(Non-Monotonic) Effects of Productivity and Credit Constraints on Equilibrium Aggregate Production in General Equilibrium Models with Heterogeneous Producers

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Abstract

In a market economy, the aggregate production level depends not only on the aggregate variables but also on the distribution of individual characteristics (e.g., productivity, credit limit, ...). We point out that, due to financial frictions, the equilibrium aggregate production may be non-monotonic in both individual productivity and credit limit. We provide conditions under which this phenomenon happens. By consequence, improving productivity or relaxing credit limit of firms may not necessarily be beneficial to economic development.

JEL Classifications: D2, D5, E44, G10, O4.
Keywords: Productivity shock, financial shock, credit constraint, heterogeneity, productivity dispersion, distributional effects, efficiency, general equilibrium.

1 Introduction

We investigate two basic questions in economics: what are the impacts of (individual and aggregate) productivity and financial changes on the aggregate output?

Looking back to the literature, on the one hand, the productivity is widely viewed as one of the most important determinants of economic growth. In economics textbooks and classical papers (Solow, 1957; Romer, 1986, 1990), an increase of productivity generates a positive effect on the aggregate output and economic growth. On the other hand, one can expect that relaxing credit limits would have positive impact

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on the aggregate output as argued by several papers (for example, Khan and Thomas (2013) (section VI. C), Midrigan and Xu (2014) (section II.B), Moll (2014) (Proposition 1), and Catherine, Chaney, Huang, Sraer, and Thesmar (2022)).

We provide a novel view: whether a rise of productivity or credit limit generates a positive (or negative) effect on the aggregate output depends on the distribution of productivity, the size of these rises and the level of financial imperfection. In order to explore our insights, we build general equilibrium models with credit constraints and heterogeneous producers (having their own productivity) and provide conditions under which the equilibrium aggregate production is decreasing (or increasing) in the producers' productivity and credit limit.

Let us first explain the role of productivity in a static framework. We prove that when the productivity of all agents increases, this change improves the aggregate production if either (1) the productivity growth rates are the same or (2) there is no financial friction. However, the more interesting and realistic case is when the productivity of producers increases at different rates and there is a credit constraint (these two styled facts are documented by several studies).[1] In this case, we argue that the aggregate production may decrease. This may happen if the TFP of less productive agents increases faster than that of more productive agents. Indeed, in such a case, less productive agents absorb more capital and produces more. Since the aggregate capital is limited, other producers (who are more productive) get less capital (because of market imperfections) and so they produce less. By consequence, the net effect may be negative. This happens if (1) the TFP of less productive agents is far from that of more productive producers, i.e., the productivity dispersion is high,[2] (2) the productivity rise is quite small, (3) the credit constraint is tight.

Regarding the role of financial shock, we argue that, while a homogeneous rise of credit limit improves the aggregate output, an asymmetric rise of credit limits can reduce the output. The intuition behind this result is similar to that in the case of productivity effects we have mentioned above: If credit limits of less productive agents increase faster that those of more productive ones, less productive agents get more capital and more productive agents get less capital, hence the aggregate output may decrease. It should be noticed that although the aggregate output is not necessarily monotonic in credit limits of producers, it does not exceed that in the frictionless economy which is in line with the existing literature.

In the second part of our paper, we investigate our above questions in infinite-horizon models à la Ramsey. Before doing this, we prove the existence of intertemporal

[1]See, for instance, Syverson (2011), Andrews, Criscuolo and Gal (2015), Kehrig (2015), Barth, Bryson, Davis and Freeman (2016), Decker, Haltiwanger, Jarmin, and Miranda (2018), Berlingieri, Blanchenay, and Chiara (2017), Bouche, Cette, and Lecat (2021), Levine and Warusawitharana (2021), Gouin-Bonenfant (2022).

[2]Andrews, Criscuolo and Gal (2015) use a harmonised cross-country dataset, based on underlying data from the OECD-ORBIS database (Gal, 2013), to analyze the characteristics of firms that operate at the global productivity frontier and their relationship with other firms in the economy. Andrews, Criscuolo and Gal (2015) document growing productivity dispersion for several developed countries over the 2000s. Bouche, Cette, and Lecat (2021) present empirical evidence showing an increase in productivity dispersion between French firms during the period 1991-2016, with a growing productivity gap between frontier and laggard firms. See Goldin, Koutroumpis, Lafond, and Winkler (2024) for an excellent review on the slowdown in productivity growth.

equilibrium. To do so, we adopt the following approach:[3] (1) we prove the existence of equilibrium for each $T-$ truncated economy $\mathcal{E}^T$; (2) we show that this sequence of equilibria converges for the product topology to an equilibrium of our original economy.

We show that the non-monotonic effect of productivity and credit limit on the aggregate output cannot appear at the steady state. The reason is that the steady state interest rate only depends on the rate of time preferences of agents. Therefore, we focus on the global dynamics of intertemporal equilibrium. Technically, this task is far from trivial and very few papers do this.[4] However, we manage to obtain several insights. First, our findings suggest that a permanent increase of productivity of less productive agents improves the aggregate output in the long run. However, when this productivity rise is quite small and credit constraints are tight, the aggregate output may decrease in the short-run and then increase from some period on.

Second, we look at the effects of credit limits. Recall that in the static model, an increase in the most productive agent's credit limit is always beneficial for the aggregate output. However, along intertemporal equilibrium, we show that an increase of the credit limit of the most productive producer may reduce the output at every period. The intuition behind is that when her(his) credit limit goes up, the equilibrium interest rate increases, and hence, her(his) repayment also increases. This in turn reduces her(his) net worth in the next period. By consequence, her(his) saving and hence the production decrease. The economic mechanism can be summarized by the following schema:

$$\begin{aligned}&\text{Credit limit} \uparrow \;\Rightarrow\; \text{Interest rate} \uparrow \;\Rightarrow\; \text{Agent's net worth} \downarrow \;\Rightarrow \\ &\Rightarrow\; \text{Saving} \downarrow \;\Rightarrow\; \text{Production} \downarrow \;\Rightarrow \cdots \end{aligned} \tag{1.1}$$

As in the static model, this mechanism can happen because the credit limit of the most productive agent remains low and the productivity dispersion is high.

Third, we show how the equilibrium interest rate and the outcomes of intertemporal equilibrium (in particular in the long run) depend on the distribution of initial endowments, credit limit and productivity as well as of the discount factors. Recall that in a standard Ramsey model with one representative producer, the most patient household owns the entire capital of the economy after some finite time - this is the so-called *Ramsey conjecture* - and the equilibrium interest rate in the long run depends only on the rate of preference time of the most patient agent (Becker and Mitra, 2012; Becker, Dubey and Mitra, 2014; Becker, Borissov and Dubey, 2015). In our models with many potential producers, along the intertemporal equilibrium, in particular in the long run, there may be several producers sharing the aggregate capital. We point out that whether an agent holds the capital depends on the distribution of discount factor, credit limit, productivity and initial capital. Precisely, the capital holding of a producer is increasing in each of these parameters.

[3] See Becker, Bosi, Le Van and Seegmuller (2015) and Le Van and Pham (2016) among others.

[4] The existing literature focuses on the balanced-growth path, recursive equilibrium or provides analyses around the steady-state equilibrium. See Le Van and Pham (2016) for intertemporal equilibrium in a model with heterogeneous households and a representative producer.

Link to the literature

Our article is related to a growing literature on general equilibrium models with heterogeneous producers and financial frictions.[5] Let us mention some of them.[6] Midrigan and Xu (2014) consider a two-sector model with a collateral constraint that requires the debt of producer does not exceed a fraction of its capital stock. They focus on balanced growth equilibrium to study the role of collateral constraint in determining TFP. Their parameterizations consistent with the data imply fairly small losses from misallocation, but potentially sizable losses from inefficiently low levels of entry and technology adoption. Khan and Thomas (2013) develop a dynamic stochastic general equilibrium with a representative household and heterogeneous firms facing a borrowing constraint (slightly different from ours) and focus on recursive equilibrium. They find that a negative shock to borrowing conditions can generate a large and persistent recession through disruptions to the distribution of capital. Buera and Shin (2013) develop a model with individual-specific technologies and collateral constraints to investigate the role of the misallocation and reallocation of resources in macroeconomic transitions. Buera and Shin (2013) find that collateral constraints have a large impact along the transition to the steady state. Moll (2014) studies the effect of collateral constraints on capital misallocation and aggregate productivity in a general equilibrium with a continuum of heterogeneous firms and financial frictions (modeled by a collateral constraint). Proposition 1 in Moll (2014) shows that the aggregate TFP is increasing in the leverage ratio which is the common across firms.[7]

Our paper differs from this literature in two points. First, the credit limit is individualized in our model while all credit parameters in the above studies are common across producers. Second, we argue that this credit heterogeneity plays an important role in the distribution of capital and of income as well as in the aggregate output. Indeed, we prove that the aggregate output and the aggregate TFP in our model may not be monotonic functions of the credit limits which are different across agents; they may display an inverted-U form.[8] However, we show that, if agents have the same credit limit, the aggregate output and the aggregate TFP are increasing functions of this common credit limit; this finding is consistent with the above literature.

Our paper is related to Baqaee and Farhi (2020) who build a general equilibrium model where productivity and wedge are exogenous parameters to study how the impact of (productivity and wedge) shocks can be decomposed into a pure technology

[5] The reader is referred to Matsuyama (2007), Quadrini (2011), Brunnermeier, Eisenbach, and Sannikov (2013) for more complete reviews on the macroeconomic effects of financial frictions and to Buera, Kaboski, and Shin (2015) for the relationship between entrepreneurship and financial frictions.

[6] While we focus on producer heterogeneity, there is a growing literature studying the roles of household heterogeneity in macroeconomics (the reader is refereed to Kaplan and Violante (2018) for an excellent review on this topic).

[7] In both Buera and Shin (2013), Moll (2014), the collateral constraint, which is slightly different from ours, states that the capital of a firm does not exceed a leverage ratio of its financial wealth.

[8] Our finding is related to Aghion, Bergeaud, and Maghin (2019). They consider a model of firm dynamics and innovation with entry, exit, and credit constraints, based on Klette and Kortum (2004), Aghion, Akcigit, and Howitt (2015). They assume that intermediate firms (monopolist) cannot invest more than μ times their current market value in innovation. They argue that the credit access may harm productivity growth because it allows less efficient incumbent firms to remain longer on the market, which discourages entry of new and potentially more efficient innovators.

effect and an allocative efficiency effect. There are some differences between Baqaee and Farhi (2020) and the present paper. First, Baqaee and Farhi (2020) model frictions by wedge while we model frictions by a credit constraint and the credit limit is our exogenous parameter. Second, Baqaee and Farhi (2020) provide a quantitative analysis by applying their approach to the firm-level markups in the U.S. but they do not provide conditions (based on exogenous parameters) under which the aggregate output is increasing or decreasing in productivity and friction level (wedge in their framework). Although we do not provide quantitative applications of our results, we show several conditions (based on exogenous parameters) under which the aggregate output is increasing or decreasing in productivity and friction level (credit limit in our framework). We also run some simulations and extend our analyses in infinite-horizon models while Baqaee and Farhi (2020) do not do this.

Our paper also concerns the literature on the welfare effects of financial constraints. Jappelli and Pagano (1994, 1999) consider overlapping generations models with liquidity constraints and households living for three periods and argue that liquidity constraints may increase or decrease welfares. The central point in Jappelli and Pagano (1994, 1999) is that liquidity constraints have two opposite effects on welfare: "they force the consumption of young below the unconstrained level but raise their permanent income by fostering capital accumulation". Obiols-Homs (2011) considers a general equilibrium with heterogeneous households (whose borrowings are bounded by an exogenous limit) and a representative firm. He argues that the borrowing limit has a negative on the welfare of borrower if its *quantity effect* dominates its *price effect*. As in Jappelli and Pagano (1994, 1999), the mechanism of Obiols-Homs (2011) relies on the role of supply of credit to households who need to smooth their consumption. By contrast, our mechanism focuses on credit to firms who need credit to finance their productive investment. Moreover, Obiols-Homs (2011) considers exogenous borrowing limits while we focus on credit constraints and our model has endogenous borrowing limits.

Catherine, Chaney, Huang, Sraer, and Thesmar (2022) build a dynamic general equilibrium model with heterogeneous firms and collateral constraints. They focus on the steady state and provide estimates suggesting that lifting financial frictions (modeled by collateral constraints) would increase aggregate welfare by 9.4% and aggregate output by 11%. Our paper differs from Catherine, Chaney, Huang, Sraer, and Thesmar (2022) in two aspects. First, although we also find that the aggregate output in the frictionless economy is higher than that in the economy with financial frictions, it is not a monotonic function of the degree of financial friction. Second, both individual and social welfares may not be monotonic in the degree of financial friction. Interestingly, lifting credit constraint may decrease the welfare of some agents.

Last but not least, our paper contributes to the debate concerning the slowdown in aggregate productivity growth that has been documented by several studies such as Andrews, Criscuolo and Gal (2015), Bouche, Cette, and Lecat (2021), Goldin, Koutroumpis, Lafond, and Winkler (2024); see Footnote 2. Our above analyses suggest that the interplay between credit constraints, high heterogeneity of productivity, asymmetry of productivity and financial shocks may generate a slowdown in aggregate productivity growth. We argue that the aggregate productivity growth rate may be far from that of most productive firms. It may be even lower than the smallest productivity growth rate of firms. Our approach, which is different from those in the literature, is based on the general equilibrium theory with financial frictions and heterogeneous producers.

The rest of our article is organized as follows. Section 3 presents a motivating example with two agents while Section 3 present a two-period general equilibrium framework with many producers to study the effects of productivity and credit limits. Section 4 explores our analyses in infinite-horizon general equilibrium models à la Ramsey. Section 5 concludes. Formal proofs are gathered in the appendices.

2 A motivating example

In this section, we consider a deterministic two-period economy with a two agents $i = 1, 2$. There is a single good (numéraire) which can be consumed or used to produce. Each agent i has exogenous initial wealth (S_i units of good) at the initial date. To keep the model as simple as possible, we assume that agents just maximize their consumption in the second period and we focus on the output in this period.

Agents have two ways for investing. On the one hand, agent i can buy k_i units of physical capital at the initial date to produce $F_i(k_i)$ units of good at the second date, where F_i is the production function. Assume that $F_i(k) = A_i k$, $\forall k \geq 0$, with $0 < A_1 < A_2$.

On the other hand, she can invest in a financial asset with real return R which is endogenous. Denote b_i the asset holding of agent i. She can also borrow and then pay back Rb_i in the next period. However, there is a borrowing constraint. The maximization problem of agent i can be described as follows:

$$(P_i): \quad \pi_i = \max_{k_i, b_i} [F_i(k_i) - Rb_i] \tag{2.1a}$$

$$\text{subject to: } 0 \leq k_i \leq S_i + b_i \text{ (budget constraint)} \tag{2.1b}$$

$$Rb_i \leq \gamma_i F_i(k_i) \text{ (borrowing constraint)} \tag{2.1c}$$

where $\gamma_i \in (0, 1)$ is an exogenous parameter. Borrowing constraint (2.1c) means that the repayment does not exceed the market value of the borrower's project.[9][10] This is similar to the collateral constraint (4) in Kiyotaki (1998) or the so-called *earnings-based constraint* in Lian and Ma (2021).[11] The better the commitment, the higher value of γ_i, the larger the set of feasible allocations of the agent i. Kiyotaki (1998) interprets γ_i

[9] Here, we follow Kiyotaki (1998) by assuming that the debtor is required to put her project as collateral in order to borrow: If she does not repay, the creditor can seize the collateral. Due to the lack of commitment (or just because the debtor is not willing to help the creditor take the whole value of the debtor's project), the creditor can only obtain a fraction γ_i of the total value of the project. Anticipating the possibility of default, the creditor limits the amount of credit so that the debt repayment will not exceed a fraction γ_i of the debtor's project value.

[10] Matsuyama (2007) (Section 2) considers a model with heterogeneous agents, which corresponds to our model with $k_i = 1$, $S_i = w, b_i = 1 - w$. However, different from our setup, investment projects in Matsuyama (2007) are non-divisible.

[11] Some authors (Buera and Shin, 2013; Moll, 2014) set $k_i \leq \theta w_i$, where $w_i \geq 0$ is the agent i's wealth and interpret that θ measures the degree of credit frictions (credit markets are perfect if $\theta = \infty$ while $\theta = 1$ corresponds to financial autarky, where all capital must be self-financed by entrepreneurs). In our framework, S_i plays a similar role of wealth w_i in Buera and Shin (2013), Moll (2014). Another way

as the collateral value of investment. In our paper, we call γ_i the *credit limit* of agent i.

The following table from the Enterprise Surveys (2018)'s panel datasets suggests that borrowing and collateral constraints matter for the development of firms.

Economy	Proportion of loans requiring collateral (%)	Value of collateral needed for a loan (% of the loan amount)	Percent of firms not needing a loan	Percent of firms whose recent loan application was rejected	Proportion of investments financed internally (%)
All Countries	79.1	205.8	46.4	11.0	71.0
East Asia & Pacific	82.6	238.4	50.7	6.4	77.8
Europe & Central Asia	78.7	191.9	54.3	10.9	72.4
Latin America & Caribbean	71.3	198.5	45.0	3.1	62.7
Middle East & North Africa	77.4	183.0	51.8	10.2	71.1
South Asia	81.1	236.0	44.7	14.4	73.9
Sub-Saharan Africa	85.3	214.8	37.4	15.3	73.9

An economy $\mathcal{E}$ with credit constraints is characterized by a list of fundamentals

$$\mathcal{E} \equiv (A_i, f_i, \gamma_i, S_i)_{i=1,2}.$$

Definition 1. *A list $(R, (k_i, b_i)_i)$ is an equilibrium if (1) for each i, given R, the allocation (b_i, k_i) is a solution of the problem (P_i), and (2) financial market clears $\sum_i b_i = 0$.*

In our example with linear production function, we can explicitly compute the equilibrium interest rate and aggregate output (see Theorem 2 in Appendix D):

Lemma 1. *In the above economy with 2 agents and linear production function, the equilibrium interest rate and aggregate output are determined by*

$$Y = \begin{cases} A_2(S_1 + S_2) & \textit{if } A_1 < \gamma_2 A_2 \frac{S_1+S_2}{S_1} \\ A_1 S_1 + A_2 S_2 \dfrac{A_1(1-\gamma_2)}{A_1 - \gamma_2 A_2} & \textit{if } A_1 \geq \gamma_2 A_2 \frac{S_1+S_2}{S_1} \end{cases} \tag{2.2}$$

$$R = \begin{cases} A_2 & \textit{if } S_1 \leq \frac{\gamma_2}{1-\gamma_2} S_2 \\ \frac{\gamma_2 A_2 (S_1+S_2)}{S_t} & \textit{if } \frac{\gamma_2}{1-\gamma_2} S_2 < S_1 < \frac{\gamma_2 A_2}{A_1 - \gamma_2 A_2} S_2 \\ A_1 & \textit{if } S_1 \geq \frac{\gamma_2 A_2}{A_1 - \gamma_2 A_2} S_2, \textit{ or, equivalently, } A_1 \geq \gamma_2 A_2 \frac{S_1+S_2}{S_1} \end{cases} \tag{2.3}$$

This allows us to fully investigate the effects of productivity changes. First, we look at the individual level.

Proposition 1 (effects of individual productivity changes). *1. The aggregate output is always increasing in A_2 - the productivity of the most productive agent.*

to introduce credit constraint is to set that $b_i \leq \theta k_i$. This corresponds to constraint (3) in Midrigan and Xu (2014). Other authors (Kocherlakota, 1992; Obiols-Homs, 2011) consider exogenous borrowing limits by imposing a short sales constraint: $b_i \leq B$ for any i. Under these three settings, the asset holding b_i is bounded from above by an upper bound which does not depend on the interest rate R. Carosi, Gori, and Villanacci (2009) present a two-period general equilibrium model with uncertainty, numéraie assets, and participation constraints described by functions of agent's choices and prices. Carosi, Gori, and Villanacci (2009) prove the existence of equilibrium and study indeterminacy but do not provide comparative statics.

2. When $A_1 < \gamma_2 A_2 \frac{S_1+S_2}{S_1}$, the aggregate output does not depend on A_1. When $\gamma_2 A_2 \frac{S_1+S_2}{S_1} < A_1$, we have $\frac{\partial Y}{\partial A_1} = S_1 - \frac{(1-\gamma_2)\gamma_2 A_2^2 S_2}{(A_1-\gamma_2 A_2)^2}$, and, by consequence,

$$\frac{\partial Y}{\partial A_1} \geq 0 \Leftrightarrow \frac{S_1}{S_2}\Big(\frac{A_1}{A_2} - \gamma_2\Big)^2 \geq (1-\gamma_2)\gamma_2 \tag{2.4}$$

So, the aggregate output displays an U-shape as a function of the least productive agent's credit limit. It is increasing in A_1 if the productivity ratio A_1/A_2 is higher than a threshold (or, equivalently, the productivity gap A_2/A_1 is lower than a threshold). Figure 1 illustrates an example. In this numerical simulation, we set $S_1 = 1, S_2 = 0.7$, $A_2 = 1, \gamma_2 = 0.2$, and let A_1 vary from $\gamma_2 A_2 \frac{S_1+S_2}{S_1} = 0.34$ to $A_2 = 2$. Then the output, as a function of A_1, is decreasing on the interval $(0.34, 0.54]$ and then increasing in the interval $(0.54, 1)$.

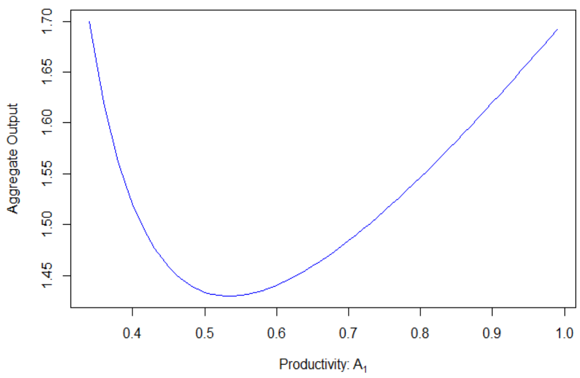


Figure 1: Non-monotonic effect of the agent 1's productivity.

We now let both productivities A_1 and A_2 vary.

Proposition 2 (effects of productivity changes)**.** *Consider a two-agent economy having linear technologies $F_i(k) = A_i k \ \forall i = 1,2$ with $\gamma_2 < A_1 < A_2$, and borrowing constraints: $Rb_i \leq \gamma_i A_i k_i$.*

Assume that there is a productivity shock that changes the productivity of agents from (A_1, A_2) to (A'_1, A'_2). Assume that $A'_2 > A'_1$. Assume that the credit constraint of agent 2 is low so that $\gamma_2 < \frac{A_1}{A_2}\frac{S_1}{S_1+S_2}$ and $\gamma_2 < \frac{A'_1}{A'_2}\frac{S_1}{S_1+S_2}$. Then, the output change is

$$Y(A'_1, A'_2) - Y(A_1, A_2) = (A'_1 - A_1)S_1 + A_2 S_2 (1-\gamma_2)\frac{A_1 A'_2 - A'_1 A_2}{(A_1 - \gamma_2 A_2)(A'_1 - \gamma_2 A'_2)} \tag{2.5}$$

(1) We have that:

$$\text{If } \frac{A'_2}{A_2} \geq \frac{A'_1}{A_1} \geq 1, \text{ then } Y(A'_1, A'_2) \geq Y(A_1, A_2) \tag{2.6}$$

(2) Assume that

$$S_2 A_2 (1-\gamma_2)\frac{\gamma_2 A_2}{(A_1 - \gamma_2 A_2)^2} - S_1 > 0, \ \text{i.e.,} \ \frac{S_1}{S_2}\Big(\frac{A_1}{A_2} - \gamma_2\Big)^2 < (1-\gamma_2)\gamma_2 \tag{2.7}$$

Then, there is a neighborhood $\mathcal{B}$ *of* (A_1, A_2) *such that*

$$\frac{Y(A_1', A_2') - Y(A_1, A_2)}{A_1' - A_1} < 0 \tag{2.8a}$$

$$\forall (A_1', A_2') \in \mathcal{B} \text{ satisfying } \frac{\frac{A_2'}{A_2} - 1}{\frac{A_1'}{A_1} - 1} < \frac{\gamma_2 A_2}{A_1} - \frac{S_1(A_1 - \gamma_2 A_2)^2}{S_2 A_1 A_2 (1-\gamma_2)} \text{ and } A_1' \neq A_1. \tag{2.8b}$$

Proof. See Appendix A. □

Condition (2.6) says that the aggregate output increases if the productivity of both producers increases and the productivity of the most productive agent increases faster than that of the less productive one.

Let us now focus on point 2 of Proposition 2. Here, condition (2.7) plays a very important role. It is satisfied if the ratio $\frac{A_1}{A_2}$ is low in the sense that $\frac{A_1}{A_2} < \gamma_2 + \left(\frac{\gamma_2(1-\gamma_2)S_2}{S_1}\right)^{0.5}$. This can be interpreted as a *high productivity dispersion.* Under this condition, we see that $\frac{\gamma_2 A_2}{A_1} - \frac{S_1(A_1-\gamma_2 A_2)^2}{S_2 A_1 A_2 (1-\gamma_2)} \in (0,1)$. According to conditions (2.7) and (2.8a), under a positive shock that improves the TFP of all agents, the aggregate output may decrease:

$$Y(A_1', A_2') < Y(A_1, A_2), \forall A_1' > A_1, A_2' > A_2, (A_1', A_2') \in \mathcal{B} \text{ satisfying (2.8b)}.$$

Let us explain the economic intuition behind this result. Assume that the productivity dispersion is high and let us consider a small positive shock (both the TFP of both agents increases). If the productivity of the less productive agent increases faster than that of the most productive agent (i.e., $\frac{A_2'}{A_2}$ is low with respect to $\frac{A_1'}{A_1}$, see condition (2.8b)), the first agent absorbs more physical capital and the most productive agent gets less capital (i.e., $k_2(A_1', A_2') < k_2(A_1, A_2)$). By consequence, the aggregate output may decrease.

3 A two-period model with many agents

We now extend the two-period model in Section 2 by allowing for a finite number (m) of heterogeneous agents and general production functions $F_i : \mathbb{R}_+ \to \mathbb{R}_+$.[12]

We require standard assumptions on the production function.

Assumption 1. *The production function* F_i *is concave, strictly increasing,* $F_i(0) = 0$. *The credit limit* γ_i *belongs the interval* $(0, 1)$ *for any* i.

We define the notion of equilibrium as in Definition 1. Under the above assumption, we can prove the equilibrium existence.

[12] We can interpret the one-factor production function F_i as a reduced form for a setting with other factors of production. Indeed, suppose that the producer has a two-factor production function, say capital and labor, $G_i(k, N)$. For a given level of capital k_i, the firm chooses labor quantity N_i to maximize its profit $\max_{N_i \geq 0}[G_i(k_i, N_i) - wN_i]$. The first order condition writes $\frac{\partial G_i}{\partial N}(k_i, N_i) = w$. This implies that $N_i = N_i(k_i, w)$. So, $G_i(k_i, N_i) = G_i(k_i, N_i(k_i, w))$. We now define $F_i(k_i) \equiv G_i(k_i, N_i(k_i, w))$.

Proposition 3. *Under Assumption 1, there exists an equilibrium.*

Proof. See Appendix. □

Given an equilibrium $(R, (k_i, b_i)_i)$, the aggregate output is $Y = \sum_i F_i(k_i)$. This depends on the forms of functions (F_i), the initial wealths (S_i), and the credit limits (γ_i).

Note that in an economy with perfect financial market, the aggregate production is simply determined by

$$Y^{perfect} \equiv \max_{(k_i)\geq 0} \sum_i F_i(k_i) \quad \text{subject to}: \quad \sum_i k_i \leq S \equiv \sum_i S_i. \tag{3.1}$$

$Y^{perfect}$ is increasing in A_i, $\forall i$, and in S.

In equilibrium, we have $\sum_i k = S$. So, we have that $Y \leq Y^{perfect}$. This is consistent with a number of studies on the macroeconomic effects of financial constraints (Buera and Shin, 2013; Karaivanov and Townsend, 2014; Midrigan and Xu, 2014; Moll, 2014; Catherine, Chaney, Huang, Sraer, and Thesmar, 2022).

However, an interesting open issue is whether the aggregate output is increasing or decreasing in agents' productivity A_i and credit limit γ_i. In the following sections, we will investigate how the aggregate output changes when productivities (A_i) and credit limits (γ_i) vary.

3.1 Effects of productivity changes

We study conditions the aggregate production is increasing or decreasing when productivity changes take place. Since we are interested in the effect of productivity changes, we assume that the production functions take the following form:

$$F_i(k) = A_i f_i(k), \tag{3.2}$$

where the parameter $A_i > 0$ represents the productivity of agent i while f_i is the original production function.

Assume that the TFP of agents depends on an exogenous variable $x \in \mathbb{R}$ in the sense that $A_i = A_i(x)$ where A_i is a differentiable function of x. Since we focus on positive changes, we assume that $A_i'(x) > 0$, $\forall i$.

We wonder how the aggregate output changes when x varies. Note that the equilibrium physical capital, denoted by $k_i(x)$, depends on $A_i(x)$ and the equilibrium interest rate R which in turn depend on all productivities $A_1(x), \ldots, A_m(x)$. Assume that we have the differentiability. So, we can compute

$$k_i'(x) = \underbrace{\frac{\partial k_i}{\partial R}}_{<0} \underbrace{\frac{\partial R}{\partial x}}_{>0} + \underbrace{\frac{\partial k_i}{\partial A_i}}_{>0} \underbrace{\frac{\partial A_i}{\partial x}}_{>0}, \quad \frac{\partial R}{\partial x} = \sum_j \frac{\partial R}{\partial A_j} \frac{\partial A_j}{\partial x} \tag{3.3}$$

Notice that $\frac{\partial k_i}{\partial A_i} \geq 0$, $\frac{\partial k_i}{\partial R} \leq 0$, $\frac{\partial A_i}{\partial x} \geq 0$, $\frac{\partial R}{\partial x} \geq 0$. By consequence, we can expect that $k_i'(x)$ may have any sign. However, we have $\sum_i k_i'(x) = 0$ because $\sum_i k_i = S$ in equilibrium.

We now look at the aggregate output:

$$Y(x) = \sum_i A_i(x) f_i\big(k_i(x)\big) = \sum_i A_i(x) f_i\Big(k_i\big(A_i(x), R(A_1(x), \dots, A_m(x))\big)\Big)$$

$$\frac{\partial Y}{\partial x} = \sum_i A_i'(x) f_i\big(k_i(x)\big) + \sum_i A_i(x) f_i'(k_i(x)) k_i'(x). \tag{3.4}$$

By using (3.3) and the fact that $\sum_i k_i'(x) = 0$, we obtain two decompositions.

Proposition 4 (effects of productivity changes - general decompositions)**.** *Consider an equilibrium and assume that the equilibrium outcomes are differentiable functions. We have*

$$\frac{\partial Y}{\partial x} = \underbrace{\sum_i A_i'(x) f_i\big(k_i(x)\big) + \sum_{i:k_i'(x)\geq 0} A_i(x) f_i'(k_i(x)) k_i'(x)}_{\textit{Added production of some agents}} + \underbrace{\sum_{i:k_i'(x)<0} A_i(x) f_i'(k_i(x)) k_i'(x)}_{\textit{Production losses of other agents}} \tag{3.5a}$$

$$\frac{\partial Y}{\partial x} = \underbrace{\sum_i A_i'(x) f_i\big(k_i(x)\big) + \sum_i A_i(x) f_i'(k_i(x)) \underbrace{\frac{\partial k_i}{\partial A_i}}_{>0} \underbrace{\frac{\partial A_i}{\partial x}}_{>0}}_{\textit{Quantity effect}} + \underbrace{\sum_i A_i(x) f_i'(k_i(x)) \underbrace{\frac{\partial k_i}{\partial R}}_{<0} \underbrace{\frac{\partial R}{\partial x}}_{>0}}_{\textit{Price effect}}. \tag{3.5b}$$

Proposition 4 provides different interpretations of the effects of productivity changes and helps us understand how the aggregate output may be increasing or decreasing in the exogenous change x. Look at (3.5a). When x increases, it generates a direct and positive effect on the productivity of agents, which is represented by the terms $\sum_i A_i'(x) f_i\big(k_i(x)\big) > 0$. However, since the capital supply is fixed, we have $\sum_i k_i'(x) = 0$. So, some agents get more input (i.e., $k_i'(x) \geq 0$) and produce more. However, others get less (i.e., $k_i'(x) < 0$) and produce less. Therefore, the aggregate production can increase or decrease. The second decomposition (3.5b) shows us the quantity and price effects. Indeed, the equilibrium physical capital k_i is increasing in the productivity $A_i(x)$ which contribute to the quantity effect. However, it is decreasing in the interest rate R; see (3.3). Since the interest rate is increasing in x, agents pay higher cost when borrowing, which generates the price effect.

Proposition 4 leads to the following result showing the effect of individual productivity change.

Corollary 1 (effect of individual productivity changes)**.** *Consider an equilibrium with $k_j > 0$. Let only A_j vary and assume that the equilibrium outcomes are differentiable functions. We have*

$$\frac{\partial Y}{\partial A_j} = \underbrace{f_j(k_j)}_{\textit{Productivity effect}} + \underbrace{\sum_{i\neq j} \big(A_j f_j'(k_j) - A_i f_i'(k_i)\big) \underbrace{\frac{-\partial k_i}{\partial R}}_{\geq 0} \underbrace{\frac{\partial R}{\partial A_j}}_{\geq 0}}_{\textit{Allocation effect}}. \tag{3.6}$$

By consequence, $\partial Y/\partial A_j \geq 0$, $\forall j \in \mathcal{I}$, where $\mathcal{I} = \arg\max_{i>n}\{A_i f_i'(k_i)\}$. The aggregate output increases in A_i if the producer i has the highest total marginal factor productivity.

Conditions in Proposition 4 and Corollary 1 are based on endogenous variables. We can go further by providing conditions based on exogenous parameters, shows the role of credit limit on the effect of productivity change.

Firstly, we consider linear production functions. The following result is a generalization of Proposition 1.

Proposition 5 (effects of productivity changes - linear technology)**.** *Assume linear production functions $F_i(k) = A_i k\ \forall i, \forall k$, where $A_1 < \cdots < A_m$.*

1. *We have $Y \leq Y^{perfect} \equiv A_m \sum_i S_i$. Moreover, $Y = Y^*$ if and only if $f_m A_m \geq A_{m-1}(1 - \frac{S_m}{S})$.*
2. *Assume that $A_n > \max_i(\gamma_i A_i)$ and $\sum_{i=n+1}^m \frac{A_n S_i}{A_n - \gamma_i A_i} \leq S \leq \sum_{i=n}^m \frac{A_n S_i}{A_n - \gamma_i A_i}$. Then, the equilibrium interest rate equals A_n[13] and the aggregate output equals $= A_n \sum_{i=1}^n S_i + \sum_{i=n+1}^m \dfrac{A_n(1-\gamma_i)}{A_n - \gamma_i A_i} A_i S_i$.*

   We also have that $\frac{\partial Y}{\partial A_j} > 0$, $\forall j > n$, and

$$\frac{\partial Y}{\partial A_n} = \sum_{i=1}^n S_i - \sum_{i=n+1}^m \frac{(1-\gamma_i)\gamma_i S_i}{(\frac{A_n}{A_i} - \gamma_i)^2} \tag{3.7}$$

We can see clearly that $\frac{\partial Y}{\partial A_n}$ may have any sign. Since $\frac{\partial Y}{\partial A_n}$ is increasing in A_n, it can be negative when A_n is low and positive when A_n is high. This is consistent with our insights mentioned in Section 2.

Secondly, we investigate the case of strictly concave production function. We require standard assumptions.

Assumption 2. *For any i, the function f_i is strictly increasing, strictly concave, twice continuously differentiable, $f_i(0) = 0$, $f_i(\infty) = \infty$, $f_i'(0) = \infty$, $f_i'(\infty) = 0$.*

Assumption 3. *For any i, the function $\frac{k f_i'(k)}{f_i(k)}$ is increasing in k.*

Definition 2. *Given R, γ_i, A_i, S_i, denote $k_i^n = k_i^n(R/A_i)$ the unique solution to the equation $A_i f_i'(k) = R$ and $k_i^b = k_i^b(\frac{R}{\gamma_i A_i}, S_i)$ the unique solution to $R(k - S_i) = \gamma_i A_i f_i(k)$.*

Under Assumption 2, k_i^n and k_i^b are uniquely defined. Observe that k_i^n (resp., k_i^b) represents the optimal physical capital of agent i when her borrowing constraint is not binding (resp., binding).

The following result explores conditions under which the equilibrium aggregate output increases or decreases in agents' productivity.

[13] In Appendix A, we present also the case where $R \in (A_{n-1}, A_n)$ for some $n \in \{1, \ldots, m\}$. In such a case, the output is increasing in the productivity of any producer.

Proposition 6 (effects of productivity changes - strictly concave technology)**.** *Consider the case of strictly concave technology and let Assumptions 2 and 3 be satisfied.*

1. *The equilibrium outcomes coincide to those in the economy without frictions, (and hence, the equilibrium aggregate output is increasing in each individual productivity A_i) if one of the two following conditions*

   (a) *The credit limit of any agent is high, in the sense that $\gamma_i > \lim_{x\to\infty} \frac{xf_i'(x)}{f_i(x)}, \forall i$.*

   (b) *$\gamma_i < \lim_{x\to\infty} \frac{xf_i'(x)}{f_i(x)}, \forall i$, $R_1 < R_2 < \cdots < R_m$, and $S < \sum_{i=1}^m k_i^n(R_m/A_i)$, where R_i is the unique value satisfying*

   $$R_i \frac{k_i^n(R_i/A_i) - S_i}{A_i f_i(k_i^n(R_i/A_i))} = \gamma_i. \tag{3.8}$$

2. *Assume now that $\gamma_i < \lim_{k\to\infty} \frac{kf_i'(k)}{f_i(k)}$, $\forall i$, and $R_2 < R_3 < \cdots < R_m$. We look at the role of A_1.*

   (a) *There exists $\bar{A}_1 > 0$ such that the equilibrium output Y is increasing in A_1 on the interval $(\bar{A}_1, \infty)$.*

   (b) *Consider the case when A_1 is small. Denote*

   $$D_2 = k_2^n(\frac{R_2}{A_2}) + \sum_{i=3}^m k_i^b(\frac{R_2}{\gamma_i A_i}, S_i), \quad D_3 = \sum_{i=2}^3 k_i^n(\frac{R_3}{A_i}) + \sum_{i=3}^m k_i^b(\frac{R_3}{\gamma_i A_i}, S_i), \ldots$$
   $$D_m = \sum_{i=2}^m k_i^n(\frac{R_m}{A_i})$$

   Since $R_2 < R_3 < \cdots < R_m$, we have $D_2 > D_3 > \cdots > D_m > 0$.

   i. *If $S < D_m$, then the output is increasing in A_1 when A_1 is small enough.*

   ii. *Assume that*

   $$D_n > S > D_{n+1} \tag{3.9a}$$
   $$\gamma_i \frac{f_i(k)}{kf_i'(k)} < \frac{S_1}{S_1 + \sum_{t\geq n+1} S_t}, \forall i = n+1, \ldots, m, \forall k \in (0, S) \tag{3.9b}$$
   $$\lim_{x\to+\infty} \frac{x}{f_1''(x)} < 0 \tag{3.9c}$$

   Then, for any A_1 small enough, we have that $\frac{\partial Y}{\partial A_1} < 0$.

