{kf'(k)}{k} > 0$, therefore $\lambda_1\lambda_2\lambda_3\lambda_4 > 0$. Equation (F4) shows that the characteristic equation must have 4 positive roots, or 4 negative roots, or two positive and two negative roots. If there are 4 positive roots, the steady state is unstable. If there are 4 negative roots, then the steady state is locally stable, if there are two positive and two negative roots, then the steady state is locally saddle-path stable. Therefore, as long as we can rule out the case of four positive roots, the equilibrium growth path is at least saddle-path stable.

We prove by contradiction that not all four roots can be positive. That is, provided $\varepsilon > 1$, claiming that equation (F3) has four positive roots results in a contradiction.

Use the Vieta theorem to obtain:

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = a_{ss} + 1 + a_{kk} \qquad (F5)$$

If equation (F3) has 4 positive roots, then from equation (F5) we can obtain $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = (a_{ss} + 1 + a_{kk}) > 0$, implying that $1 + a_{kk} > -a_{ss} > 0$. From the Vieta theorem also the following equation holds:

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = a_{ss}(1 + a_{kk}) + (a_{kk} + a_{ck}) - a_{sk}a_{ks} \qquad (F6)$$

Owing to $a_{ss} < 0$, if $1 + a_{kk} > 0$, then $a_{ss}(1 + a_{kk}) < 0$; and $a_{kk} + a_{ck} = M\frac{\beta}{\theta}f''(k) < 0$; $a_{ks} = \frac{1-\beta}{\beta}\left(b_k\phi(S_k) + b_l\phi(S - S_k)\right) > 0$, but when $\varepsilon > 1$, $a_{sk} = G'.\frac{\varepsilon-1}{\varepsilon}\frac{b_k\phi(S_k)}{b_l\phi(S-S_k)}\frac{(1-\gamma)}{\gamma}k^{\frac{-1}{\varepsilon}} > 0$. It follows that $-a_{sk}a_{ks} < 0$. Therefore, the RHS of equation (F6) is less than 0. However, if all four roots are positive, then the RHS of equation (F6) should be greater than zero. Therefore, equation (F3) cannot possess four positive roots, and can either have two positive roots and two negative roots, or four negative roots. In the former case, the steady-state equilibrium is locally saddle-path stable. In the latter case, the steady-state equilibrium is locally stable. In summary, the steady-state equilibrium is at least saddle-path stable.

When $\varepsilon < 1$, $a_{sk} = G'.\frac{\varepsilon-1}{\varepsilon}\frac{b_k\phi(S_k)}{b_l\phi(S-S_k)}\frac{(1-\gamma)}{\gamma}k^{\frac{-1}{\varepsilon}} < 0$, $a_{ms} > 0$ and $(a_{km} + a_{cm}) = \frac{\beta}{\theta}\frac{kf'(k)}{k} > 0$ then $\lambda_1\lambda_2\lambda_3\lambda_4 = a_{sk}a_{ms}(a_{km} + a_{cm}) < 0$, so the equation must have negative roots and the steady-state equilibrium is also saddle-path stable. Whether there is just one or three negative roots, the equilibrium growth path is saddle-path stable.

Therefore, the model's BGP equilibrium is saddle-stable regardless of whether $\varepsilon < 1$ or $\varepsilon > 1$.

Appendix G: Proof of Revised Proposition 6

Revised Proposition 6: When $\varepsilon < 1$, the steady state is locally stable, and when $\varepsilon > 1$, it is locally saddle-path stable.

Proof.

Linearizing equations (16) near the equilibrium point yields:

$$\begin{cases} \frac{\dot{S_k}}{S_k} = a_{ss}(S_k - {S_k}^*) + a_{sm}(M - M^*) \\ \frac{\dot{M}}{M} = a_{ms}(S_k - {S_k}^*) \end{cases} \qquad (G1)$$

where $a_{ss} \equiv \frac{\partial\frac{\dot{S_k}}{S_k}}{\partial S_k} = G'.\frac{b_k}{b_l}\frac{(1-\gamma)\nabla}{\gamma}k(M)^{\frac{\varepsilon-1}{\varepsilon}}\frac{\phi'(S_k)\phi(S-S_k)+\phi(S_k)\phi'(S-S_k)}{[\phi(S-S_k)]^2} < 0$, $a_{sm} \equiv \frac{\partial\frac{\dot{S_k}}{S_k}}{\partial M} = G'.\frac{\varepsilon-1}{\varepsilon}\frac{b_k\phi(S_k)}{b_l\phi(S-S_k)}\frac{(1-\gamma)\nabla}{\gamma}k(M)^{\frac{-1}{\varepsilon}}\frac{dk}{dM}$ the sign of which depends on the value of ε, and $a_{ms} \equiv \frac{\partial\frac{\dot{M}}{M}}{\partial S_k} = \frac{1-\beta}{\beta}b_k\phi(S_k) > 0$.[9]

The characteristic equation of the model is given by:

[9] The sign of these coefficients holds under all circumstances, not just in the steady state.

$$det\begin{vmatrix} a_{ss}-\lambda & a_{sm} \\ a_{ms} & -\lambda \end{vmatrix} = 0 \quad , \tag{G2}$$

leading to:

$$\lambda^2 - \lambda a_{ss} - a_{sm}a_{ms} = 0\,, \tag{G3}$$

By using the Vieta theorem we obtain:

$$\begin{cases} \lambda_1\lambda_2 = -a_{sm}a_{ms} \\ \lambda_1 + \lambda_2 = a_{ss} < 0 \end{cases} \tag{G4}$$

When $\varepsilon < 1$， $a_{sm} < 0$ and $a_{ms} > 0$, so that $-a_{sm}a_{ms} > 0$. Equation (G4) shows that equation (G3) must have two negative roots, $\lambda_1 < 0$ and $\lambda_2 < 0$. In this case, the steady-state equilibrium of the model is locally stable.

When $\varepsilon > 1$, then $a_{sm} > 0$ and $a_{ms} > 0$ Since $\lambda_1\lambda_2 = -a_{sm}a_{ms} < 0$, and there must be one positive root and one negative root, so the steady-state equilibrium of the model is locally saddle-path stable.[10]

Therefore, the proposition is proved.

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[10] Notice that because the Euler equation disappears, there is nothing to guarantee that the economy is on the saddle path initially. However, this problem disappears when one considers $\theta > 0$ and impose the transversality conditions.

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