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17 October 2022

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January 24, 2025

Abstract

Some important contests have participation costs and 'default allocations' where the contest prize is still awarded even when no-one actively competes. This paper incorporates flexible forms of these features into a general (single-prize) all-pay contest model under arbitrary asymmetry. We offer a tractable equilibrium characterization that fundamentally rests on only two measures (per-player): 'reach' and a new concept, 'strength'. We then i) analyze how participation costs and default allocations can be employed as novel tools in contest design, ii) solve 'clearinghouse' models of price competition under full asymmetry for the first time, and iii) offer a new equilibrium refinement for symmetric multi-player all-pay contests. Throughout, the *combination* of participation costs and default allocations is key and often reverses otherwise familiar intuitions.

Keywords: All-Pay Contests; Participation Costs; Default Allocations; Endogenous Participation; Contest Design; Clearinghouse

JEL Codes: C72; L13; D43

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1 Introduction

The burgeoning literature on contests analyzes situations where players compete with sunk resource investments in order to win some form of prize. This successful literature has considered many applications including R+D, rent-seeking, political campaigns, rewards in organizations, litigation, contract tendering, and conflict.[1] Typically, such contests are modeled with an exogenous number of participants. However, in practice, in addition to their potential investment costs, players often face (fixed) costs of active participation, such as entry fees, set-up costs, foregone outside options or minimum required outlays. As such, players face a non-trivial decision of whether or not to actively participate in a contest.

In these cases, as we later show, the contest outcome can depend upon what we term as the 'default allocation' - what happens to the prize in the event that no player actively participates. As reviewed below, the existing literature contains relatively few models with endogenous, costly participation, and all such models (implicitly) assume that the prize is withheld when all players refrain from active participation. This prevents any analysis of some common situations where the prize must always be allocated, or where the contest organizer cannot commit to withholding it. Important examples include i) tendering processes where a contract is renewed with an incumbent unless a bid is received from an entrant, ii) policy decisions where an outcome remains unless it is contested by a lobbyist, iii) legal disputes where, unless a party starts litigation, some default outcome applies, iv) political settings where the electorate votes in favor of different candidates on the ballot list with exogenous probabilities unless a candidate engages in active campaigning, or v) market settings where, unless a rival firm lists on a digital platform (or 'clearinghouse'), consumers trade with their default firms in given proportions depending on geographical location or past experience.

To help address this gap, our paper makes three main contributions. First, it provides a general framework that can explicitly characterize all potential equilibria in a full information (single prize) all-pay contest with endogenous participation while allowing for flexible forms of participation costs and default allocations, under arbitrary asymmetry. Despite the complexity of the problem, we offer a tractable characterization that fundamentally rests on only two measures (per-player): 'reach' and a new concept which we introduce as 'strength'.

Second, after using our framework to further understand how participation

[1] For reviews, see Konrad (2009), Dechenaux et al. (2015), Corchón and Serena (2018), and Fu and Wu (2019).

costs and default allocations affect equilibrium behavior, we analyze how they can be used as novel, practical tools in contest design. Such tools have remained underexplored within the literature (as reviewed by Fu and Wu 2019 and Chowdhury et al. 2023) and so our results are striking. For instance, contrary to the usual motivation for handicapping stronger players to 'level the playing field' (e.g. Baye et al. 1993, Szech 2015, Franke et al. 2018), we show how *asymmetric* participation costs or default allocation probabilities can optimally stimulate competition even in otherwise symmetric settings. This implies that participation costs and default allocations may arise endogenously in practice; further underlining the importance of our framework.