Proof. See Appendix A. □

Proposition 6 explores the role of two important factors: credit limits (γ_i) and productivity A_1.[14]

Look at firstly on part 1 of Proposition 6. Condition $\gamma_i > \lim_{x\to\infty} \frac{xf_i'(x)}{f_i(x)}$ ensures that agent i's borrowing constraint is not binding (see Lemma 14 in Appendix D).

[14] In Appendix C.2.1, we provide more detailed analyses for the case of two agents with strictly concave technologies.

By consequence, the equilibrium coincides to that in the economy without frictions. Therefore, the output is increasing in each productivity.

Under condition 1.(b) of Proposition 6, Theorem 3 in Appendix A implies that the equilibrium coincides to that in the economy without frictions (this is similar to part 1 of Proposition 5). Our proof is based on the key result: Agent i's borrowing constraint is binding if and only if $R \leq R_i$ (see Lemma 13 in Online Appendix 1).

Observe that $\sum_{i=1}^{m} k_i^n(R_m/A_i)$ is increasing in γ_m because R_i/A_i does not depend on A_i and $k_i^n(R/A_i)$ is decreasing in R/A_i. So, condition $S < \sum_{i=1}^{m} k_i^n(R_m/A_i)$ is more likely to be satisfied if the credit limit γ_m of the agent m who needs the credit the most (in the sense that $R_m > R_i$, $\forall i$) is quite high, then the credit constraints of this agent and of all other ones are not binding.

To better understand point 1.b, we look at the case where $F_i(k) = A_i k^{\alpha}$, $\forall i, \forall k$, with $\alpha > \gamma_i$ $\forall i$. We can compute that $R_m = \alpha A_m S_m^{\alpha-1}(1 - \frac{\gamma_m}{\alpha})^{1-\alpha}$,[15] and hence

$$\sum_{i=1}^{m} k_i^n(\frac{R_m}{A_i}) = \sum_{i=2}^{m}(\frac{\alpha A_i}{R_m})^{\frac{1}{1-\alpha}} = \sum_{i=1}^{m}(\frac{A_i}{A_m S_m^{\alpha-1}(1-\frac{\gamma_m}{\alpha})^{1-\alpha}})^{\frac{1}{1-\alpha}} = \sum_{i=1}^{m}(\frac{A_i}{A_m})^{\frac{1}{1-\alpha}}\frac{S_m}{1-\frac{\gamma_m}{\alpha}}.$$

So, we get that:

$$S < \sum_{i=1}^{m} k_i^n(R_m/A_i) \Leftrightarrow \sum_{i=1}^{m} S_i < \sum_{i=1}^{m}(\frac{A_i}{A_m})^{\frac{1}{1-\alpha}}\frac{S_m}{1-\frac{\gamma_m}{\alpha}}.$$

This can be satisfied if γ_m is high in the sense that it is closed to α.[16]

We now explain part 2 of Proposition 6. According to point 2.a, when the productivity A_1 is high, a positive productivity change is good for the aggregate output. The intuition behind is that when A_1 is high enough, the marginal productivity $A_1 f_1'(k_1)$ of this agent is the highest total marginal factor productivity, and hence, decomposition (3.6) ensures that $\frac{\partial Y}{\partial A_1} > 0$.

Regarding point $2.b.i$ of Proposition 6, condition $S < D_m$ is non-empty and it can be satisfied with a large class of parameter.[17] Observe that D_m is increasing in $A_2, \ldots, A_{m-1}$ but decreasing in A_m because R_i/A_i does not depend on A_i and $k_i^n(R/A_i)$ is decreasing in R/A_i. Moreover, D_m is increasing in agent m's credit limit γ_m. In other words, condition $S < D_m$ is likely to be satisfied if γ_m is quite high. In such a case, point 2.a ensures that, when A_1 is small enough, the credit constraints of all agents are not binding and hence the aggregate output is increasing in A_i, $\forall i \geq 1$.

Let us now look at point 2.b.ii. Condition $D_n > S > D_{n+1}$ ensures that when A_1 is low enough, the credit constraint of any agent $i \geq n+1$ is binding while that of any agent $i \leq n$ is not. Condition (3.9b) means that agents whose credit constraints

[15] See Remark 4 in Online Appendix 1.

[16] For instance, we can take $\gamma_i = \gamma < \alpha$, $S_i = s$, $\forall i$, and $A_1 < \cdots < A_m$. Then $R_1 < \cdots < R_m$. Moreover, $S < \sum_{i=1}^{m} k_i^n(R_m/A_i)$ becomes $m(1-\frac{\gamma_m}{\alpha}) < \sum_{i=1}^{m}(\frac{A_i}{A_m})^{\frac{1}{1-\alpha}}$, which is satisfied if γ is closed to α.

[17] Indeed, let $F_i(k) = A_i k^{\alpha}$, $\forall i, \forall k$, with $\alpha > \gamma_i$. We have $R_m = \alpha A_m S_m^{\alpha-1}(1-\frac{\gamma_m}{\alpha})^{1-\alpha}$, and hence $D_m = \sum_{i=2}^{m}(\frac{A_i}{A_m})^{\frac{1}{1-\alpha}}\frac{S_m}{1-\frac{\gamma_m}{\alpha}}$. When $S_i = s, \gamma_i = \gamma$, $\forall i$, and $A_2 < \cdots < A_m$, then we have $R_2 < \cdots < R_m$. Condition $S < D_m$ is equivalent to $m(1-\frac{\gamma}{\alpha}) < \sum_{i=2}^{m}(\frac{A_i}{A_m})^{\frac{1}{1-\alpha}}$ which can be satisfied.

are binding have a very low credit limit. In such a case, the aggregate output may be decreasing in productivity A_1 when A_1 is small enough. The fact that A_1 is very small ensures that the productivity dispersion is high. This is consistent with condition (2.4) in our motivating example.

3.1.1 Homogeneous versus heterogeneous productivity changes

When the TFP of all producers changes at the same rate, we have the following result.

Proposition 7 (homogeneous productivity changes)**.** *Consider an equilibrium. Assume that an exogenous change makes the individual TFP vary from A_i to $A_i(x) = xA_i$, $\forall i$, where $x > 0$. Then, for this new economy, there is an equilibrium where $Y(x) = xY$, i.e., the aggregate output changes at the same rate.*

Proof. Denote $(R, (k_i, b_i))$ an equilibrium for the economy $\mathcal{E} \equiv (A_i, f_i, \gamma_i, S_i)_{i=1,...,m}$ with borrowing constraints: $Rb_i \leq \gamma_i A_i f_i(k_i)$. We can check that $(R(x), (k_i, b_i))$, where $R(x) \equiv xR$, is an equilibrium for the new economy $\mathcal{E}(x) \equiv (A_i(x), f_i, \gamma_i, S_i)_{i=1,...,m}$. In equilibrium, the new aggregate output is $Y(x) = \sum_i A_i(x) f_i(k_i) = xY$. □

Next, we consider the case where productivity changes are not proportional. In such a case, we argue that positive productivity changes may reduce the aggregate output. Indeed, by using Taylor's theorem and Proposition 6, we obtain the following result.

Proposition 8 (asymmetric productivity changes)**.** *Consider an economy which satisfies conditions in case 2.(b) in Proposition 6, and $A_1 > 0$ small enough. Then, there exist $g \in (0,1)$ and a neighborhood $\mathcal{G}$ of $(A_1, \dots, A_m)$ such that*

$$\frac{Y(A'_1, \dots, A'_m) - Y(A_1, \dots, A_m)}{A'_1 - A_1} < 0, \tag{3.10}$$
$$\forall (A'_1, \dots, A'_m) \in \mathcal{G} \text{ satisfying } \left|\frac{A'_i - A_i}{A'_1 - A_1}\right| < g, \forall j.$$

Proof. Denote $A \equiv (A_1, \dots, A_m)$ and $A' \equiv (A'_1, \dots, A'_m)$. By Taylor's theorem, we have

$$Y(A') - Y(A) = \frac{\partial Y(A)}{\partial A_1}(A'_1 - A_1) + \sum_{i \geq 2} \frac{\partial Y(A)}{\partial A_i}(A'_i - A_i) + \sum_i h_i(A, A')(A'_i - A_i)$$

where $\lim_{A' \to A} h_i(A, A') = 0$.

We can choose $\epsilon < 0, g < 1$ and (A'_i) such that $\frac{A'_i - A_i}{A'_1 - A_1}| < g$ and $\frac{\partial Y(A)}{\partial A_1} + \sum_{i \geq 2} \frac{\partial Y(A)}{\partial A_i} \frac{A'_i - A_i}{A'_1 - A_1} < \epsilon < 0$. In this case, we get (3.10). □

There are two key points that ensure (3.10). The first condition is $\frac{\partial Y(A)}{\partial A_1} < 0$, i.e., the output is decreasing in A_1 in a neighborhood of $(A_1, \dots, A_m)$; notice that this may happen only if A_1 is small enough. Of course, we have $\frac{Y(A'_1, \dots, A'_m) - Y(A_1, \dots, A_m)}{A'_1 - A_1} > 0$ if $\frac{\partial Y(A)}{\partial A_i} > 0$, $\forall i$. The second condition is $|\frac{A'_i - A_i}{A'_1 - A_1}| < g$, i.e., the productivity does not change at the same rate and that the productivity of the less productive agent (agent 1) increases faster than that of the most productive agents. This implies that agent 1 absorbs more capital than other ones.

3.2 Effects of credit limits

In this section, we investigate the effects of credit limits (γ_t) on the aggregate production, which help us to understand better the relationship between finance and economic growth. A meaningful question is whether financial development has positive effects on the economic growth. In our model, relaxing credit limit (i.e., increasing γ_i) can be interpreted as reduction of financial friction or improvement of the financial sector.

Assume that the credit limit of all agents depends on an exogenous variable $x \in \mathbb{R}$ in the sense that $\gamma_i = \gamma_i(x)$ where γ_i is a differentiable function of x and $\gamma_i'(x) > 0$.

We wonder how the aggregate output changes when x varies. The equilibrium physical capital of agent i, which depends on x, is denoted by $k_i(x)$. We write $k_i(x) = k_i\big(\gamma_i(x), R(\gamma_1(x), \dots, \gamma_m(x))\big)$, where $R = R(\gamma_1(x), \dots, \gamma_m(x))$ is the equilibrium interest rate which depends on the credit limit $(\gamma_i(x))_i$. We can write the aggregate output as follows:

$$Y(x) = \sum_i F_i\big(k_i(x)\big) = \sum_i F_i\Big(k_i\big(\gamma_i(x), R(\gamma_1(x), \dots, \gamma_m(x))\big)\Big). \tag{3.11}$$

Assume the differentiability, we have

$$k_i'(x) = \frac{\partial k_i}{\partial \gamma_i}\frac{\partial \gamma_i}{\partial x} + \frac{\partial k_i}{\partial R}\frac{\partial R}{\partial x}, \quad \frac{\partial R}{\partial x} = \sum_j \frac{\partial R}{\partial \gamma_j}\frac{\partial \gamma_j}{\partial x} \tag{3.12}$$

Recall that $\frac{\partial k_i}{\partial \gamma_i} \geq 0$, $\frac{\partial k_i}{\partial R} \leq 0$, $\frac{\partial \gamma_i}{\partial x} \geq 0$, $\frac{\partial R}{\partial x} \geq 0$ because $\frac{\partial R}{\partial x} = \sum_j \frac{\partial R}{\partial \gamma_j}\frac{\partial \gamma_j}{\partial x}$ and $\frac{\partial R}{\partial \gamma_j} \geq 0$, $\forall j$. So, we see that $k_i'(x)$ may have any sign. However, we know $\sum_i k_i'(x) = 0$ because $\sum_i k_i = S$ in equilibrium. By consequence, we obtain two decompositions which help us to understand why the aggregate output may be increasing or decreasing in the exogenous change x.

Proposition 9 (effects of credit changes)**.** *Consider an equilibrium.*

1. *The equilibrium outcomes do not depend on credit limits $\gamma_i(x)$ of agents whose borrowing constraints are not binding.*

2. *For any agent j whose borrowing constraint is binding, let x vary and assume that the equilibrium outcomes are differentiable functions. Then, we have decompositions:*

$$\frac{\partial Y}{\partial x} = \underbrace{\sum_{i: k_i'(x) \geq 0} F_i'(k_i(x))k_i'(x)}_{\textit{Added production of agent } j} + \underbrace{\sum_{i: k_i'(x) < 0} F_i'(k_i(x))k_i'(x)}_{\textit{Production losses of other agents}} \tag{3.13}$$

$$= \underbrace{\sum_i F_i'(k_i(x)) \underbrace{\frac{\partial k_i}{\partial \gamma_i}}_{>0} \underbrace{\frac{\partial \gamma_i}{\partial x}}_{>0}}_{\textit{Quantity effect}} + \underbrace{\sum_i F_i'(k_i(x)) \underbrace{\frac{\partial k_i}{\partial R}}_{<0} \underbrace{\frac{\partial R}{\partial x}}_{>0}}_{\textit{Price effect}} \tag{3.14}$$

3. *Consider a particular case where only A_j varies (other being fixed). We have that*

$$\frac{\partial Y}{\partial \gamma_j} = \underbrace{\frac{\partial R}{\partial \gamma_j}}_{\geq 0} \sum_{i \neq j} \left(F_i'(k_j) - F_i'(k_i)\right) \underbrace{\frac{-\partial k_i}{\partial R}}_{\geq 0} \tag{3.15}$$

While we directly get (3.13) and (3.14) by taking the derivative of x with respect to γ_j, condition (3.15) is a consequence of (3.13) and the fact that $\sum_i k_i = S$.[18] Proposition 9 has a similar insight as in Proposition 4 and Corollary 1. This directly leads to the following result.

Corollary 2. *Denote $\mathcal{I}_n = \arg\max_i\{F_i'(k_i)\}$. Thus, we have that $\partial Y/\partial \gamma_j \geq 0 \; \forall j \in \mathcal{I}_n$, i.e., the aggregate output is increasing in the credit limit of agents having the highest marginal productivity.*

We now provide conditions under which the aggregate output may be decreasing in credit limits.

Proposition 10 (effects of individual credit limit)**.** *Assume that $F_i(k) = A_i k \; \forall i, k$. Assume that $\max_i(\gamma_i A_i) < A_1 < \cdots < A_m$. Consider the case where the equilibrium interest rate is belong to the interval (A_{n-1}, A_n). Then, we have that:*

1. $\frac{\partial Y}{\partial \gamma_n} < 0 < \frac{\partial Y}{\partial \gamma_m}$ *if $n < m$.*[19]
2. *Consider an entrepreneur i with $n < i < m$, we have that:*

$$\frac{\partial Y}{\partial \gamma_i} > 0 \text{ if } A_i \text{ is high enough, i.e., } \frac{A_i - A_{i-1}}{A_m - A_i} > \frac{\sum_{t=i+1}^{m} \frac{\gamma_t A_t S_t}{(A_{n-1} - \gamma_t A_t)^2}}{\sum_{t=n}^{i-1} \frac{\gamma_t A_t S_t}{(A_n - \gamma_t A_t)^2}} \tag{3.16a}$$

$$\frac{\partial Y}{\partial \gamma_i} < 0 \text{ if } A_i \text{ is low enough, i.e., } \frac{A_i - A_n}{A_{i+1} - A_i} < \frac{\sum_{t=i+1}^{m} \frac{\gamma_t A_t S_t}{(A_n - \gamma_t A_t)^2}}{\sum_{t=n}^{i-1} \frac{\gamma_t A_t S_t}{(A_{n-1} - \gamma_t A_t)^2}}. \tag{3.16b}$$

Proof. See Appendix B. □

Condition $\frac{\partial Y}{\partial \gamma_{n+1}} < 0$ indicates that an increasing of the credit limit of the least productive producer harms the aggregate output while condition $\frac{\partial Y}{\partial \gamma_m} > 0$) has a similar interpretation as in Corollary 2.

According to (4.29a) and (4.29b), the aggregate output is more likely to be increasing (resp., decreasing) in the credit limit of an agent if the TFP of this producer is quite close to those of more productive entrepreneurs (resp., that of the least productive entrepreneur) or/and credit limits and initial wealths of more productive agents $(\gamma_t)_{t>i}$ are low.

We complement our above points by a numerical example.

[18] Indeed, notice that k_i depends on R and γ_i, taking the derivative of both sides of $\sum_i k_i = S$ with respect to γ_j, we have $\left(\sum_{i=1}^m \frac{\partial k_i}{\partial R}\right)\frac{\partial R}{\partial \gamma_j} + \frac{\partial k_j}{\partial \gamma_j} = 0$, which imply that $\frac{\partial k_j}{\partial R}\frac{\partial R}{\partial \gamma_j} + \frac{\partial k_j}{\partial \gamma_j} = -\sum_{i \neq j} \frac{\partial k_i}{\partial R}\frac{\partial R}{\partial \gamma_j} \geq 0$. Combining this with the equation $Y = \sum_i F_i(k_i)$, we get (3.15).

[19] Moreover, if $n = m$ (i.e., only agent m produces), we have $\frac{\partial Y}{\partial \gamma_m} = 0$.

Numerical simulation 1. Consider a three-agent economy with linear production functions $F_i(k) = A_i k$, $\forall i, \forall k$, and borrowing constraints are $Rb_i \leq \gamma_i A_i k_i$. In Appendix B, we completely compute the equilibrium. Assume now that fundamentals are given by $S_1 = 4$, $S_2 = 4$, $S_3 = 3$, $A_1 = 1$, $A_2 = 1.2$, $A_3 = 1.5$, $\gamma_1 = 0.2$.

First, we set $\gamma_3 = 0.3$ and we let γ_2 vary. Figure 2 shows the effects of the agent 2's credit limit γ_2 on the equilibrium interest rate and the aggregate output. When γ_2 varies from 0.15 to 0.45, the interest rate varies from $A_1 = 1$ to $A_2 = 1.2$. The aggregate output is not monotonic functions of γ_2. Indeed, it is increasing in γ_2 in the regime $\mathcal{A}_1$ where the interest rate $R = A_1$, but decreasing in γ_2 in the regime $\mathcal{R}_1$ where the interest rate $R = R_1$ (consistent with Proposition 10), and then constant in the regime $\mathcal{A}_1$ where $R = A_2$.

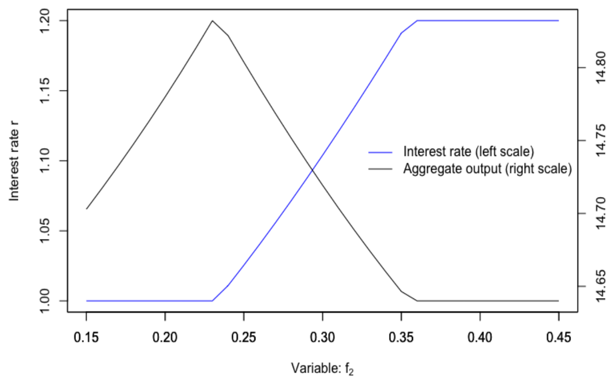


Figure 2: Non-monotonic effects of credit limit γ_2.

Second, we set $\gamma_2 = 0.3$ and let γ_3 vary. Figure 3 shows the effects of the most productive agent's credit limit γ_3 on the equilibrium interest rate and the aggregate output. The output is increasing in γ_3 (this is consistent with point 1 of Proposition 10).

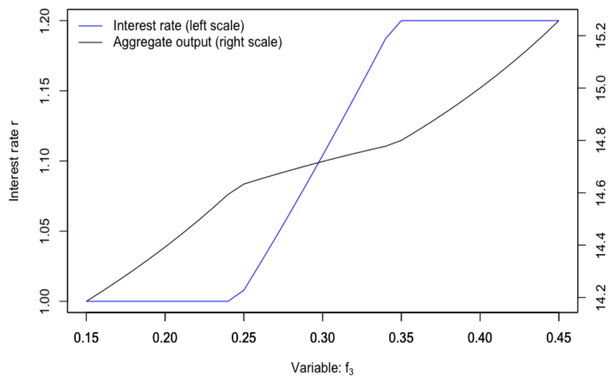


Figure 3: Monotonic effects of credit limit γ_3.

3.2.1 Homogeneous versus heterogeneous credit changes

We firstly consider the case of homogeneous credit change.

Proposition 11 (homogeneous credit change). *Assume either $F_i(k) = A_i k$, $\forall i, \forall k$ or Assumption 2 is satisfied. Assume also that $\gamma_i = \gamma \in (0,1)$, $\forall i$. Then the equilibrium aggregate output is an increasing function of the credit limit γ.*

Proof. See Appendix B. □

The intuition of the result is simple: all credit-constrained producers, who have higher marginal productivity, can borrow more from other agents who have lower marginal productivity, and hence produce more. This point is consistent with those in in Khan and Thomas (2013) (section VI. C), Midrigan and Xu (2014) (section II.B), Moll (2014) (Proposition 1), and Catherine, Chaney, Huang, Sraer, and Thesmar (2022).

We now assume that there is an aggregate change on credit limits under which the new credit limits are $(\gamma'_i)_i$. Our novel point is that, even $\gamma'_i > \gamma_i$ $\forall i$, the new aggregate output $Y' = Y(\gamma'_1, \dots, \gamma'_m)$ may be lower than $Y = Y(\gamma_1, \dots, \gamma_m)$. Formally, we have the following result.

Proposition 12 (general credit changes). *Assume that $F_i(k) = A_i k$ $\forall i, k$, and $\max_i(\gamma_i A_i) < A_1 < \cdots < A_m$. Consider the case where the equilibrium interest rate is in the interval $[A_n, A_{n+1})$. Consider an agent i such that $n+1 < i < m$ and assume that condition (4.29b) holds. Then there exist $g \in (0,1)$ and a neighborhood $\mathcal{G}$ of $(\gamma_1, \dots, \gamma_m)$ such that*

$$\frac{Y(\gamma'_1, \dots, \gamma'_m) - Y(\gamma_1, \dots, \gamma_m)}{\gamma'_i - \gamma_i} < 0, \tag{3.17}$$

$$\forall (\gamma'_1, \dots, \gamma'_m) \in \mathcal{G} \text{ satisfying } \left|\frac{\gamma'_j - \gamma_j}{\gamma'_i - \gamma_i}\right| < g, \forall j \neq i.$$

We can apply the same argument used in Proposition 8 to prove Proposition 12.

Proposition 12 shows that the aggregate output may be reduced even the credit limits of all agents increase (i.e., $\gamma'_i > \gamma_i, \forall i$). It complements Proposition 11, Proposition 10, and those in Buera and Shin (2013), Khan and Thomas (2013), Midrigan and Xu (2014), Moll (2014), Catherine, Chaney, Huang, Sraer, and Thesmar (2022). Recall that these studies provide conditions under which relaxing credit limits has positive impact on the aggregate output.

3.3 Productivity growth, productivity dispersion and credit constraint

Definition 3 (aggregate production function and aggregate TFP). *If we assume that $F_i(k) = A_i f(k)$ where A_i represents the individual productivity of agent i and f is a production function, then we can define the aggregate production function G and the aggregate TFP A by*

$$\text{the aggregate TFP: } A \equiv \frac{Y}{f(S)} \tag{3.18a}$$

$$\text{the aggregate production function: } G(S) \equiv Y = Af(S). \tag{3.18b}$$

Consider the case $F_i(k) = A_i f(k)$, $\forall i, \forall k$. The aggregate productivity TFP is defined by $TFP = Y/f(S)$. Assume that there is a shock (technical progress, for instance) that changes productivity from A_i to A'_i and credit limit from γ_i to γ'_i. The new TFP of the economy is $TFP' = Y'/f(S)$. We have

$$\frac{TFP'}{TFP} = \frac{\frac{Y(A'_1,\ldots,A'_m)}{f(S)}}{\frac{Y(A_1,\ldots,A_2)}{f(S)}} = \frac{Y(A'_1,\ldots,A'_m)}{Y(A_1,\ldots,A_2)}$$

We aim to understand the relationship between the aggregate productivity growth $\frac{TFP'}{TFP}$ and individual ones $\frac{A'_1}{A_1},\ldots,\frac{A'_m}{A_m}$.

In the economy without frictions, by using the definition (3.1) we have that

$$\frac{TFP'}{TFP} = \frac{\max\{\sum_i A'_i f(k_i) : k_i \geq 0, \sum_i k_i \leq S\}}{\max\{\sum_i A_i f(k_i) : k_i \geq 0, \sum_i k_i \leq S\}}$$

Observe that $min_i\{\frac{A'_i}{A_i}\}A_i f(k_i) \leq A'_i f(k_i) \leq max_i\{\frac{A'_i}{A_i}\}A_i f(k_i)$. So, obtain that $min_i\{\frac{A'_i}{A_i}\} \leq \frac{TFP'}{TFP} \leq max_i\{\frac{A'_i}{A_i}\}$.

However, when we consider economies with credit constraints, our above analyses (see Propositions 2, 6, 8, 12) show that the aggregate productivity growth $\frac{TFP'}{TFP}$ may be less than $min_i\{\frac{A'_i}{A_i}\}$. Indeed, for instance, we can choose (A_i) and (A'_i) so that all conditions in Proposition 8 are satisfied and $min_i\{\frac{A'_i}{A_i}\} > 1$. In this case, we have $Y(A'_1,\ldots,A'_m) - Y(A_1,\ldots,A_m) < 0$, or, equivalently, $\frac{TFP'}{TFP} < 1$. We summarize our points in the following result.

Proposition 13 (productivity growth, productivity dispersion and credit constraint)**.** *Consider the case $F_i(k) = A_i f(k)$, $\forall i, \forall k$. Assume that there is a shock that changes productivity from A_i to A'_i and credit limit from γ_i to γ'_i.*

1. *In the economy without frictions, we always have*

$$min_i\{\frac{A'_i}{A_i}\} \leq \frac{TFP'}{TFP} \leq max_i\{\frac{A'_i}{A_i}\} \tag{3.19}$$

2. *Consider economies with credit constraints $\mathcal{E} \equiv (A_i, f_i, \gamma_i, S_i)_{i=1,\ldots,m}$.*

   (a) *If $\frac{A'_i}{A_i} = g > 0$, $\forall i$, then Proposition 7 implies that $\frac{TFP'}{TFP} = g$.*

   (b) *However, under some situations as in Propositions 2, 6, 8, 12, we may have that*

$$\frac{TFP'}{TFP} < min_i\{\frac{A'_i}{A_i}\}. \tag{3.20}$$

By consequence, the aggregate productivity growth rate may be far from that of most productive firms. It may be even lower than the smallest productivity growth rate of firms.

Our points contribute to the debate concerning the slowdown in aggregate productivity growth. For instance, by using data in 23 OECD countries over the 2000s, Andrews, Criscuolo and Gal (2015) document a slowdown in aggregate productivity growth, a rising productivity gap between the global frontier and other firms, and that productivity growth at the global frontier remained robust.

The following graphic from Bouche, Cette, and Lecat (2021)shows the median productivity level of frontier firms and laggard firms, over the period 1991-2016 in France, productivity being measured by TFP. We see that the productivity dispersion tends to increase over time.

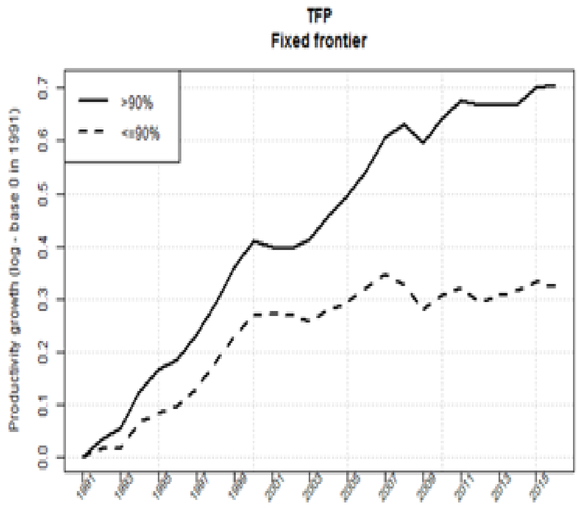


Figure 4: TFP growth. Source: Bouche, Cette, and Lecat (2021)

As recognized by Goldin, Koutroumpis, Lafond, and Winkler (2024), there is no single reason for the slowdown in aggregate productivity growth. We provide a supply-side point of view by using a general equilibrium model with credit constraint. Our above analyses suggest that the interplay between credit constraints, high heterogeneity of productivity, asymmetry of productivity and financial shocks may generate a slowdown in the aggregate productivity growth, and eventually a decrease in the aggregate productivity.

4 Extension: Infinite-horizon models à la Ramsey

We now extend our previous models by considering infinite-horizon models à la Ramsey. Each agent i maximizes her intertemporal utility subject to budget and borrowing constraints:

$$\max_{(c_i,k_i,b_i)} \sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) \tag{4.1a}$$

$$\text{subject to: } c_{i,t} + k_{i,t} - (1-\delta)k_{i,t-1} + R_t b_{i,t-1} \leq f_{i,t}(k_{i,t-1}) + b_{i,t} \tag{4.1b}$$

$$R_{t+1} b_{i,t} \leq \gamma_i \Big(f_{i,t}(k_{i,t}) + (1-\delta)k_{i,t} \Big), \tag{4.1c}$$

$$c_{i,t} \geq 0, \quad k_{i,t} \geq 0, \tag{4.1d}$$

where $\delta \in [0,1]$ is the depreciation rate. We assume that $b_{i,-1} = 0$, $\forall i$, and denote the exogenous initial wealth $w_{i,0} = F_{i,0}(k_{i,-1})$.

Note that we allow for non-stationary production functions. Let us define the function $F_{i,t} : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$F_{i,t}(k) = f_{i,t}(k) + (1-\delta)k.$$

Definition 4. *An intertemporal equilibrium is a list $((c_{i,t}, k_{i,t}, b_{i,t})_i, R_t)_{t\geq 0}$ satisfying two conditions: (1) given (R_t), the allocation $(c_{i,t}, k_{i,t}, b_{i,t})$ is a solution of the above maximization problem, and (2) markets clear: $\sum_i b_{i,t} = 0$, $\sum_i (c_{i,t} + k_{i,t}) = \sum_i F_{i,t}(k_{t-1})$, $\forall t$.*

In this section, we require standard assumptions.

Assumption 4 (endowments)**.** *$k_{i,-1} > 0$ and $b_{i,-1} = 0$ for any i.*

Assumption 5 (borrowing limits)**.** *$\gamma \in (0,1)$ for any i.*

Assumption 6 (production functions)**.** *For each i, the function $F_{i,t}$ is concave, continuously differentiable, $f'_{i,t} > 0$, $f_{i,t}(0) = 0$.*

Assumption 7 (utility functions)**.** *For each i and for each $t \geq 0$, the function $u_i : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable, concave, strictly increasing.*

Assumption 8 (finite utility)**.** *For each $i \in \{1, \cdots, m\}$,*

$$\max_{c_{i,t}, k_{i,t} \geq 0} \Big\{ \sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) : c_{i,t} + k_{i,t} \leq F_{i,t}(k_{i,t-1}) \Big\} > -\infty \tag{4.2}$$

$$\sum_{t\geq 0} \beta_i^t u_i(B_{K,t}) < \infty. \tag{4.3}$$

where we define the exogenous sequence $(B_{K,t})$ as follows:

$$B_{K,-1} = \max_{(k_i):\sum_i k_i \leq \sum_i k_{i,-1}; k_i \geq 0, \forall i} \sum_i F_{i,0}(k_i) \tag{4.4}$$

$$B_{K,t} = \max_{(k_i):\sum_i k_i \leq B_{K,t-1};; k_i \geq 0, \forall i} \sum_i F_{i,t}(k_i). \tag{4.5}$$

Theorem 1. *Under the above assumptions, there exists an intertemporal equilibrium.*

The detailed proof is presented in Online Appendix E. Let us explain the main idea. First, we prove the existence of equilibrium for each $T-$ truncated economy $\mathcal{E}^T$ where there is no activity from date $T+1$. Second, we show that this sequence of equilibria converges for the product topology to an equilibrium of our economy. The main difficulty is to bound the volume of financial asset holding of agents. Thanks to borrowing constraint (4.1c), we can do this.

4.1 Effects of productivity changes

We firstly look at the steady state.

Proposition 14 (steady state analysis)**.** *Consider the above infinite-horizon model. Assume that* $F_{i,t} = F_i$*, i.e., does not depend on time. Consider a steady state equilibrium with* $k_i > 0, \forall i$*.*

1. *The steady state interest rate is* $R = 1/\max_i\{\beta_i\}$*.*
2. *Assume, in addition, that* $\beta_1 > \beta_i, \forall i \geq 2$*. Then* $A_1 F_1'(k_1) = R = 1/\beta_1$*, agent* 1*'s borrowing constraint is not binding, and for any* $i \geq 2$*,*

$$\frac{R - \gamma_i F_i'(k_i)}{R} = F_i'(k_i)(1 - \gamma_i)$$

   Hence, k_i *is increasing in* A_i*. Since* $R\beta_i \leq 1$*, the value* k_i *is increasing in credit limit* γ_i*. By consequence, the steady state output* $Y = \sum_i F_i(k_i)$ *is increasing in TFP* A_i *and credit limit* γ_i *for any* i*.*

Proof. See Appendix C. □

In the long run, the interest rate is determined by the time preference rate of the most patient agent.

According to Proposition 14, the non-monotonic effect of productivity and credit limit on the aggregate output can only be appeared along the global dynamics of the economy. Therefore, we will focus on global dynamics, i.e., the dynamic properties of the intertemporal equilibrium.

In general, it is very difficult to provide comparative statics of intertemporal equilibrium in infinite-horizon models. For the sake of tractability, we assume that $u_i(c) = ln(c)$ and $F_{i,t}(k) = A_{i,t}k$. Thanks to this specification, we can, in some cases, explicitly compute the equilibrium.

Firstly, we look at the economy without financial frictions. It is easy to prove the following result.

Lemma 2 (economy without credit constraint)**.** *Assume that* $u_i(c) = ln(c)$ *and* $F_{i,t}(k) = A_{i,t}k$ *where* $A_{1,t} < A_{2,t} \cdots < A_{m,t}, \forall t$*. Consider an economy without credit constraints. Then, in equilibrium, we must have* $R_t = A_{m,t}$ *and the output equals denoted by* Y_t^* *and growth rate* (G_t^*) *of this economy are determined by*

$$Y_t^* = A_{m,t} \cdots A_{m,1} \sum_{i=1}^{m} \beta_i^{t-1} s_{i,0}, \tag{4.6a}$$

$$G_{t+1}^* = A_{m,t+1} \frac{\sum_{i=1}^{m} \beta_i^t}{\sum_{i=1}^{m} \beta_i^{t-1}}. \tag{4.6b}$$

where we denote $s_{i,0} \equiv \beta_i w_{i,0}, \forall i$*.*

For the economy with credit constraints, the following result provides conditions under which the equilibrium interest rate equals the TFP of some agent. Other cases will be presented latter.

Lemma 3 (economy with credit constraint)**.** *Consider an infinite-horizon model with utility function* $u_i(c) = ln(c)\ \forall c, \forall t, \forall i$ *and production functions* $F_{i,t}(k) = A_{i,t}k, \forall k, \forall t, \forall i$. *Assume that* $\max_i \gamma_i A_{i,t} < A_{1,t} < \cdots < A_{m,t}\ \forall t$.

1. *Assume that there is an agent h so that*

$$\frac{\beta_h^t s_{h,0}}{1-\gamma_h} \geq \sum_{i\leq h}\beta_i^t s_{i,0} - \sum_{j>h}\beta_j^t \frac{\gamma_j A_{j,t+1}}{A_{h,t+1}-\gamma_j A_{j,t+1}}\frac{(1-\gamma_j)A_{j,t}}{A_{h,t}-\gamma_j A_{j,t}}\cdots\frac{(1-\gamma_j)A_{j,1}}{A_{h,1}-\gamma_j A_{j,1}}s_{j,0} > 0. \tag{4.7}$$

   where $s_{i,0} = \beta_i w_{i,0}$.

   Then there exists an equilibrium with $R_t = A_{h,t}, \forall t$. *In such an equilibrium, the aggregate output at date t,* $(t \geq 1)$, *is*

$$Y_t = A_{h,t}\cdots A_{h,1}\sum_{i\leq h}\beta_i^{t-1}s_{i,0} \tag{4.8}$$
$$+ A_{h,t}\cdots A_{h,1}\sum_{j>h}\beta_j^{t-1}(1-\gamma_j)^t\frac{A_{j,t}}{A_{h,t}-\gamma_j A_{j,t}}\frac{A_{j,t-1}}{A_{h,t-1}-\gamma_j A_{j,t-1}}\cdots\frac{A_{j,1}}{A_{h,1}-\gamma_j A_{j,1}}s_{j,0}.$$

2. *In particular, when*

$$\frac{\beta_m^t s_{m,0}}{1-\gamma_m} \geq \sum_{i\leq m}\beta_i^t s_{i,0}, \forall t, \tag{4.9}$$

   then there exists an equilibrium which coincides with the equilibrium in the economy without credit constraints: the interest rate equals $R_t = A_{m,t}, \forall t$ *and the aggregate output is* $Y_t = A_{h,t}\cdots A_{h,1}\sum_{i\leq h}\beta_i^{t-1}s_{i,0}$.

Proof. See Appendix C. □

The right hand side of condition (4.7) ensures that agent h produces, i.e., $k_{h,t} > 0$ while the left hand side ensures agent h's borrowing constraint. Under these conditions, we can compute the equilibrium outcome.

4.1.1 Effect of permanent productivity changes

Lemma 3 allows us to investigate the effects of productivity changes. First, assume that for some reasons like technical progress, the productivity of producers increase (or decrease) at any date. We explore how this change affect the aggregate output and the growth rate along the intertemporal equilibrium.

Proposition 15. *Assume that* $F_{i,t}(k) = A_i k, \forall i, \forall k \geq 0$ *with* $\max_i \gamma_i A_i < A_1 < A_2 < \ldots < A_m$, *and utility function* $u_i(c) = ln(c)\ \forall i$. *Assume that*

$$\frac{\beta_h^t s_{h,0}}{1-\gamma_h} \geq \sum_{i\leq h}\beta_i^t s_{i,0} - \sum_{j>h}\frac{\gamma_j A_j}{A_h-\gamma_j A_j}\Big(\frac{\beta_j(1-\gamma_j)A_j}{A_h-\gamma_j A_j}\Big)^t s_{j,0} > 0, \forall t \geq 0 \tag{4.10}$$

$$\beta_h = \max_{i\leq h}\beta_i > \max_{j>h}\frac{\beta_j(1-\gamma_j)A_j}{A_h-\gamma_j A_j}. \tag{4.11}$$

for some agent h.

