Response to "A note on Brandl and Brandt's axiomatic characterization of Nash equilibrium"

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Brandl and Brandt (2024) characterize Nash equilibrium as the unique total solution concept satisfying consequentialism, consistency, and rationality. Schroeder (2025) claims that the solution concept f^*, which, for each n-player game, returns the set of all strategy profiles where each player's strategy has full support, also satisfies the given axioms and thus constitutes a counter-example to the characterization by Brandl and Brandt (2024). We show that this is not true because f^* violates consequentialism.

More generally, we provide a simple proof showing that any solution concept that returns a superset of the strategy profiles returned by f^* for each game violates one of the three axioms in the characterization of Nash equilibrium. We adopt the notation of Brandl and Brandt (2024) throughout.

First, consider the 1-player games G and G' below.

$$G \qquad \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad\qquad G' \qquad \begin{pmatrix} 0 \end{pmatrix}$$

In G', the player has a single action, say a, with payoff 0, and in G the player has two actions, say a and a', each with payoff 0. Hence, G is a blowup of G', which is witnessed by the unique map $\phi\colon \{a, a'\} \to \{a\}$. The set of strategy profiles for G' is $\Delta\{a\} = \{p'\}$, where p' is the profile for which the player's strategy assigns probability 1 to a. Hence, for any total solution concept f, $f(G') = \{p'\}$, and in particular, $f^*(G') = \{p'\}$. The set of strategy profiles for G is $\Delta\{a, a'\} = \{p \in \mathbb{R}^U_+ \colon p(a) + p(a') = 1\}$. Note that for each $p \in \Delta\{a, a'\}$, $\phi_*(p) = p'$. Thus, if f satisfies consequentialism, then $f(G) = \phi_*^{-1}(\{p'\}) = \Delta\{a, a'\}$. It follows that f^* violates consequentialism since $f^*(G) = \{p \in \Delta\{a, a'\} \colon p(a) > 0 \text{ and } p(a') > 0\}$.

Second, we observe that any solution concept returning for each game a superset of the strategy profiles returned by f^* violates one of the three axioms in the characterization of Nash equilibrium. Let f be a solution concept such that $f^*(G) \subseteq f(G)$ for each game G. Assume for contradiction that f satisfies consequentialism, consistency, and rationality, and consider the 1-player games G and G' below.

$$\underset{G}{{}^1\!/_2 \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}} \quad \underset{G'}{+ {}^1\!/_2 \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}} \quad \underset{{}^1\!/_2\, G + {}^1\!/_2\, G'}{= \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}$$

Let $\{a, b, c\}$ be the action set of the player for G and G'. We have ${}^1\!/_4\ a + {}^1\!/_4\ b + {}^1\!/_2\ c \in f^*(G) \subseteq f(G)$ and ${}^1\!/_2\ a + {}^1\!/_4\ b + {}^1\!/_4\ c \in f^*(G') \subseteq f(G')$. Consequentialism then implies that $p = {}^1\!/_2\ a + {}^1\!/_2\ c \in f(G) \cap f(G')$ since a and b are clones in G and b and c are clones in G'. Consistency thus implies that $p \in f({}^1\!/_2\ G + {}^1\!/_2\ G')$. But this contradicts rationality since b is a dominant action in ${}^1\!/_2\ G + {}^1\!/_2\ G'$ and $p(b) = 0$.

Schroeder (2025) claims that an "important axiomatic requirement [for Nash equilibrium] should focus on the aspect of maximizing the expected value of player i, given that all other players $-i$ also follow this optimization principle and this is common knowledge." He emphasizes that "a maximization criterion is neither explicitly nor implicitly reflected in any of the axioms in Brandl and Brandt (2024)."

As noted in the original paper and witnessed by the example above, the absence of an axiom that requires expected utility maximization is a special feature of the characterization by Brandl and Brandt (2024). In fact, the characterization is nontrivial even for 1-player games, where Nash equilibrium coincides with expected utility maximization.

References

Florian Brandl and Felix Brandt. An axiomatic characterization of Nash equilibrium. *Theoretical Economics*, 19(4):1473–1504, 2024.

Andreas Schroeder. A note on Brandl and Brandt's axiomatic characterization of Nash equilibrium. Technical Report 123069, MPRA, posted 01 Jan 2025.