Finally, we use our framework to solve the broad family of 'clearinghouse' models (e.g. Baye and Morgan 2001, Baye et al. 2004, Baye et al. 2006) under arbitrary asymmetry for the first time - something that was previously intractable under conventional approaches within the associated literature. As further explained below, clearinghouse models are commonly used within industrial organization and marketing to study the role of price comparison platforms (or 'clearinghouses') on pricing and advertising.[2] By doing this, we also open up the ability to derive equilibrium uniqueness in an n-player symmetric all-pay contest. Thus, participation costs alongside default allocations, can offer a new equilibrium refinement in a setting which is well-known to otherwise suffer from equilibrium multiplicity (Baye et al. 1996). Within the unique equilibrium, we further provide some comparative statics related to the 'competitiveness' of a contest and show how they can differ markedly to previous results (e.g. Hillman and Samet 1987, Fang et al. 2020). In this, and throughout the paper, we demonstrate how the *combination* of participation costs and default allocations is key. Together, they can often reverse otherwise familiar intuitions.

In more detail, Sections 2-4 present our main framework with default allocations and a general form of participation costs that can incorporate both 'direct' participation costs, such as entry fees, set-up costs, or forgone outside options, and 'indirect' participation costs, such as minimum required bids or reservation offers. By building on some popular tools in the study of all-pay contests (e.g. Siegel

[2]Clearinghouse models are also used as a foundation to study wider issues such as consumer search, obfuscation, choice complexity and even some macroeconomic topics. For reviews and recent examples, see Guimaraes and Sheedy (2011), Moraga-González and Wildenbeest (2012), Armstrong (2015), Spiegler (2015), Kaplan and Menzio (2016), Burdett and Menzio (2017), Bergemann et al. (2021), Armstrong and Vickers (2022) and Ronayne and Taylor (2022).

2009, 2010, 2014), the framework shows how one can flexibly introduce default allocations and participation costs into a full-information (single prize) all-pay contest under arbitrary asymmetry.

Within the framework, each player must simultaneously decide whether to be 'active' or 'passive'. Unlike a passive player, an active participant incurs a participation cost but is able to submit a bid or 'offer'. The prize is then awarded to the player with the highest active offer. However, rather than assuming that the prize is withheld in the event that no player is active, we allow for the possibility that the prize is still awarded. Specifically, in such a scenario, we assume that any player i wins according to some tie-break rule, equivalent to a 'default allocation probability', x_i.

Under arbitrary asymmetry, the paper derives a two-player equilibrium that is unique (apart from some knife-edge parameter cases). The resulting equilibrium is tractable and neatly depends on only two measures, 'reach' and a new concept which we refer to as 'strength'. Broadly speaking, a player's reach determines their willingness to be active when their rival is also active, whereas a player's strength determines their willingness to be active when their rival is passive. In the previous literature, these measures would have been equivalent to each other and consistent with Siegel's (2009) definition of reach. However, in our context with participation costs and default allocations, the two measures differ and prove sufficient for determining the form of equilibrium in what would otherwise be a complex problem. For instance, in equilibrium, we find that i) neither player actively competes if they both have low strength, ii) only one player actively competes if one player has high strength while the other has low reach, or iii) both players actively compete with positive probability (in several possible forms) if one player has high strength and the other has sufficient reach.

Section 5 further examines how the framework's novel features, participation costs and default allocations, influence the equilibrium and how they can be used as new, practical tools in contest design. In particular, we study how a contest organizer would select participation costs, $\{A_1, A_2\}$, and default allocation probabilities, $\{x_1, x_2\}$. Throughout, we assume the organizer has an 'offer-based objective' involving any combination of expected offers, expected total offers, or expected winning offers. Despite the existing literature suggesting that greater player heterogeneity typically lowers competition (e.g. Baye et al. 1993, Szech 2015, Franke et al. 2018), our results demonstrate that an offer-orientated organizer will find it optimal to use an *asymmetric* contest design even when the players are otherwise symmetric. Specifically, the organizer will strictly prefer to give one player i a strictly lower participation cost, $A_i = 0 < A_j$, or default allocation probability, $x_i < x_j$.[3]