Then, there is an equilibrium with the interest rate $R_t = A_h$, $\forall t$. In this equilibrium, we have that:

1. *The aggregate output equals*

$$Y_t = A_h^t \sum_{i\leq h} \beta_i^{t-1} s_{i,0} + \sum_{j>h} \beta_j^{t-1}(1-\gamma_j)^t A_j^t \Big(\frac{A_h}{A_h-\gamma_j A_j}\Big)^t s_{j,0}. \quad (4.12)$$

This is increasing in A_j for any $j > h$. However, for agent h, we have that:

$$\frac{\partial Y_t}{\partial A_h} > 0 \Leftrightarrow \sum_{i\leq h} \beta_i^{t-1} s_{i,0} - \sum_{j>h} \frac{(1-\gamma_j)\gamma_j A_j^2}{(A_j-\gamma_j A_j)^2}\Big(\frac{\beta_j(1-\gamma_j)A_j}{A_h-\gamma_j A_j}\Big)^{t-1} s_{j,0} > 0 \quad (4.13)$$

and this condition is non empty.[20]

   (a) *If $\sum_{i:\beta_i=\beta_h} s_{i,0} > \sum_{j>h} \frac{(1-\gamma_j)\gamma_j A_j^2}{(A_j-\gamma_j A_j)^2} s_{j,0}$, then $\frac{\partial Y_t}{\partial A_h} > 0$ for any t.*

   (b) *If $\sum_{i\leq h} s_{i,0} < \sum_{j>h} \frac{(1-\gamma_j)\gamma_j A_j^2}{(A_j-\gamma_j A_j)^2} s_{j,0}$, then $\frac{\partial Y_1}{\partial A_h} < 0$ at date 1 but there exists a date t_0 such that $\frac{\partial Y_t}{\partial A_h} \geq 0$, $\forall t > t_0$.*

2. *The growth rate $G_{t+1} \equiv \frac{Y_{t+1}}{Y_t}$ equals*

$$G_{t+1} \equiv \frac{Y_{t+1}}{Y_t} = A_h \frac{\sum_{i\leq h} \beta_i^t s_{i,0} + \sum_{j>h}\Big(\frac{\beta_j(1-\gamma_j)A_j}{A_h-\gamma_j A_j}\Big)^{t+1}\frac{s_{j,0}}{\beta_j}}{\sum_{i\leq h} \beta_i^{t-1} s_{i,0} + \sum_{j>h}\Big(\frac{\beta_j A_j(1-\gamma_j)}{A_h-\gamma_j A_j}\Big)^{t}\frac{s_{j,0}}{\beta_j}}. \quad (4.14)$$

and it converges to $A_h\beta_h$. Moreover, for agent j, with $j > h$, there exists a date t_1 such that the growth rate G_{t+1} is decreasing in the productivity A_j for any date $t \geq t_1$.

Proof. See Appendix C. □

Observe that the right hand side of (4.13) is increasing in A_h. So, the aggregate output Y_t is more likely to be increasing in the TFP A_h (i.e., $\frac{\partial Y_t}{\partial A_h} > 0$) if (1) the productivity gap $\frac{A_j}{A_h}$ (for $j > h$) is low or (2) the initial income gap $\frac{s_{j,0}}{s_{i,0}}$ (for $j > h$, $i \leq h$) is low or (3) the time preference gap $\frac{\beta_j}{\beta_i}$ (for $j > h$, $i \leq h$) is low.

Condition (4.10) ensures that agent h still produces, i.e., $k_{h,t} > 0$, $\forall t$.[21] This happens if its TFP A_h is not too low and the rate of time preference β_h is high enough. Notice that Condition (4.11) ensures imply that $\beta_h A_h > \beta_j A_j$, $\forall j > h$. This ensures that agent 1 still produces and the growth rate $\frac{Y_{t+1}}{Y_t}$ converges to $\beta_h A_h$.

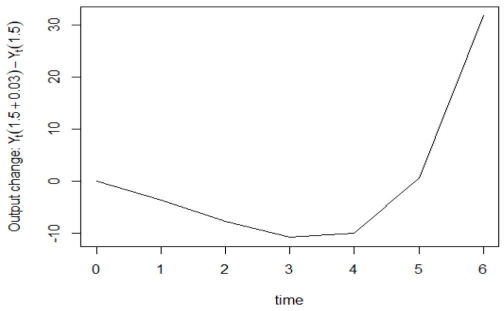
Proposition 15 allows us to understand the impact of a shock on the TFP of the less productive agent. Observe that, if A_h increases, then the output will increase in the long run. However, point 1.b of Proposition 15 indicates that, if A_h increases but it is still low, the output may decrease in the short run and then increase in the long run.

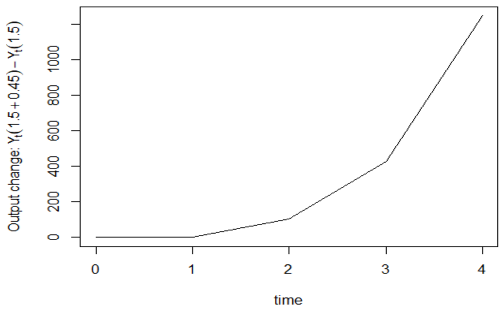
[20] In the sense that there exists $(s_{i,0}, \gamma_i, A_i)_i$ satisfying this condition.

[21] Notice that condition (4.10) requires that $\max_{i\geq h} \beta_i \geq \max_{j>h} \frac{\beta_j(1-\gamma_j)A_j}{A_h-\gamma_j A_j}$ and $\max\Big(\beta_h, \max_{j>h} \frac{\beta_j(1-\gamma_j)A_j}{A_h-\gamma_j A_j}\Big) \geq \max_{i<h} \beta_i$.

Numerical simulation 2. To complement our theoretical findings presented above, we run a simulation in a two-agent model with linear production function $F_i(k) = A_i k$, and $s_{1,0} = 200, s_{2,0} = 100, \beta_1 = 0.99, \beta_2 = 0.4, A_1 = 1.5, A_2 = 2.25$. The credit limit of agent 2 is $\gamma_2 = 0.4$. Let us denote $Y_t(A_1)$ the equilibrium aggregate output of the economy when the productivity of agent 1 is A_1. The following graphics show how the difference between $Y_t(A_1 + h) - Y_t(A_1)$ changes over time, where h is a productivity change.

First, when the productivity of agent 1 increases from 1.5 to 1.53 (a small productivity change), the output goes down and then goes up. Precisely, $Y_t(1.5+0.03) - Y_t(1.5) < 0$ for $t = 1, 2, 3, 4$ and then $Y_t(1.5 + 0.03) - Y_t(1.5) > 0$, $\forall t \geq 5$.





Second, when there is a high productivity change so that the productivity of agent 1 increases from 1.5 to 1.95, the output goes up at any period: $Y_t(1.5+0.45) - Y_t(1.5) > 0$, $\forall t \geq 1$. This is consistent with the insights in Proposition 2.

4.1.2 Effect of temporary productivity changes

Let us look at the effects of temporary productivity changes. Assume that there is a productivity change only at date 1, which affects the TFP of agent h. We would like to understand how the aggregate output changes when $A_{h,1}$ varies.

According to Lemma 3, we have

$$\begin{aligned}
\frac{\partial Y_t}{\partial A_{h,1}} =& A_{h,t} \cdots A_{h,2} \sum_{i \leq h} \beta_i^{t-1} s_{i,0} \\
&+ A_{h,t} \cdots A_{h,2} \sum_{j>h} \beta_j^{t-1} (1-\gamma_j)^t \frac{A_{j,t}}{A_{h,t} - \gamma_j A_{j,t}} \cdots \frac{A_{j,2}}{A_{h,2} - \gamma_j A_{j,2}} \frac{\partial \left(\frac{A_{h,1}}{A_{h,1} - \gamma_j A_{j,1}}\right)}{\partial A_{h,1}} A_{j,1} s_{j,0} \\
=& A_{h,t} \cdots A_{h,2} \sum_{i \leq h} \beta_i^{t-1} s_{i,0} \\
&- A_{h,t} \cdots A_{h,2} \sum_{j>h} \beta_j^{t-1} (1-\gamma_j)^t \frac{A_{j,t}}{A_{h,t} - \gamma_j A_{j,t}} \cdots \frac{A_{j,2}}{A_{h,2} - \gamma_j A_{j,2}} \frac{\gamma_j A_{j,1}^2}{(A_{h,1} - \gamma_j A_{j,1})^2} s_{j,0}
\end{aligned}$$

For the sake of simplicity, we focus on the case where $A_{i,t} = A_i$, $\forall t$, $\forall i \neq h$ and $A_{h,t} = A_h$, $\forall t \neq 1$. We only let $A_{h,1}$ - the productivity of agent h at date 1 vary. In this case, we have that:

$$Y_t = A_h^{t-1} A_{h,1} \Big(\sum_{i \leq h} \beta_i^{t-1} s_{i,0} + \sum_{j>h} \beta_j^{t-1} (1-\gamma_j)^t \big(\frac{A_j}{A_h - \gamma_j A_j} \big)^{t-1} \frac{A_{j,1}}{A_{h,1} - \gamma_j A_{j,1}} s_{j,0} \tag{4.15}$$

$$\frac{\partial Y_t}{\partial A_{h,1}} = A_h^{t-1} \Big(\sum_{i \leq h} \beta_i^{t-1} s_{i,0} - \sum_{j>h} \beta_j^{t-1} (1-\gamma_j)^t \big(\frac{A_j}{A_h - \gamma_j A_j} \big)^{t-1} \frac{\gamma_j A_{j,1}^2}{(A_{h,1} - \gamma_j A_{j,1})^2} s_{j,0} \Big). \tag{4.16}$$

The growth rate $\frac{Y_{t+1}}{Y_t}$ again converges to $\beta_h A_h$. However, the output can decrease when $A_{h,1}$ increases. According to (4.16), the output is more likely to be increasing in $A_{h,1}$ is decrease when the productivity dispersion $\frac{A_j}{A_h}$ is low. The insights is consistent with the effects of permanent productivity shocks.

4.2 Effects of credit limits

In this section, we explore the effects of credit limits on the aggregate output in intertemporal equilibrium. To simplify our exposition, we focus on the case of stationary linear technology $F_i(k) = A_i k$. Since $A_1 < A_2 < \cdots < A_m$, the equilibrium interest rate is between A_1 and A_m. We distinguish two cases: (t) the interest rate equals the TFP of some producer and (2) the interest rate is between the TFPs of two producers. The following result considers an equilibrium in the first case.

Proposition 16. *Assume that the technology is stationary: $A_{i,t} = A_i, \forall i, \forall t$. Let assumptions in Lemma 3 be satisfied. Then, there exists an equilibrium with $R_t = A_h$, $\forall t$. In equilibrium, we have that:*

1. *The aggregate output Y_t is increasing in the credit limit γ_j of agent j for any $j > h$. Moreover, the output in (4.12) in the economy with credit constraints is lower that the output in the economy without credit constraints.*

2. *However, the growth rate determined by (4.14) is not necessarily increasing in the credit limit γ_j. It converges to $A_h \beta_h$ which is higher than $A_m \beta_m$ - the growth rate of the economy without credit constraint.*

Proof. Observe that $\frac{1-\gamma_j}{A_{h,t} - \gamma_j A_{j,t}}$ is increasing in γ_j.[22] So, according to (4.8), the aggregate output Y_t is increasing in each $\gamma_j, \forall j > h$. □

Point 1 is consistent with the insights in the literature concerning the macroeconomic effects of credit constraint (Khan and Thomas (2013) (section VI. C), Midrigan and Xu (2014) (section II.B), Moll (2014) (Proposition 1), and Catherine, Chaney, Huang, Sraer, and Thesmar (2022)).

[22] Indeed, we have

$$\frac{\partial \big(\frac{1-\gamma_j}{A_{h,t} - \gamma_j A_{j,t}} \big)}{\partial \gamma_j} = \frac{-1(A_{h,t} - \gamma_j A_{j,t}) + (1-\gamma_j) A_{j,t}}{(A_{h,t} - \gamma_j A_{j,t})^2} = \frac{A_{j,t} - A_{h,t}}{(A_{h,t} - \gamma_j A_{j,t})^2} > 0.$$

However, the insight of point 2 of Proposition 16 is new. It indicates when producers' credit limits are low, a rise in credit limit may decrease the growth rate of the economy. This is consistent with the empirical fact: the rate of growth of developing countries (with more severe credit constraints of firms) is in general higher than the grow rate of developed countries).

We now investigate a question: Along an intertemporal equilibrium, does relaxing credit limit always improve or, in some cases, reduce the aggregate output? The full answer is complicated. The following result provides the first part of our answer: conditions (based on exogenous parameters) under which the aggregate output is a decreasing function of the credit limit.

Proposition 17 (intertemporal equilibrium with $R_1 \in (A_{m-1}, A_m)$, $R_t = A_m, \forall t \geq 2$)**.** *Assume that $u_i(c) = ln(c)$, $\forall i, \forall c > 0$, $F_{i,t}(k) = A_i k$, $\forall i, \forall k \geq 0$ with $\max_i \gamma_i A_i < A_1 < A_2 < \ldots < A_m$, and*

$$\frac{\sum_{i<m} \beta_i^t s_{i,0}}{\sum_{i<m} s_{i,0}} \leq \beta_m^t, \forall t, \tag{4.17a}$$

$$\gamma_m < \frac{\sum_{i \neq m} s_{i,0}}{S_0} \tag{4.17b}$$

$$\frac{A_{m-1}}{A_m} < \gamma_m \frac{S_0}{\sum_{i \neq m} s_{i,0}}. \tag{4.17c}$$

Then, there exists an equilibrium where the interest rates are determined by

$$R_1 = \gamma_m A_m \frac{S_0}{\sum_{i \neq m} s_{i,0}} \in (A_{m-1}, A_m), \quad R_t = A_m, \forall t \geq 2, \tag{4.18}$$

where $S_0 \equiv \sum_{i=1}^m s_{i,0}$.

The aggregate capital is

$$K_0 \equiv \mathcal{S} \sum_i s_{i,0} = \sum_i \beta_i w_{i,0} \tag{4.19a}$$

$$K_t = k_{m,t} = S_0 A_m^t \Big(\gamma_m \frac{\sum_{i \neq m} \beta_i^t s_{i,0}}{\sum_{j \neq m} s_{j,0}} + \beta_m^t (1 - \gamma_m) \Big), \forall t \geq 1 \tag{4.19b}$$

and the aggregate output

$$Y_1 = A_m k_{m,0} = A_m S_0 \tag{4.20a}$$

$$Y_t = A_m k_{m,t-1} = S_0 A_m^t \Big(\gamma_m \frac{\sum_{i \neq m} \beta_i^{t-1} s_{i,0}}{\sum_{j \neq m} s_{j,0}} + \beta_m^{t-1} (1 - \gamma_m) \Big). \tag{4.20b}$$

Proof. See Appendix C. □

In such an equilibrium, only the most productive agent produces. Notice that her borrowing constraint at date 1 is binding but her borrowing constraints from date 2 on are not necessarily binding.[23] From date 2 on, the equilibrium interest rate equals the

[23] In this equilibrium, borrowing constraint $R_{t+1} b_{m,t} \leq \gamma_m A_m k_{m,t}$ is equivalent to $\sum_{i \neq m} \beta_i^t \frac{s_{i,0}}{\sum_{j \neq m} s_{j,0}} \leq \beta_m^t$.

productivity of the most productive agent: $R_t = A_m$, $\forall t \geq 2$. However, the interest rate between the initial date and date 1 equals R_1 which is lower than A_m because the credit limit γ_m of agent m is not so high (in the sense that $\gamma_m < \frac{\sum_{i\neq m} s_{i,0}}{S_0}$) and the productivity gap is high (in the sense that $\frac{A_{m-1}}{A_m} < \gamma_m \frac{S_0}{\sum_{i\neq m} s_{i,0}}$). Notice that Kiyotaki (1998)'s Section 2 only focuses on the case where the equilibrium interest rate equals the rate of return on investment of unproductive agents, i.e., $R_t = A_1$, $\forall t$.

We now look at the equilibrium aggregate output.

1. First, according to Lemma 2, the output in the economy without credit constraints is $Y_t^* = A_m^t \sum_{i=1}^m \beta_i^{t-1} s_{i,0}$. So, the output at date 1 in our economy coincides to Y_1^*. However, we can verify, by using $\gamma_m < \frac{\sum_{i\neq m} s_{i,0}}{S_0}$ and $\frac{\sum_{i<m} \beta_i^t s_{i,0}}{\sum_{i<m} s_{i,0}} \leq \beta_m^t$, that $Y_t < Y_t^*$ for any $t \geq 2$.[24] It means that, the output in the economy with credit constraints is lower than the output in the economy without credit constraints. This is consistent with the existing literature.

2. Second, according to (4.20b) and our assumption (4.17a), we have that:

$$\frac{\partial Y_t}{\partial \gamma_m} < 0, \forall t \geq 2. \tag{4.21}$$

   It means that, from date 2 on, the aggregate output decreases when the most productive agent's credit limit increases.[25] This interesting result is new with respect to the standard view on the effects of financial constraints as shown in Buera and Shin (2013), Khan and Thomas (2013), Midrigan and Xu (2014), Moll (2014), Catherine, Chaney, Huang, Sraer, and Thesmar (2022).

Let us explain the intuition of our finding (4.21). Denote $W_{i,t} \equiv F_i(k_{i,t-1}) - R_t b_{i,t-1}$ the net worth of agent i at date t. In equilibrium in Proposition 17, the net worth of the most productive agent is given by

$$W_{m,1} = A_m k_{m,0} - R_1 b_{m,0} = (1-\gamma_m)\sum_i s_{i,0}$$

$$t \geq 2:\ W_{m,t} = A_m k_{m,t-1} - R_t b_{m,t-1} = A_m s_{m,t-1} = A_m(\beta_m A_m)^{t-1} s_1 = (\beta_m A_m)^t (1-\gamma_m) A_m S_0.$$

where we denote the individual saving: $s_{i,t} \equiv k_{i,t} - b_{i,t}$.

We see that the net worth is decreasing in the credit limit γ_m. The reason behind is that when γ_m increases, the interest rate R_1 goes up which makes the repayment $R_1 b_{m,0}$ increase. However, the capital $k_{m,0}$ of agent m is already equal to the aggregate savings $\sum_i s_{i,0}$ which can no longer increase. By consequence, the net worth $W_{m,1} =$

[24] Indeed, we have $Y_t = S_0 A_m^t \Big(\frac{\sum_{i\neq m} s_{i,0}}{S_0} \frac{\sum_{i\neq m} \beta_i^{t-1} s_{i,0}}{\sum_{i\neq m} s_{i,0}} + \beta_m^{t-1}(1 - \frac{\sum_{i\neq m} s_{i,0}}{S_0}) \Big) = A_m^t \sum_{i=1}^m \beta_i^{t-1} s_{i,0} = Y_t^*$.

[25] Notice that the aggregate output at date 1 does not depend on the credit limit γ_m of the most productive agent. The equilibrium in Proposition 17 does not depend on the credit limits $(\gamma_i)_{i<m}$ of less productive agents because these agents neither borrow nor produce.

$A_m k_{m,0} - R_1 b_{m,0}$ decreases. This makes the saving of agent m go down, and, hence, the output decreases. The mechanism can be summarized by the following schema:

$$\begin{aligned}&\textbf{Credit limit } \gamma_m \uparrow \Rightarrow \textbf{Interest rate} \uparrow \Rightarrow \textbf{Agent m's net worth} \downarrow \Rightarrow \\ &\Rightarrow \textbf{Saving} \downarrow \Rightarrow \textbf{Production} \downarrow \Rightarrow \cdots \end{aligned} \tag{4.22}$$

However, this mechanism does not happen when the credit limit γ_m of agent m is high enough (if this happens, we recover the equilibrium in part 2 of Lemma 3 where the output of our economy coincides to the output of the economy without credit constraints).

In Proposition 17, the most productive agent is the unique producer at date 1 and, thanks to this, the output at date 1 equals the output in the economy without credit constraints. When there are more than 2 producers, the effects of credit limits (γ_i) of different agents become more interesting. We attempt to understand what would happen in this case. Let us start with an intermediate step.

Lemma 4 (intertemporal equilibrium with $R_1 \in (A_{n-1}, A_n), R_t = A_h, \forall t \geq 2, h \geq n$). *Assume that $u_i(c) = ln(c)$, $\forall i, \forall c > 0$, $F_{i,t}(k) = A_i k$, $\forall i, \forall k \geq 0$ with $\max_i \gamma_i A_i < A_1 < A_2 < \ldots < A_m$, and*

$$\sum_{j\geq n} \frac{\gamma_j A_j}{A_n - \gamma_j A_j} s_{j,0} < \sum_{i<n} s_{i,0} < \sum_{j\geq n} \frac{\gamma_j A_j}{A_{n-1} - \gamma_j A_j} s_{j,0} \tag{4.23}$$

$$\begin{aligned}\beta_h^t \frac{A_h}{R_1 - \gamma_h A_h} s_{j,0} \geq &\sum_{i<n} \beta_i^t s_{i,0} + \sum_{n\leq i\leq h} \beta_j^t (1-\gamma_j) \frac{A_j}{R_1 - \gamma_j A_j} s_{j,0} \\ &- \sum_{j>h} \Big(\beta_j \frac{(1-\gamma_j)A_j}{A_h - \gamma_j A_j}\Big)^t \frac{\gamma_j A_j}{R_1 - \gamma_j A_j} s_{j,0} \quad \geq 0, \quad \forall t \geq 1.\end{aligned} \tag{4.24}$$

for some agent h with $n \leq h \leq m$. Then, there exists an equilibrium with the interest rates

$$R_1 \in (A_{n-1}, A_n) \text{ is determined by } \sum_{i<n} s_{i,0} = \sum_{j\geq n} \frac{\gamma_j A_j}{R_1 - \gamma_j A_j} s_{j,0} \tag{4.25a}$$

$$R_t = A_h, \forall t \geq 2. \tag{4.25b}$$

Proof. See Appendix C. □

Condition (4.23) ensures that the equilibrium interest rate R_1 is determined by (4.25a) while (4.24) ensures (4.25b). The first inequality in (4.24) means that the borrowing constraint of agents h are satisfied while the second inequality is equivalent to $k_{h,t} \geq 0$. Note that condition (4.24) requires that[26]

$$\beta_h \geq \max\big(\max_{i<h} \beta_i, \max_{j>h} \beta_j \frac{(1-\gamma_j)A_j}{A_h - \gamma_j A_j}\big) > \max_{j>h} \beta_i. \tag{4.26}$$

So, agent h has the highest discount factor.

[26]Because $\max\Big(\beta_h, \max_{j>h} \beta_j \frac{(1-\gamma_j)A_j}{A_h - \gamma_j A_j}\Big) \geq \max_{i<h} \beta_i$ and $\max_{i\leq h} \beta_i \geq \max_{j>h} \beta_j \frac{(1-\gamma_j)A_j}{A_h - \gamma_j A_j}$.

In such an equilibrium in Lemma 4, the capital of producers and the aggregate output are determined by

$$k_{j,0} = \begin{cases} 0, & \forall j < n \\ \frac{R_1}{R_1-\gamma_j A_j} s_{j,0}, & \forall j \geq n \end{cases} \tag{4.27}$$

$$k_{j,t} = \begin{cases} 0, & \forall j < h \\ \sum_{i<n} \beta_i^t A_h^{t-1} R_1 s_{i,0} + \sum_{n \leq j \leq h} \beta_j^t A_h^{t-1}(1-\gamma_j) \frac{A_j R_1}{R_1-\gamma_j A_j} s_{j,0} & \\ \qquad - \sum_{j>h} A_h^{t-1} \Big(\beta_j \frac{(1-\gamma_j)A_j}{A_h-\gamma_j A_j}\Big)^t \frac{\gamma_j A_j R_1}{R_1-\gamma_j A_j} s_{j,0}, & \text{for } j = h, \\ \frac{A_h}{A_h-\gamma_j A_j} \Big(\beta_j \frac{(1-\gamma_j)A_j A_h}{A_h-\gamma_j A_j}\Big)^{t-1} \Big(\beta_j \frac{(1-\gamma_j)A_j R_1}{R_1-\gamma_j A_j}\Big) s_{j,0}, & \forall j > h. \end{cases} \tag{4.28}$$

We are now ready to state our result showing the effects of credit limits.

Proposition 18. *Let assumptions in Lemma 4 be satisfied.*

1. For date $t = 1$, the aggregate output equals $Y_1 = \sum_{j \geq n} A_j k_{j,0}$.

1.1. $\frac{\partial Y_j}{\partial \gamma_n} < 0 < \frac{\partial Y_j}{\partial \gamma_m}$ *if $n < m$.*[27]

1.2. Consider any producer i with $n < i < m$, we have that:

$$\frac{\partial Y_1}{\partial \gamma_i} > 0 \text{ if } A_i \text{ is high enough, i.e., } \frac{A_i - A_{i-1}}{A_m - A_i} > \frac{\sum_{j=i+1}^{m} \frac{\gamma_j A_j s_{j,0}}{(A_{n-1}-\gamma_j A_j)^2}}{\sum_{j=n}^{i-1} \frac{\gamma_j A_j s_{j,0}}{(A_n-\gamma_j A_j)^2}} \tag{4.29a}$$

$$\frac{\partial Y_1}{\partial \gamma_i} < 0 \text{ if } A_i \text{ is low enough, i.e., } \frac{A_i - A_n}{A_{i+1} - A_i} < \frac{\sum_{j=i+1}^{m} \frac{\gamma_j A_j s_{j,0}}{(A_n-\gamma_j A_j)^2}}{\sum_{j=n}^{i-1} \frac{\gamma_j A_j s_{j,0}}{(A_{n-1}-\gamma_j A_j)^2}}. \tag{4.29b}$$

2. From second date.

2.1. For $v \in \{n, \dots, h\}$, this agent produces only at date 1. We have that

$$\frac{1}{A_h^t} \frac{\partial Y_{t+1}}{\partial \gamma_v} \frac{1}{\frac{\partial R_1}{\partial \gamma_v}} = \sum_{i<n} \beta_i^t s_{i,0} - \sum_{n \leq j \leq h} \beta_j^t \frac{(1-\gamma_j)\gamma_j A_j^2}{(R_1-\gamma_j A_j)^2} s_{j,0} - \sum_{j>h} \Big(\beta_j \frac{(1-\gamma_j)A_j}{A_h-\gamma_j A_j}\Big)^t \frac{(1-\gamma_j)\gamma_j A_j^2}{(R_1-\gamma_j A_j)^2} s_{j,0}\Big) + \beta_v^t (A_v - R_1) \sum_{j \geq n} \frac{\gamma_j A_j}{(R_1-\gamma_j A_j)^2} s_{j,0}, \forall t \geq 1.$$

By consequence, if $\beta_j > \max_{i \neq j} \beta_i$, then there exists t_0 such that $\frac{\partial Y_{t+1}}{\partial \gamma_v} < 0, \forall t \geq t_0$.

2.2. For agent $v > h$, this agent produces at any date. We have, for any $t \geq 1$, that

$$\frac{1}{A_h^t} \frac{\partial Y_{t+1}}{\partial \gamma_v} \frac{1}{\frac{\partial R_1}{\partial \gamma_v}} = \sum_{i<n} \beta_i^t s_{i,0} - \sum_{n \leq j \leq h} \beta_j^t \frac{(1-\gamma_j)\gamma_j A_j^2}{(R_1-\gamma_j A_j)^2} s_{j,0} - \sum_{j>h} \Big(\beta_j \frac{(1-\gamma_j)A_j}{A_h-\gamma_j A_j}\Big)^t \frac{(1-\gamma_j)\gamma_j A_j^2}{(R_1-\gamma_j A_j)^2} s_{j,0}\Big) + \Big(\frac{\beta_v(1-\gamma_v)A_v}{A_h-\gamma_v A_v}\Big)^t \Big(\frac{t(R_1-\gamma_v A_v)(A_v-A_h)}{(A_h-\gamma_v A_v)} + A_v - R_1\Big)\Big(\sum_{j \geq n} \frac{\gamma_j A_j}{(R_1-\gamma_j A_j)^2} s_{j,0}\Big)$$

[27] Moreover, if $n = m$ (i.e., only agent m produces), we have $\frac{\partial Y}{\partial \gamma_m} = 0$.

By consequence, if $\beta_h > \max\{\frac{\beta_v(1-\gamma_v)A_v}{A_h-\gamma_v A_v}, \max_{i<n}\beta_i\}$*, then there exists* t_0 *such that* $\dfrac{\partial Y_{t+1}}{\partial \gamma_v} < 0, \forall t \geq t_0$.

Proposition 10 allows us to understand why the aggregate output is decreasing or increasing in the credit limits of producers. It depends not only on the distribution of productivity and of credit limits but also on the distribution of initial capital of agents.

Since the insight of part 1 is similar to Proposition 10 in the two-period model, let us explain the intuition of part 2. Note that the aggregate output does not depend on the credit limits of non-producers. So, we only look at the producers in equilibrium. At date 1, producers are any agent $v \geq n$. From date 2 on, producers are any agent $v \geq h$. In both cases of part 2, from some date on, the output will be decreasing in the credit limit of any producer if the discount factor β_h is high. This finding is consistent with (4.21). The basic intuition behind is the input is used by less productive agents. Indeed, at the date 1, because low credit limits and high productivity dispersion (see condition (4.23)), we have $R_1 < A_n < A_h$, so we have a capital misallocation. When agent h has the highest discount factor β_h, this agent absorbs capital in the long run which makes the misallocation persistent over time and the output decrease.

Remark 1 (additional analyses)**.** *In Appendix C.2.1, we present two additional results. Proposition 20 shows that the aggregate output is increasing in the credit limits of producers for the case* $R_1 \in (A_{m-1}, A_m), R_t = A_h, \forall t \geq 2$*, with* $h < m$*.*

Proposition 21 provides conditions under which there exists an equilibrium with $R_1 \in (A_{m-1}, A_m), \forall t \geq 1$*. In this case, there is only one producer in equilibrium and the output is increasing in the credit limit of this agent.*

The intuition behind these two results is that the equilibrium interest rate is not so high low (it is lower that A_m*). Hence, the borrowing cost of producers is not so high. This helps producers borrow more and produce more.*

Numerical simulation 3. *We complement our theoretical result by a numerical simulation (Figure 5). Consider a model with 3 agents. In this simulation, we set that* $\beta_1 = 0.2, \beta_2 = 0.2, \beta_3 = 0.95$, $s_{1,0} = 4 = \beta_1 w_{1,0}, s_{2,0} = 4 = \beta_2 w_{2,0}, s_{3,0} = 3 = \beta_3 w_{3,0}$, $\gamma_1 = 0.2, \gamma_3 = 0.3$*. Productivity:* $A_1 = 1, A_2 = 1.2, A_3 = 1.5$*. We draw the output path for two cases:* $\gamma_2 = 0.3$ *and* $\gamma_2 = 0.35$*. We observe that*

$$Y_t(\gamma_2 = 0.35) < Y_t(\gamma_2 = 0.30), \forall t \geq 1.$$

It means that when the credit limit γ_2 *of agent 2 increases from 0.30 to 0.35, the aggregate output will be lower at any period of time.*

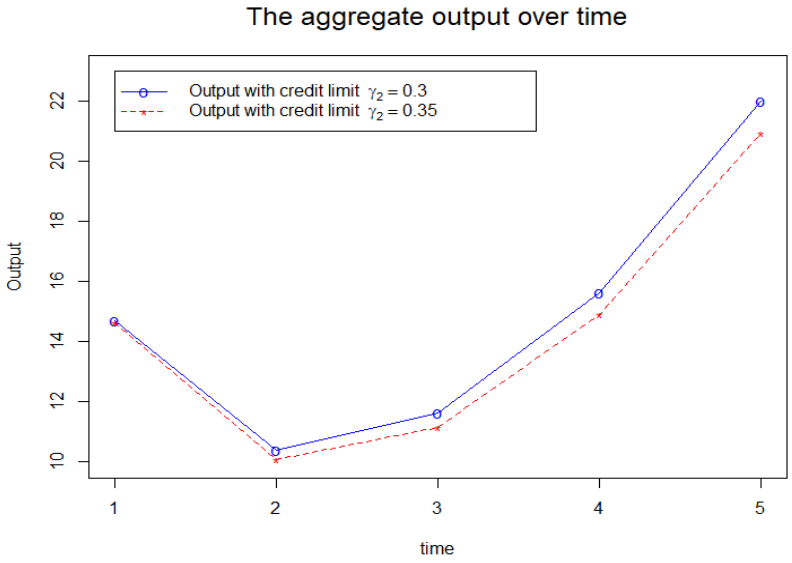


Figure 5: Effects of credit limits γ_2 on the aggregate output.

5 Conclusion

We have build general equilibrium models with borrowing constraints to explain why the aggregate output may be decreasing (increasing, respectively) when the productivity or credit limit of producers increases (decreases, respectively). A positive homogeneous (productivity or financial) shock has a positive impact on the aggregate output. This is consistent with the insights in economic textbooks and several articles. Our new insight is that positive asymmetric (productivity or financial) shocks may reduce the aggregate production. Overall, not only productivity but also financial frictions and the productivity gap (or dispersion of productivity distribution) matter for the economic development.

The contribution of the present paper is primarily theoretical. A promising avenue for future research would be to develop a quantitative model calibrated with empirical data to reassess the effects of asymmetric (productivity and financial) shocks and the persistence of shocks on equilibrium dynamics.

Appendices

A Proofs for Section 3.1

A.1 Characterization of general equilibrium

A.1.1 Linear technology

When the production functions are linear, it is easy to compute the optimal allocation of agents as a function of the interest rate (see Lemma 7 in Appendix D). Therefore, the key point is to determine the equilibrium interest rate. To state our characterization of equilibrium, we introduce some notations.

$$\mathbb{D}_n \equiv \sum_{i=n}^{m} \frac{A_n S_i}{A_n - \gamma_i A_i} \ \forall n \geq 1, \quad \mathbb{B}_n \equiv \sum_{i=n+1}^{m} \frac{A_n S_i}{A_n - \gamma_i A_i} \ \forall n \geq 1. \tag{A.1}$$

where by convention, $\sum_{i=n}^{m} x_i = 0$ if $n > m$.

Denote R_n^L the greatest solution of the following equation:[28]

$$\underbrace{\sum_{i=n+1}^{m} \frac{\gamma_i A_i}{R - \gamma_i A_i} S_i}_{\text{Asset demand}} = \underbrace{\sum_{i=1}^{n} S_i}_{\text{Asset supply}} \quad \text{or equivalently} \quad \underbrace{\sum_{i=n+1}^{m} \frac{R S_i}{R - \gamma_i A_i}}_{\text{Capital demand}} = \underbrace{S}_{\text{Capital supply}} \tag{A.2}$$

Definition 5. 1. *the regime $\mathcal{A}_n$ (with $n \in \{1, \dots, m\}$) is the set of all economies satisfying $A_n > \max_i(\gamma_i A_i)$ and $\mathbb{B}_n \leq S \leq \mathbb{D}_n$*

2. *the regime $\mathcal{R}_n$ (with $n \in \{1, \dots, m-1\}$) is the set of all economies satisfying*

   (a) *either $\max_i(\gamma_i A_i) < A_n < R_n^L < A_{n+1}$ (or equivalently $\max_i(\gamma_i A_i) < A_n$ and $\mathbb{D}_{n+1} < S < \mathbb{B}_n$)*

   (b) *or $A_n \leq \max_i(\gamma_i A_i) < R_n^L < A_{n+1}$ (or equivalently $A_n \leq \max_i(\gamma_i A_i) < R_n^L$ and $\mathbb{D}_{n+1} < S$).*

We now provide a characterization of general equilibrium.

Theorem 2 (characterization of general equilibrium with linear technologies)**.** *Assume that $F_i(K) = A_i K \ \forall i$ and $A_1 < \dots < A_m$. Then, there exists a unique equilibrium. The equilibrium interest rate is determined by the following:*

$$R = \begin{cases} A_i & \text{in the regime } \mathcal{A}_i. \\ R_i^L & \text{in the regime } \mathcal{R}_i. \end{cases} \tag{A.3}$$

Proof. See Appendix D. □

[28] It should be noticed that the function $f(x) \equiv \sum_{i=n+1}^{m} \frac{x S_i}{x - \gamma_i A_i}$ is not continuous at point $\gamma_i A_i$ with $i \geq n+1$. However, it is continuous and decreasing in the interval $(\max_{i \geq n+1}(\gamma_i A_i), \infty)$. Then, the equation $f(x) = S$ has a unique solution in such interval.

A.1.2 Strictly concave technology

Before providing the characterization of equilibrium, we state an assumption about the credit limit.

Assumption 9. $\gamma_i < \lim_{k\to\infty} \frac{kf_i'(k)}{f_i(k)}$, $\forall i$.

As proved in Lemma 14 in Appendix D, under Assumptions 2 and 3, if agent i's borrowing constraint is binding, we must have $\gamma_i \leq \lim_{x\to\infty} \frac{xF_i'(x)}{F_i(x)}$.

We are now ready to state the characterization of equilibrium.

Theorem 3 (characterization of general equilibrium: strictly concave technologies)**.** *Under Assumption 2, there exists a unique equilibrium. Assume, in addition, that Assumption 3 and 9 hold and $R_1 < R_2 < \ldots < R_m$, where R_i is the unique value satisfying*

$$H_i(R_i) \equiv R_i \frac{k_i^n(R_i/A_i) - S_i}{A_i f_i(k_i^n(R_i/A_i))} = \gamma_i. \tag{A.4}$$

Then the unique equilibrium is determined as follows:

1. *In the regime $\mathcal{R}_m$, i.e., when $S < \sum_{i=1}^m k_i^n(R_m/A_i)$, credit constraint of any agent is not binding. In this case, the equilibrium coincides to that of the economy without credit constraints, and the interest rate is $R = R^* > R_m$. Agent i borrows ($k_i \geq S_i$) if and only if $F_i'(S_i) \geq R^*$.*

2. *In the regime $\mathcal{R}_n$ (with $1 \leq n \leq m-1$), i.e., when*

$$\sum_{i=1}^n k_i^n(\frac{R_n}{A_i}) + \sum_{i=n+1}^m k_i^b(\frac{R_n}{\gamma_i A_i}, S_i) > S \geq \sum_{i=1}^{n+1} k_i^n(\frac{R_{n+1}}{A_i}) + \sum_{i=n+2}^m k_i^b(\frac{R_{n+1}}{\gamma_i A_i}, S_i), *$$

then the equilibrium interest rate is determined by the following equation

$$\sum_{i=1}^n k_i^n(\frac{R}{A_i}) + \sum_{i=n+1}^m k_i^b(\frac{R}{\gamma_i A_i}, S_i) = S \equiv \sum_i S_i \tag{A.5}$$

while agents' capital is

$$k_i = \begin{cases} k_i^n(\frac{R}{A_i}) & \text{if } i \leq n \\ k_i^b(\frac{R}{\gamma_i A_i}, S_i) & \text{if } i \geq n+1. \end{cases}$$

Notice that $R_n < R \leq R_{n+1}$ in this case. Any agent i ($i \geq n+1$) borrows and her credit constraint is binding. The credit constraint of any agent $i \leq n$ is not binding. Moreover, agent i ($i \leq n$) borrows if and only if $F_i'(S_i) \geq R$.

Proof. See Appendix D. □

Proof of Proposition 2. In the case $\gamma_2 < \frac{A_1}{\lambda A_2}\frac{S_1}{S_1+S_2} = \frac{A_1'}{A_2'}\frac{S_1}{S_1+S_2}$ and $\gamma_2 < \frac{A_1}{A_2}\frac{S_1}{S_1+S_2}$, we have that

$$\begin{aligned} Y(A_1', A_2') - Y(A_1, A_2) &= A_1'S_1 + A_2'S_2\frac{A_1'(1-\gamma_2)}{A_1' - \gamma_2 A_2'} - A_1S_1 - A_2S_2\frac{A_1(1-\gamma_2)}{A_1 - \gamma_2 A_2} \\ &= (A_1' - A_1)S_1 + A_2S_2(1-\gamma_2)\frac{A_1A_2' - A_1'A_2}{(A_1 - \gamma_2 A_2)(A_1' - \gamma_2 A_2')} \end{aligned}$$

Point 1. When $\frac{A_2'}{A_2} \geq \frac{A_1'}{A_1} \geq 1$, we have $(A_1' - A_1)S_1 > 0$ and $A_1A_2' - A_1'A_2 \geq 0$. By consequence, we get that $Y(A_1', A_2') - Y(A_1, A_2) > 0$.
Point 2. We can compute that

$$\frac{\partial Y}{\partial A_1} = S_1 - A_2S_2(1-\gamma_2)\frac{\gamma_2 A_2}{(A_1-\gamma_2A_2)^2}$$
$$\frac{\partial Y}{\partial A_2} = \frac{S_2A_1^2(1-\gamma_2)}{(A_1-\gamma_2A_2)^2}$$

So, we have that

$$\begin{aligned}
&\frac{\partial Y}{\partial A_1}(A_1, A_2)(A_1' - A_1) + \frac{\partial Y}{\partial A_1}(A_1, A_2)(A_2' - A_2) \\
=&\Big(S_1 - A_2S_2(1-\gamma_2)\frac{\gamma_2A_2}{(A_1-\gamma_2A_2)^2}\Big)(A_1' - A_1) + \Big(\frac{S_2A_1^2(1-\gamma_2)}{(A_1-\gamma_2A_2)^2}\Big)(A_2' - A_2) \\
=&(A_1' - A_1)\Big(S_1 - A_2S_2(1-\gamma_2)\frac{\gamma_2A_2}{(A_1-\gamma_2A_2)^2} + \frac{S_2A_1^2(1-\gamma_2)}{(A_1-\gamma_2A_2)^2}\frac{A_2'-A_2}{A_1'-A_1}\Big) \\
=&(A_1' - A_1)\Big(\frac{S_2A_1A_2(1-\gamma_2)}{(A_1-\gamma_2A_2)^2}\frac{\frac{A_2'}{A_2}-1}{\frac{A_1'}{A_1}-1} - \Big(S_2(1-\gamma_2)\frac{\gamma_2A_2^2}{(A_1-\gamma_2A_2)^2} - S_1\Big)\Big).
\end{aligned}$$

Therefore, we have

$$\frac{\frac{\partial Y}{\partial A_1}(A_1, A_2)(A_1' - A_1) + \frac{\partial Y}{\partial A_1}(A_1, A_2)(A_2' - A_2)}{A_1' - A_1} < 0$$

if $A_1' \neq A_1$, $S_2(1-\gamma_2)\frac{\gamma_2A_2^2}{(A_1-\gamma_2A_2)^2} - S_1 > 0$ and

$$\frac{\frac{A_2'}{A_2}-1}{\frac{A_1'}{A_1}-1} < \frac{\gamma_2A_2}{A_1} - \frac{S_1(A_1-\gamma_2A_2)^2}{S_2A_1A_2(1-\gamma_2)}.$$

By Taylor's theorem, we get point 2.

□

Proof of Proposition 6. Part 1. Point (a) is a direct consequence of Lemma 14 in Appendix D. Point (b) is a direct consequence of Theorem 3.

Part 2. Since the production functions satisfy Inada's condition, all agents produce in equilibrium. According to (3.6), we have

$$\frac{\partial Y}{\partial A_1} = \underbrace{f_1(k_1)}_{\text{Productivity effect}} + \underbrace{\sum_{i\neq 1}\big(A_1f_1'(k_1) - A_if_i'(k_i)\big)\underbrace{\frac{-\partial k_i}{\partial R}}_{\geq 0}\underbrace{\frac{\partial R}{\partial A_1}}_{\geq 0}}_{\text{Allocation effect}}. \tag{A.6}$$

According to FOCs, we have

$$\begin{aligned}
&[k] : (1+\mu_i\gamma_i)F_i'(k) = \lambda_i \\
&[a] : (1+\mu_i)R = \lambda_i, \quad \mu_i \geq 0, \text{ and } \mu_i(\gamma_iF_i(k_i) - R_ib_i) = 0.
\end{aligned}$$

These equations imply that:

$$\gamma_iA_if_i'(k_i) \leq R = A_if_i'(k_i)\frac{1+\gamma_i\mu_i}{1+\mu_i} \leq A_if_i'(k_i), \forall i. \tag{A.7}$$

This implies that $R \geq \max_j \gamma_jF_j'(k_j) \geq \max_j \gamma_jF_j'(S))$. Thus, $R \geq \max_j \gamma_jF_j'(S)) > 0$, $\forall A_1$.

1. When A_1 is high enough. Note that $\lim_{A_1\to\infty} R_1 = \infty$. Hence, for A_1 high enough, we have that $R_1 > S$. We prove that the equilibrium interest rate goes to infinity when A_1 goes to infinity. Indeed, if agent 1's borrowing constraint is not binding, we have $R = A_1 f_1'(k_1) > A_1 f_1'(S)$. If agent 1's borrowing constraint is binding, we have $R(k_1 - S_1) = \gamma_1 A_1 f_1(k_1)$ which implies that

$$R = \frac{\gamma_1 A_1 f_1(k_1)}{k_1 - S_1} \geq \frac{\gamma_1 A_1 f_1(S_1)}{S - S_1}$$

Hence, $R \geq \min\left(A_1 f_1'(S), \frac{\gamma_1 A_1 f_1(S_1)}{S-S_1}\right)$. From this, we obtain that $\lim_{A_1\to\infty} R = \infty$.

Now, condition $\lim_{A_1\to\infty} R = \infty$ implies that borrowing constraint of any agent $i \geq 2$ is not binding for A_1 high enough. So, $A_1 f_1'(k_1) \geq R = A_i f_i'(k_i)$, $\forall i \geq 1$. By combining this and condition (A.6), we get that $\frac{\partial Y}{\partial A_1} > 0$ for A_1 high enough.

2. We will prove that when A_1 is small enough, the productivity effect is smaller than the allocation effect. To show $\frac{\partial Y}{\partial A_1} < 0$ for A_1 small enough, we will prove that $\lim_{A_1\to 0} k_1 = 0$, $\lim_{A_1\to 0} A_1 f_1'(k_1) - A_2 f_2'(k_2) < 0$, $\lim_{A_1\to 0} \frac{-\partial k_2}{\partial R} > 0$, and $\lim_{A_1\to 0} \frac{\partial R}{\partial A_1} > 0$.

   Since $A_1 f_1'(k_1) \geq R \geq \max_j \gamma_j F_j'(S)) > 0$, we have $\lim_{A_1\to 0} f_1'(k_1) = \infty$. Therefore, we have

$$\lim_{A_1\to 0} k_1 = 0, \text{ and } \lim_{A_1\to 0} \sum_{i\neq 1} k_i = S. \tag{A.8}$$

   Since $\lim_{A_1\to 0} k_1 = 0$, we get that $\gamma_1 A_1 f_1(k_1) - Rb_1 = \gamma_1 A_1 f_1(k_1) - Rk_1 + RS_1 > 0$ for A_1 small enough. It means that the borrowing constraint of agent 1 is not binding. To sum up, we have

$$R = A_1 f_1'(k_1) \geq \max_j \gamma_j F_j'(S)) > 0, \text{ for } A_1 \text{ small enough.}$$

   Denote

$$B_1 = B_1(R_1) \equiv k_1^n(R_1) + \sum_{i=2}^{m} k_i^b(R_1), \quad B_2 = B_2(R_2) = \sum_{i=1}^{2} k_i^n(R_2) + \sum_{i=3}^{m} k_i^b(R_2)$$
$$B_m = B_m(R_m) = \sum_{i=1}^{m} k_i^n(R_m)$$

   where, to simplify notations, we write $k_i^n(R)$ and $k_i^b(R)$ instead of $k_i^n(\frac{R}{A_i})$ and $k_i^b(\frac{R}{\gamma_i A_i}, S_i)$ (see Definition 6). We see that $D_i \equiv B_i - k_i^n(R_i)$, $\forall i$. Notice also that $B_1, \ldots, B_m$ depend on A_1 but $D_2, D_3, \ldots, D_m$ do not. Moreover, $\lim_{A_1\to 0}(B_i - D_i) = 0$, $\forall i \geq 2$ because $\lim_{A_1\to 0} k_1^n(R_i) = 0$, $\forall i \geq 2$.

   Condition $R_2 < R_3 < \cdots < R_m$ implies that $D_2 > \cdots > D_m$. Since $\lim_{A_1\to 0} B_1 = +\infty$ and $\lim_{A_1\to 0}(B_i - D_i) = 0$, $\forall i \geq 2$, we have $B_1 > B_2 > \cdots > B_m$ for A_1 small enough.

   (a) $S < D_m$. Then we have $S < B_m$. According to Theorem 3, the equilibrium coincides to that of the economy without frictions. Therefore, the output is increasing in A_1.

(b) Let $D_n > S > D_{n+1}$. In this case, we have $B_n > S > B_{n+1}$ for any A_1 small enough. According to Theorem 3, the equilibrium interest rate R is in the interval $(R_n, R_{n+1}]$ and determined by

$$\sum_{i=1}^{n} k_i^n(R) + \sum_{i=n+1}^{m} k_i^b(R) = S \equiv \sum_i S_i \tag{A.9}$$

Denote $Z_2(R) = \sum_{i=2}^{n} k_i^n(R) + \sum_{i=n+1}^{m} k_i^b(R)$. When A_1 tends to zero, we have $\lim_{A_1 \to 0} k_1^n(R) = 0$ and $\lim_{A_1 \to 0} R = R(0)$ where $R(0) > 0$ is uniquely determined by $Z_2(R(0)) = S$.

For $i \geq n+1$, agent i's borrowing constraint is binding: $R(k_i - S_i) = \gamma_i A_i f_i(k_i)$ for any A_1 small enough. Let A_1 tend to zero, we have k_i tends to $k_i(0)$, R tends to $R(0)$, and

$$\gamma_i A_i f_i(k_i(0)) = R(0)(k_i(0) - S_i).$$

Let σ be such that

$$\gamma_i \frac{f_i(k)}{k f_i'(k)} < \sigma < \frac{S_1}{S_1 + S_{n+1} + \cdots + S_m}, \forall i \geq n+1, \forall k \in (0, S). \tag{A.10}$$

According to condition (3.9b), we have

$$R(0) - A_i f_i'(k_i(0)) = \frac{\gamma_i A_i f_i(k_i(0))}{k_i(0) - S_i} - A_i f_i'(k_i(0)) \tag{A.11}$$

$$\leq \frac{A_i f_i'(k_i(0))}{k_i(0) - S_i}\big(\sigma k_i(0) - (k_i(0) - S_i)\big) \tag{A.12}$$

By market clearing condition, we have

$$\sum_{i=n+1}^{m} k_i = \sum_{i=2}^{m}(S_i - k_i) + S_1 - k_1 + \sum_{i=n+1}^{m} S_i \geq S_1 - k_1 + \sum_{i=n+1}^{m} S_i$$

Let A_1 tend to zero, we get that $\sum_{i=n+1}^{m} k_i(0) \geq S_1 + \sum_{i=n+1}^{m} S_i$. Thus,

$$\sum_{i=n+1}^{m} \Big(\sigma k_i(0) - (k_i(0) - S_i)\Big) = \sum_{i=n+1}^{m} \Big(S_i - (1-\sigma)k_i(0)\Big)$$

$$\leq \sum_{i=n+1}^{m} S_i - (1-\sigma)\big(S_1 + \sum_{i=n+1}^{m} S_i\big) < 0$$

Therefore, there exists $j \in \{n+1, \ldots, m\}$ such that $\sigma k_j(0) - (k_j(0) - S_j) < 0$, and hence

$$R(0) - A_j f_j'(k_j(0)) \leq \frac{A_j f_j'(k_j(0))}{k_j(0) - S_j}\big(\sigma k_j(0) - (k_j(0) - S_j)\big) < 0. \tag{A.13}$$

Now, by noting that $A_1 f'(k_1) = R$, we have

$$\frac{\partial Y}{\partial A_1} \leq f_1(k_1) + \big(R - A_j f_j'(k_j)\big)\frac{-\partial k_j}{\partial R}\frac{\partial R}{\partial A_1} \tag{A.14}$$

Again, by the market clearing condition

$$k_1^n(\frac{R}{A_1}) + \sum_{i\neq 2} k_i(R) = S \tag{A.15}$$

we have that

$$(k_1^n)'(\frac{R}{A_1})\frac{R'(A_1)A_1 - R}{A_1^2} + \sum_{i\neq 2}\frac{\partial k_i}{\partial R}R'(A_1) = 0 \tag{A.16}$$

$$\Leftrightarrow R'(A_1)\Big(\frac{1}{A_1}(k_1^n)'(\frac{R}{A_1}) + \sum_{i\neq 2}\frac{\partial k_i}{\partial R}\Big) = (k_1^n)'(\frac{R}{A_1})\frac{R}{A_1^2}$$

$$\Leftrightarrow R'(A_1)A_1\Big(\frac{1}{R} + \frac{A_1}{R}\frac{\sum_{i\neq 2}\frac{\partial k_i}{\partial R}}{(k_1^n)'(\frac{R}{A_1})}\Big) = 1 \tag{A.17}$$

Since $\frac{\partial k_i}{\partial R} < 0$, $\forall i \neq 1$, and $(k_1^n)'\big(\frac{R}{A_1}\big) < 0$, we have $R'(A_1) > 0$.
By definition of k_1^n, we have $f_1'(k_1^n(x)) = x$. So, $(k_1^n)'(x)f_1''(k_1^n(x)) = 1$, and hence,

$$\lim_{A_1\to 0}\frac{R}{A_1}(k_1^n)'(\frac{R}{A_1}) = \lim_{A_1\to 0}\frac{\frac{R}{A_1}}{f_1''(\frac{R}{A_1})} = \lim_{x\to\infty}\frac{x}{f_1''(x)} < 0.$$

By combining this with (A.17), $\lim_{R\to R(0)}\frac{\partial k_i}{\partial R} < 0$, $\forall i$, and $\lim_{A_1\to 0} R = R(0) > 0$, we get that

$$\lim_{A_1\to 0} R'(A_1) = +\infty. \tag{A.18}$$

By combining (A.14), (A.13), (A.18), and $\lim_{R\to R(0)}\frac{\partial k_j}{\partial R} < 0$, we get that $\frac{\partial Y}{\partial A_1} < 0$ for any $A_1 > 0$ small enough.

□

A.2 Additional results

In the case of a two-agent model, we have the following result with more details and intuitive conditions.

Proposition 19. *Consider a two-agent model.*

1. *Let Assumptions 2, 3 and 9 be satisfied. Assume also that*

$$k_2^n(\frac{R_2}{A_2}) < S, \quad \gamma_2 < \frac{S_1}{S_1+S_2}\frac{Sf_2'(S)}{f_2(S)}, \quad \lim_{x\to+\infty}\frac{x}{f_1''(x)} < 0$$

   Then, for any A_1 small enough, we have that $\frac{\partial Y}{\partial A_1} < 0$.

2. *By consequence, in a two-agent economy with Cobb-Douglas production functions ($F_i(k) = A_i k^\alpha$) and $\gamma_2 < \alpha\frac{S_1}{S_1+S_2}$, we have that: $\frac{\partial Y}{\partial A_1} < 0$ for A_1 small enough.*

Proof of Proposition 19. First, we state a corollary of Theorem 3.

Corollary 3. *Let Assumptions 2, 3 and 9 be satisfied. Consider a two-agent model and assume that $R_1 < R_2$.*

1. *In the regime $\mathcal{R}_2$, i.e., when $S < k_1^n(\frac{R_2}{A_1}) + k_2^n(\frac{R_2}{A_2})$, credit constraint of any agent is not binding.*

2. *In the regime $\mathcal{R}_1$, i.e., when $S \geq k_1^n(\frac{R_2}{A_1}) + k_2^n(\frac{R_2}{A_2})$,[29] the equilibrium interest rate R is determined by*

$$k_1^n(\frac{R}{A_1}) + k_2^b(\frac{R}{\gamma_2 A_2}, S_2) = S \equiv \sum_i S_i. \tag{A.19}$$

In this regime, $R_1 < R \leq R_2$, agent 2 borrows and her credit constraint is binding while agent 1 is lender.

Now, we prove part 1 of Proof of Proposition 19. Since Inada condition holds, all agents produce in equilibrium. According to (3.6), we have

$$\frac{\partial Y}{\partial A_1} = f_1(k_1) + \big(A_1 f_1'(k_1) - A_2 f_2'(k_2)\big) \underbrace{\frac{-\partial k_2}{\partial R}}_{>0} \underbrace{\frac{\partial R}{\partial A_1}}_{>0}. \tag{A.20}$$

To show $\frac{\partial Y}{\partial A_1} < 0$ for A_1 small enough, we will prove that $\lim_{A_1 \to 0} k_1 = 0$, $\lim_{A_1 \to 0} A_1 f_1'(k_1) - A_2 f_2'(k_2) < 0$, $\lim_{A_1 \to 0} \frac{-\partial k_2}{\partial R} > 0$, and $\lim_{A_1 \to 0} \frac{\partial R}{\partial A_1} > 0$.

According to FOCs, we have

$$\begin{aligned}
&[k] : (1 + \mu_i \gamma_i) F_i'(k) = \lambda_i \\
&[a] : (1 + \mu_i) R = \lambda_i, \quad \mu_i \geq 0, \text{ and } \mu_i(\gamma_i F_i(k_i) - R_i b_i) = 0.
\end{aligned}$$

These equations imply that:

$$\gamma_i A_i f_i'(k_i) \leq R = A_i f_i'(k_i) \frac{1 + \gamma_i \mu_i}{1 + \mu_i} \leq A_i f_i'(k_i), \forall i. \tag{A.21}$$

This implies that $R \geq \gamma_2 F_2'(k_2) \geq \gamma_2 F_2'(S)$. Thus, $R \geq \gamma_2 F_2'(S)$, $\forall A_1$. Since $R \leq A_1 f_1'(k_1)$. So, we have $\lim_{A_1 \to 0} f_1'(k_1) = \infty$. Therefore, we have

$$\lim_{A_1 \to 0} k_1 = 0, \text{ and } \lim_{A_1 \to 0} k_2 = S. \tag{A.22}$$

By consequence, we get that $\gamma_1 A_1 f_1(k_1) - R b_1 = \gamma_1 A_1 f_1(k_1) - R k_1 + R S_1 > 0$ for A_1 small enough. It means that the borrowing constraint of agent 1 is not binding. To sum up, we have $R = A_1 f_1'(k_1) \geq \gamma_2 F_2'(S)$ for A_1 small enough.

Since R_2 does not depend on A_1, we observe that $\lim_{A_1 \to 0} k_1^n(R_2/A_1) = 0$. So, by combining with the assumption $k_2^n(\frac{R_2}{A_2}) < S$, we have $k_1^n(\frac{R}{A_1}) + k_2^n(\frac{R}{A_2}) < S$ for A_1 small enough. According to point 3 of Lemma 15, we have $R_1 < R_2$ for A_1 small enough. Hence, we can apply Corollary 3 to obtain that the borrowing constraint of agent 2 is binding in equilibrium. It means that $\gamma_2 A_2 f_2(k_2) - R k_2 + R S_2 = 0$.

Look at the market clearing condition: $k_1^n(\frac{R}{A_1}) + k_2^b(\frac{R}{\gamma_2 A_2}, S_2) = S \equiv \sum_i S_i$. When A_1 converges to 0, we have $k_1^n(\frac{R}{A_1})$ converges to 0. So, R converges to $R(0)$ satisfying $k_2^b(\frac{R(0)}{\gamma_2 A_2}, S_2) = S$. So, we have

$$\lim_{A_1 \to \infty} \big(A_1 f_1'(k_1) - A_2 f_2'(k_2)\big) = \lim_{A_1 \to \infty} \big(R - A_2 f_2'(k_2)\big) = R(0) - A_2 f_2'(S). \tag{A.23}$$

[29] Notice that we always have that $k_1^n(R_1) = k_1^b(R_1)$, $k_2^n(R_2) = k_2^b(R_2)$, and $k_1^n(R_1) + k_2^b(R_1) = k_1^b(R_1) + k_2^b(R_1) > S$.

Since agent 2's borrowing constraint is binding: $R(k_2 - S_2) = \gamma_2 A_2 f_2(k_2)$ for any A_1 small enough. Let A_1 tend to zero, we have $\gamma_2 A_2 f_2(S) = R(0)(S - S_2) = R(0)S_1$. So, we have

$$R(0) - A_2 f_2'(S) = \frac{\gamma_2 A_2 f_2(S)}{S_1} - A_2 f_2'(S) < 0 \tag{A.24}$$

because we assume that $\gamma_2 < \frac{S_1}{S_1+S_2}\frac{Sf_2'(S)}{f_2(S)}$.

Again, by the market clearing condition $k_1^n(\frac{R}{A_1}) + k_2^b(\frac{R}{\gamma_2 A_2}, S_2) = S$, we have

$$(k_1^n)'(\frac{R}{A_1})\frac{R'(A_1)A_1 - R}{A_1^2} + \frac{\partial k_2^b}{\partial x_1}(\frac{R}{\gamma_2 A_2}, S_2)\frac{R'(A_1)}{(\gamma_2 A_2)^2} = 0$$

$$\Leftrightarrow R'(A_1)\Big(\frac{1}{A_1}(k_1^n)'(\frac{R}{A_1}) + \frac{1}{(\gamma_2 A_2)^2}\frac{\partial k_2^b}{\partial x_1}(\frac{R}{\gamma_2 A_2}, S_2)\Big) = (k_1^n)'(\frac{R}{A_1})\frac{R}{A_1^2}$$

$$\Leftrightarrow R'(A_1)A_1\Big(\frac{1}{R} + \frac{A_1}{(\gamma_2 A_2)^2 R}\frac{\frac{\partial k_2^b}{\partial x_1}(\frac{R}{\gamma_2 A_2}, S_2)}{(k_1^n)'(\frac{R}{A_1})}\Big) = 1 \tag{A.25}$$

First, since $\frac{\partial k_2^b}{\partial x_1} < 0$ and $(k_1^n)'(\frac{R}{A_1}) < 0$, we have $R'(A_1) > 0$.

Recall that $f_1'(k_1^n(x)) = x$. So, we have $(k_1^n)'(x)f_1''(k_1^n(x)) = 1$, and hence,

$$\lim_{A_1\to 0}\frac{R}{A_1}(k_1^n)'(\frac{R}{A_1}) = \lim_{A_1\to 0}\frac{\frac{R}{A_1}}{f_1''(\frac{R}{A_1})} = \lim_{x\to+\infty}\frac{x}{f_1''(x)} < 0.$$

By combining this with (A.25) and $\lim_{A_1\to 0} R = R(0) > 0$, we get that

$$\lim_{A_1\to 0} R'(A_1) = +\infty. \tag{A.26}$$

It is easy to see that, when A_1 is small enough, the $\frac{\partial k_2}{\partial R} = \frac{\partial k_2^b}{\partial R}(\frac{R}{\gamma_2 A_2}, S_2)$. Thus,

$$\lim_{A_1\to 0}\frac{\partial k_2}{\partial R} = \lim_{R\to R(0)}\frac{\partial k_2^b}{\partial R}(\frac{R}{\gamma_2 A_2}, S_2) < 0. \tag{A.27}$$

By combining (A.22), (A.24), (A.26), (A.27) and (A.20), we conclude that $\frac{\partial Y}{\partial A_1} < 0$ for any $A_1 > 0$ small enough.

We now consider the Cobb-Douglas production functions. In such a case, condition $k_2^n(\frac{R_2}{A_2}) < S$ becomes $\gamma_2 < \alpha\frac{S_1}{S_1+S_2}$. For the sake of simplicity, we write k_2^n instead of $k_2^n(\frac{R_2}{A_2})$. Recall that $R_2 = A_2 f_2'(k_2^n) = A_2\alpha(k_2^n)^{\alpha-1}$. Hence,

$$\begin{aligned}&(k_2^n - S_2)R_2 = \gamma_2 A_2 f_2(k_2^n) \Leftrightarrow (k_2^n - S_2)R_2 = \gamma_2 A_2 (k_2^n)^\alpha\\ &\Leftrightarrow (k_2^n - S_2)A_2\alpha(k_2^n)^{\alpha-1} = \gamma_2 A_2(k_2^n)^\alpha\\ &\Leftrightarrow (\alpha-\gamma_2)A_2(k_2^n)^\alpha = S_2 A_2\alpha(k_2^n)^{\alpha-1} \Leftrightarrow (\alpha-\gamma_2)k_2^n = \alpha S_2.\end{aligned}$$

Therefore, condition $S_1 + S_2 > k_2^n(R_2/A_2)$ becomes $(S_1+S_2)(\alpha-\gamma_2) > \alpha S_2$, or, equivalently, $\alpha\frac{S_1}{S_1+S_2} > \gamma_2$. □

Proof of Proposition 5. We make use of Theorem 2. We firstly consider the regime $\mathcal{R}_n$ with $n \le m-1$. In this regime, we have

$$Y = Y_n = \sum_{i=n+1}^{m}\frac{rA_iS_i}{r - f_iA_i} \le \sum_{i=n+1}^{m} A_m\frac{rS_i}{r - f_iA_i} = A_mS. \tag{A.28}$$

Notice that $Y = A_m S$ if and only if $n + 1 = m$.

We now consider the regime $\mathcal{A}_n$ with $n \leq m$. In this regime, we have

$$\begin{aligned} Y &= A_n \sum_{i=1}^{n} S_i + \sum_{i=n+1}^{m} \frac{A_n(1-f_i)A_i S_i}{A_n - f_i A_i} = A_n \sum_{i=1}^{m} S_i + A_n \sum_{i=n+1}^{m} \frac{A_n(A_i - A_n)}{A_n - f_i A_i} \\ &\leq A_n S + (A_m - A_n) \sum_{i=n+1}^{m} \frac{A_n}{A_n - f_i A_i} \leq A_n S + (A_m - A_n) S = A_m S. \end{aligned}$$

where the last inequality is from the condition $\sum_{i=n+1}^{m} \frac{A_n S_i}{A_n - f_i A_i}$ in the regime $\mathcal{A}_n$.

It is easy to see that $Y = A_m S$ if and only if either (i) $n+1 > m$ or (ii) $n+1 = m$ and $\frac{A_{m-1}}{A_{m-1} - f_m A_m} S_m = S$. Combining these two cases, we obtain point 1 of our result.

□

B Proofs for Section 3.2

Proof of Proposition 10. Under assumptions in Proposition 10, we can prove that the equilibrium interest rate is in (A_{n-1}, A_n) if and only if

$$\sum_{i=n}^{m} \frac{A_n S_i}{A_n - \gamma_i A_i} < \sum_{i} S_i < \sum_{i=n}^{m} \frac{A_{n-1} S_i}{A_{n-1} - \gamma_i A_i} \tag{B.1}$$

Then, when $R \in (A_{n-1}, A_n)$, it is determined by

$$\underbrace{\sum_{i=n}^{m} \frac{\gamma_i A_i}{R - \gamma_i A_i} S_i}_{\text{Asset demand}} = \underbrace{\sum_{i=1}^{n-1} S_i}_{\text{Asset supply}} \quad \text{or equivalently} \quad \underbrace{\sum_{i=n}^{m} \frac{R S_i}{R - \gamma_i A_i}}_{\text{Capital demand}} = \underbrace{S}_{\text{Capital supply}} \tag{B.2}$$

Agents $1, \ldots, n-1$ are lenders while agents $n, \ldots, m$ are borrowers. It is easy to see that $\frac{\partial Y}{\partial \gamma_i} = 0, \forall i \leq n-1$. For $i \geq n$, by using condition $\sum_{i=n}^{m} \dfrac{R S_i}{R - \gamma_i A_i} = \sum_i S$, we get that

$$\frac{\partial R}{\partial \gamma_j} = \frac{\frac{R A_j S_j}{(R - \gamma_j A_j)^2}}{\left(\sum_{i=n}^{m} \frac{\gamma_i A_i S_i}{(R - \gamma_i A_i)^2}\right)} > 0, \text{ and notice that } \sum_{j} \frac{\partial R}{\partial \gamma_j} \frac{\gamma_j}{R} = 1 \tag{B.3}$$

Then, we can compute that

$$\begin{aligned} \frac{\partial Y}{\partial \gamma_i} &= \sum_{j=n}^{m} A_j S_j \frac{\partial \left(\frac{R}{R - \gamma_j A_j} \right)}{\partial \gamma_i} = \sum_{j=n}^{m} A_j S_j \frac{-\gamma_j A_j}{(R - \gamma_j A_j)^2} \frac{\partial R}{\partial \gamma_i} + \frac{R S_i A_i^2}{(R - \gamma_i A_i)^2} \\ &= \sum_{j=n}^{m} A_j S_j \frac{-\gamma_j A_j}{(R - \gamma_j A_j)^2} \frac{\frac{R A_i S_i}{(R - \gamma_i A_i)^2}}{\left(\sum_{j=n}^{m} \frac{\gamma_j A_j S_j}{(R - \gamma_j A_j)^2}\right)} + \frac{R S_i A_i^2}{(R - \gamma_i A_i)^2} \\ &= \frac{\partial R}{\partial \gamma_i} \left(A_i \sum_{j=n}^{m} \frac{\gamma_j A_j S_j}{(R - \gamma_j A_j)^2} - \sum_{j=n}^{m} \frac{\gamma_j S_j A_j^2}{(R - \gamma_j A_j)^2} \right). \end{aligned}$$

The first point is a direct consequence of this expression and the fact that $A_m > \cdots > A_{n+1}$. Let us prove the second point. We have, by noticing that $R \in (A_{n-1}, A_n)$ and $A_{t+1} > A_t$, $\forall t$,

$$A_i \sum_{t=n}^{m} \frac{\gamma_t A_t S_t}{(R-\gamma_t A_t)^2} - \sum_{t=n}^{m} \frac{\gamma_t S_t A_t^2}{(R-\gamma_t A_t)^2} \tag{B.4}$$

$$= \sum_{t=n}^{i-1} \frac{\gamma_t A_t S_t}{(R-\gamma_t A_t)^2}(A_i - A_t) - \sum_{t=i+1}^{m} \frac{\gamma_t A_t S_t}{(R-\gamma_t A_t)^2}(A_t - A_i) \tag{B.5}$$

$$\geq \sum_{t=n}^{i-1} \frac{\gamma_t A_t S_t}{(A_n-\gamma_t A_t)^2}(A_i - A_{i-1}) - \sum_{t=i+1}^{m} \frac{\gamma_t A_t S_t}{(A_{n-1}-\gamma_t A_t)^2}(A_m - A_i). \tag{B.6}$$

Combining this with the expression of $\frac{\partial Y}{\partial \gamma_i}$, we obtain (4.29a).

We also have

$$A_i \sum_{t=n}^{m} \frac{\gamma_t A_t S_t}{(R-\gamma_t A_t)^2} - \sum_{t=n}^{m} \frac{\gamma_t S_t A_t^2}{(R-\gamma_t A_t)^2} \tag{B.7}$$

$$= \sum_{t=n}^{i-1} \frac{\gamma_t A_t S_t}{(R-\gamma_t A_t)^2}(A_i - A_t) - \sum_{t=i+1}^{m} \frac{\gamma_t A_t S_t}{(R-\gamma_t A_t)^2}(A_t - A_i) \tag{B.8}$$

$$< \sum_{t=n}^{i-1} \frac{\gamma_t A_t S_t}{(A_{n-1}-\gamma_t A_t)^2}(A_i - A_n) - \sum_{t=i+1}^{m} \frac{\gamma_t A_t S_t}{(A_n-\gamma_t A_t)^2}(A_{i+1} - A_i). \tag{B.9}$$

Combining this with the expression of $\frac{\partial Y}{\partial \gamma_i}$, we obtain (4.29b). □

Proof of Example 1. We focus here on the case $max(\gamma_2 A_2, \gamma_3 A_3) < A_1$ (in this case the interest rate R may take any value in $[A_1, A_m]$). Applying Theorem 2, we can check that the interest rate is uniquely determined by

$$R = \begin{cases} A_1 \text{ if } S_1 \geq \frac{\gamma_3 A_3}{A_1-\gamma_3 A_3} S_3 + \frac{\gamma_2 A_2}{A_1 - \gamma_2 A_2} S_2 \\ R_1 \text{ if } \frac{\gamma_3 A_3}{A_2-\gamma_3 A_3} S_3 + \frac{\gamma_2}{1-\gamma_2} S_2 < S_1 < \frac{\gamma_3 A_3}{A_1-\gamma_3 A_3} S_3 + \frac{\gamma_2 A_2}{A_1 - \gamma_2 A_2} S_2 \\ A_2 \text{ if } \frac{\gamma_3 A_3}{A_2-\gamma_3 A_3} S_3 - S_2 \leq S_1 \leq \frac{\gamma_3 A_3}{A_2-\gamma_3 A_3} S_3 + \frac{\gamma_2}{1-\gamma_2} S_2 \\ R_2 \text{ if } \frac{\gamma_3}{1-\gamma_3} S_3 - S_2 < S_1 < \frac{\gamma_3 A_3}{A_2-\gamma_3 A_3} - S_2 \\ A_3 \text{ if } S_1 \leq \frac{\gamma_3}{1-\gamma_3} S_3 - S_2 \end{cases} \tag{B.10}$$

where $R_2 = \gamma_3 A_3 \Big(1 + \frac{S_3}{S_1+S_2}\Big)$ and R_1 is the highest solution of the equation:

$$\frac{\gamma_2 A_2}{R - \gamma_2 A_2} S_2 + \frac{\gamma_3 A_3}{R - \gamma_3 A_3} S_3 = S_1. \tag{B.11}$$

This equation implies that $R(S_2(R - \gamma_3 A_3) + S_3(R - \gamma_2 A_2)) = S(R - \gamma_2 A_2)(R - \gamma_3 A_3)$, or equivalently

$$S_1 R^2 - R\Big((S_1 + S_2)\gamma_2 A_2 + (S_1 + S_3)\gamma_3 A_3\Big) + S\gamma_2 A_2 \gamma_3 A_3 = 0. \tag{B.12a}$$

So, the rate R_1 is computed by

$$R = \frac{(S_1 + S_2)\gamma_2 A_2 + (S_1 + S_3)\gamma_3 A_3 + \sqrt{\Delta}}{2S_1} \tag{B.12b}$$

$$\text{where } \Delta \equiv \big((S_1 + S_2)\gamma_2 A_2 + (S_1 + S_3)\gamma_3 A_3\big)^2 - 4S_1 S\gamma_2 A_2 \gamma_3 A_3 \tag{B.12c}$$

There are 5 different cases. In each case, we can explicitly compute equilibrium outcomes thanks to Lemma 7.

□

Proof of Proposition 11 (homogeneous credit limit). Since $F_i'(k_t) \geq R$, there are two cases. (1) If $F_i'(k_i) = R$, then we have hence $\frac{\partial k_i}{\partial \gamma} < 0$. (2) If $F_i'(k_i) > R$, then borrowing constraint of this agent is binding.

The market clearing condition $\sum_i k_i = \sum_i S_i$ implies that

$$\sum_{i:F_i'(k_i)=R} \frac{\partial k_i}{\partial \gamma} + \sum_{i:F_i'(k_i)>R} \frac{\partial k_i}{\partial \gamma} = 0.$$

So, we have $\sum_{i:F_i'(k_i)>R} \frac{\partial k_i}{\partial \gamma} > 0$.

We now claim that $\frac{\partial k_i}{\partial \gamma} > 0$ for any agent with $F_i'(k_i) > R$. For such agents we have $\gamma F_i(k_i^n) - R(k_i^n - S_i)$. Taking the derivative with respect to γ of both sides of this equation, we have

$$F_i(k_i) + \gamma F_i'(k_i)\frac{\partial k_i}{\partial \gamma} = \frac{\partial R}{\partial \gamma}(k_i - S_i) + R\frac{\partial k_i}{\partial \gamma} \tag{B.13}$$

$$i.e., \quad \frac{\partial k_i}{\partial \gamma} = \Big(\frac{\partial R}{\partial \gamma}\frac{\gamma}{R} - 1\Big)\frac{F_i(k_i)}{R - \gamma F_i'(k_i)}. \tag{B.14}$$

By summing with respect to i such that $F_i'(k_i) > R$ and noticing that $\sum_{i:F_i'(k_i)>R} \frac{\partial k_i}{\partial \gamma} > 0$ and $R - \gamma F_i'(k_i) > 0$ $\forall i$, we get that $\frac{\partial R}{\partial \gamma}\frac{\gamma}{R} - 1 \geq 0$. From this and (B.14), we obtain $\frac{\partial k_i}{\partial \gamma} > 0$ $\forall i$ such that $F_i'(k_i) > R$.

We now observe that

$$\frac{\partial Y}{\partial \gamma} = \sum_{i:F_i'(k_i)=R} F_j'(k_i)\frac{\partial k_i}{\partial \gamma} + \sum_{i:F_i'(k_i)>R} F_i'(k_i)\frac{\partial k_i}{\partial \gamma} \geq R\Big(\sum_{i=1}^{m} \frac{\partial k_i}{\partial \gamma}\Big) = 0. \tag{B.15}$$

□

C Proofs of Section 4

Firstly, we provide a sufficient condition to check whether a sequence of prices and allocations is an intertemporal equilibrium.