Next, Section 6 shows how our framework can be used to solve the broad family of ‘clearinghouse’ models (e.g. Baye and Morgan 2001, Baye et al. 2004, Baye et al. 2006) under arbitrary asymmetry. Specifically, by using tools from contest theory, we are able to allow for a full set of player asymmetries in a way that should open up new theoretical and empirical research where asymmetry is important, such as platform design or the regulation of platform steering and fees. This extends the initial work by Baye et al. (1996) and Baye et al. (2012) who demonstrate how a special (but widely used) case of the clearinghouse family, Varian’s (1980) model of sales, is equivalent to a form of all-pay contest (with zero participation costs) under full symmetry.[4] Our framework extends this initial link to encompass the full clearinghouse family beyond Varian by a) incorporating endogenous, costly advertising at a digital platform or ’clearinghouse’ (via participation costs) and the possibility of winning the shoppers’ custom even when no firm actively competes at the platform (via default allocation probabilities), and b) allowing for a full set of player asymmetries.

Finally, Section 7 then uses this connection to derive equilibrium uniqueness in n-player symmetric all-pay contests, while also exploring a number of comparative statics related to the ‘competitiveness’ of a contest. Interest in such competition effects within full information all-pay contests has been rejuvenated due to Fang et al.’s (2020) recent analysis. However, we show how the combination of participation costs and default allocations can reverse some standard results.[5]

[3]The optimality of asymmetric contest designs in symmetric situations has also been documented in a few other papers but our setting and contest design tools are distinct. For instance, Drugov and Ryvkin (2017) and Barbieri and Serena (2022) show how a biased contest success function can be optimal for a general family of pure-strategy contests or dynamic contest settings respectively, while Pérez-Castrillo and Wettstein (2016) show how identity-dependent prizes can be optimal under private information.

[4]To understand the intuition of this initial link, note i) each firm’s price implies an associated surplus offer to consumers, ii) the firm with the highest offer wins the ‘prize’ - equivalent to the custom of ‘shopper’ consumers who buy from the firm with best offer, iii) each firm’s offer involves a sunk (opportunity) cost in the form of reduced revenues from its loyal ‘non-shopper’ consumers, and iv) a firm’s value of winning is dependent upon the level of its surplus offer.

[5]Fang et al. (2020) show the effects of a range of competitiveness measures in all-pay contests under a different setting with multiple prizes and convex effort costs. In our context, we study the effects of some parallel measures, including increases in the number of players and (single-prize) scaling where the number of

Related Literature

Our focus is on full-information all-pay contests with single prizes and (potentially) asymmetric players. Aside from the seminal contributions by Hillman and Samet (1987), Hillman and Riley (1989) and Baye et al. (1996), the more recent works by Siegel (2009, 2010, 2014) are most relevant. Within these papers, Siegel develops a popular, tractable approach involving the concept of 'reach' to analyze a broad category of all-pay contests with general payoff functions and arbitrary asymmetry, but without endogenous costly participation or default allocations. In contrast, we bring some elements of these papers together within our context while allowing for flexible forms of costly participation and default allocation probabilities. We explicitly characterize all potential equilibria and show how they depend on each player's reach, and a new measure, 'strength'.

To our knowledge, we are the first to study default allocations, and even the role of costly, endogenous participation has not received a lot of previous attention. However, maybe confusingly, standard models without explicit participation costs are sometimes framed in terms of participation. For instance, following Hillman and Riley (1989) or Gradstein (1995), equilibria in asymmetric models often exhibit a player selecting a zero bid with positive probability in a way that is sometimes interpreted as non-participation rather than active participation with a zero bid. Our model with participation costs has no such ambiguity because it explicitly distinguishes between the two. This distinction is consistent with some of our applications including the clearinghouse setting where firms can participate with an offer of zero consumer surplus (e.g. by advertising the monopoly price under unit demand) in a way that is qualitatively distinct from not participating (by refraining from advertising).