Lemma 5. *If the sequences $(R_t, (c_{i,t}, k_{i,t}, b_{i,t})_i)_t$ and $(\lambda_{i,t}, \mu_{i,t}, \eta_{i,t})_{i,t}$ satisfy the following conditions:*

1. *$c_{i,t}, l_{i,t}, \lambda_{i,t}, \eta_{i,t}, \mu_{i,t+1}$ are non-negative and $R_t > 0$ for any t.*
2. *$c_{i,t} + k_{i,t} + R_t b_{i,t-1} = F_{i,t}(k_{i,t-1}) + b_{i,t}$, and $R_{t+1}b_{i,t} - \gamma_i F_{i,t}(k_{i,t}) = 0$, $\forall i, \forall t$.*
3. *$\sum_i b_{i,t} = 0$, $\forall t$.*
4. *$\sum\limits_{t=0}^{\infty} \lambda_{i,t}c_{i,t} < \infty$, $\sum\limits_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) < \infty$.*
5. *TVCs: $\lim_{T\to\infty} \beta_i^t u_i'(c_{i,t})(k_{i,t} - b_{i,t}) = 0$, $\forall i$.*

6. *FOCs:* $\forall i, \forall t$,

$$\beta_i^t u_i'(c_{i,t}) = \lambda_{i,t}$$
$$\lambda_{i,t} = \lambda_{i,t+1} F_{i,t+1}'(k_{i,t}) + \mu_{i,t+1}\gamma_i F_{i,t+1}'(k_{i,t}) + \eta_{i,t}, \quad \eta_{i,t}k_{i,t} = 0$$
$$\lambda_{i,t} = R_{t+1}\lambda_{i,t+1} + \mu_{i,t+1}R_{t+1}, \quad \mu_{i,t+1}\big(R_{t+1}b_{i,t} - \gamma_i F_{i,t}(k_{i,t})\big) = 0,$$

then the list $(R_t, (c_{i,t}, k_{i,t}, b_{i,t})_i)$ *is an intertemporal equilibrium.*

Proof of Lemma 5. Before presenting our proof, we should notice that this result requires neither $u_i(0) = 0$ nor $u_i'(0) = \infty$. Let us now prove our result. It is sufficient to prove the optimality of (c_i, k_i, b_i) for all i. Let (c_i', k_i', b_i') be a plan satisfying all budget and borrowing constraints and $b_{i,-1}' - b_{i,-1} = 0 = k_{i,-1}' - k_{i,-1}$. We have $\sum_{t=0}^{T} \beta_i^t (u_i(c_{i,t}) - u_i(c_{i,t}')) \geq \sum_{t=0}^{T} \beta_i^t u_i'(c_{i,t})(c_{i,t} - c_{i,t}') = \sum_{t=0}^{T} \lambda_{i,t}(c_{i,t} - c_{i,t}')$.

Budget constraints imply that $c_{i,t} = F_{i,t}(k_{i,t-1}) + b_{i,t} - k_{i,t} - R_t b_{i,t-1}$ and $c_{i,t}' \leq F_{i,t}(k_{i,t-1}') + b_{i,t}' - k_{i,t}' - R_t b_{i,t-1}'$, and hence,

$$\begin{aligned}\lambda_{i,t}(c_{i,t} - c_{i,t}') \geq& \lambda_{i,t}(F_{i,t}(k_{i,t-1}) + b_{i,t} - k_{i,t} - R_t b_{i,t-1} - F_{i,t}(k_{i,t-1}') - b_{i,t}' + k_{i,t}' + R_t b_{i,t-1}') \\ =& \lambda_{i,t}\big(F_{i,t}(k_{i,t-1}) - F_{i,t}(k_{i,t-1}')\big) - \lambda_{i,t}(k_{i,t} - k_{i,t}') + \lambda_{i,t}(b_{i,t} - b_{i,t}') - \lambda_{i,t}R_t(b_{i,t-1} - b_{i,t-1}').\end{aligned}$$

According to FOCs, we have

$$\lambda_{i,t}k_{i,t}' = \lambda_{i,t+1}F_{i,t+1}'(k_{i,t})k_{i,t}' + \gamma_i\mu_{i,t+1}F_{i,t+1}'(k_{i,t})k_{i,t}' + \eta_{i,t}k_{i,t}'$$
$$\lambda_{i,t}b_{i,t}' = R_{t+1}\lambda_{i,t+1}b_{i,t}' + R_{t+1}\mu_{i,t+1}b_{i,t}'$$

This implies that

$$\lambda_{i,t}(k_{i,t} - k_{i,t}') = \lambda_{i,t+1}F_{i,t+1}'(k_{i,t})(k_{i,t} - k_{i,t}') + \gamma_i\mu_{i,t+1}F_{i,t+1}'(k_{i,t})(k_{i,t} - k_{i,t}') + \eta_{i,t}(k_{i,t} - k_{i,t}') \tag{B.1}$$

$$\lambda_{i,t}(b_{i,t} - b_{i,t}') = R_{t+1}\lambda_{i,t+1}(b_{i,t} - b_{i,t}') + R_{t+1}\mu_{i,t+1}(b_{i,t} - b_{i,t}') \tag{B.2}$$

Therefore, we have that

$$\begin{aligned}\sum_{t=0}^{T}\lambda_{i,t}(c_{i,t} - c_{i,t}') \geq& \sum_{t=0}^{T}\Big(\lambda_{i,t}\big(F_{i,t}(k_{i,t-1}) - F_{i,t}(k_{i,t-1}')\big) - \lambda_{i,t}(k_{i,t} - k_{i,t}')\Big) \\ &+ \sum_{t=0}^{T}\Big(\lambda_{i,t}(b_{i,t} - b_{i,t}') - \lambda_{i,t}R_t(b_{i,t-1} - b_{i,t-1}')\Big) \\ \geq& \sum_{t=0}^{T-1}\big(\lambda_{i,t+1}F_{i,t+1}'(k_{i,t}) - \lambda_{i,t}\big)(k_{i,t} - k_{i,t}') - \lambda_{i,T}(k_{i,T} - k_{i,T}') \\ &+ \sum_{t=0}^{T-1}\big(\lambda_{i,t} - \lambda_{i,t+1}R_{t+1}\big)(b_{i,t} - b_{i,t}') + \lambda_{i,T}(b_{i,T} - b_{i,T}') \\ =& \lambda_{i,T}\big(k_{i,T}' - b_{i,T}' - (k_{i,T} - b_{i,T})\big) + \sum_{t=0}^{T-1}\eta_{i,t}(k_{i,t}' - k_{i,t}) \\ &+ \sum_{t=0}^{T-1}\mu_{i,t+1}\Big(-\gamma_i F_{i,t+1}'(k_{i,t})(k_{i,t} - k_{i,t}') + R_{t+1}(b_{i,t} - b_{i,t}')\Big)\end{aligned}$$

We consider $\mu_{i,t+1}\Big(- \gamma_i F'_{i,t+1}(k_{i,t})(k_{i,t}-k'_{i,t}) + R_{t+1}(b_{i,t}-b'_{i,t})\Big)$.

$$\begin{aligned}
&\mu_{i,t+1}\Big(- \gamma_i F'_{i,t+1}(k_{i,t})(k_{i,t}-k'_{i,t}) + R_{t+1}(b_{i,t}-b'_{i,t})\Big)\\
=&\mu_{i,t+1}(R_{t+1}b_{i,t}-\gamma_i F_{i,t}(k_{i,t}) - (R_{t+1}b'_{i,t}-\gamma_i F_{i,t}(k'_{i,t})))\\
&+\mu_{i,t+1}\Big(- (R_{t+1}b_{i,t}-\gamma_i F_{i,t}(k_{i,t})) + (R_{t+1}b'_{i,t}-\gamma_i F_{i,t}(k'_{i,t})) \qquad \text{(B.3)}\\
&\qquad - \gamma_i F'_{i,t+1}(k_{i,t})(k_{i,t}-k'_{i,t}) + R_{t+1}(b_{i,t}-b'_{i,t})\Big)\\
\geq&\mu_{i,t+1}(R_{t+1}b_{i,t}-\gamma_i F_{i,t}(k_{i,t}) - (R_{t+1}b'_{i,t}-\gamma_i F_{i,t}(k'_{i,t}))) \qquad \text{(B.4)}\\
&= \mu_{i,t+1}(\gamma_i F_{i,t}(k'_{i,t}) - R_{t+1}b'_{i,t}) \geq 0. \qquad \text{(B.5)}
\end{aligned}$$

It remains to prove that $\liminf_{T\to\infty} \lambda_{i,T}\big(k'_{i,T}-b'_{i,T}-(k_{i,T}-b_{i,T})\big) \geq 0$.

According to (B.1) and (B.2), we have

$$\begin{aligned}
&\lambda_{i,t}\big(k'_{i,t}-b'_{i,t}-(k_{i,t}-b_{i,t})\big)\\
=&R_{t+1}\lambda_{i,t+1}(b_{i,t}-b'_{i,t}) + \mu_{i,t+1}R_{t+1}(b_{i,t}-b'_{i,t})\\
&-\Big(\lambda_{i,t+1}F'_{i,t+1}(k_{i,t})(k_{i,t}-k'_{i,t}) + \gamma_i\mu_{i,t+1}F'_{i,t+1}(k_{i,t})(k_{i,t}-k'_{i,t}) + \eta_{i,t}(k_{i,t}-k'_{i,t})\Big)\\
=&R_{t+1}\lambda_{i,t+1}(b_{i,t}-b'_{i,t}) - \lambda_{i,t+1}F'_{i,t+1}(k_{i,t})(k_{i,t}-k'_{i,t}) + \eta_{i,t}(k_{i,t}-k'_{i,t})\\
&+\mu_{i,t+1}R_{t+1}(b_{i,t}-b'_{i,t}) + \mu_{i,t+1}\gamma_i F'_{i,t+1}(k_{i,t})(k_{i,t}-k'_{i,t})\\
\geq&R_{t+1}\lambda_{i,t+1}(b_{i,t}-b'_{i,t}) - \lambda_{i,t+1}F'_{i,t+1}(k_{i,t})(k_{i,t}-k'_{i,t}).
\end{aligned}$$

where we use (B.5) the fact that $\eta_{i,t}(k_{i,t}-k'_{i,t}) = -\eta_{i,t}k'_{i,t} \leq 0$ for the last inequality.

Since $F_{i,t+1}$ is concave, we have $F'_{i,t+1}(k_{i,t})(k_{i,t}-k'_{i,t}) \leq F_{i,t+1}(k_{i,t}) - F_{i,t+1}(k'_{i,t})$. So, we get that

$$\begin{aligned}
\lambda_{i,t}\big(k'_{i,t}-b'_{i,t}-(k_{i,t}-b_{i,t})\big) \geq& R_{t+1}\lambda_{i,t+1}(b_{i,t}-b'_{i,t}) - \lambda_{i,t+1}\big(F_{i,t+1}(k_{i,t}) - F_{i,t+1}(k'_{i,t})\big)\\
=&\lambda_{i,t+1}\big(R_{t+1}b_{i,t} - F_{i,t+1}(k_{i,t})\big) + \lambda_{i,t+1}\big(F_{i,t+1}(k'_{i,t}) - R_{t+1}b'_{i,t}\big)
\end{aligned}$$

We have $F_{i,t+1}(k'_{i,t}) - R_{t+1}b'_{i,t} \geq 0$ because $\gamma_i F_{i,t+1}(k'_{i,t}) - R_{t+1}b'_{i,t} \geq 0$.

The budget constraint at date t implies that $\lambda_{i,t}(c_{i,t}+k_{i,t}-b_{i,t}) = \lambda_{i,t}\big(F_{i,t}(k_{i,t-1}) - R_t b_{i,t-1}\big)$. Since $\lim_{t\to\infty}\lambda_{i,t}c_{i,t} = 0 = \lim_{t\to\infty}\lambda_{i,t}(k_{i,t}-b_{i,t})$, we get that $\lim_{t\to\infty}\lambda_{i,t}\big(F_{i,t}(k_{i,t-1}) - R_t b_{i,t-1}\big) = 0$. By consequence, we obtain that $\liminf_{T\to\infty}\lambda_{i,T}\big(k'_{i,T}-b'_{i,T}-(k_{i,T}-b_{i,T})\big) \geq 0$. □

C.1 Proofs for Section 4.1

Proof of Proposition 14. Steady state analysis. Let us focus on an interior equilibrium (i.e., $k_{i,t} > 0$, $\forall i, t$), we can write the FOCs

$$\begin{aligned}
\beta_i^t u'_i(c_{i,t}) &= \lambda_{i,t}\\
\lambda_{i,t} &= F'_{i,t}(k_{i,t})(\lambda_{i,t+1} + \gamma_i\mu_{i,t+1})\\
\lambda_{i,t} &= R_{t+1}(\lambda_{i,t+1} + \mu_{i,t+1})\\
\mu_{i,t+1}(R_{t+1}b_{i,t} - \gamma_i F_i(k_{i,t})) &= 0
\end{aligned}$$

where $\mu_{i,t} \geq 0$ is the multiplier with respect to the constraint $R_t b_{i,t-1} - \gamma_i F_{i,t}(k_{i,t-1}) \leq 0$.

According to FOCs, we have that $1 \geq R_{t+1}\max_i \frac{\beta_i u'_i(c_{i,t+1})}{u'_i(c_{i,t})}$, $\forall i$. Since $k_{i,t} > 0$, $\forall i, \forall t$, there exists an agent, say agent i, whose borrowing constraint at date $t+1$ is not binding. It means that $\mu_{i,t+1} = 0$. By consequence, we have $1 = R_{t+1}\frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})} = R_{t+1}\max_j \frac{\beta_j u_j'(c_{j,t+1})}{u_j'(c_{j,t})}$. Therefore, we have $R = 1/\max_i\{\beta_i\}$ at steady state.

The first-order conditions imply that $\lambda_{i,t}\frac{R_{t+1}-\gamma_i F_{i,t}'(k_{i,t})}{R_{t+1}} = F_{i,t}'(k_{i,t})\lambda_{i,t+1}(1-\gamma_i)$. By consequence, we obtain point 2.

□

Proof of Lemma 3. The maximization problem of agent i is

$$\max_{(c_i,k_i,b_i)} \sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t})$$
$$\text{subject to: } c_{i,t} + k_{i,t} + R_t b_{i,t-1} \leq A_{i,t}k_{i,t-1} + b_{i,t}$$
$$R_t b_{i,t-1} \leq \gamma_i A_{i,t}(k_{i,t-1})$$

Denote $s_{i,t} = k_{i,t} - b_{i,t}$ the net saving of agent i at date t.

Let $R_t = A_{h,t}$, $\forall t$, for some agent h.

For agent h, we have $c_{h,t} + (k_{h,t} - b_{h,t}) \leq A_{h,t}(k_{h,t-1} - b_{h,t-1})$. We can compute that

$$s_{h,0} = \beta_h w_{h,0}, \quad s_{h,t} = \beta_h A_{h,t} s_{h,t-1} \ \forall t \geq 1$$
$$s_{h,t} = \beta_h^t A_{h,t} \cdots A_{h,1} s_{i,0}$$

For agent $i < h$, since $A_{i,t} < R_t = A_{h,t}$, $\forall t$, we have $k_{i,t} = 0$ and hence we find that

$$s_{i,0} = \beta_i w_{i,0}, \quad s_{i,t} = \beta_i R_t s_{i,t-1} \ \forall t \geq 1$$
$$s_{i,t} = \beta_i^t R_t \cdots R_1 s_{i,0}.$$

For agent $j > h$, since $A_{j,t} > R_t = A_{h,t}$, $\forall t$, her borrowing constraint is always binding: $R_t b_{j,t-1} = \gamma_j A_{j,t} k_{j,t-1}$. Therefore, we have

$$s_{j,t} = k_{j,t}\Big(1 - \frac{\gamma_j A_{j,t+1}}{R_{t+1}}\Big), \quad A_{j,t}k_{j,t-1} - R_t b_{j,t-1} = (1-\gamma_j)A_{j,t}k_{j,t-1}, \forall t \geq 1.$$

From this, we can compute that

$$s_{j,t} = \beta_j \frac{(1-\gamma_j)A_{j,t}R_t}{R_t - \gamma_j A_{j,t}} s_{j,t-1} = \Big(\beta_j \frac{(1-\gamma_j)A_{j,t}R_t}{R_t - \gamma_j A_{j,t}}\Big) \cdots \Big(\beta_j \frac{(1-\gamma_j)A_{j,1}R_1}{R_1 - \gamma_j A_{j,1}}\Big) s_{j,0}$$
$$k_{j,t} = \frac{1}{1 - \frac{\gamma_j A_{j,t+1}}{R_{t+1}}} s_{j,t} = \frac{R_{t+1}}{R_{t+1} - \gamma_j A_{j,t+1}} s_{j,t}$$
$$b_{j,t} = \frac{\gamma_j A_{j,t+1}}{R_{t+1}} k_{j,t} = \frac{\gamma_j A_{j,t+1}}{R_{t+1} - \gamma_j A_{j,t+1}} s_{j,t}$$

Therefore, we can find the capital of the agent h

$$\begin{aligned} k_{h,t} =& s_{h,t} + b_{h,t} = s_{h,t} - \sum_{i<h} b_{i,t} - \sum_{j>h} b_{j,t} \\ =& \beta_h^t A_{h,t} \cdots A_{h,1} s_{h,0} + \sum_{i<h} \beta_i^t A_{h,t} \cdots A_{h,1} s_{i,0} \\ & - \sum_{j>h} \frac{\gamma_j A_{j,t+1}}{R_{t+1} - \gamma_j A_{j,t+1}} \Big(\beta_j \frac{(1-\gamma_j)A_{j,t}R_t}{R_t - \gamma_j A_{j,t}}\Big) \cdots \Big(\beta_j \frac{(1-\gamma_j)A_{j,1}R_1}{R_1 - \gamma_j A_{j,1}}\Big) s_{j,0} \end{aligned}$$

In order to keep $k_{h,t} > 0$, $\forall t$, we impose that

$$\sum_{i\leq h}\beta_i^t s_{i,0} - \sum_{j>h}\beta_j^t \frac{\gamma_j A_{j,t+1}}{A_{h,t+1}-\gamma_j A_{j,t+1}}\frac{(1-\gamma_j)A_{j,t}}{A_{h,t}-\gamma_j A_{j,t}}\cdots\frac{(1-\gamma_j)A_{j,1}}{A_{h,1}-\gamma_j A_{j,1}}s_{j,0} > 0.$$

which is actually condition (4.7).

The borrowing constraint of agent h at date t becomes $k_{h,t} \leq \frac{R_{t+1}s_{h,t}}{R_{t+1}-\gamma_h A_{h,t}} = \frac{s_{h,t}}{1-\gamma_h}$. This is equivalent to

$$\sum_{i\leq h}\beta_i^t A_{h,t}\cdots A_{h,1}s_{i,0} - \sum_{j>h}\frac{\gamma_j A_{j,t+1}}{R_{t+1}-\gamma_j A_{j,t+1}}\Big(\beta_j\frac{(1-\gamma_j)A_{j,t}R_t}{R_t-\gamma_j A_{j,t}}\Big)\cdots\Big(\beta_j\frac{(1-\gamma_j)A_{j,1}R_1}{R_1-\gamma_j A_{j,1}}\Big)s_{j,0}$$
$$\leq \beta_h^t A_{h,t}\cdots A_{h,1}s_{i,0}\frac{1}{1-\gamma_h}$$
$$\Leftrightarrow \sum_{i\leq h}\beta_i^t s_{i,0} - \sum_{j>h}\frac{\gamma_j A_{j,t+1}}{R_{t+1}-\gamma_j A_{j,t+1}}\Big(\beta_j\frac{(1-\gamma_j)A_{j,t}}{R_t-\gamma_j A_{j,t}}\Big)\cdots\Big(\beta_j\frac{(1-\gamma_j)A_{j,1}}{R_1-\gamma_j A_{j,1}}\Big)s_{j,0} \leq \beta_h^t s_{h,0}\frac{1}{1-\gamma_h}.$$

Under these conditions, by applying Lemma 5, we can check that the above list $(R_t,(c_{i,t},k_{i,t},b_{i,t})_i)$ is an equilibrium.

We now compute the aggregate production

$$Y_t = A_{h,t}k_{h,t-1} + \sum_{j>h}A_{h,t}k_{j,t-1}$$
$$= A_{h,t}\cdots A_{h,1}\Big(\beta_h^{t-1}s_{h,0} + \sum_{i<h}\beta_i^{t-1}s_{i,0} - \sum_{j>h}\beta_j^{t-1}\frac{\gamma_j A_{j,t}}{A_{h,t}-\gamma_j A_{j,t}}\frac{(1-\gamma_j)A_{j,t-1}}{A_{h,t-1}-\gamma_j A_{j,t-1}}\cdots\frac{(1-\gamma_j)A_{j,1}}{A_{h,1}-\gamma_j A_{j,1}}s_{j,0}\Big)$$
$$+\sum_{j>h}A_{j,t}\frac{R_t}{R_t-\gamma_j A_{j,t}}\Big(\beta_j\frac{(1-\gamma_j)A_{j,t-1}R_{t-1}}{R_{t-1}-\gamma_j A_{j,t-1}}\Big)\cdots\Big(\beta_j\frac{(1-\gamma_j)A_{j,1}R_1}{R_1-\gamma_j A_{j,1}}\Big)s_{j,0}$$
$$= A_{h,t}\cdots A_{h,1}\sum_{i\leq h}\beta_i^{t-1}s_{i,0}$$
$$+ A_{h,t}\cdots A_{h,1}\sum_{j>h}\beta_j^{t-1}(1-\gamma_j)^t\frac{A_{j,t}}{A_{h,t}-\gamma_j A_{j,t}}\frac{A_{j,t-1}}{A_{h,t-1}-\gamma_j A_{j,t-1}}\cdots\frac{A_{j,1}}{A_{h,1}-\gamma_j A_{j,1}}s_{j,0}.$$

1. When there are 2 agents and $h=2$, i.e., only the most productive agent produces, this condition is obviously satisfied.

2. When there are 2 agents and $h=1$. This condition becomes

$$\beta_1^t s_{1,0} - \beta_2^t\frac{\gamma_2 A_{2,t+1}}{A_{1,t+1}-\gamma_2 A_{2,t+1}}\frac{(1-\gamma_2)A_{2,t}}{A_{1,t}-\gamma_2 A_{2,t}}\cdots\frac{(1-\gamma_2)A_{2,1}}{A_{1,1}-\gamma_2 A_{2,1}}s_{2,0} > 0$$

or equivalently

$$\frac{s_{2,0}}{s_{1,0}}\frac{\beta_1}{\beta_2}\frac{\gamma_2}{1-\gamma_2}\Big(\frac{\beta_2}{\beta_1}\frac{(1-\gamma_2)A_{2,1}}{A_{1,1}-\gamma_2 A_{2,1}}\cdots\frac{\beta_2}{\beta_1}\frac{(1-\gamma_2)A_{2,t+1}}{A_{1,t+1}-\gamma_2 A_{2,t+1}}\Big) < 1\forall t.$$

This happens if $\sup_t \frac{\beta_2}{\beta_1}\frac{1-\gamma_2}{\frac{A_{1,t}}{A_{2,t}}-\gamma_2} < 1$ and $\frac{s_{2,0}}{s_{1,0}}\frac{\beta_1}{\beta_2}\frac{\gamma_2}{1-\gamma_2} \leq 1$.

□

Proof of Proposition 15 (m agents). To investigate the properties of the output and the growth rate, we need an useful lemma whose proof is left for the reader.

Lemma 6. *Let $N \geq 1$ be an integer. For each integer $t \geq 1$, we denote $X_t \equiv \sum_{i=1}^{N} \alpha_i a_i^t$, where $\alpha_i > 0, a_i > 0$ for any i. We have*

$$\lim_{t\to\infty} \frac{X_{t+1}}{X_t} = \max_{1\leq i\leq N} a_i \tag{B.6}$$

According to Lemma 3, we have that

$$Y_t = A_{h,t}\cdots A_{h,1}\sum_{i\leq h}\beta_i^{t-1}s_{i,0} + A_{h,t}\cdots A_{h,1}\sum_{j>h}\beta_j^{t-1}(1-\gamma_j)^t\frac{A_{j,t}}{A_{h,t}-\gamma_j A_{j,t}}\frac{A_{j,t-1}}{A_{h,t-1}-\gamma_j A_{j,t-1}}\cdots\frac{A_{j,1}}{A_{h,1}-\gamma_j A_{j,1}}s_{j,0}.$$

When $A_{i,t} = A_i$, $\forall t, \forall i,$, we have that

$$Y_t = A_h^t\sum_{i\leq h}\beta_i^{t-1}s_{i,0} + \sum_{j>h}\beta_j^{t-1}(1-\gamma_j)^t A_j^t\Big(\frac{A_h}{A_h-\gamma_j A_j}\Big)^t s_{j,0}.$$

1. By consequence, we can compute that

$$\frac{1}{tA_h^{t-1}}\frac{\partial Y_t}{\partial A_h} = \sum_{i\leq h}\beta_i^{t-1}s_{i,0} - \sum_{j>h}(1-\gamma_j)\frac{\gamma_j A_j^2}{(A_j-\gamma_j A_j)^2}\Big(\frac{\beta_j(1-\gamma_j)A_j}{A_h-\gamma_j A_j}\Big)^{t-1}s_{j,0}.$$

   This implies (4.13).

   Since $\max_{i\leq h}\beta_i > \max_{j>h}\frac{\beta_j(1-\gamma_j)A_j}{A_h-\gamma_j A_j}$, Lemma 6 implies that

$$\lim_{t\to\infty}\frac{\Big(\sum_{i\leq h}\beta_i^{t-1}s_{i,0} - \sum_{j>h}(1-\gamma_j)\frac{\gamma_j A_j^2}{(A_j-\gamma_j A_j)^2}\Big(\frac{\beta_j(1-\gamma_j)A_j}{A_h-\gamma_j A_j}\Big)^{t-1}s_{j,0}\Big)}{\beta_h^t} = \sum_{i\leq h:\beta_i=\beta_h} s_{i,0} > 0$$

   By consequence, there exists a date t_0 such that $\frac{\partial Y_t}{\partial A_1} \geq 0$, $\forall t > t_0$.

2. Since $\max_{i\leq h}\beta_i > \max_{j>h}\frac{\beta_j(1-\gamma_j)A_j}{A_h-\gamma_j A_j}$, Lemma 6 directly implies that $G_{t+1} \equiv \frac{Y_{t+1}}{Y_t}$ converges to $A_h \max_{1\leq i\leq m}\beta_i$.

   We now look at the formula of $G_{t+1} \equiv \frac{Y_{t+1}}{Y_t}$

$$G_{t+1} \equiv \frac{Y_{t+1}}{Y_t} = \frac{A_h^{t+1}\sum_{i\leq h}\beta_i^t s_{i,0} + \sum_{j>h}\beta_j^t(1-\gamma_j)^{t+1}A_j^{t+1}\Big(\frac{A_h}{A_h-\gamma_j A_j}\Big)^{t+1}s_{j,0}}{A_h^t\sum_{i\leq h}\beta_i^{t-1}s_{i,0} + \sum_{j>h}\beta_j^{t-1}(1-\gamma_j)^t A_j^t\Big(\frac{A_h}{A_h-\gamma_j A_j}\Big)^t s_{j,0}}$$
$$= A_h\frac{\sum_{i\leq h}\beta_i^t s_{i,0} + \sum_{j>h}\Big(\frac{\beta_j(1-\gamma_j)A_j}{A_h-\gamma_j A_j}\Big)^{t+1}\frac{s_{j,0}}{\beta_j}}{\sum_{i\leq h}\beta_i^{t-1}s_{i,0} + \sum_{j>h}\Big(\frac{\beta_j A_j(1-\gamma_j)}{A_h-\gamma_j A_j}\Big)^t\frac{s_{j,0}}{\beta_j}}.$$

   Let us denote $g(x_{h+1},\dots,x_m) \equiv \frac{\sum_{i\leq h}\beta_i^t s_{i,0}+\sum_{j>h}x_j^{t+1}\frac{s_{j,0}}{\beta_j}}{\sum_{i\leq h}\beta_i^{t-1}s_{i,0}+\sum_{j>h}x_j^t\frac{s_{j,0}}{\beta_j}}$, where $x_j \equiv \frac{\beta_j A_j(1-\gamma_j)}{A_h-\gamma_j A_j}$, for $j > h$.

Denote $B_h \equiv \max_{j>h} x_j$. Recall that we assume that $M < \beta_h < 1$.

For $d \in \{h+1, \dots, m\}$, we compute that

$$\frac{\partial g}{\partial x_d} = \frac{(t+1)x_d^t \frac{s_{d,0}}{\beta_d}\left(\sum_{i\le h}\beta_i^{t-1}s_{i,0} + \sum_{j>h}x_j^t\frac{s_{j,0}}{\beta_j}\right) - tx_d^{t-1}\frac{s_{d,0}}{\beta_d}\left(\sum_{i\le h}\beta_i^t s_{i,0} + \sum_{j>h}x_j^{t+1}\frac{s_{j,0}}{\beta_j}\right)}{\left(\sum_{i\le h}\beta_i^{t-1}s_{i,0} + \sum_{j>h}x_j^t\frac{s_{j,0}}{\beta_j}\right)^2}$$
$$= A\Big((t+1)x_d\big(\sum_{i\le h}\beta_i^{t-1}s_{i,0} + \sum_{j>h}x_j^t\frac{s_{j,0}}{\beta_j}\big) - t\big(\sum_{i\le h}\beta_i^t s_{i,0} + \sum_{j>h}x_j^{t+1}\frac{s_{j,0}}{\beta_j}\big)\Big).$$

where $A \equiv \frac{x_d^{t-1}\frac{s_{d,0}}{\beta_d}}{\left(\sum_{i\le h}\beta_i^{t-1}s_{i,0}+\sum_{j>h}x_j^t\frac{s_{j,0}}{\beta_j}\right)^2}$. Applying Lemma 6, we have that

$$\lim_{t\to\infty}\frac{(t+1)x_d\left(\sum_{i\le h}\beta_i^{t-1}s_{i,0} + \sum_{j>h}x_j^t\frac{s_{j,0}}{\beta_j}\right)}{t\left(\sum_{i\le h}\beta_i^t s_{i,0} + \sum_{j>h}x_j^{t+1}\frac{s_{j,0}}{\beta_j}\right)} = \frac{x_d}{\beta_h} < 1$$

which implies that there exists a date t_1 such that $\frac{\partial g}{\partial x_d} < 0$ for any $t \ge t_1$. Since $x_j \equiv \frac{\beta_j A_j(1-\gamma_j)}{A_h-\gamma_j A_j}$ is increasing in A_j, we get our result.

□

C.2 Proofs for Section 4.2

Proof of Proposition 17. Let us focus on an equilibrium where only the most productive agent produces. The interest rate $R_1 \in (A_{m-1}, A_m)$ and $R_t = A_m, \forall t \ge 2$.

Denote the individual saving $s_{i,t} \equiv k_{i,t} - b_{i,t}$. For $i < m$, agent i is lender, $k_{i,t} = 0$, $s_{i,t} = -b_{i,t}$, $\forall t$. We can compute that

$$s_{i,0} = \beta_i w_{i,0}, \quad s_{i,t} = \beta_i R_t s_{i,t-1}\ \forall t \ge 1$$
$$s_{i,t} = \beta_i^t R_t \cdots R_1 s_{i,0}.$$

For agent m, since $A_m > R_1$, her borrowing constraints at date 0 is binding: $R_1 b_{m,0} = \gamma_m A_m k_{m,0}$. Therefore, we have

$$A_m k_{m,0} - R_1 b_{m,0} = (1-\gamma_m)A_m k_{m,0}$$
$$s_{m,0} = k_{m,0} - b_{m,0} = k_{m,0}\Big(1 - \frac{\gamma_m A_m}{R_1}\Big)$$
$$k_{m,0} = \frac{R_1}{R_1 - \gamma_m A_m}s_{m,0}, \quad b_{m,0} = \frac{\gamma_m A_m}{R_1 - \gamma_m A_m}s_{m,0}$$

The budget constraints of agent m write

$$c_{m,0} + s_{m,0} = w_{m,0}$$
$$c_{m,1} + s_{m,1} = (1-\gamma_m)A_m k_{m,0} = \frac{(1-\gamma_m)A_m R_1}{R_1 - \gamma_m A_m}s_{m,0}$$
$$c_{m,t} + s_{m,t} = A_m s_{m,t}, \forall t \ge 2$$
$$s_{m,t} = k_{m,t} - b_{m,t}, \forall t \ge 2.$$

From this and the FOCs, we can compute the individual saving

$$\begin{aligned}
s_{i,0} &= \beta_i w_{i,0}, \forall i\\
s_{m,1} &= \beta_m \frac{(1-\gamma_m)A_m R_1}{R_1 - \gamma_m A_m} s_{m,0}\\
s_{i,t} &= \beta_i A_m s_{i,t-1}, \forall t \geq 2, \forall i = 1, \cdots, m.
\end{aligned}$$

We now look at equilibrium. From the market clearing condition $\sum_i b_{i,t} = 0$, we have that

$$\begin{aligned}
-\sum_{i\neq m} b_{i,0} &= b_{m,0}\\
\Leftrightarrow \sum_{i\neq m} s_{i,0} &= \frac{\gamma_m A_m}{R_1 - \gamma_m A_m} s_{m,0}\\
\Leftrightarrow R_1 &= \gamma_m A_m \Big(1 + \frac{s_{m,0}}{\sum_{i\neq m} s_{i,0}}\Big) = \gamma_m A_m \frac{S_0}{\sum_{i\neq m} s_{i,0}}.
\end{aligned}$$

By consequence, we find the saving of all agents: $s_{i,0} = \beta_i w_{i,0}, \forall i$, and

$$\begin{aligned}
s_{i,1} &= \beta_i R_1 s_{i,0} = \beta_i \gamma_m A_m \Big(1 + \frac{s_{m,0}}{\sum_{i\neq m} s_{i,0}}\Big) s_{i,0} = \beta_i \gamma_m A_m S_0 \frac{s_{i,0}}{\sum_{j\neq m} s_{i,0}}, \forall i \neq m\\
s_{m,1} &= \beta_m \frac{(1-\gamma_m)A_m R_1}{R_1 - \gamma_m A_m} s_{m,0} = \beta_m \frac{(1-\gamma_m)A_m \gamma_m A_m \big(1 + \frac{s_{m,0}}{\sum_{i\neq m} s_{i,0}}\big)}{\gamma_m A_m \big(1 + \frac{s_{m,0}}{\sum_{i\neq m} s_{i,0}}\big) - \gamma_m A_m} s_{m,0}\\
&= \beta_m (1-\gamma_m) A_m S_0\\
s_{i,t} &= \beta_i A_m s_{i,t-1} = (\beta_i A_m)^{t-1} s_{i,1}, \forall t \geq 2, \forall i = 1, 2.
\end{aligned}$$

where $S_0 \equiv \sum_{i=1}^m s_{i,0}$.

It remains to find the sequence of capital $(k_{i,t})$. We have, $\forall t \geq 1$

$$\begin{aligned}
k_{m,0} &= \sum_{i=1}^m s_{i,0}, \quad k_{m,t} = s_{m,t} + b_{m,t} = s_{m,t} - \sum_{i\neq m} b_{i,t} = \sum_{i=1}^m s_{i,t}, \forall t \geq 1\\
k_{m,1} &= \sum_{i\neq m} \beta_i R_1 s_{i,0} + s_{m,1}\\
&= \sum_{i\neq m} \beta_i \gamma_m A_m S_0 \frac{s_{i,0}}{\sum_{j\neq m} s_{j,0}} + \beta_m (1-\gamma_m) A_m \sum_{i=1}^m s_{i,0}\\
k_{m,t} &= \sum_i s_{i,t} = \sum_i (\beta_i A_m)^{t-1} s_{i,1}, \forall t \geq 1\\
&= \sum_{i\neq m} (\beta_i A_m)^{t-1} \beta_i \gamma_m A_m S_0 \frac{s_{i,0}}{\sum_{j\neq m} s_{j,0}} + (\beta_m A_m)^{t-1} \beta_m (1-\gamma_m) A_m S_0, , \forall t \geq 1\\
&= S_0 A_m^t \Big(\gamma_m \sum_{i\neq m} \beta_i^t \frac{s_{i,0}}{\sum_{j\neq m} s_{j,0}} + \beta_m^t (1-\gamma_m)\Big).
\end{aligned}$$

We now check that the above list $((c_{i,t}, k_{i,t}, b_{i,t})_i, R_t)_t$ is an equilibrium. We use Lemma 5. It is easy to verify the market clearing conditions and the FOCs.

Condition $R_1 \in (A_{m-1}, A_m)$ is ensured by the assumption that

$$A_{m-1} < \gamma_m A_m \Big(1 + \frac{s_{m,0}}{\sum_{i\neq m} s_{i,0}}\Big) < A_m.$$

- We verify borrowing constraints: $R_{t+1}b_{m,t} \leq \gamma_m A_m k_{m,t}$. This is satisfied for $t = 0$. Let us consider $t \geq 1$. Since $R_{t+1} = A_m$, we get that $k_{m,t} - s_{m,t} = b_{m,t} \leq \gamma_m k_{m,t}$, or, equivalently, $(1-\gamma_m)k_{m,t} \leq s_{m,t}$. So, we must prove, for any $t \geq 1$,

$$(1-\gamma_m)S_0 A_m^t\Big(\gamma_m \sum_{i\neq m}\beta_i^t \frac{s_{i,0}}{\sum_{j\neq m}s_{j,0}} + \beta_m^t(1-\gamma_m)\Big) \leq (\beta_m A_m)^t(1-\gamma_m)S_0$$
$$\Leftrightarrow \Big(\gamma_m \sum_{i\neq m}\beta_i^t \frac{s_{i,0}}{\sum_{j\neq m}s_{j,0}} + \beta_m^t(1-\gamma_m)\Big) \leq \beta_m^t$$
$$\Leftrightarrow \sum_{i\neq m}\beta_i^t \frac{s_{i,0}}{\sum_{j\neq m}s_{j,0}} \leq \beta_m^t$$

which is satisfied under our assumption.

- Tranversality conditions: $\lim_{T\to\infty}\beta_i^T u_i'(c_{i,T})(k_{i,T} - b_{i,T}) = 0$. It is easy to verify these conditions because $\beta_i \in (0,1)$ and $u'(c) = 1/c$.

□

Proof of Lemma 4. Let us focus on an equilibrium where only the most productive agent produces. The interest rate $R_1 \in (A_{n-1}, A_n)$ and $R_t = A_h$, $\forall t \geq 2$, where $h \geq n$.

Denote the individual saving $s_{i,t} \equiv k_{i,t} - b_{i,t}$. For $i < n$, Agent i is lender, $k_{i,t} = 0$, $s_{i,t} = -b_{i,t}$, $\forall t$. We can compute that

$$s_{i,0} = \beta_i w_{i,0}, \quad s_{i,t} = \beta_i R_t s_{i,t-1}\ \forall t \geq 1$$
$$s_{i,t} = \beta_i^t R_t \cdots R_1 s_{i,0}.$$

For agent $j \geq n$, since $A_n > R_1$, her borrowing constraints at date 0 is binding: $R_1 b_{j,0} = \gamma_j A_j k_{j,0}$. Therefore, we have

$$Ak_{j,0} - R_1 b_{j,0} = (1-\gamma_j)A_j k_{j,0}$$
$$s_{j,0} = k_{j,0} - b_{j,0} = k_{j,0}\Big(1 - \frac{\gamma_j A_j}{R_1}\Big)$$
$$k_{j,0} = \frac{R_1}{R_1 - \gamma_j A_j}s_{j,0}, \quad b_{j,0} = \frac{\gamma_j A_j}{R_1 - \gamma_j A_j}s_{j,0}$$

The budget constraints of agent $j = n$ write

$$c_{j,0} + s_{j,0} = w_{j,0}, \forall j$$
$$c_{j,1} + s_{j,1} = (1-\gamma_j)A_j k_{j,0} = \frac{(1-\gamma_j)A_j R_1}{R_1 - \gamma_j A_j}s_{j,0}, \forall j \geq n$$
$$c_{n,t} + s_{n,t} = A_h s_{n,t}, \forall t \geq 2$$
$$s_{n,t} = k_{n,t} - b_{n,t}, \forall t \geq 2.$$

From this and the FOCs, we can compute that

$$s_{i,0} = \beta_i w_{i,0}, \forall i$$
$$s_{j,1} = \beta_j \frac{(1-\gamma_j)A_j R_1}{R_1 - \gamma_j A_j}s_{j,0}, \forall j \geq n$$
$$s_{j,t} = \beta_j A_h s_{j,t-1}, \forall t \geq 2$$
$$= \beta_j^{t-1}A_h^{t-1}s_{j,1} = \beta_j^{t-1}A_h^{t-1}\beta_j \frac{(1-\gamma_j)A_j R_1}{R_1 - \gamma_j A_j}s_{n,0}$$
$$\text{for } j \leq h, \quad s_{j,t} = \beta_j^t A_h^{t-1}\frac{(1-\gamma_j)A_j R_1}{R_1 - \gamma_j A_j}s_{j,0}, \forall t \geq 1.$$

We now look at the equilibrium R_1. From the market clearing condition $\sum_i b_{i,0} = 0$, we have that

$$
\begin{aligned}
&-\sum_{i<n} b_{i,0} = \sum_{j\geq n} b_{j,0} \\
&\Leftrightarrow \sum_{i<n} s_{i,0} = \sum_{j\geq n} \frac{\gamma_j A_j}{R_1 - \gamma_j A_j} s_{j,0} \Leftrightarrow S_0 = \sum_{j\geq n} \frac{R_1}{R_1 - \gamma_j A_j} s_{j,0}.
\end{aligned}
$$

Since $R_1 \in (A_{n-1}, A_n)$, this condition requires that

$$
\sum_{j\geq n} \frac{\gamma_j A_j}{A_n - \gamma_j A_j} s_{j,0} < \sum_{i<n} s_{i,0} < \sum_{j\geq n} \frac{\gamma_j A_j}{A_{n-1} - \gamma_j A_j} s_{j,0}.
$$

In a particular case where $n = m$, we find that $R_1 = \gamma_m A_m (1 + \frac{s_{m,0}}{\sum_{i\neq m} s_{i,0}}) = \gamma_m A_m \frac{S_0}{\sum_{i\neq m} s_{i,0}}$.

Now, consider agent $j > h$ and date $t \geq 1$. For agent $j > h$, since $A_{j,t} > R_t = A_{h,t}$, $\forall t$, her borrowing constraint is always binding: $R_t b_{j,t-1} = \gamma_j A_{j,t} k_{j,t-1}$. Therefore, we have

$$
s_{j,t} = k_{j,t}\Big(1 - \frac{\gamma_j A_{j,t+1}}{R_{t+1}}\Big), \quad A_{j,t}k_{j,t-1} - R_t b_{j,t-1} = (1-\gamma_j)A_{j,t}k_{j,t-1}, \forall t \geq 1.
$$

From this, we can compute that

$$
\begin{aligned}
s_{j,0} &= \beta_j w_{j,0}, \\
s_{j,t} &= \beta_j \frac{(1-\gamma_j)A_{j,t}R_t}{R_t - \gamma_j A_{j,t}} s_{j,t-1}, \forall t \geq 2 \\
&= \Big(\beta_j \frac{(1-\gamma_j)A_{j,t}R_t}{R_t - \gamma_j A_{j,t}}\Big) \cdots \Big(\beta_j \frac{(1-\gamma_j)A_{j,1}R_1}{R_1 - \gamma_j A_{j,1}}\Big) s_{j,0} \\
k_{j,t} &= \frac{1}{1 - \frac{\gamma_j A_{j,t+1}}{R_{t+1}}} s_{j,t} = \frac{R_{t+1}}{R_{t+1} - \gamma_j A_{j,t+1}} s_{j,t} \\
&= \frac{R_{t+1}}{R_{t+1} - \gamma_j A_{j,t+1}} \Big(\beta_j \frac{(1-\gamma_j)A_{j,t}R_t}{R_t - \gamma_j A_{j,t}}\Big) \cdots \Big(\beta_j \frac{(1-\gamma_j)A_{j,1}R_1}{R_1 - \gamma_j A_{j,1}}\Big) s_{j,0} \\
b_{j,t} &= \frac{\gamma_j A_{j,t+1}}{R_{t+1}} k_{j,t} = \frac{\gamma_j A_{j,t+1}}{R_{t+1} - \gamma_j A_{j,t+1}} s_{j,t} \\
&= \frac{\gamma_j A_{j,t+1}}{R_{t+1} - \gamma_j A_{j,t+1}} \Big(\beta_j \frac{(1-\gamma_j)A_{j,t}R_t}{R_t - \gamma_j A_{j,t}}\Big) \cdots \Big(\beta_j \frac{(1-\gamma_j)A_{j,1}R_1}{R_1 - \gamma_j A_{j,1}}\Big) s_{j,0}.
\end{aligned}
$$

From the market clearing condition $\sum_i b_{i,t} = 0$, we have $\sum_i s_{i,t} = \sum_i k_{i,t}$ which implies that

$$k_{h,t} = \sum_i s_{i,t} - \sum_{i \neq h} k_{i,t} = \sum_i s_{i,t} - \sum_{i>h} k_{i,t} \text{ (since } k_{i,t} = 0, \forall i < h) \tag{B.7}$$

$$= \sum_{i<n} s_{i,t} + \sum_{n \leq j \leq h} s_{i,t} - \sum_{j>h} b_{j,t} \tag{B.8}$$

$$= \sum_{i<n} \beta_i^t R_t \cdots R_1 s_{i,0} + \sum_{n \leq j \leq h} \beta_j^{t-1} A_h^{t-1} \beta_j (1-\gamma_j) \frac{A_j R_1}{R_1 - \gamma_j A_j} s_{j,0} \tag{B.9}$$

$$- \sum_{j>h} \frac{\gamma_j A_{j,t+1}}{R_{t+1} - \gamma_j A_{j,t+1}} \Big(\beta_j \frac{(1-\gamma_j) A_{j,t} R_t}{R_t - \gamma_j A_{j,t}} \Big) \cdots \Big(\beta_j \frac{(1-\gamma_j) A_{j,1} R_1}{R_1 - \gamma_j A_{j,1}} \Big) s_{j,0} \tag{B.10}$$

$$= \sum_{i<n} \beta_i^t A_h^{t-1} R_1 s_{i,0} + \sum_{n \leq j \leq h} \beta_j^t A_h^{t-1} (1-\gamma_j) \frac{A_j R_1}{R_1 - \gamma_j A_j} s_{j,0} \tag{B.11}$$

$$- \sum_{j>h} A_h^{t-1} \Big(\beta_j \frac{(1-\gamma_j) A_j}{A_h - \gamma_j A_j} \Big)^t \frac{\gamma_j A_j R_1}{R_1 - \gamma_j A_j} s_{j,0}, \quad \forall t \geq 1. \tag{B.12}$$

We then compute the output as in the statement of our result.

Since $k_{h,t} \geq 0, \forall t$, we must have

$$\sum_{i<n} \beta_i^t R_1 s_{i,0} + \sum_{n \leq j \leq h} \beta_j^t (1-\gamma_j) \frac{A_j R_1}{R_1 - \gamma_j A_j} s_{j,0} - \sum_{j>h} \Big(\beta_j \frac{(1-\gamma_j) A_j}{A_h - \gamma_j A_j} \Big)^t \frac{\gamma_j A_j R_1}{R_1 - \gamma_j A_j}) s_{j,0} \geq 0, \forall t \geq 1.$$

We now check that the above list $((c_{i,t}, k_{i,t}, b_{i,t})_i, R_t)_t$ is an equilibrium. We use Lemma 5. It is easy to verify the market clearing conditions and the FOCs.

- Condition $R_1 \in (A_{n-1}, A_n)$ is ensured by (4.23). Condition $k_{h,t} \geq 0$ is ensured by (4.24).
- We verify borrowing constraints: $R_{t+1} b_{h,t} \leq \gamma_h A_h k_{h,t}$. This is satisfied for $t = 0$. Let us consider $t \geq 1$. Since $R_{t+1} = A_h$, $\forall t \geq 1$, this becomes $k_{h,t} - s_{h,t} = b_{h,t} \leq \gamma_h k_{h,t}$, or, equivalently, $(1-\gamma_h) k_{h,t} \leq s_{h,t}$. So, we must prove, for any $t \geq 1$,

$$\sum_{i<n} \beta_i^t A_h^{t-1} R_1 s_{i,0} + \sum_{n \leq j \leq h} \beta_j^t A_h^{t-1} (1-\gamma_j) \frac{A_j R_1}{R_1 - \gamma_j A_j} s_{j,0}$$
$$- \sum_{j>h} A_h^{t-1} \Big(\beta_j \frac{(1-\gamma_j) A_j}{A_h - \gamma_j A_j} \Big)^t \frac{\gamma_j A_j R_1}{R_1 - \gamma_j A_j}) s_{j,0}, \quad \forall t \geq 1$$
$$< \frac{1}{1-\gamma_h} \beta_h^t A_h^{t-1} (1-\gamma_h) \frac{A_h R_1}{R_1 - \gamma_h A_h} s_{j,0} = \beta_h^t A_h^{t-1} \frac{A_h R_1}{R_1 - \gamma_h A_h} s_{j,0}$$

  which is satisfied under our assumption.

- Tranversality conditions: $\lim_{T \to \infty} \beta_i^T u_i'(c_{i,T})(k_{i,T} - b_{i,T}) = 0$. It is easy to verify these conditions because $\beta_i \in (0,1)$ and $u'(c) = 1/c$.

□

Proof of Proposition 18. Part 1: The aggregate output at date 1 equals $Y_1 = \sum_{j \geq n} A_j k_{j,0}$. By using the same technique in Proposition 10, we can provide conditions under which the aggregate output Y_1 is increasing or decreasing in the credit limit of producers.

Part 2: We now look at the output from second date on. For any $t \geq 1$, the aggregate output is computed by

$$\frac{1}{A_h^t} Y_{t+1} = \sum_{i<n} \beta_i^t R_1 s_{i,0} + \sum_{n \leq j \leq h} \beta_j^t \frac{(1-\gamma_j) A_j R_1}{R_1 - \gamma_j A_j} s_{j,0} + \sum_{j>h} \Big(\beta_j \frac{(1-\gamma_j)A_j}{A_h - \gamma_j A_j}\Big)^t \frac{(1-\gamma_j) A_j R_1}{R_1 - \gamma_j A_j} s_{j,0}$$

Note that R_1 does not depend on γ_i with $i < n$. So, the output does not depend on any agent $i < n$, who are not producer in equilibrium.

From (4.25a), we get that

$$\Big(\sum_{j \geq n} \frac{\gamma_j A_j}{(R_1 - \gamma_j A_j)^2} s_{j,0}\Big) \frac{\partial R_1}{\partial \gamma_v} = \frac{A_v R_1}{(R_1 - \gamma_v A_v)^2} s_{v,0}. \tag{B.13}$$

Thus, $\frac{\partial R_1}{\partial \gamma_v} > 0$.

Part 2.1. For $v \in \{n, \dots, h\}$, we compute that

$$\begin{aligned}\frac{1}{A_h^t} \frac{\partial Y_{t+1}}{\partial \gamma_v} =& \frac{\partial R_1}{\partial \gamma_v} \Big(\sum_{i<n} \beta_i^t s_{i,0} - \sum_{n \leq j \leq h} \beta_j^t \frac{(1-\gamma_j)\gamma_j A_j^2}{(R_1 - \gamma_j A_j)^2} s_{j,0} - \sum_{j>h} \Big(\beta_j \frac{(1-\gamma_j)A_j}{A_h - \gamma_j A_j}\Big)^t \frac{(1-\gamma_j)\gamma_j A_j^2}{(R_1 - \gamma_j A_j)^2} s_{j,0}\Big) \\ &+ \beta_v^t \frac{A_v R_1 (A_v - R_1)}{(R_1 - \gamma_v A_v)^2} s_{v,0}\end{aligned}$$

Combining with (B.13), we get that

$$\begin{aligned}&\frac{1}{A_h^t} \frac{\partial Y_{t+1}}{\partial \gamma_v} \frac{1}{\frac{\partial R_1}{\partial \gamma_v}} = \Big(\sum_{i<n} \beta_i^t s_{i,0} - \sum_{n \leq j \leq h} \beta_j^t \frac{(1-\gamma_j)\gamma_j A_j^2}{(R_1 - \gamma_j A_j)^2} s_{j,0} - \sum_{j>h} \Big(\beta_j \frac{(1-\gamma_j)A_j}{A_h - \gamma_j A_j}\Big)^t \frac{(1-\gamma_j)\gamma_j A_j^2}{(R_1 - \gamma_j A_j)^2} s_{j,0}\Big) \\ &+ \beta_v^t \frac{A_v R_1 (A_v - R_1)}{(R_1 - \gamma_v A_v)^2} s_{v,0} \frac{\sum_{j \geq n} \frac{\gamma_j A_j}{(R_1 - \gamma_j A_j)^2} s_{j,0}}{\frac{A_v R_1}{(R_1 - \gamma_v A_v)^2} s_{v,0}} \\ &= \sum_{i<n} \beta_i^t s_{i,0} - \sum_{n \leq j \leq h} \beta_j^t \frac{(1-\gamma_j)\gamma_j A_j^2}{(R_1 - \gamma_j A_j)^2} s_{j,0} - \sum_{j>h} \Big(\beta_j \frac{(1-\gamma_j)A_j}{A_h - \gamma_j A_j}\Big)^t \frac{(1-\gamma_j)\gamma_j A_j^2}{(R_1 - \gamma_j A_j)^2} s_{j,0}\Big) \\ &+ \beta_v^t (A_v - R_1) \sum_{j \geq n} \frac{\gamma_j A_j}{(R_1 - \gamma_j A_j)^2} s_{j,0} \\ &= \sum_{i<n} \beta_i^t s_{i,0} - \sum_{n \leq j \leq h} \beta_j^t \frac{(1-\gamma_j)\gamma_j A_j^2}{(R_1 - \gamma_j A_j)^2} s_{j,0} - \sum_{j>h} \Big(\beta_j \frac{(1-\gamma_j)A_j}{A_h - \gamma_j A_j}\Big)^t \frac{(1-\gamma_j)\gamma_j A_j^2}{(R_1 - \gamma_j A_j)^2} s_{j,0}\Big) \\ &+ \beta_v^t (A_v - R_1) \sum_{j \geq n} \frac{\gamma_j A_j}{(R_1 - \gamma_j A_j)^2} s_{j,0}.\end{aligned}$$

We now assume that $\beta_h > \max_{i \neq h} \beta_i$. When $v \neq h$, it is easy to see that $\dfrac{\partial Y_{t+1}}{\partial \gamma_v} < 0$ for t high enough.

When $v = h$, observe that

$$\beta_v^t \frac{(1-\gamma_v)\gamma_v A_v^2}{(R_1 - \gamma_v A_v)^2} s_{v,0} - \beta_v^t (A_v - R_1) \frac{\gamma_v A_v}{(R_1 - \gamma_v A_v)^2} s_{v,0} \tag{B.14}$$

$$= \frac{\beta_v^t s_{v,0}}{(R_1 - \gamma_v A_v)^2} \gamma_v A_v \Big((1-\gamma_v)A_v - (A_v - R_1)\Big) \tag{B.15}$$

$$= \frac{\beta_v^t s_{v,0}}{(R_1 - \gamma_v A_v)^2} \gamma_v A_v \Big(R_1 - \gamma_v A_v\Big) = \frac{\beta_v^t s_{v,0}}{(R_1 - \gamma_v A_v)} \gamma_v A_v. \tag{B.16}$$

By consequence, if $\beta_h > \max_{i\neq h}\beta_i$, then there exists t_0 such that $\frac{\partial Y_{t+1}}{\partial \gamma_h} < 0, \forall t \geq t_0$.

Part 2.2. For agent $v > h$, we compute that

$$\frac{1}{A_h^t}\frac{\partial Y_{t+1}}{\partial \gamma_v} = \frac{\partial R_1}{\partial \gamma_v}\Big(\sum_{i<n}\beta_i^t s_{i,0} - \sum_{n\leq j\leq h}\beta_j^t\frac{(1-\gamma_j)\gamma_j A_j^2}{(R_1-\gamma_jA_j)^2}s_{j,0} - \sum_{j>h}\Big(\beta_j\frac{(1-\gamma_j)A_j}{A_h-\gamma_jA_j}\Big)^t\frac{(1-\gamma_j)\gamma_jA_j^2}{(R_1-\gamma_jA_j)^2}s_{j,0}\Big)$$
$$+\Big(\frac{\beta_v(1-\gamma_v)A_v}{A_h-\gamma_vA_v}\Big)^t\frac{A_vR_1\big[t(R_1-\gamma_vA_v)(A_v-A_h)+(A_v-R_1)(A_h-\gamma_vA_v)\big]}{(A_h-\gamma_vA_v)(R_1-\gamma_vA_v)^2}s_{v,0}$$

$$\frac{1}{A_h^t}\frac{\partial Y_{t+1}}{\partial \gamma_v}\frac{1}{\frac{\partial R_1}{\partial\gamma_v}} = \Big(\sum_{i<n}\beta_i^t s_{i,0} - \sum_{n\leq j\leq h}\beta_j^t\frac{(1-\gamma_j)\gamma_j A_j^2}{(R_1-\gamma_jA_j)^2}s_{j,0} - \sum_{j>h}\Big(\beta_j\frac{(1-\gamma_j)A_j}{A_h-\gamma_jA_j}\Big)^t\frac{(1-\gamma_j)\gamma_jA_j^2}{(R_1-\gamma_jA_j)^2}s_{j,0}\Big)$$
$$+\Big(\frac{\beta_v(1-\gamma_v)A_v}{A_h-\gamma_vA_v}\Big)^t\frac{A_vR_1\big[t(R_1-\gamma_vA_v)(A_v-A_h)+(A_v-R_1)(A_h-\gamma_vA_v)^2\big]}{(A_h-\gamma_vA_v)(R_1-\gamma_vA_v)}s_{v,0}\frac{\sum_{j\geq n}\frac{\gamma_jA_j}{(R_1-\gamma_jA_j)^2}s_{j,0}}{\frac{A_vR_1}{(R_1-\gamma_vA_v)^2}s_{v,0}}$$
$$=\Big(\sum_{i<n}\beta_i^t s_{i,0} - \sum_{n\leq j\leq h}\beta_j^t\frac{(1-\gamma_j)\gamma_j A_j^2}{(R_1-\gamma_jA_j)^2}s_{j,0} - \sum_{j>h}\Big(\beta_j\frac{(1-\gamma_j)A_j}{A_h-\gamma_jA_j}\Big)^t\frac{(1-\gamma_j)\gamma_jA_j^2}{(R_1-\gamma_jA_j)^2}s_{j,0}\Big)$$
$$+\Big(\frac{\beta_v(1-\gamma_v)A_v}{A_h-\gamma_vA_v}\Big)^t\frac{A_vR_1\big[t(R_1-\gamma_vA_v)(A_v-A_h)+(A_v-R_1)(A_h-\gamma_vA_v)\big]}{(A_h-\gamma_vA_v)(R_1-\gamma_vA_v)^2\frac{A_vR_1}{(R_1-\gamma_vA_v)^2}}\Big(\sum_{j\geq n}\frac{\gamma_jA_j}{(R_1-\gamma_jA_j)^2}s_{j,0}\Big)$$
$$=\Big(\sum_{i<n}\beta_i^t s_{i,0} - \sum_{n\leq j\leq h}\beta_j^t\frac{(1-\gamma_j)\gamma_j A_j^2}{(R_1-\gamma_jA_j)^2}s_{j,0} - \sum_{j>h}\Big(\beta_j\frac{(1-\gamma_j)A_j}{A_h-\gamma_jA_j}\Big)^t\frac{(1-\gamma_j)\gamma_jA_j^2}{(R_1-\gamma_jA_j)^2}s_{j,0}\Big)$$
$$+\Big(\frac{\beta_v(1-\gamma_v)A_v}{A_h-\gamma_vA_v}\Big)^t\Big(\frac{t(R_1-\gamma_vA_v)(A_v-A_h)}{(A_h-\gamma_vA_v)}+A_v-R_1\Big)\Big(\sum_{j\geq n}\frac{\gamma_jA_j}{(R_1-\gamma_jA_j)^2}s_{j,0}\Big)$$

□

C.2.1 Additional results

Proposition 20 (**equilibrium with $R_1 \in (A_{m-1}, A_m), R_t = A_h, \forall t \geq 2$, $h < m$**)**.** *Assume that $u_i(c) = ln(c)$, $\forall i, \forall c > 0$, $F_{i,t}(k) = A_ik$, $\forall i, \forall k \geq 0$ with $\max_i \gamma_iA_i < A_1 < A_2 < \ldots < A_m$, and*

$$\frac{\beta_h^t s_{h,0}}{1-\gamma_h} \geq \sum_{i\leq h}\beta_i^t s_{i,0} - \sum_{j=h+1}^{m-1}\frac{\gamma_jA_j}{A_h-\gamma_jA_j}\Big(\beta_j\frac{(1-\gamma_j)A_j}{A_h-\gamma_jA_j}\Big)^{t-1}\beta_js_{j,0} - \Big(\beta_m\frac{(1-\gamma_m)A_m}{A_h-\gamma_mA_m}\Big)^t(S_0-s_{m,0}) \quad \geq 0 \tag{B.17a}$$

$$\gamma_m < \frac{\sum_{i\neq m}s_{i,0}}{S_0} \tag{B.17b}$$

$$\frac{A_{m-1}}{A_m} < \gamma_m\frac{S_0}{\sum_{i\neq m}s_{i,0}}. \tag{B.17c}$$

where $h \leq m-1$.

Then, there exists an equilibrium where the interest rates are determined by

$$R_1 = \gamma_mA_m\frac{S_0}{\sum_{i\neq m}s_{i,0}} \in (A_{m-1}, A_m), \quad R_t = A_h, \forall t \geq 2, \tag{B.18}$$

where $S_0 \equiv \sum_{i=1}^m s_{i,0}$.

1. *In such an equilibrium, only agent m produces at date 1 but (m-h+1) agents produces from date 2 on. The individual capital is given by*

$$k_{j,0} = \begin{cases} 0, & \forall j < m \\ S_0, & \forall j = m \end{cases}$$

$$k_{j,t} = \begin{cases} 0, & \forall j < h \\ \sum_{i \le h} \beta_i^t A_h^{t-1} R_1 s_{i,0} - \sum_{j=h+1}^{m-1} \frac{\gamma_j A_j}{A_h - \gamma_j A_j} \left(\beta_j \frac{(1-\gamma_j) A_j A_h}{A_h - \gamma_j A_j}\right)^{t-1} \beta_j R_1 s_{j,0} & \\ -\frac{\gamma_m A_m}{A_h - \gamma_m A_m} \left(\beta_m \frac{(1-\gamma_m) A_m A_h}{A_h - \gamma_m A_m}\right)^{t-1} \left(\beta_m \frac{(1-\gamma_m) A_m R_1}{R_1 - \gamma_m A_m}\right) s_{m,0}, & \text{for } j=h \\ \frac{A_h}{A_h - \gamma_j A_j} \left(\beta_j \frac{(1-\gamma_j) A_j A_h}{A_h - \gamma_j A_j}\right)^{t-1} \beta_j R_1 s_{j,0} & \forall h < j < m \\ \frac{A_h}{A_h - \gamma_m A_m} \left(\beta_m \frac{(1-\gamma_m) A_m A_h}{A_h - \gamma_m A_m}\right)^{t-1} \left(\beta_m \frac{(1-\gamma_m) A_m R_1}{R_1 - \gamma_m A_m}\right) s_{m,0} & \text{for } j = m. \end{cases}$$

2. *The aggregate output is increasing in the credit limit γ_j of each producer j.*

Proof of Proposition 20. Let us focus on an equilibrium where only the most productive agent produces. The interest rate $R_1 \in (A_{m-1}, A_m)$ and $R_t = A_h$, $\forall t \geq 2$.

Denote the individual saving $s_{i,t} \equiv k_{i,t} - b_{i,t}$.

First, we observe that

$$s_{i,0} = \beta_i w_{i,0}, \forall i. \tag{B.19}$$

At date 0, since $R_1 \in (A_{m-1}, A_m)$, we have

$$\begin{aligned} k_{i,0} = 0, \quad b_{i,0} &= -s_{i,0}, \forall i < m \\ A_m k_{m,0} - R_1 b_{m,0} &= (1 - \gamma_m) A_m k_{m,0} \\ s_{m,0} = k_{m,0} - b_{m,0} &= k_{m,0}\Big(1 - \frac{\gamma_m A_m}{R_1}\Big) \\ k_{m,0} = \frac{R_1}{R_1 - \gamma_m A_m} s_{m,0}, \quad b_{m,0} &= \frac{\gamma_m A_m}{R_1 - \gamma_m A_m} s_{m,0} \end{aligned}$$

We now look at equilibrium. From the market clearing condition $\sum_i b_{i,0} = 0$, we have that

$$\begin{aligned} -\sum_{i \neq m} b_{i,0} = b_{m,0} \Leftrightarrow \sum_{i \neq m} s_{i,0} &= \frac{\gamma_m A_m}{R_1 - \gamma_m A_m} s_{m,0} \\ \Leftrightarrow R_1 = \gamma_m A_m \Big(1 + \frac{s_{m,0}}{\sum_{i \neq m} s_{i,0}}\Big) &= \gamma_m A_m \frac{S_0}{\sum_{i \neq m} s_{i,0}}. \end{aligned}$$

Let $h \leq m - 1$.

For each agent $i < h$, we have $k_{i,t} = 0$, $s_{i,t} = -b_{i,t}$, $\forall t$.

Since $R_t = A_h, \forall t \geq 2$, we can compute that, for any $i \leq h$,

$$\begin{aligned} s_{i,0} = \beta_i w_{i,0}, \quad s_{i,t} &= \beta_i R_t s_{i,t-1} \ \forall t \geq 1 \\ s_{i,t} &= \beta_i^t A_h^{t-1} R_1 s_{i,0}, t \geq 1. \end{aligned}$$

The capital $k_{h,t}$ will be determined by the market clearing condition.

For any m, we have

For each agent j with $h < j < m$ and , their borrowing constraints bind at any date $t \geq 1$: $R_{t+1}b_{j,t} = \gamma_j A_j k_{j,t}$. Therefore, we have $s_{j,t} = k_{j,t}\Big(1 - \frac{\gamma_j A_{j,t+1}}{R_{t+1}}\Big), \forall t \geq 1$. From this, we can compute that

$$\begin{aligned}
s_{j,0} &= \beta_j w_{j,0}, \quad k_{j,0} = 0,\\
s_{j,1} &= \beta_j R_1 s_{j,0},\\
s_{j,t} &= \beta_j \frac{(1-\gamma_j)A_{j,t}R_t}{R_t - \gamma_j A_{j,t}} s_{j,t-1}, \forall t \geq 2\\
s_{j,t} &= \Big(\beta_j \frac{(1-\gamma_j)A_j A_h}{A_h - \gamma_j A_j}\Big)^{t-1} \beta_j R_1 s_{j,0}, \forall t \geq 1\\
k_{j,t} &= \frac{R_{t+1}}{R_{t+1} - \gamma_j A_{j,t+1}} s_{j,t} = \frac{A_h}{A_h - \gamma_j A_j} s_{j,t} = \frac{A_h}{A_h - \gamma_j A_j}\Big(\beta_j \frac{(1-\gamma_j)A_j A_h}{A_h - \gamma_j A_j}\Big)^{t-1} \beta_j R_1 s_{j,0}\\
b_{j,t} &= \frac{\gamma_j A_{j,t+1}}{R_{t+1} - \gamma_j A_{j,t+1}} s_{j,t} = \frac{\gamma_j A_j}{A_h - \gamma_j A_{j,t}} s_{j,t}, \forall t \geq 1.
\end{aligned}$$

For each agent $j = m$, we have

$$\begin{aligned}
s_{m,0} &= \beta_m w_{m,0}, \quad s_{m,t} = \beta_m \frac{(1-\gamma_m)A_{m,t}R_t}{R_t - \gamma_m A_{m,t}} s_{m,t-1}, \forall t \geq 1\\
s_{m,t} &= \Big(\beta_m \frac{(1-\gamma_m)A_m A_h}{A_h - \gamma_m A_m}\Big)^{t-1} \Big(\beta_m \frac{(1-\gamma_m)A_m R_1}{R_1 - \gamma_m A_m}\Big) s_{m,0}, \forall t \geq 1\\
k_{m,t} &= \frac{R_{t+1}}{R_{t+1} - \gamma_m A_{m,t+1}} s_{m,t} = \frac{A_h}{A_h - \gamma_m A_m} s_{j,t}\\
&= \frac{A_h}{A_h - \gamma_m A_m}\Big(\beta_m \frac{(1-\gamma_m)A_m A_h}{A_h - \gamma_m A_m}\Big)^{t-1} \Big(\beta_m \frac{(1-\gamma_m)A_m R_1}{R_1 - \gamma_m A_m}\Big) s_{m,0}\\
b_{m,t} &= \frac{\gamma_m A_{m,t+1}}{R_{t+1} - \gamma_m A_{m,t+1}} s_{m,t} = \frac{\gamma_m A_m}{A_h - \gamma_m A_m} s_{m,t}, \forall t \geq 1.
\end{aligned}$$

From the market clearing condition $\sum_i b_{i,t} = 0$, we have $\sum_i s_{i,t} = \sum_i k_{i,t}$ which implies that

$$k_{h,t} = \sum_i s_{i,t} - \sum_{i \neq h} k_{i,t} = \sum_i s_{i,t} - \sum_{i>h} k_{i,t} \text{ (since } k_{i,t} = 0, \forall i < h) \tag{B.20}$$

$$= \sum_{i \leq h} s_{i,t} - \sum_{j>h} b_{j,t} \tag{B.21}$$

$$= \sum_{i \leq h} \beta_i^t A_h^{t-1} R_1 s_{i,0} - \sum_{j=h+1}^{m-1} \frac{\gamma_j A_j}{A_h - \gamma_j A_j}\Big(\beta_j \frac{(1-\gamma_j)A_j A_h}{A_h - \gamma_j A_j}\Big)^{t-1} \beta_j R_1 s_{j,0} \tag{B.22}$$

$$- \frac{\gamma_m A_m}{A_h - \gamma_m A_m}\Big(\beta_m \frac{(1-\gamma_m)A_m A_h}{A_h - \gamma_m A_m}\Big)^{t-1} \Big(\beta_m \frac{(1-\gamma_m)A_m R_1}{R_1 - \gamma_m A_m}\Big) s_{m,0}. \tag{B.23}$$

We will verify that $0 \leq k_{h,t}$ and $R_{t+1}b_{h,t} \leq \gamma_h A_h k_{h,t}$. We now check that the above list $((c_{i,t}, k_{i,t}, b_{i,t})_i, R_t)_t$ is an equilibrium. We use Lemma 5. It is easy to verify the market clearing conditions and the FOCs.

Condition $R_1 \in (A_{m-1}, A_m)$ is ensured by the assumption that

$$A_{m-1} < \gamma_m A_m \Big(1 + \frac{s_{m,0}}{\sum_{i \neq m} s_{i,0}}\Big) < A_m.$$

- We verify borrowing constraints: $R_{t+1}b_{i,t} \leq \gamma_i A_i k_{i,t}$ and $k_{i,t} \geq 0$, $\forall t \geq 0$. It is clear for any $j > m$. Let us consider agent h. This is satisfied for $t = 0$. Let us consider $t \geq 1$. Since $R_{t+1} = A_h$, $\forall t \geq 1$, this becomes $k_{h,t} - s_{h,t} = b_{h,t} \leq \gamma_h k_{h,t}$, or, equivalently, $(1-\gamma_h)k_{h,t} \leq s_{h,t}$. Note that $0 \leq k_{h,t} \leq \frac{s_{h,t}}{1-\gamma_h}$ becomes

$$\frac{\beta_h^t s_{h,0}}{1-\gamma_h} \geq \sum_{i\leq h} \beta_i^t s_{i,0} - \sum_{j=h+1}^{m-1} \frac{\gamma_j A_j}{A_h - \gamma_j A_j}\Big(\beta_j \frac{(1-\gamma_j)A_j}{A_h-\gamma_j A_j}\Big)^{t-1}\beta_j s_{j,0} \tag{B.24}$$

$$- \frac{\gamma_m A_m}{A_h - \gamma_m A_m}\Big(\beta_m \frac{(1-\gamma_m)A_m}{A_h-\gamma_m A_m}\Big)^{t-1}\Big(\beta_m \frac{(1-\gamma_m)A_m}{R_1-\gamma_m A_m}\Big)s_{m,0} \geq 0 \tag{B.25}$$

$$\Leftrightarrow \frac{\beta_h^t s_{h,0}}{1-\gamma_h} \geq \sum_{i\leq h} \beta_i^t s_{i,0} - \sum_{j=h+1}^{m-1} \frac{\gamma_j A_j}{A_h - \gamma_j A_j}\Big(\beta_j \frac{(1-\gamma_j)A_j}{A_h-\gamma_j A_j}\Big)^{t-1}\beta_j s_{j,0} \tag{B.26}$$

$$- \Big(\beta_m \frac{(1-\gamma_m)A_m}{A_h-\gamma_m A_m}\Big)^{t}\Big(\frac{\gamma_m A_m}{R_1-\gamma_m A_m}\Big)s_{m,0} \geq 0. \tag{B.27}$$

Since $R_1 = \gamma_m A_m(1 + \frac{s_{m,0}}{\sum_{i\neq m} s_{i,0}})$, this is equivalent to

$$\frac{\beta_h^t s_{h,0}}{1-\gamma_h} \geq \sum_{i\leq h} \beta_i^t s_{i,0} - \sum_{j=h+1}^{m-1} \frac{\gamma_j A_j}{A_h - \gamma_j A_j}\Big(\beta_j \frac{(1-\gamma_j)A_j}{A_h-\gamma_j A_j}\Big)^{t-1}\beta_j s_{j,0} \tag{B.28}$$

$$- \frac{\gamma_m A_m}{A_h - \gamma_m A_m}\Big(\beta_m \frac{(1-\gamma_m)A_m}{A_h-\gamma_m A_m}\Big)^{t-1}\Big(\beta_m \frac{(1-\gamma_m)A_m}{R_1-\gamma_m A_m}\Big)s_{m,0} \geq 0 \tag{B.29}$$

$$\Leftrightarrow \frac{\beta_h^t s_{h,0}}{1-\gamma_h} \geq \sum_{i\leq h} \beta_i^t s_{i,0} - \sum_{j=h+1}^{m-1} \frac{\gamma_j A_j}{A_h - \gamma_j A_j}\Big(\beta_j \frac{(1-\gamma_j)A_j}{A_h-\gamma_j A_j}\Big)^{t-1}\beta_j s_{j,0} \tag{B.30}$$

$$- \Big(\beta_m \frac{(1-\gamma_m)A_m}{A_h-\gamma_m A_m}\Big)^{t}(S_0 - s_{m,0}) \geq 0. \tag{B.31}$$

- Tranversality conditions: $\lim_{T\to\infty} \beta_i^T u_i'(c_{i,T})(k_{i,T} - b_{i,T}) = 0$. It is easy to verify these conditions because $\beta_i \in (0,1)$ and $u'(c) = 1/c$.

The aggregate output at date 1 is $Y_1 = A_m S_0$. We now compute

$$\begin{aligned}
Y_{t+1} = &\sum_{i\geq h} A_i k_{i,t} = A_h\Big(\sum_{i\leq h} \beta_i^t A_h^{t-1} R_1 s_{i,0} - \sum_{j=h+1}^{m-1} \frac{\gamma_j A_j}{A_h-\gamma_j A_j}\Big(\beta_j \frac{(1-\gamma_j)A_j A_h}{A_h-\gamma_j A_j}\Big)^{t-1}\beta_j R_1 s_{j,0} \\
&- \frac{\gamma_m A_m}{A_h-\gamma_m A_m}\Big(\beta_m \frac{(1-\gamma_m)A_m A_h}{A_h-\gamma_m A_m}\Big)^{t-1}\Big(\beta_m \frac{(1-\gamma_m)A_m R_1}{R_1-\gamma_m A_m}\Big)s_{m,0}\Big) \\
&+ \sum_{j=h+1}^{m-1} A_j \frac{A_h}{A_h-\gamma_j A_j}\Big(\beta_j \frac{(1-\gamma_j)A_j A_h}{A_h-\gamma_j A_j}\Big)^{t-1}\beta_j R_1 s_{j,0} \\
&+ A_m \frac{A_h}{A_h-\gamma_m A_m}\Big(\beta_m \frac{(1-\gamma_m)A_m A_h}{A_h-\gamma_m A_m}\Big)^{t-1}\Big(\beta_m \frac{(1-\gamma_m)A_m R_1}{R_1-\gamma_m A_m}\Big)s_{m,0}.
\end{aligned}$$

Hence, we get

$$\begin{aligned}\frac{Y_{t+1}}{A_h^t} =&\sum_{i\le h}\beta_i^t R_1 s_{i,0} + \sum_{j=h+1}^{m-1}\frac{A_j(1-\gamma_j)}{A_h-\gamma_j A_j}\Big(\beta_j\frac{(1-\gamma_j)A_j}{A_h-\gamma_j A_j}\Big)^{t-1}\beta_j R_1 s_{j,0}\\ &+\frac{A_m(1-\gamma_m)}{A_h-\gamma_m A_m}\Big(\beta_m\frac{(1-\gamma_m)A_m}{A_h-\gamma_m A_m}\Big)^{t-1}\Big(\beta_m\frac{(1-\gamma_m)A_m R_1}{R_1-\gamma_m A_m}\Big)s_{m,0}\\ =&\sum_{i\le h}\beta_i^t R_1 s_{i,0} + \sum_{j=h+1}^{m-1}\Big(\beta_j\frac{(1-\gamma_j)A_j}{A_h-\gamma_j A_j}\Big)^{t} R_1 s_{j,0} + \Big(\beta_m\frac{(1-\gamma_m)A_m}{A_h-\gamma_m A_m}\Big)^{t}\frac{(1-\gamma_m)A_m R_1}{R_1-\gamma_m A_m}s_{m,0}.\end{aligned}$$

Since $R_1 = \gamma_m A_m \frac{S_0}{\sum_{i\neq m}s_{i,0}}$, we have that

$$\frac{R_1}{R_1-\gamma_m A_m}s_{m,0} = \frac{\gamma_m A_m \frac{S_0}{\sum_{i\neq m}s_{i,0}}}{\gamma_m A_m \frac{S_0}{\sum_{i\neq m}s_{i,0}}-\gamma_m A_m}s_{m,0} = S_0.$$

Therefore, we get that

$$\frac{Y_{t+1}}{A_h^t A_m S_0} =\gamma_m\frac{\sum_{i\le h}\beta_i^t s_{i,0}+\sum_{j=h+1}^{m-1}\Big(\beta_j\frac{(1-\gamma_j)A_j}{A_h-\gamma_j A_j}\Big)^t s_{j,0}}{\sum_{i<m}s_{i,0}} + \Big(\beta_m\frac{(1-\gamma_m)A_m}{A_h-\gamma_m A_m}\Big)^t(1-\gamma_m).$$

From this, we can see that $\frac{Y_{t+1}}{\partial\gamma_j} > 0, \forall h+1 < j < m-1$ since $\beta_j\frac{(1-\gamma_j)A_j}{A_h-\gamma_j A_j}$ is increasing in γ_j. The intuition is simple: the credit limits of these agents do not affect the equilibrium interest rate while it allows these producers to borrow more and produce more.