Aside from our paper, a small contest literature considers endogenous participation more explicitly with either direct participation costs (e.g. Fu and Lu 2010, Fu et al. 2015), or indirect participation costs (e.g. Hillman and Samet 1987, Morgan et al. 2012, Bertoletti 2016, Chowdhury 2017, Boosey et al. 2020). While much of this literature considers a broader range of contests beyond our all-pay setting, all of these papers assume that participation costs are symmetric and that the prize is withheld if no player actively participates. In contrast, within our all-pay setting with arbitrary asymmetry, we introduce a general form of participation costs and allow the prize to be awarded even when no player actively participates. In addition, we demonstrate how asymmetric participation costs can

players and the prize value are both increased proportionately.

arise endogenously in terms of contest design.[6][7]

The clearinghouse sales framework (e.g. Baye and Morgan 2001, Baye et al. 2004, Baye et al. 2006) encompasses a large range of sales models, including Varian (1980) as a special case with zero participation costs and no default allocations. Within the special case of Varian (1980), some older literature studies a limited form of asymmetries in simplified settings (e.g. Narasimhan 1988, Baye et al. 1992, Wildenbeest 2011, Shelegia 2012), whilst Myatt and Ronayne (2024) study a more general form of asymmetries. Outside the special case of Varian, there is very little work. Arnold et al. (2011) allow for one form of asymmetry, whilst Shelegia and Wilson (2021) allow for wider asymmetries by employing a specific 'equilibrium' tie-break rule (or default allocation) that acts to equalize players' strengths. In contrast, the current paper uses tools from contest theory to offer a general clearinghouse characterization for *any* default allocation probabilities under a flexible form of participation costs and arbitrary asymmetry. As noted above, this substantially expands the initial work pioneered by Baye et al. (1996) and Baye et al. (2012) who made the connection between between all-pay contests and the special case of Varian (1980) under symmetry.

Beyond our paper, Montez and Schutz (2021) explore another connection between all-pay contests and pricing in a very different context where firms simultaneously source unobservable inventories before setting prices. Their paper focuses on inventory behavior and associated public policy, but as a side result, they show how their equilibrium can tend to a version of the asymmetric clearinghouse equilibrium as inventory costs become fully recoverable. However, contrary to the full clearinghouse literature and our framework, they assume that the informed 'shopper' consumers do not buy if neither firm advertises (implying that default allocation probabilities are zero). Our results highlight the importance of this assumption and derive the equilibrium for all default allocation probabilities in order to fully connect the literatures on all-pay contests and clearinghouse sales price competition.

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[6] Some of these listed papers further differ to our base set-up due to their assumptions of sequential participation decisions (e.g. Fu and Lu 2010 and Morgan et al. 2012) or private information (e.g. Hammond et al. 2019 and Liu and Lu 2019). The two latter papers also assume participation costs in the form of entry fees that can be used to supplement the prize fund. We exclude this possibility to focus solely on the role of costly participation.

[7] As we later show, participation costs create a discontinuity in the players' payoffs. Duvocelle and Mourmans (2021) study some wider forms of payoff discontinuities and show how Siegel's equilibrium payoff results can still apply.

2 Model

2.1 Assumptions

Consider two risk-neutral players, $i = \{1, 2\}$, and a contest to win a single prize. Each player must decide whether to actively participate in the contest, and if so, how much to bid. We assume that the players' participation and bidding decisions are simultaneous. Specifically, we model this as a one-stage decision where each player i chooses a bid or 'offer', $u_i \in \{\phi\} \bigcup [0, \infty)$. Under our terminology, we class player i as an 'active' participant if she submits an explicit offer $u_i \in [0, \infty)$. On the other hand, if player i selects $u_i = \phi$, she makes no explicit offer and is termed as only a 'passive' participant.[8]

Given the players' chosen strategies, $S = \{u_1, u_2\}$, player i's probability of winning is then given by the following contest success function, $\Psi_i(\cdot)$:

$$\Psi_i(\cdot) = \begin{cases} 1 & \text{if } u_i \geq 0 \text{ and } u_j \in \{\phi\} \bigcup [0, u_i) \\ y_i & \text{if } u_i = u_j \text{ with } u_i \geq 0 \text{ and } u_j \geq 0 \\ x_i & \text{if } u_i = u_j = \phi \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Intuitively, as consistent with the wider literature on participation costs, player i wins outright if she submits an active offer, $u_i \geq 0$, and player j either submits a lower active offer or only participates passively. If both players submit the same active offer, then player i wins with a tie-break probability, y_i. As the exact level of y_i will prove irrelevant, we allow any $y_i \in [0, 1]$ such that $y_1 + y_2 = 1$. However, in contrast to the literature, we allow the prize to still be awarded even when both players are passive, $u_i = u_j = \phi$. In such an event, player i wins with a 'default allocation probability', $x_i \geq 0$, where $x_1 + x_2 \equiv X \in [0, 1]$.

For a given set of strategies, S, and contest success function, $\Psi_i(\cdot)$, player i's

[8] Our 'simultaneous' set-up implies that players make their bidding decisions without knowing the exact number of active participants. This assumption helps provide a clean analysis and is consistent with the entire clearinghouse literature and much of the contest literature (e.g. Fu et al. 2015, Hammond et al. 2019 and Liu and Lu 2019). However, the Supplementary Online Appendix considers an alternative two-stage set-up where active players learn the number of other active participants before bidding. It verifies how the equilibrium still shares many features with that in the main model, including its dependence on our measure of strength.

expected payoff can be described as

$$E(\Pi_i(S;\Psi_i(\cdot))) = \Psi_i(\cdot)W_i(u_i) + [1-\Psi_i(\cdot)]L_i(u_i) \tag{2}$$

where $W_i(u_i)$ and $L_i(u_i)$ provide general descriptions of player i's (net) payoffs from winning and losing respectively for any given offer, including passive participation where $W_i(\phi) \equiv W_i^\phi$ and $L_i(\phi) \equiv L_i^\phi$. In addition, we make the following assumptions for each player i:

A1) $W_i(u_i) > L_i(u_i)$ for any given $u_i \in \{\phi\} \bigcup [0,\infty)$.

A2) For any $u_i \in [0,\infty)$, both $W_i(u_i)$ and $L_i(u_i)$ have the same unique finite maximizer, $u_i^m \in [0,\infty)$, and are strictly decreasing in $u_i > u_i^m$.

A3) $c(u_i^m) \equiv L_i^\phi - L_i(u_i^m) > 0$ and $W_i^\phi > W_i(u_i^m)$.

A4) $x_i > 0$.

A1 simply assumes that the payoffs from winning are always larger than those from losing for any given offer (including passive participation, $W_i^\phi > L_i^\phi$).

A2 assumes that player i's payoffs from winning and losing both have a unique finite maximizer and are strictly decreasing in the player's offer thereafter. Moreover, although not always required, A2 also assumes that $W_i(u_i)$ and $L_i(u_i)$ have the *same* such maximizer, u_i^m. To allow for a form of non-monotonicity in player i's payoffs (or headstarts in the sense of Siegel 2009), this maximizer can be non-zero, $u_i^m \in [0,\infty)$. The maximizer can also differ across players $u_i^m \neq u_j^m$, to reflect potentially different technologies, preferences, or prior investments.[9]

A3 assumes the existence of participation costs in a flexible way such that $L_i^\phi > L_i(u_i^m)$ and $W_i^\phi > W_i(u_i^m)$. To understand this, and as later formalized, note that player i will never want to select an active offer lower than u_i^m. A3 then assumes that losing (or winning) under passive participation to gain L_i^ϕ (or W_i^ϕ) is always strictly preferred to losing (winning) under active participation to gain, at most, $L_i(u_i^m)$ (or $W_i(u_i^m)$). Without A3, the distinction between passive and active participation becomes blurred and the default allocation probabilities become ill-defined.