We now look at the effect of γ_m. In terms of interest, this credit limit positively affects the interest rate R_1 and hence the savings of any agents.

$$\frac{\frac{\partial Y_{t+1}}{\partial\gamma_m}}{A_h^t A_m S_0} =\Big(\frac{\sum_{i\le h}\beta_i^t s_{i,0}}{\sum_{i<m}s_{i,0}} + \frac{\sum_{j=h+1}^{m-1}\Big(\beta_j\frac{(1-\gamma_j)A_j}{A_h-\gamma_j A_j}\Big)^t s_{j,0}}{\sum_{i<m}s_{i,0}}\Big) + (\beta_m A_m)^t\Big(\frac{(1-\gamma_m)^{t+1}}{(A_h-\gamma_m A_m)^t}\Big)$$

We compute

$$\begin{aligned}\frac{\partial\Big(\frac{(1-\gamma_m)^{t+1}}{(A_h-\gamma_m A_m)^t}\Big)}{\partial\gamma_m} &= \frac{-(t+1)(1-\gamma_m)^t(A_h-\gamma_m A_m)^t + tA_m(A_h-\gamma_m A_m)^{t-1}(1-\gamma_m)^{t+1}}{(A_h-\gamma_m A_m)^{2t}}\\ =&(1-\gamma_m)^t\frac{-(t+1)(A_h-\gamma_m A_m)+tA_m(1-\gamma_m)}{(A_h-\gamma_m A_m)^{t+1}}\\ =&(1-\gamma_m)^t\frac{-(t+1)A_h-\gamma_m A_m+tA_m}{(A_h-\gamma_m A_m)^{t+1}} = (1-\gamma_m)^t\frac{t(A_m-A_h)-(A_h-\gamma_m A_m)}{(A_h-\gamma_m A_m)^{t+1}}\\ =&(1-\gamma_m)^t\frac{-(t+1)A_h-\gamma_m A_m+tA_m}{(A_h-\gamma_m A_m)^{t+1}} = \frac{t(A_m-A_h)(1-\gamma_m)^t}{(A_h-\gamma_m A_m)^{t+1}} - \frac{(1-\gamma_m)^t}{(A_h-\gamma_m A_m)^t}.\end{aligned}$$

Therefore, we get that

$$\begin{aligned}\Big(\frac{\sum_{i<m}s_{i,0}}{A_h^t A_m S_0}\Big)\frac{\partial Y_{t+1}}{\partial\gamma_m} =&\sum_{i\le h}\beta_i^t s_{i,0} + \sum_{j=h+1}^{m-1}\Big(\beta_j\frac{(1-\gamma_j)A_j}{A_h-\gamma_j A_j}\Big)^t s_{j,0} - \Big(\beta_m\frac{(1-\gamma_m)A_m}{A_h-\gamma_m A_m}\Big)^t\sum_{i<m}s_{i,0}\\ &+\frac{(\beta_m(1-\gamma_m)A_m)^t}{(A_h-\gamma_m A_m)^{t+1}}(A_m-A_h)t\sum_{i<m}s_{i,0}.\end{aligned}$$

This is strictly positive thanks to the assumption (B.17a).

□

Proposition 21 (**equilibrium with $R_t \in (A_{m-1,t}, A_{m,t}), \forall t$**). *Assume that $F_{i,t}(k) = A_i k$, $\forall i, \forall k \geq 0$ with $\max_i \gamma_i A_i < A_1 < A_2 < \ldots < A_m$, and utility function $u_i(c) = ln(c)\ \forall i$.*

Assume also that

$$\frac{A_{m-1,1}}{A_{m,1}} < \gamma_m \frac{S_0}{\sum_{i<m} s_{i,0}} < 1 \tag{B.32}$$

$$\frac{A_{m-1,t+1}}{A_{m,t+1}} < \gamma_m + (1-\gamma_m)\beta_m \frac{\sum_{i<m} \beta_i^{t-1} s_{i,0}}{\sum_{i\neq m} \beta_i^t s_{i,0}}, \forall t \geq 1 \tag{B.33}$$

$$\beta_m \frac{\sum_{i<m} \beta_i^{t-1} s_{i,0}}{\sum_{i\neq m} \beta_i^t s_{i,0}} < 1, \forall t \geq 1. \tag{B.34}$$

Then, there exists an equilibrium whose the interest rates are

$$R_1 = \gamma_m A_{m,1} \frac{S_0}{\sum_{i<m} s_{i,0}} \tag{B.35}$$

$$R_{t+1} = A_{m,t+1}\Big(\gamma_m + (1-\gamma_m)\beta_m \frac{\sum_{i<m} \beta_i^{t-1} s_{i,0}}{\sum_{i<m} \beta_i^t s_{i,0}}\Big), \forall t \geq 1, \tag{B.36}$$

Observe that $R_t \in (A_{m-1,t}, A_{m,t}), \forall t$. When $A_{m,t}$ converges to A_m, then we have

$$R = A_m\Big(\gamma_m + \frac{\beta_m}{\beta_{i_0}}(1-\gamma_m)\Big). \tag{B.37}$$

In this equilibrium, the aggregate output is increasing in the credit limit γ_m at any date.

Proof of Proposition 21. Consider an equilibrium with $R_t \in (A_{m-1,t}, A_{m,t}), \forall t$.

For agent $i < m$, since $A_{i,t} < R_t$, $\forall t$, we have $k_{i,t} = 0$ and hence we find that

$$s_{i,0} = \beta_i w_{i,0}, \quad s_{i,t} = \beta_i R_t s_{i,t-1}\ \forall t \geq 1$$
$$-b_{i,t} = s_{i,t} = \beta_i^t R_t \cdots R_1 s_{i,0}.$$

For agent $j > h$, since $A_{j,t} > R_t = A_{h,t}$, $\forall t$, her borrowing constraint is always binding: $R_t b_{j,t-1} = \gamma_j A_{j,t} k_{j,t-1}$. Therefore, we have

$$s_{j,t} = k_{j,t}\Big(1 - \frac{\gamma_j A_{j,t+1}}{R_{t+1}}\Big), \quad A_{j,t}k_{j,t-1} - R_t b_{j,t-1} = (1-\gamma_j)A_{j,t}k_{j,t-1}, \forall t \geq 1.$$

From this, we can compute that

$$\begin{aligned}
s_{j,0} &= \beta_j w_{j,0},\\
s_{j,t} &= \beta_j \frac{(1-\gamma_j)A_{j,t}R_t}{R_t - \gamma_j A_{j,t}} s_{j,t-1} = \Big(\beta_j \frac{(1-\gamma_j)A_{j,t}R_t}{R_t - \gamma_j A_{j,t}}\Big) \cdots \Big(\beta_j \frac{(1-\gamma_j)A_{j,1}R_1}{R_1 - \gamma_j A_{j,1}}\Big) s_{j,0}\\
k_{j,t} &= \frac{1}{1 - \frac{\gamma_j A_{j,t+1}}{R_{t+1}}} s_{j,t} = \frac{R_{t+1}}{R_{t+1} - \gamma_j A_{j,t+1}} s_{j,t}\\
&= \frac{R_{t+1}}{R_{t+1} - \gamma_j A_{j,t+1}} \Big(\beta_j \frac{(1-\gamma_j)A_{j,t}R_t}{R_t - \gamma_j A_{j,t}}\Big) \cdots \Big(\beta_j \frac{(1-\gamma_j)A_{j,1}R_1}{R_1 - \gamma_j A_{j,1}}\Big) s_{j,0}\\
b_{j,t} &= \frac{\gamma_j A_{j,t+1}}{R_{t+1}} k_{j,t} = \frac{\gamma_j A_{j,t+1}}{R_{t+1} - \gamma_j A_{j,t+1}} s_{j,t}\\
&= \frac{\gamma_j A_{j,t+1}}{R_{t+1} - \gamma_j A_{j,t+1}} \Big(\beta_j \frac{(1-\gamma_j)A_{j,t}R_t}{R_t - \gamma_j A_{j,t}}\Big) \cdots \Big(\beta_j \frac{(1-\gamma_j)A_{j,1}R_1}{R_1 - \gamma_j A_{j,1}}\Big) s_{j,0}.
\end{aligned}$$

The market clearing condition writes $\sum_i b_{i,t} = 0$, i.e., $\sum_{i>h} b_{i,t} = -\sum_{i<h} b_{i,t} = 0$. This becomes

$$\sum_{j>h} \frac{\gamma_j A_{j,t+1}}{R_{t+1} - \gamma_j A_{j,t+1}} \Big(\beta_j \frac{(1-\gamma_j)A_{j,t}R_t}{R_t - \gamma_j A_{j,t}}\Big) \cdots \Big(\beta_j \frac{(1-\gamma_j)A_{j,1}R_1}{R_1 - \gamma_j A_{j,1}}\Big) s_{j,0} = \sum_{i<h} \beta_i^t R_t \cdots R_1 s_{i,0}$$

$$\Leftrightarrow \sum_{j>h} \frac{\gamma_j A_{j,t+1}}{R_{t+1} - \gamma_j A_{j,t+1}} \Big(\beta_j \frac{(1-\gamma_j)A_{j,t}}{R_t - \gamma_j A_{j,t}}\Big) \cdots \Big(\beta_j \frac{(1-\gamma_j)A_{j,1}}{R_1 - \gamma_j A_{j,1}}\Big) s_{j,0} = \sum_{i<h} \beta_i^t s_{i,0}$$

$$\text{Date 1: } \sum_{j>h} \frac{\gamma_j A_{j,1}}{R_1 - \gamma_j A_{j,1}} s_{j,0} = \sum_{i<h} s_{i,0}$$

$$\text{Date 2: } \sum_{j>h} \frac{\gamma_j A_{j,2}}{R_2 - \gamma_j A_{j,2}} \Big(\beta_j \frac{(1-\gamma_j)A_{j,1}}{R_1 - \gamma_j A_{j,1}}\Big) s_{j,0} = \sum_{i<h} \beta_i s_{i,0}$$

$$\text{Date 3: } \sum_{j>h} \frac{\gamma_j A_{j,3}}{R_3 - \gamma_j A_{j,3}} \Big(\beta_j \frac{(1-\gamma_j)A_{j,2}}{R_2 - \gamma_j A_{j,2}}\Big) \Big(\beta_j \frac{(1-\gamma_j)A_{j,1}}{R_1 - \gamma_j A_{j,1}}\Big) s_{j,0} = \sum_{i<h} \beta_i^2 s_{i,0}$$

Let us focus on the equilibrium with $R_t \in (A_{m,t-1}, A_{m,t})$, i.e., $j = m$. We have that

$$\frac{\gamma_m A_{m,t+1}}{R_{t+1} - \gamma_m A_{m,t+1}} \Big(\beta_m \frac{(1-\gamma_m)A_{m,t}}{R_t - \gamma_m A_{m,t}}\Big) \cdots \Big(\beta_m \frac{(1-\gamma_m)A_{m,1}}{R_1 - \gamma_m A_{m,1}}\Big) s_{m,0} = \sum_{i<m} \beta_i^t s_{i,0}$$

$$R_1\text{: } \frac{\gamma_m A_{m,1}}{R_1 - \gamma_m A_{m,1}} s_{m,0} = \sum_{i<m} s_{i,0}$$

$$R_2\text{: } \frac{\gamma_m A_{m,2}}{R_2 - \gamma_m A_{m,2}} \Big(\beta_m \frac{(1-\gamma_m)A_{m,1}}{R_1 - \gamma_m A_{m,1}}\Big) s_{m,0} = \sum_{i<m} \beta_i s_{i,0}.$$

Therefore, we can find the interest rate. First, the interest rate $R_1 = \gamma_m A_{m,1} \frac{S_0}{\sum_{i<m} s_{i,0}}$. For date $t \geq 1$, we have

$$\begin{aligned}
\frac{\sum_{i \neq m} \beta_i^t s_{i,0}}{\sum_{i \neq m} \beta_i^{t-1} s_{i,0}} &= \frac{\frac{\gamma_m A_{m,t+1}}{R_{t+1} - \gamma_m A_{m,t+1}} \Big(\beta_m \frac{(1-\gamma_m)A_{m,t}}{R_t - \gamma_m A_{m,t}}\Big) \cdots \Big(\beta_m \frac{(1-\gamma_m)A_{m,1}}{R_1 - \gamma_m A_{m,1}}\Big) s_{m,0}}{\frac{\gamma_m A_{m,t}}{R_t - \gamma_m A_{m,t}} \Big(\beta_m \frac{(1-\gamma_m)A_{m,t-1}}{R_{t-1} - \gamma_m A_{m,t-1}}\Big) \cdots \Big(\beta_m \frac{(1-\gamma_m)A_{m,1}}{R_1 - \gamma_m A_{m,1}}\Big) s_{m,0}} \\
&= \frac{\frac{\gamma_m A_{m,t+1}}{R_{t+1} - \gamma_m A_{m,t+1}} \Big(\beta_m \frac{(1-\gamma_m)A_{m,t}}{R_t - \gamma_m A_{m,t}}\Big)}{\frac{\gamma_m A_{m,t}}{R_t - \gamma_m A_{m,t}}} = \frac{\gamma_m A_{m,t+1}}{R_{t+1} - \gamma_m A_{m,t+1}} \frac{\beta_m (1-\gamma_m)}{\gamma_m} \\
&= \frac{\beta_m (1-\gamma_m) A_{m,t+1}}{R_{t+1} - \gamma_m A_{m,t+1}}.
\end{aligned}$$

To sum up, we obtain that:

$$R_1 = \gamma_m A_{m,1} \frac{S_0}{\sum_{i<m} s_{i,0}} \tag{B.38}$$

$$R_{t+1} = A_{m,t+1} \Big(\gamma_m + (1-\gamma_m)\beta_m \frac{\sum_{i \neq m} \beta_i^{t-1} s_{i,0}}{\sum_{i \neq m} \beta_i^t s_{i,0}}\Big), \forall t \geq 1. \tag{B.39}$$

We need to check that $A_{m-1,t} < R_t < A_{m,t}$, $\forall t$. It means that

$$\frac{A_{m-1,1}}{A_{m,1}} < \gamma_m \frac{S_0}{\sum_{i<m} s_{i,0}} < 1 \tag{B.40}$$

$$\frac{A_{m-1,t+1}}{A_{m,t+1}} < \gamma_m + (1-\gamma_m)\beta_m \frac{\sum_{i\neq m} \beta_i^{t-1} s_{i,0}}{\sum_{i\neq m} \beta_i^t s_{i,0}} < 1, \forall t \geq 1 \tag{B.41}$$

$$\beta_m \frac{\sum_{i\neq m} \beta_i^{t-1} s_{i,0}}{\sum_{i\neq m} \beta_i^t s_{i,0}} < 1, \forall t \geq 1. \tag{B.42}$$

So, we can see that R_t is increasing in γ_m for any t.

Let $\frac{A_{m-1,t+1}}{A_{m,t+1}}$ converges to $\frac{A_{m-1}}{A_m}$, $\forall t$ and $\beta_{i_0} = \max_{i<m} \beta_i$. We must have

$$\frac{A_{m-1}}{A_m} \leq \gamma_m + (1-\gamma_m)\frac{\beta_m}{\beta_{i_0}} \leq 1. \tag{B.43}$$

We now find the capital at date t. We have

$$\begin{aligned}
k_{m,0} &= \sum_i s_{i,0} \\
k_{m,1} &= \sum_{i<m} s_{i,1} + s_{m,1} = \sum_{i<m} \beta_i R_1 s_{i,0} + \beta_m \frac{(1-\gamma_m)A_{m,1}R_1}{R_1 - \gamma_m A_{m,1}} s_{m,0} \\
&= R_1\Big(\sum_{i<m} \beta_i s_{i,0} + \beta_m \frac{(1-\gamma_m)A_{m,1}}{R_1 - \gamma_m A_{m,1}} s_{m,0}\Big) = R_1\Big(\sum_{i<m} \beta_i s_{i,0} + \beta_m \frac{1-\gamma_m}{\gamma_m} \sum_{i<m} s_{i,0}\Big) \\
&= \gamma_m A_{m,1} \frac{S_0}{\sum_{i<m} s_{i,0}} \Big(\sum_{i<m} \beta_i s_{i,0} + \beta_m \frac{1-\gamma_m}{\gamma_m} \sum_{i<m} s_{i,0}\Big) \\
&= A_{m,1} S_0 \Big(\gamma_m \frac{\sum_{i<m} \beta_i s_{i,0}}{\sum_{i<m} s_{i,0}} + \beta_m (1-\gamma_m)\Big).
\end{aligned}$$

Since $\beta_m \frac{\sum_{i\neq m} \beta_i^{t-1} s_{i,0}}{\sum_{i\neq m} \beta_i^t s_{i,0}} < 1, \forall t \geq 1$, we see that $k_{m,1}$ is increasing in γ_m.

For any date $t \geq 2$, by using $\frac{\gamma_m A_{m,t+1}}{R_{t+1} - \gamma_m A_{m,t+1}}\Big(\beta_m \frac{(1-\gamma_m)A_{m,t}}{R_t - \gamma_m A_{m,t}}\Big) \cdots \Big(\beta_m \frac{(1-\gamma_m)A_{m,1}}{R_1 - \gamma_m A_{m,1}}\Big) s_{m,0} = \sum_{i<m} \beta_i^t s_{i,0}$, we get that

$$\begin{aligned}
k_{m,t} &= \frac{R_{t+1}}{R_{t+1} - \gamma_m A_{m,t+1}} \Big(\beta_m \frac{(1-\gamma_m)A_{m,t}R_t}{R_t - \gamma_m A_{m,t}}\Big) \cdots \Big(\beta_m \frac{(1-\gamma_m)A_{m,1}R_1}{R_1 - \gamma_m A_{m,1}}\Big) s_{m,0} \\
&= R_{t+1} \dots R_1 \frac{1}{\gamma_m A_{m,t+1}} \frac{\gamma_m A_{m,t+1}}{R_{t+1} - \gamma_m A_{m,t+1}} \Big(\beta_m \frac{(1-\gamma_m)A_{m,t}R_t}{R_t - \gamma_m A_{m,t}}\Big) \cdots \Big(\beta_m \frac{(1-\gamma_m)A_{m,1}R_1}{R_1 - \gamma_m A_{m,1}}\Big) s_{m,0} \\
&= R_{t+1} \dots R_1 \frac{1}{\gamma_m A_{m,t+1}} \sum_{i<m} \beta_i^t s_{i,0} \\
&= R_{t+1} \dots R_2 \frac{1}{A_{m,t+1}} \Big(\sum_{i<m} \beta_i^t s_{i,0}\Big) A_{m,1} \frac{S_0}{\sum_{i<m} s_{i,0}} \\
&= R_t \dots R_2 A_{m,1} S_0 \frac{\sum_{i<m} \beta_i^t s_{i,0}}{\sum_{i<m} s_{i,0}} \Big(\gamma_m + (1-\gamma_m)\beta_m \frac{\sum_{i\neq m} \beta_i^{t-1} s_{i,0}}{\sum_{i\neq m} \beta_i^t s_{i,0}}\Big), \forall t \geq 1,
\end{aligned}$$

where the last equality follows $R_{t+1} = A_{m,t+1}\Big(\gamma_m + (1-\gamma_m)\beta_m \frac{\sum_{i\neq m} \beta_i^{t-1} s_{i,0}}{\sum_{i\neq m} \beta_i^t s_{i,0}}\Big)$. We see that this is increasing in the credit limit γ_m, and so is the aggregate output. This is also increasing in agents' productivity.

□

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D Online appendix 1: Characterization of equilibrium in a two-period model

Proof of Theorem 1

We have the following result which characterizes the optimal solution of agents.

Lemma 7 (individual choice - linear production function)**.** *Assume that $F_i(K) = A_iK$. Let $R > 0$ be given. The solution for agent i's maximization problem is described as follows.*

1. *If $R \leq \gamma_i A_i$, then there is no solution ($k_i = \infty$).*

2. *If $A_i > R > \gamma_i A_i$, then agent i borrows from the financial market and the borrowing constraint is binding. We have $k_i = \frac{R}{R-\gamma_i A_i} S_i, a_i = \frac{\gamma_i A_i}{R-\gamma_i A_i} S_i, \pi_i = A_i k_i - R b_i = \frac{R(1-\gamma_i)}{R-\gamma_i A_i} A_i S_i$.*

3. *If $A_i = R$, then the solutions for the agent's problem include all pairs (k_i, b_i) such that $-S_i \leq b_i \leq \frac{\gamma_i}{1-\gamma_i} S_i$ and $k_i = b_i + S_i$.*

4. *If $A_i < R$, then agent i does not produce goods and invest all her initial wealth in the financial market: $k_i = 0, b_i = -S_i$.*

According to Definition of $\mathbb{D}_n, \mathbb{B}_n$,

$$\mathbb{D}_n \equiv \sum_{i=n}^{m} \frac{A_n S_i}{A_n - \gamma_i A_i}, \ \forall n \geq 1, \quad \mathbb{B}_n \equiv \sum_{i=n+1}^{m} \frac{A_n S_i}{A_n - \gamma_i A_i}, \ \forall n \geq 1, \tag{A.1}$$

we observe that

$$\frac{S_m}{1-\gamma_m} = \mathbb{D}_m < \cdots < \mathbb{D}_{n+1} < \mathbb{B}_n < \mathbb{D}_n < \mathbb{B}_{n-1} < \cdots < \mathbb{B}_1 = \sum_{i=2}^{m} \frac{A_1 S_i}{A_1 - \gamma_i A_i}. \tag{A.2}$$

Theorem 2 is a direct consequence of the existence of equilibrium and Lemmas 8-12 below. First, the following result is a direct consequence of Lemma 7.

Lemma 8. *Assume that $A_1 < A_2 < \cdots < A_m$. If $\max_i(\gamma_i A_i) \geq A_n$ and there exists an equilibrium, then $R > A_n$.*

By comparing $\mathbb{B}_n, \mathbb{D}_n$ with the aggregate capital supply $S \equiv \sum_{i=1}^m S_i$, we obtain the following result.

Lemma 9. *Assume that $A_1 < A_2 < \cdots < A_m$. Denote $S \equiv \sum_{i=1}^m S_i$ the aggregate capital. Consider an equilibrium.*

1. *If $A_n > \max_i(\gamma_i A_i)$ and $R > A_n$, then $\mathbb{B}_n > S$. Consequently, if $A_n > \max_i(\gamma_i A_i)$ and $\mathbb{B}_n \leq S$, then $R \leq A_n$.*

2. *If $A_n > \max_i(\gamma_i A_i)$ and $R < A_n$, then $S > \mathbb{D}_n$. Consequently, if $A_n > \max_i(\gamma_i A_i)$ and $S \leq \mathbb{D}_n$, then $R \geq A_n$.*

Proof. 1. Since $R > A_i$ for any $i = 1, \dots, n$, Lemma 7 implies that $k_i = 0, a_i = -S_i$ $\forall i = 1, \dots, n$. Hence, we have, by using market clearing condition,

$$\sum_{i=1}^{n} S_i = -\sum_{i=1}^{n} a_i = \sum_{i=n+1}^{m} a_i \leq \sum_{i=n+1}^{m} \frac{\gamma_i A_i}{R - \gamma_i A_i} S_i < \sum_{i=n+1}^{m} \frac{\gamma_i A_i}{A_n - \gamma_i A_i} S_i \tag{A.3}$$

where the first inequality follows $b_i \leq \frac{\gamma_i A_i S_i}{R - \gamma_i A_i}$ while the last inequality follows $R > A_n > \max_i(\gamma_i A_i)$ and the fact that the function $Func(R) \equiv \sum_{i=n+1}^{m} \frac{\gamma_i A_i}{R - \gamma_i A_i} S_i$ is decreasing in $(\max_i(\gamma_i A_i), +\infty)$. Notice that this function is not decreasing in the interval $(0, \infty)$.

2. Since $R < A_n$, again Lemma 7 implies that $k_i = \frac{R}{R - \gamma_i A_i} S_i$ and $a_i = \frac{\gamma_i A_i}{R - \gamma_i A_i} S_i \forall i \geq n$. We have

$$\sum_{i=n}^{m} \frac{A_n S_i}{A_n - \gamma_i A_i} S < \sum_{i=n}^{m} \frac{R S_i}{R - \gamma_i A_i} = \sum_{i=n}^{m} k_i \leq \sum_{i=1}^{m} S_i = S \tag{A.4}$$

where the first inequality follows $A_n > R > \max_i(\gamma_i A_i)$.

□

Lemma 10. *$R = A_n$ if and only if $A_n > \max_i(\gamma_i A_i)$ and $\mathbb{B}_n \leq S \leq \mathbb{D}_n$.*[30]

Proof. If $R = A_n$, we have $k_i = 0$ $\forall i \leq n-1$ and $k_i = \frac{R S_i}{R - \gamma_i A_i}$ $\forall i \geq n+1$. This implies that $A_n = R > \max_i(\gamma_i A_i)$. Since $0 \leq k_n \leq \frac{R S_i}{R - f_n A_n}$, we have

$$\sum_{i=n+1}^{m} \frac{R S_i}{R - \gamma_i A_i} \leq \sum_{i} k_i = \sum_{i=n}^{m} k_i \leq \sum_{i=n}^{m} \frac{R S_i}{R - \gamma_i A_i} = \sum_{i=n} \frac{A_n S_i}{A_n - \gamma_i A_i} \tag{A.5}$$

By converse, suppose that $A_n > \max_i(\gamma_i A_i)$ and $\sum_{i=n+1}^{m} \frac{A_n S_i}{A_n - \gamma_i A_i} \leq S \leq \sum_{i=n}^{m} \frac{A_n S_i}{A_n - \gamma_i A_i}$. Applying points 1 and 2 of Lemma 9, we have $R \geq A_n$ and $R \leq A_n$. Hence $R = A_n$. □

By combining Lemma 9 and the fact that $R > \max_i(\gamma_i A_i)$, we obtain the following result.

Lemma 11. *Assume that $A_1 < A_2 < \cdots < A_m$. Consider an equilibrium. If $R \in (A_n, A_{n+1})$, then $A_{n+1} > \max_i(\gamma_i A_i)$ and $R = R_n^L$ (hence $R_n^L \in (A_n, A_{n+1})$).*

We now identify the necessary and sufficient conditions under which $R = R_n^L$.

Lemma 12. *$R = R_n^L \neq A_n$ if and only if one of the following conditions is satisfied:*

1. *$\max_i(\gamma_i A_i) < A_n < r_n^L < A_{n+1}$, or equivalently $\max_i(\gamma_i A_i) < A_n$ and $\mathbb{D}_{n+1} < S < \mathbb{B}_n$*
2. *$A_n \leq \max_i(\gamma_i A_i) < R_n^L < A_{n+1}$, or equivalently $A_n \leq \max_i(\gamma_i A_i) < R_n^L$ and $\mathbb{D}_{n+1} < S$.*

In any case, we have that $R_n^L \in [A_n, A_{n+1})$.

[30] We need condition $A_n > M \equiv \max_i(\gamma_i A_i)$ because that $R > \max_i(\gamma_i A_i)$. Condition $\sum_{i=n+1}^{m} \frac{A_n S_i}{A_n - \gamma_i A_i} \leq S$ ensures that $R \leq A_n$ while condition $S \leq \sum_{i=n}^{m} \frac{A_n S_i}{A_n - \gamma_i A_i}$ ensures that $R \geq A_n$.

Proof. **Part 1**. Assume that $R = R_n^L \neq A_n$. By definition of R and R_n^L, we have $\sum_{i=n+1}^m \frac{RS_i}{R-\gamma_i A_i} = S$, and $R_n^L > \max_i(\gamma_i A_i)$. We will prove that $R = R_n^L \in (A_n, A_{n+1})$.

If $R \leq A_n$, then $R < A_{n+1}$, and hence $k_i = \frac{RS_i}{R-\gamma_i A_i}$ $\forall i \geq n+1$. Since $\sum_{i=n+1}^m \frac{RS_i}{R-\gamma_i A_i} = S = \sum_i k_i$. We have $k_i = 0$ $\forall i \leq n$, and hence $k_n = 0$. This implies that $R \geq A_n$. Therefore, we have $R = A_n$, a contradiction. Thus, we have $R > A_n$.

If $R \geq A_{n+1}$, we have $k_i = 0$ $\forall i \leq n$. Hence $S = \sum_i k_i \leq \sum_{i=n+1}^m \frac{RS_i}{R-\gamma_i A_i}$. Since $\sum_{i=n+1}^m \frac{RS_i}{R-\gamma_i A_i} = S$, we have $k_i = \frac{RS_i}{R-\gamma_i A_i}$ $\forall i \geq n+1$. Hence $A_{n+1} \geq R$. So, $R = A_{n+1}$. We have just proved that $R \leq A_{n+1}$. By definition of R, we get that $A_{n+1} > \max_i(\gamma_i A_i)$. If $R_n^L = A_{n+1}$, then applying Lemma 10, we have $\sum_{i=n+2}^m \frac{A_{n+1}S_i}{A_{n+1}-\gamma_i A_i} = \mathbb{B}_{n+1} \leq S$. However, by definition of R_n^L, we have $\sum_{i=n+1}^m \frac{A_{n+1}S_i}{A_{n+1}-\gamma_i A_i} = S$, contradiction. Therefore, we obtain $R_n^L < A_{n+1}$.

We have just proved that $R_n^L \in (A_n, A_{n+1})$. Applying point 2 of Lemma 9, we have $S > \mathbb{D}_{n+1}$. There are two cases:

1. $\max_i(\gamma_i A_i) \geq A_n$. In this case, we have $A_n \leq \max_i(\gamma_i A_i) < R_n^L < A_{n+1}$.

2. $\max_i(\gamma_i A_i) < A_n$. We get $\max_i(\gamma_i A_i) < A_n < R_n^L < A_{n+1}$. Notice that, in this case, $R_n^L \in (A_n, A_{n+1})$ is equivalent to $\mathbb{D}_{n+1} < S < \mathbb{B}_n$.

Part 2. Conversely, assume that (i) $A_n \leq \max_i(\gamma_i A_i) < R_n^L < A_{n+1}$ or (ii) $\max_i(\gamma_i A_i) < A_n < R_n^L < A_{n+1}$.

1. If $A_n \leq \max_i(\gamma_i A_i) < R_n^L < A_{n+1}$. Condition $A_n \leq \max_i(\gamma_i A_i)$ implies that $R > A_n$. Then $k_i = 0$ $\forall i \leq n$, and hence $S = \sum_{i=n+1}^m k_i \leq \sum_{i=n+1}^m \frac{RS_i}{R-\gamma_i A_i}$

   By definition R_n^L, we have $S = \sum_{i=n+1}^m \frac{R_n^L S_i}{R_n^L-\gamma_i A_i}$. Since the function $f(X) \equiv \sum_{i=n+1}^m \frac{XS_i}{X-\gamma_i A_i}$ is decreasing in the interval $(\max_{i\geq n+1}(\gamma_i A_i), \infty)$ and $R, R_n^L > \max_i(\gamma_i A_i)$, we have $R \leq R_n^L$. This implies that $R \in (A_n, A_{n+1})$. Therefore, Lemma 11 implies that $R = R_n^L$.

2. If $\max_i(\gamma_i A_i) < A_n$ and $\mathbb{D}_{n+1} < S < \mathbb{B}_n$. We have $S < \mathbb{D}_n$ because $\mathbb{D}_n > \mathbb{B}_n$. According to point 2 of Lemma 9, we have $R \geq A_n$.

   Condition $S > \mathbb{D}_{n+1}$ implies that $S > \mathbb{B}_{n+1}$ because $\mathbb{D}_{n+1} > \mathbb{B}_{n+1}$. According to point 1 of Lemma 9, we have $R \leq A_{n+1}$.

   If $R = A_{n+1}$, then Lemma 10 implies that $S \leq \mathbb{D}_{n+1}$. This is a contradiction because $S > \mathbb{D}_{d+1}$.

   If $R = A_n$, Lemma 10 implies that $S \in [\mathbb{B}_n, \mathbb{D}_n]$. However, $S \leq \mathbb{B}_n$. Thus, we have $S = \mathbb{B}_n = \sum_{i=n+1}^m \frac{A_n S_i}{A_n-\gamma_i A_i}$. Since $A_n > \max_i(\gamma_i A_i)$, then $A_n = R_n^L$, a contradiction.

   Summing up, we have $R \in (A_n, A_{n+1})$. By applying point 3 of Lemma 11, we have $R = R_n^L$.

□

Remark 2. We can check that the regimes in Theorem 2 are not overlap, and the union of these regimes is equal to the set of economies satisfying $A_1 < \cdots < A_m$, or, formally,

$$\mathbf{E} = \cup_{i=1}^m \mathcal{A}_i \cup \cup_{i=1}^{m-1} \mathcal{R}_i \tag{A.6a}$$

$$\mathcal{X} \cap \mathcal{Y} = \emptyset \ \forall X, Y \in \{\mathcal{A}_1, \ldots, \mathcal{A}_m, \mathcal{R}_1, \ldots, \mathcal{R}_{m-1}\} \text{ and } X \neq Y. \tag{A.6b}$$

Denote $M \equiv \max_i(\gamma_i A_i)$. By definition, we see that:

1. The economy $\mathcal{E} \equiv (F_i, \gamma_i, S_i)_{i=1,\dots,m} \in \mathcal{A}_1$ if and only if $A_1 > \max_i(\gamma_i A_i)$ and $S > \mathbb{B}_1$.

2. $\mathcal{E} \in \mathcal{A}_m$ if and only if $S \leq \mathbb{D}_m$.

3. $\mathcal{E} \in \mathcal{A}_n$ with $n \in \{2, \dots, m-1\}$ if and only if $A_n > \max_i(\gamma_i A_i)$ and $\mathbb{B}_n \leq S \leq \mathbb{D}_n$.

4. $\mathcal{R}_n \equiv \mathcal{R}_{n,1} \cup \mathcal{R}_{n,2}$ with $n \in \{1, \dots, m-1\}$ where

   (a) $\mathcal{R}_{n,1}$ is the set of economies such that $A_n > \max_i(\gamma_i A_i)$ and $\mathbb{D}_{n+1} < S < \mathbb{B}_n$.

   (b) $\mathcal{R}_{n,2}$ is the set of economies such that $A_{n+1} > \max_i(\gamma_i A_i) \geq A_n$ and $\mathbb{D}_{n+1} < S$.

We now prove (A.6a) which implies the existence of equilibrium. It suffices to verify that $\mathbf{E} \subset \cup_{i=1}^m \mathcal{A}_i \cup \cup_{i=1}^{m-1} \mathcal{R}_i$. Let us consider an economy $\mathcal{E}$. There are only two cases.

1. $\max_i(\gamma_i A_i) < A_1$. In this case, we have $\max_i(\gamma_i A_i) < A_n \ \forall n$. Therefore, it is easy to see that $\mathcal{E} \in \cup_{i=1}^m \mathcal{A}_i \cup \cup_{i=1}^{m-1} \mathcal{R}_{i,1} \subset \cup_{i=1}^m \mathcal{A}_i \cup \cup_{i=1}^{m-1} \mathcal{R}_i$.

2. There exists $n \in \{1, \dots, m-1\}$ such that $A_{n+1} > \max_i(\gamma_i A_i) \geq A_n$. There are two sub-cases.

   (a) $S > \mathbb{D}_{n+1}$. In this case, $\mathcal{E} \in \mathcal{R}_{n+1,2}$.

   (b) $S \leq \mathbb{D}_{n+1}$. Recall that $M < A_{n+1}$. In this case, we will prove that $\mathcal{E} \in \cup_{i=n+1}^m \mathcal{A}_i \cup \cup_{i=n+1}^{m-1} \mathcal{R}_i$. Indeed, since $S \leq \mathbb{D}_{n+1}$, there are $2(m-n)-1$ cases.

      i. If there exists $i \in \{n+1, m-1\}$ such that $\mathbb{B}_i \leq S \leq \mathbb{D}_i$. Then $\mathcal{E} \in \mathcal{A}_i$ because $A_i \geq A_{n+1} > \max_i(\gamma_i A_i)$.

      ii. If there exists $i \in \{n+1, m-1\}$ such that $\mathbb{D}_{i+1} \leq S \leq \mathbb{B}_i$. Then $\mathcal{E} \in \mathcal{R}_{i,1}$ because $A_i \geq A_{n+1} > \max_i(\gamma_i A_i)$.

      iii. Last, if $S \leq \mathbb{D}_m$, then $\mathcal{E} \in \mathcal{R}_m$.

Proof of (A.6b). Observe that the equilibrium interest rate is unique if (A.6b) holds. We have to prove that:

$$\mathcal{A}_n \cap \mathcal{A}_h = \emptyset \ \forall n \neq h \tag{A.7a}$$
$$\mathcal{A}_n \cap \mathcal{R}_{h,1} = \emptyset \ \forall n, h \tag{A.7b}$$
$$\mathcal{A}_n \cap \mathcal{R}_{h,2} = \emptyset \ \forall n, h \tag{A.7c}$$
$$\mathcal{R}_n \cap \mathcal{R}_h = \emptyset \ \forall n \neq h. \tag{A.7d}$$

Following (A.2), it is easy to see that the two first equalities hold.

We now prove that $\mathcal{A}_n \cap \mathcal{R}_{h,2} = \emptyset \ \forall n, h$. Suppose that there exists $\mathcal{E} \in \mathcal{A}_n \cap \mathcal{R}_{h,2}$. It means that (1) $A_n > \max_i(\gamma_i A_i)$ and $\mathbb{B}_n \leq S \leq \mathbb{D}_n$, and (ii) $A_{h+1} > \max_i(\gamma_i A_i) \geq A_h$ and $\mathbb{D}_{h+1} < S$. From these conditions we get $A_n > A_h$, and hence $n \geq h+1$. Thus, we obtain $S > \mathbb{D}_{h+1} \geq \mathbb{D}_n \geq S$, a contradiction. Therefore, we have $\mathcal{A}_n \cap \mathcal{R}_{h,2} = \emptyset \ \forall n, h$.

Last, we prove $\mathcal{R}_n \cap \mathcal{R}_h = \emptyset$, or equivalently $\mathcal{R}_{n,i} \cap \mathcal{R}_{h,j} = \emptyset \ \forall i, j \in \{1, 2\}, \ \forall n \neq h$. Without loss of generality, we can assume that $n < h$. It is easy to see that $\mathcal{R}_{n,1} \cap \mathcal{R}_{h,1} = \emptyset$ and $\mathcal{R}_{n,2} \cap \mathcal{R}_{h,2} = \emptyset$. We now prove that $\mathcal{R}_{n,1} \cap \mathcal{R}_{h,2} = \emptyset$ and $\mathcal{R}_{n,2} \cap \mathcal{R}_{h,1} = \emptyset$.

1. Suppose that there exists $\mathcal{E} \in \mathcal{R}_{n,1} \cap \mathcal{R}_{h,2}$. It means that $A_n > \max_i(\gamma_i A_i)$; $\mathbb{D}_{n+1} < S < \mathbb{B}_n$; $A_{h+1} > \max_i(\gamma_i A_i) \geq A_h$; $\mathbb{D}_{h+1} < S$. Since $h > n$, then $A_h > A_n > \max_i(\gamma_i A_i)$. This is a contradiction because $\max_i(\gamma_i A_i) \geq A_h$. So, we have $\mathcal{R}_{n,1} \cap \mathcal{R}_{h,2} = \emptyset$.

2. Suppose that there exists $\mathcal{E} \in \mathcal{R}_{n,2} \cap \mathcal{R}_{h,1}$. It means that $A_{n+1} > \max_i(\gamma_i A_i) \geq A_n$; $\mathbb{D}_{n+1} < S; A_h > \max_i(\gamma_i A_i)$; $\mathbb{D}_{h+1} < S < \mathbb{B}_h$.

   Since $h \geq n+1$, we have $\mathbb{B}_h \leq \mathbb{B}_{n+1} < \mathbb{D}_{n+1} < S < \mathbb{B}_h$, a contradiction.

Remark 3. *In Theorem 2, we assume that $A_1 < \cdots < A_m$. However, we can characterize the set of equilibria in the general case where some agents have the same productivity. Indeed, without lost of generality, we can (1) rank that $A_i \leq A_{i+1}$, $\forall i$, and assume that (2) the set $\{A_i : i \in \{1, \ldots, m\}\}$ has the cardinal p, $p \leq m$ and its distinct values are $(A_{i_t})_{t=1}^p$, where $A_1 = A_{i_1} < A_{i_2} < \cdots < A_{i_p} = A_m$. We can decompose that*

$$A_1, A_2, \ldots, A_m = \underbrace{A_1, \ldots, A_1}_{i_1 \ times}, \underbrace{A_{i_1+1}, \ldots, A_{i_1+i_2}}_{i_2 \ times}, \ldots, \underbrace{A_{i_1+\cdots+i_{p-1}}, \ldots, A_m}_{i_m \ times}$$

Let us denote $\mathbb{A}_t \equiv A_{i_t}$, $\mathbb{S}_t \equiv \sum_{i:A_i=A_{i_t}} S_i$. Then, we can use the same argument in Theorem 2 (but we replace m by p, A_i by $\mathbb{A}_i$, S_i by $\mathbb{S}_i$) to determine the unique equilibrium interest rate. However, there may be multiple equilibrium allocations when one of the sets $\{i : A_i = A_{i_1}\}, \ldots, \{i : A_i = A_{i_p}\}$ has multiple elements.

Proof of Theorem 2 (economy with strictly concave technologies)

Individual optimal choice

Before present the properties of individual optimal choice, we introduce some notations:

Definition 6. *Given R, γ_i, A_i, S_i, denote $k_i^n = k_i^n(R/A_i)$ the unique solution to the equation $A_i f_i'(k) = R$ and $k_i^b = k_i^b(\frac{R}{\gamma_i A_i}, S_i)$ the unique solution to $R(k - S_i) = \gamma_i A_i f_i(k)$.*

k_i^b (k_i^n, respectively) represents the capital level of agent i when her borrowing constraint is binding (not binding, respectively). Under assumptions in Lemma 2, we can verify that: (1) k_i^n is strictly decreasing in R/A_i. Moreover, $\lim_{R/A_1 \to 0} k_i^n = +\infty$, and $\lim_{R/A_1 \to \infty} k_i^n = 0$. (2) k_i^b is strictly increasing in S_i but strictly decreasing in $\frac{R}{\gamma_i A_i}$. Moreover, $\lim_{R/A_i \to 0} k_i^b = +\infty$, and $\lim_{R/A_i \to \infty} k_i^b = S_i$.

The following result characterizes the solution of the problem (P_i).

Lemma 13 (individual choice - strictly concave production function)**.** *Under Assumption 2, there exists a unique solution to the problem (P_i). The optimal capital k_i is increasing in TFP A_i, credit limit γ_i but decreasing in the interest rate R.*

1. *If $R\frac{k_i^n(R/A_i) - S_i}{A_i f_i(k_i^n(R/A_i))} \geq \gamma_i$, then credit constraint is binding and the capital level is $k_i = k_i^b$. Moreover, $k_i = k_i^b \leq k_i^n$.*

2. *If $R\frac{k_i^n(R/A_i) - S_i}{A_i f_i(k_i^n(R/A_i))} < \gamma_i$, then credit constraint is not binding and $k = k_i^n$. In this case, we have $k_i = k_i^n < k_i^b$.*

Proof of Lemma 13. Since $F_i'(0) = \infty$, we have $k_i > 0$ at optimum. The Lagrange function is

$$L = F_i(k_i) - Rb_i + \lambda_i(S_i + b_i - k_i) + \mu_i(\gamma_i F_i(k_i) - Rb_i)$$

It is easy to see that (k_i, b_i) is a solution if and only if there exists (λ_i, μ_i) such that

$$[k] : (1 + \mu_i\gamma_i)F_i'(k_i) = \lambda_i$$
$$[a] : (1 + \mu_i)R = \lambda_i, \quad \mu_i \geq 0, \text{ and } \mu_i(\gamma_i F_i(k_i) - R_i b_i) = 0.$$

These equations imply that:

$$A_i f_i(k_i) = F_i'(k_i) = R\frac{1 + \mu_i}{1 + \gamma_i\mu_i} \geq R. \tag{A.8}$$

Since F_i' is decreasing, we have $k_i \leq k_i^n(R/A_i)$.

We consider two cases.

Case 1: The credit constraint is binding: $\gamma F_i(k_i) = Rb_i$. In this case, (k_i, b_i) is the solutions of the following equations:

$$b_i = k_i - S_i \tag{A.9}$$

$$\gamma F_i(k_i) = R(k_i - S_i), \quad i.e., \quad \frac{\gamma_i}{R} = \frac{k_i}{F_i(k_i)} - \frac{S_i}{F_i(k_i)}. \tag{A.10}$$

Consider the function $k/F_i(k)$. Its derivative equals $\frac{F_i(k) - kF_i'(k)}{(F_i(k))^2}$ which is non-negative because F is concave. So, the function $G_i(k) \equiv \frac{k - S_i}{F_i(k)}$ is strictly increasing in k. Moreover, $\lim_{k\to 0} G_i(k) < \gamma_i/R$ and $G_i(\infty) > \gamma_i/R$ (because $F_i'(\infty) < 1$). Therefore, there exists a unique solution k_i of equation (A.10), and this is positive. It is actually k_i^b.

We now investigate condition $k_i \leq k_i^n$. Since $G_i(k_i) = \gamma_i/R$, condition $k_i \leq k_i^n$ is equivalent to $G_i(k_i^n) \geq \gamma_i/R$ (because $G_i(k_i) = \gamma_i/R$) or, equivalently, $R\frac{k_i^n(R/A_i) - S_i}{F_i(k_i^n(R/A_i))} \geq \gamma_i$.

Conversely, assume that $R\frac{k_i^n(R/A_i) - S_i}{F_i(k_i^n(R/A_i))} \geq \gamma_i$. We choose $k_i = k_i^b$. Then, by definition of k_i^b, we have $k_i \in (S_i, \infty)$. Therefore, we have

$$R > R(1 - \frac{S_i}{k_i}) = \gamma_i\frac{F_i(k_i)}{k_i} \geq \gamma_i F_i'(k_i)$$

where the last inequality follows the fact that F_i is concave. It means that $R > \gamma_i F_i'(k_i)$. So, we can define μ_i, λ_i by

$$1 - \frac{F_i'(k_i)}{R} = \mu_i\Big(\frac{F_i'(k_i)}{R} - \gamma_i\Big), \quad \lambda_i = R(1 + \mu_i).$$

Therefore, (λ_i, μ_i) and (k_i, b_i) satisfy conditions $[k]$ and $[b]$ above. It means that (k_i, b_i) is a solution.

Case 2: $\gamma_i F_i(k_i) > Rb_i$. In this case, we have $\mu_i = 0$, and hence $F_i'(k_i) = R$, i.e, $k_i = k_i^n$. It remains to check that this value of k_i satisfies the condition: $\gamma_i F_i(k_i) > Rb_i = R(S_i - k_i)$, i.e., $\gamma_i/R > G_i(k_i^n)$.

Observe that if $RG_i(k_i^n) < (\geq)\gamma_i$, then $G_i(k_i^n) < (\geq)\gamma_i/R = G_i(k_i^b)$, which implies that $k_i^n < (\geq)k_i^b$.

The converse is easy. Notice that, in this case, agent borrows (i.e., $b_i > 0$) if and only if $k_i > S$ or equivalently $k_i^n > S$. This means that her wealth is low and/or interest rate is low and/or her productivity is high.

□

Under Assumptions 2 and 3, the function $\frac{(k - S_i)f_i'(k)}{f_i(k)}$ is strictly increasing in k. Therefore, the function $G_i(x) \equiv \frac{(k_i^n(x) - S_i)x}{f_i(k_i^n(x))}$ is strictly decreasing in x. Moreover, we can check that $\lim_{x\to+\infty} G_i(x) = -\infty$, $\lim_{x\to 0} G_i(x) = \lim_{k\to\infty} \frac{kf_i'(k)}{f_i(k)}$. By consequence, we obtain the following result.

Lemma 14. *Let Assumptions 2 and 3 be satisfied. Then, if agent i's borrowing constraint is binding, we must have* $\gamma_i \leq \lim_{x\to\infty} \frac{xF_i'(x)}{F_i(x)}$. *By consequence, when* $F_i(k) = A_i k^{\alpha_i}$ *and* $\gamma_i > \alpha_i$*, then agent i's borrowing constraint is not binding.*

The following result show the interaction between interest rate, credit limit γ_i and borrowing constraint.

Lemma 15. *Let Assumptions 2, 3 and 9 be satisfied. We can define* R_i *the unique value satisfying*

$$H_i(R_i) \equiv R_i \frac{k_i^n(R_i/A_i) - S_i}{A_i f_i(k_i^n(R_i/A_i))} = \gamma_i. \tag{A.11}$$

Then, we have that:

1. *Agent i's borrowing constraint is binding if and only if* $H_i(R) \geq \gamma_i$ *which is equivalent to* $R \leq R_i \equiv H_i^{-1}(\gamma_i)$.
2. R_i/A_i *does not depend on* A_i*, and* $\lim_{A_i\to\infty} R_i = \infty$, $\lim_{A_i\to 0} R_i = 0$. R_i *is increasing in productivity* A_i *but decreasing in* γ_i *and in* S_i.
3. *We also have* $k_i^n(R_i/A_i) = k_i^b(R_i/A_i)$.

The threshold R_i is exogenous. It represents the subjective interest rate of agent below which agent borrows so that her(his) borrowing constraint is binding. Point 2 of Lemma 15 indicates that the credit constraint of agent i is more likely to bind if the interest rate, her initial wealth and credit limit are low, and/or her productivity is high.

Remark 4. *Under Cobb-Douglas technology, i.e.,* $F_i(k) = A_i k^\alpha$*, we can compute that* $H_i(R) = \alpha\big(1 - \big(\frac{R}{\alpha A_i S_i^{\alpha-1}}\big)^{\frac{1}{1-\alpha}}\big)$ *is decreasing in R and* $H_i(0) = \alpha$*. So, if* $\alpha_i < \gamma_i$*, then borrowing constraint is not binding, whatever the level of interest rate R. When* $H_i(0) > \gamma_i$*, i.e.,* $\alpha > \gamma_i$*, we have* $R_i = \alpha A_i S_i^{\alpha-1}\big(1 - \frac{\gamma_i}{\alpha}\big)^{1-\alpha}$.

Proof of Theorem 3

To simplify notations, we write $k_i^n(R)$ and $k_i^b(R)$ instead of $k_i^n(\frac{R}{A_i})$ and $k_i^b(\frac{R}{\gamma_i A_i}, S_i)$. We also introduce the so-called aggregate capital demand function:

$$B_n(R) \equiv \begin{cases} \sum_{i=1}^n k_i^n(R) + \sum_{i=n+1}^m k_i^b(R) & \text{if } n \leq m-1 \\ \sum_{i=1}^m k_i^n(R) & \text{if } n = m. \end{cases}$$

Lemma 16. $B_n(R_n) > B_{n+1}(R_{n+1}) = B_n(R_{n+1})$.

Proof. Indeed, since $R_n < R_{n+1}$, we notice that

$$\begin{aligned} B_n(R_n) &\equiv \sum_{i=1}^n k_i^n(R_n) + \sum_{i=n+1}^m k_i^b(R_n) \\ &> B_n(R_{n+1}) = \sum_{i=1}^n k_i^n(R_{n+1}) + \sum_{i=n+1}^m k_i^b(R_{n+1}) = \sum_{i=1}^{n+1} k_i^n(R_{n+1}) + \sum_{i=n+2}^m k_i^b(R_{n+1}) \end{aligned}$$

where the last equality follows $k_{n+1}^b(R_{n+1}) = k_{n+1}^n(R_{n+1})$. Therefore, $B_n(R_n) > B_{n+1}(R_{n+1}) = B_n(R_{n+1})$ $\forall n$. □

We state an intermediate step whose proof is based on Lemma 13 and Lemma 15.

Lemma 17. *Let assumptions in Theorem 3 be satisfied. Consider an equilibrium* $((k_i, b_i)_i, R)$ *and an index* $n \in \{1, \dots, m-1\}$*.*

1. *If* $R > R_m$*, Lemma 13 implies that credit constraint of any agent is not binding. So, the equilibrium coincides to that of the economy without credit constraints. Therefore, we have* $R = R^* > R_m$*.*

2. *If* $R > R_n$*, then credit constraint of any agent* $i \leq n$ *is not binding. Hence* $k_i = k_i^n(R) < k_i^n(R_n)$ $\forall i \leq n$*. Condition* $R > R_n$ *also implies that* $k_i^b(R) < k_i^b(R_n)$*. Therefore, we have* $\sum_i S_i < B_n(R_n)$*.*

3. *If* $R \leq R_{n+1}$*, then credit constraint of any agent* $i \geq n+1$ *is binding, and hence* $k_i = k_i^b(R) \geq k_i^b(R_{n+1})$ $\forall i \geq n+1$*. Moreover, we have* $k_i \geq k_i^n(R) \geq k_i^n(R_{n+1})$*. Therefore, we have* $\sum_i S_i \geq B_n(R_{n+1})$*.*

We now prove Theorem 3. Let us consider an equilibrium. Since there is at least one agent whose credit constraint is not binding, we have $R > R_1$.

Step 1. Suppose that $R \in (R_n, R_{n+1}]$. So, credit constraint of any agent $i \geq n+1$ is binding and that of any agent $i \leq n$ is not binding. Hence, the capital demand is

$$\sum_i k_i = \sum_{i=1}^{n} k_i^n(R) + \sum_{i=n+1}^{m} k_i^b(R). \tag{A.12}$$

Therefore, the equilibrium interest rate is determined by

$$\sum_{i=1}^{n} k_i^n(R) + \sum_{i=n+1}^{m} k_i^b(R) = S \equiv \sum_i S_i. \tag{A.13}$$

The left-hand side is decreasing in r, and hence this equation has a unique solution.

Since $R \in (R_n, R_{n+1}]$, we have

$$\sum_{i=1}^{n} k_i^n(R_n) + \sum_{i=n+1}^{m} k_i^b(R_n) > \sum_i S_i \geq \sum_{i=1}^{n} k_i^n(R_{n+1}) + \sum_{i=n+1}^{m} k_i^b(R_{n+1}).$$

Conversely, if this condition holds, by using properties of functions k_i^b, k_i^n, we can easily prove that $R \in (R_n, R_{n+1}]$. Indeed, if $R > R_{n+1}$, then point 2 of Lemma 17 implies that $S < B_{n+1}(R_{n+1})$. This contradicts to $S \geq B_{n+1}(R_{n+1})$. If $R \leq R_n$, then point 3 of Lemma 17 implies that $S \geq B_{n-1}(R_n) = B_n(R_n)$. This contradicts to $S < B_n(R_n)$. Therefore, we obtain $R \in (R_n, R_{n+1}]$.

Step 2. We now suppose that $R^* > R_m$. We will prove that credit constraint of any agent is not binding. Suppose that the set

$$\mathcal{B} = \{i \in \{1, \dots, m\} : \text{ agent i's borrowing constraint is binding}\}$$

is not empty. Let $n : 1 \leq n \leq m-1$ be the highest element in $\mathcal{B}$, i.e., credit constraint of any agent $i \geq n+1$ is binding while that of any agent $i \leq n$ is not. We have $R \in (R_n, R_{n+1}]$. So, $k_i^b(R) \geq k_i^b(R_n + 1) > k(R_m)$ and $k_i^n(R) \geq k_i^n(R_{n+1}) \geq k_i^n(R_m)$. Hence, we get that

$$\sum_i S_i = \sum_{i=1}^{n} k_i^n(R) + \sum_{i=n+1}^{m} k_i^b(R) \geq \sum_{i=1}^{m} k_i^n(R_m). \tag{A.14}$$

However, by definition of R^*, we have

$$\sum_i S_i = \sum_{i=1}^{m} k_i^n(R^*) < \sum_{i=1}^{m} k_i^n(R_m). \tag{A.15}$$

This is a contradiction.

Step 3. We now prove that $R_n \leq R^*$ $\forall n \leq m-1$. Indeed, in the regime $\mathcal{R}_n$, for any $i \geq n+1$, agent i's credit constraint is binding. Hence, Lemma 13 follows that $k_i^b(R_n) \leq k_i^n(R_n)$ $\forall i \geq n+1$. Consequently, we get that

$$\sum_{i=1}^{m} k_i^n(R^*) = S = \sum_{i=1}^{n} k_i^n(R_n) + \sum_{i=n+1}^{m} k_i^b(R_n) \leq \sum_{i=1}^{m} k_i^n(R_n)$$

which implies that $R^* \geq R_n$.

E Online appendix: the existence of intertemporal equilibrium

The proof is similar to the one in Bosi, Le Van, and Pham (2018). But our added-value is that we do not need that $u_{i,t}(0) = 0, \forall c$. Notice that we cannot directly use a method of Becker, Bosi, Le Van and Seegmuller (2015) or Le Van and Pham (2016) because the financial asset in our model is a short-lived asset with zero supply.

The idea is that we can bound the individual demand for the financial asset, and so can prove the existence of equilibrium by adapting the method of Becker, Bosi, Le Van and Seegmuller (2015) and Le Van and Pham (2016): (1) we prove the existence of equilibrium for each $T-$ truncated economy $\mathcal{E}^T$; (2) we show that this sequence of equilibria converges for the product topology to an equilibrium of our economy $\mathcal{E}$.

E.1 Existence of equilibrium for truncated economies

For each $T \geq 1$, we define $T-$truncated economy $\mathcal{E}^T$ as $\mathcal{E}$ but there are no activities from period $T+1$ to the infinity, i.e., $c_{i,t+1} = k_{i,t} = b_{i,t} = 0$ for every $i = 1, \dots, m$ and for any $t \geq T$.

We then define the bounded economy $\mathcal{E}_b^T$ as $\mathcal{E}^T$ but consumption level $(c_{i,t})_{t\leq T}$, physical capital $(k_{i,t})_{t\leq T}$, and asset holding $(b_{i,t})_{t\leq T}$ are respectively bounded in the following sets:

$$\mathcal{C} \ := \ [-B_c, B_c]^{T+1}, \quad \mathcal{K} := [-B_k, B_k]^{T+1}, \quad \mathcal{B} := [-B_b, B_b]^{T+1},$$

where $|S|$ denotes the cardinal of the set S and the bounds satisfy

$$B_c, B_k > \max_{t\leq T} B_{K,t}; \quad B_b = mB_c. \tag{A.1}$$

The economy $\mathcal{E}_b^T$ depends on bounds B_c, B_k, B_b, so we write $\mathcal{E}_b^T(B_c, B_k, B_b)$.

Let us define

$$\mathcal{X}_b \equiv \mathcal{C} \times \mathcal{K} \times \mathcal{B}, \qquad \mathcal{X} \equiv (\mathcal{X}_b)^m \tag{A.2}$$

$$\mathcal{P} \equiv \{z_0 = (p_t, R_t)_{t\leq T} : R_0 = 0; \quad p_t, R_t \geq 0; \quad p_t + R_t = 1, \forall t \leq T\} \tag{A.3}$$

$$\Phi \equiv \mathcal{P} \times \mathcal{X}. \tag{A.4}$$

An element $z \in \Phi$ is in the form $z = (z_i)_{i=0}^m$ where $z_0 = (p_t, R_t)_{t\leq T}, z_i = (c_{i,t}, k_{i,t}, b_{i,t})_{t\leq T}$ for each $i = 1, \dots, m$.

The following remark is to ensure that the asset volume $(b_{i,t})$ is bounded.

Remark 5. *If $z \in \Phi$ is an equilibrium for the economy $\mathcal{E}^T$, then, by using the fact that $p_t + R_t = 1$, we obtain that $b_{i,t} \leq B_c$ for any i, t. Indeed, this is true for $t = 0$ because*

$$-b_{i,0} \leq p_0 F_{i,0}(k_{i,-1}) \leq p_0 B_c = B_c \tag{A.5}$$

and then, for any $t \geq 0$, we have

$$-b_{i,t} \leq p_t F_{i,t}(k_{i,t-1}) - R_t b_{i,t-1} \leq (p_t + R_t) B_c = B_c \tag{A.6}$$

Since $\sum_{i=1}^{m} b_{i,t} = 0$, we get that $b_{i,t} \in [-B_b, B_b]$ for any i and any t, where $B_b \equiv mB_c > (m-1)B_c$.

Proposition 22. *Under Assumptions (1-4), there exists an equilibrium* $(p, R, (c_i, k_i, b_i)_{i=1}^m)$*, with* $p_t + R_t = 1$*,* $\forall t$*, for the economy* $\mathcal{E}_b^T(B_c, B_k, B_b)$*. This is actually an equilibrium for the economy* $\mathcal{E}^T(B_c, B_k, B_b)$

Proof. We firstly define

$$\begin{aligned} B_i^T(p,R) := \big\{&(c_{i,t}, k_{i,t}, b_{i,t})_{t\leq T} \in \mathcal{X}_b : \text{(a) } k_{i,t} = b_{i,t} = 0\ \forall t \in D_T, \\ &\text{(b) } p_0(c_{i,0} + k_{i,0}) \leq p_0 F_{i,0}(k_{i,-1}) + b_{i,0} \\ &\text{(c) for each } t : 1 \leq t(t) \leq T : \\ &\quad R_t b_{i,t-1} \leq \gamma_i p_t F_{i,t}(k_{i,t-1}) \\ &\quad p_t(c_{i,t} + k_{i,t}) + R_t b_{i,t-1} \leq p_t F_{i,t}(k_{i,t-1}) + b_{i,t}\big\}. \end{aligned}$$

We also define $C_i^T(p,R)$ as follows.

$$\begin{aligned} C_i^T(p,R) := \big\{&(c_{i,t}, k_{i,t}, b_{i,t})_{t\leq T} \in \mathcal{X} : \text{(a) } k_{i,t} = b_{i,t} = 0\ \forall t \in D_T, \\ &\text{(b) } p_0(c_{i,0} + k_{i,0}) < p_0 F_{i,0}(k_{i,-1}) + b_{i,0} \\ &\text{(c) for each } t : 1 \leq t(t) \leq T : \\ &\quad R_t b_{i,t-1} < \gamma_i p_t F_{i,t}(k_{i,t-1}) \\ &\quad p_t(c_{i,t} + k_{i,t}) + R_t b_{i,t-1} < p_t F_{i,t}(k_{i,t-1}) + b_{i,t}\big\}. \end{aligned}$$

Lemma 18. $C_i^T(p,R) \neq \emptyset$ *and* $\bar{C}_i^T(p,R) = B_i^T(p,R)$*.*

Proof. Since $k_{i,-1} > 0$ and $p_0 = 1$, we always have $p_0 F_{i,0}(k_{i,-1}) > 0$. Therefore, we can choose $(c_{i,0}, k_{i,0}, b_{i,0}) \in \mathbb{R}_+^2 \times \mathbb{R}_-$, and then $(c_{i,t}, k_{i,t}, b_{i,t}) \in \mathbb{R}_+^2 \times \mathbb{R}_-$ such that this plan belongs to $C_i^T(p,q,r)$. Note that $p_t F_{i,t}(k_{i,t-1}) - R_t b_{i,t-1} > 0$ if $k_{i,t-1} > 0, b_{i,t-1} < 0$ and $(p_t, R_t) \neq (0,0,0)$. □

Lemma 19. $C_i^T(p,R)$ *is lower semi-continuous correspondence on* $\mathcal{P}$*.* $B_i^T(p,R)$ *is continuous on* $\mathcal{P}$ *with compact convex values.*

Proof. It is clear since $C_i^T(p,R)$ is nonempty and has open graph. □

We now define correspondences. First, we define φ_0 (for additional agent 0) : $\mathcal{X} \to 2^{\mathcal{P}}$ by

$$\varphi_0((z_i)_{i=1}^m) := \arg\max_{(p,R)\in\mathcal{P}} \Big\{ \sum_{t\leq T} \Big[p_t \sum_{i=1}^m \big(c_{i,t} + k_{i,t} - F_{i,t}(k_{i,t-1})\big)\Big] + \sum_{t\leq T} \Big[R_t \sum_{i=1}^m b_{i,t-1}\Big]\Big\}.$$

Second, for each $i = 1, \ldots, m$, we define $\varphi_i : \mathcal{P} \to 2^{\mathcal{X}}$

$$\varphi_i((p,R)) := \arg\max_{(c_i,k_i,b_i)\in C_i^T(p,R)} \Big\{\sum_{t=0}^T u_{i,t}(c_{i,t})\Big\}.$$

Lemma 20. *The correspondence* φ_i *is upper semi-continuous and non-empty, convex, compact valued for each* $i = 0, 1, \ldots, m+1$*.*

Proof. This is a direct consequence of the Maximum Theorem. □

According to the Kakutani Theorem, there exists $(\bar{p}, \bar{R}, (\bar{c}_i, \bar{k}_i, \bar{b}_i)_{i=1}^m)$ such that

$$(\bar{p}, \bar{R}) \in \varphi_0((\bar{c}_i, \bar{k}_i, \bar{b}_i)_{i=1}^m) \tag{A.7}$$

$$(\bar{c}_i, \bar{k}_i, \bar{b}_i) \in \varphi_i((\bar{p}, \bar{R})). \tag{A.8}$$

Denote, for each $t \geq 0$,

$$\bar{X}_t := \sum_{i=1}^m (\bar{c}_{i,t} + k_{i,t} - F_{i,t}(\bar{k}_{i,t-1})), \quad \bar{Z}_t := \sum_{i=1}^m \bar{b}_{i,t}.$$

Therefore, for every $(p, q, r) \in \mathcal{P}$, we have

$$\sum_{t \leq T} (p_t - \bar{p}_t)\bar{X}_t + \sum_{t \leq T} (R_t - \bar{R}_t)\bar{Z}_{t-1} \leq 0 \tag{A.9}$$

Consider date t, by summing budget constraints over i, we get that

$$\bar{p}_t \bar{X}_t + \bar{R}_t \bar{Z}_{t-1} \leq \bar{Z}_t.$$

By consequence, we have, for each $t \leq T$ and for every $(p_t, R_t) \geq 0$ with $p_t + R_t = 1$,

$$p_t \bar{X}_t + R_t \bar{Z}_{t-1} \leq \bar{p}_t \bar{X}_t + \bar{R}_t \bar{Z}_{t-1} \leq \bar{Z}_t.$$

Since at date T, we have $\bar{Z}_T = 0$. So, $\bar{p}_T \bar{X}_T + \bar{R}_T \bar{Z}_{T-1} \leq 0$. Hence, $p_T \bar{X}_T + R_T \bar{Z}_{T-1} \leq 0$ for any $(p_T, R_T) \geq 0$ with $p_T + R_T = 1$. This implies that $\bar{X}_T, \bar{Z}_{T-1} \leq 0$. Repeating this argument, we obtain that $\bar{X}_t, \bar{Z}_t \leq 0$ $\forall t \leq T$ which means that

$$\sum_{i=1}^m (\bar{c}_{i,t} + \bar{k}_{i,t}) \leq \sum_{i=1}^m F_{i,t}(\bar{k}_{i,t})$$

$$\sum_{i=1}^m \bar{b}_{i,t} \leq 0.$$

Lemma 21. $\bar{p}_t > 0, \bar{R}_t > 0$ *for any* $t \leq T$.

Proof. By definition of $B_{K,t}$, we see that $\sum_{i=1}^m \bar{c}_{i,t} \leq B_{K,t} < B_c$, or any t. This allows us to prove that $\bar{p}_t > 0$ for any t. Indeed, if $\bar{p}_t = 0$ then $c_{i,t} = B_c > B_{K,t}$, a contradiction.

If $\bar{R}_t = 0$, then $\bar{b}_{i,t-1} = -B_a$ for any i, which implies that $\sum_{i=1}^m \bar{b}_{i,t-1} < 0$, contradiction. Therefore, $\bar{R}_t > 0$.

□

Lemma 22. $\bar{X}_t = \bar{Z}_t = 0$.

Proof. Using $\bar{p}_t \bar{X}_t + \bar{R}_t \bar{Z}_{t-1} \leq 0$ and Lemma 21. □

Lemma 23. *The optimality of* $(\bar{c}_i, \bar{k}_i, \bar{b}_i)$.

Proof. It is clear since $(\bar{c}_i, \bar{k}_i, \bar{b}_i) \in \varphi_i((\bar{p}, \bar{q}, \bar{r}))$. □

We have just proved that $(\bar{p}, \bar{R}, (\bar{c}_i, \bar{k}_i, \bar{b}_i)_{i=1}^m)$ is an equilibrium for the economy $\mathcal{E}_b^T$. We now prove that this equilibrium for the economy $\mathcal{E}^T(B_c, B_k, B_b)$. The market clearing conditions are obviously satisfied. It remains to prove the optimality of the allocation $\bar{z}_i = (\bar{c}_i, \bar{k}_i, \bar{b}_i)$. Suppose the contra

Let $z_i \equiv (c_i, k_i, b_i)$ be in the budget set of the $T-$truncated economy. Since $(\bar{c}_i, \bar{k}_i, \bar{b}_i)$ belongs the interior of $\mathcal{X}_b$, there exists $\lambda \in (0,1)$ such that $\lambda z^i + (1-\lambda)\bar{z}_i \in \mathcal{X}_b$. Of course, $\lambda z^i + (1-\lambda)\bar{z}_i$ is in the budget set of the economy $\mathcal{E}_b^T$. Denote $U_i(c) \equiv \sum_{t \leq T} u_{i,t}(c_t)$.

We have $\lambda U^i(c^i) + (1-\lambda)U^i(\bar{c}_i) \leq U^i(\lambda c^i + (1-\lambda)\bar{c}_i) \leq U^i(\bar{c}_i)$, which implies that $U^i(c^i) \leq U^i(\bar{c}_i)$.

□

E.2 Existence of equilibrium for the infinite-horizon economy

For simplicity of notation, in what follows, we write F_i instead of $F_{i,t}$.

Proposition 23. *Under Assumptions (1-4) and 7, there exists an equilibrium for the economy $\tilde{\mathcal{E}}$.*

Proof. We present a proof in the spirit Le Van and Pham (2016).

We have shown that there exists an equilibrium, say $\left(\bar{p}^T, \bar{R}^T, (\bar{c}_i^T, \bar{k}_i^T, \bar{b}_i^T)_i\right)$, for each $T-$horizon truncated economy $\mathcal{E}^T$. Recall that $\bar{p}_t^T + \bar{R}_t^T = 1$ for any $t \leq T$.

It is clear that the sequence $\left(\bar{p}^T, \bar{R}^T, (\bar{c}_i^T, \bar{k}_i^T, \bar{b}_i^T)_i\right)_T$ is bounded for the product topology. Since the set of time is a countable set, we can assume that, without loss of generality,

$$\left(\bar{p}^T, \bar{R}^T, (\bar{c}_i^T, \bar{k}_i^T, \bar{b}_i^T)_i)\right) \xrightarrow{T\to\infty} \left(\bar{p}, \bar{R}, (\bar{c}_i, \bar{k}_i, \bar{b}_i)_i\right)$$

for the product topology.

We will prove that $\left(\bar{p}, \bar{R}, (\bar{c}_i, \bar{k}_i, \bar{b}_i)_i\right)$ is an equilibrium for the economy $\mathcal{E}$. The market clearing conditions are trivial. We will prove that all prices are strictly positive and the allocation $(\bar{c}_i, \bar{k}_i, \bar{b}_i)$ is optimal.

Let (c_i, k_i, b_i) be a feasible allocation of the problem $P_i(\bar{p}, \bar{R})$. We prove that $U_i(c_i) \leq U_i(\bar{c}_i)$. Let define $(c'_{i,t}, k'_{i,t}, b'_{i,t})_{t \leq T}$ as follows:

$$(c'_{i,t}, k'_{i,t}, b'_{i,t}) = (c_{i,t}, k_{i,t}, b_{i,t}) \text{ if } t \leq T-1$$
$$(c'_{i,t}, k'_{i,t}, b'_{i,t}) = (F_{i,t}(k_{i,t-1}), 0, 0) \text{ if } t = T.$$

We see that $(c'_{i,t}, k'_{i,t}, b'_{i,t})_{t \leq T}$ belongs to $B_i^T(\bar{p}, \bar{R})$.

Since $k_{i,-1} > 0$ and $\bar{p}_0 = 1$, we have $\bar{p}_0 F_{i,0}(k_{i,-1}) > 0$, and hence $C_i^T(\bar{p}, \bar{R}) \neq \emptyset$. Therefore there exists a sequence $\left((c_{i,t}^n, k_{i,t}^n, b_{i,t}^n)_{t \leq T}\right)_{n=0}^{\infty} \in C_i^T(\bar{p}, \bar{R})$, with $k_{i,T}^n = 0$, $b_{i,T}^n = 0$, and this sequence converges to $(c'_{i,t}, k'_{i,t}, b'_{i,t})_{t \leq T}$ when n tends to infinity. We have, for each $t \leq T$,

$$\bar{p}_t(c_{i,t}^n + k_{i,t}^n) + b_{i,t}^n < \bar{p}_t F_{i,t}(k_{i,t-1}^n) + R_t b_{i,t-1}^n$$
$$f_i \bar{p}_t F_{i,t}(k_{i,t-1}^n) + \bar{R}_t b_{i,t-1}^n > 0.$$

Since $(\bar{p}^T, \bar{R}^T)$ converges to $(\bar{p}, \bar{R})$, we can chose s_0 high enough such that (i) $s_0 > T$ and (ii) for every $s \geq s_0$, we have

$$\bar{p}_t^s(c_{i,t}^n + k_{i,t}^n) + b_{i,t}^n < \bar{p}_t^s F_{i,t}(k_{i,t-1}^n) + R_t^s b_{i,t-1}^n$$
$$f_i \bar{p}_t^s F_{i,t}(k_{i,t-1}^n) + \bar{R}_t^s b_{i,t-1}^n > 0.$$

Condition (ii) implies that $(c^n_{i,t}, k^n_{i,t}, b^n_{i,t})_{t\leq T} \in C^T_i(\bar{p}^s, \bar{R}^s)$. Therefore, by the definition of equilibrium in the T-truncated economy, we get that

$$\sum_{t\leq T} \beta_i^t u_i(c^n_{i,t}) \leq \sum_{t\leq T} \beta_i^t u_i(\bar{c}^s_{i,t}).$$

Let s tend to infinity, we obtain $\sum\limits_{t\leq T} \beta_i^t u_i(c^n_{i,t}) \leq \sum\limits_{t\leq T} \beta_i^t u_i(\bar{c}_{i,t})$ for any n and for any T high enough .

Let n tend to infinity, we have $\sum\limits_{t\leq T} \beta_i^t u_i(c'_{i,t}) \leq \sum\limits_{t\leq T} \beta_i^t u_i(\bar{c}_{i,t})$ for any T. We write clearly this as follows:

$$\sum_{t\leq T-1} \beta_i^t u_i(c_{i,t}) + \beta_i^T u_i(F_{i,T}(k_{T,t-1})) \leq \sum_{t\leq T} \beta_i^t u_i(\bar{c}_{i,t}).$$

Let T tend to infinity and note that $\lim_{T\to\infty} \beta_i^T u_i(F_{i,T}(k_{i,T-1})) = 0$, we get that[31]

$$\sum_{t\geq 0} \beta_i^t u_i(c_{i,t}) \leq \sum_{t\geq 0} \beta_i^t u_i(\bar{c}_{i,t}).$$

So, we have proved the optimality of $(\bar{c}_i, \bar{k}_i, \bar{b}_i)$.

Now, we prove that $\bar{p}_t > 0$. Indeed, if $\bar{p}_t = 0$, the agent i can freely improve her consumption to obtain a level of utility, which is higher than $\sum_{t\geq 0} u_{i,t}(\bar{c}_{i,t})$. This contradicts the optimality of $(\bar{c}_i, \bar{k}_i, \bar{b}_i)$.

We have $\bar{R}_t$ is strictly positive because otherwise we can choose another allocation such that $b_{i,t-1} = \infty$ and at the date t, we have the consumption $c'_{i,t+1} = \infty$, which make the utility of agent i infinity, contradiction.

□

[31] Here, we do not need that $u_i(0) = 0$.