[9]A2 is consistent with past research and many applications. We believe that all previous contest papers (implicitly) assume that the related functions are both maximized at zero or some positive constant, e.g. Siegel (2009). Within the clearinghouse literature, $W_i(u_i)/L_i(u_i)$ is effectively a constant ratio related to the proportions of different types of consumers, and so both functions also have a common maximizer as later detailed in Section 6.

A4 assumes that each player has a strictly positive default allocation probability as consistent with the organizer being unable to perfectly commit to withholding the prize from either player in the event that both players are passive. Without A4, some economically uninteresting equilibrium multiplicity can arise.

We now characterize the Nash equilibria for any permitted set of default allocation probabilities, $\{x_1, x_2\}$, and payoff functions, $W_i(u_i)$ and $L(u_i)$ for $i = \{1, 2\}$. To allow for mixed strategies, we define i) $(1 - \alpha_i) \in [0, 1]$ as player i's probability of passive participation (with $u_i = \phi$), ii) $\alpha_i \in [0, 1]$ as player i's probability of active participation on some support $u_i \in [\underline{u}_i, \bar{u}_i]$ where $0 \leq \underline{u}_i \leq \bar{u}_i$, and iii) $F_i(u)$ as player i's *overall* (unconditional) offer distribution on $u_i \in \{\phi\} \bigcup [0, \infty)$.[10] Lastly, we denote $u^m = max\{u_1^m, u_2^m\} \geq 0$.

2.2 Definitions

Despite the complexity of our framework, we will show how the equilibria will fundamentally depend on only two measures for each player, 'reach' and a new concept, 'strength'. These two measures will drive players' participation decisions. Broadly speaking, a player's reach determines their willingness to be active when their rival is also active, whereas a player's strength determines their willingness to be active when their rival is passive.

Definition 1. For a given contest, the reach of player i, r_i, is the unique value of $u_i \geq u_i^m$ that solves

$$W_i(u_i) = L_i^{\phi} \tag{3}$$

if such a solution exists, and $r_i = -\infty$ otherwise. When $W_i(u_i^m) \geq L_i^{\phi}$, a unique solution always exists with $r_i \geq u_i^m$. When $W_i(u_i^m) < L_i^{\phi}$, no solution exists.[11]

Intuitively, player i will never find it optimal to provide an active offer above her 'reach', r_i, because it is defined as the active offer $u_i \geq u_i^m$ at which player i's

[10] To facilitate the use of $F_i(u)$, we abuse notation slightly and treat ϕ as if it were a number less than 0. Player i then sets $u_i = \phi$ with probability mass $(1 - \alpha_i) = F_i(\phi)$, and submits an active offer on $u_i \in [\underline{u}_i, \bar{u}_i]$ with aggregate probability $\alpha_i = 1 - F_i(\phi)$ where $F_i(u) = 0$ for $u < \phi$, $F_i(u) = 1$ for $u \geq \bar{u}_i$, and $F_i'(u) \geq 0$ for all u.

[11] A solution exists and is unique iff $W_i(u_i^m) \geq L_i^{\phi}$ because, for $u_i \geq u_i^m$, the LHS of (3) is i) at most $W_i(u_i^m)$ and ii) strictly decreasing for $u_i \geq u_i^m$, while iii) L_i^{ϕ} is a constant unbounded above.

payoff from winning for sure, $W_i(u_i)$, equals her payoff from losing for sure under passive participation, L_i^ϕ. Further, when player $j \neq i$ is active, player i can never win under passive participation and can therefore only guarantee L_i^ϕ from being passive. Hence, when player j is active, player i will prefer to submit an active offer $u_i \geq u_i^m$ only if $W_i(u_i^m) \geq L_i^\phi$ or equivalently, only if $r_i \geq u_i^m$.

Definition 2. For a given contest, the strength of player i, s_i, is the unique value of $u_i \geq u_i^m$ that solves

$$W_i(u_i) = \Omega_i \equiv L_i^\phi + x_i(W_i^\phi - L_i^\phi)\frac{c_i(u_i^m)}{b_i(u_i^m)} \tag{4}$$

if such a solution exists, and $s_i = -\infty$ otherwise, where $c_i(u_i^m) \equiv L_i^\phi - L_i(u_i^m) > 0$ and $b_i(u_i^m) \equiv W_i(u_i^m) - L_i(u_i^m) - x_i(W_i^\phi - L_i^\phi)$. When $W_i(u_i^m) \geq L_i^\phi + x_i(W_i^\phi - L_i^\phi)$ (or equivalently when $W_i(u_i^m) \geq \Omega_i$ or $b_i(u_i^m) \geq c_i(u_i^m)$), a unique solution always exists with $s_i \geq u_i^m$. When $W_i(u_i^m) < L_i^\phi + x_i(W_i^\phi - L_i^\phi)$, no solution exists.[12]

While the definition of strength is more involved, it provides clear implications for player i's participation decision when player $j \neq i$ is passive. If player j is passive, player i's expected payoff from being passive equals $L_i^\phi + x_i(W_i^\phi - L_i^\phi)$. Hence, player i will prefer to submit an active offer $u_i \geq u_i^m$ only if $W_i(u_i^m) \geq L_i^\phi + x_i(W_i^\phi - L_i^\phi)$ or equivalently, only if $s_i \geq u_i^m$. In more detail, when $s_i \geq u_i^m$, player i's strength is the level of active offer, $u_i \geq u_i^m$, at which her payoff from winning for sure, $W_i(u_i)$, is equal to an expression that we denote by Ω_i. Where relevant, Ω_i, can be understood as player i's expected payoff at the point where she is indifferent between being passive and submitting an active offer of u_i^m.[13]

Note the following important remarks about reach and strength. First, our use of the term 'reach' broadly parallels the existing literature, e.g. Siegel (2009). However, in the previous literature, a player's reach would always equal our measure of strength. To see this, note from (3) and (4) that $r_i = s_i$ if player i has

[12]A solution exists and is unique iff $W_i(u_i^m) \geq \Omega_i$ because, for $u_i \geq u_i^m$, the LHS of (4) is i) at most $W_i(u_i^m)$ and ii) strictly decreasing for $u_i \geq u_i^m$, while iii) Ω_i is a constant unbounded above.

[13]More precisely, note that player i's expected payoff from being passive equals $L_i^\phi + x_i(1 - \alpha_j)(W_i^\phi - L_i^\phi)$. If, as will be true in equilibrium, i) player j never selects any active offers below u_i^m, such that $(1 - \alpha_j) = F_j(u_i^m)$, and ii) there are no ties at u_i^m, then player i's expected payoff from submitting u_i^m is $L_i(u_i^m) + (1 - \alpha_j)(W_i(u_i^m) - L_i(u_i^m))$. Whenever player i is indifferent between these two payoffs, it must be that $(1 - \alpha_j) = c_i(u_i^m)/b_i(u_i^m)$, where $(1 - \alpha_j) \in (0, 1]$ if $s_i \geq u_i^m$. Hence, after substituting $(1 - \alpha_j) = c_i(u_i^m)/b_i(u_i^m)$ back in, Ω_i represents the expected payoff whenever player i is indifferent between ϕ and u_i^m.

i) zero participation costs, $c_i(u_i^m) \equiv L_i^\phi - L_i(u_i^m) = 0$, and/or ii) a zero default allocation probability, $x_i = 0$. This highlights the important interaction between participation costs in A3, $c_i(u_i^m) > 0$, and default allocation probabilities in A4, $x_i > 0$; when *combined*, they imply that each player's reach is strictly larger than their strength, $r_i > s_i$ for $i = \{1, 2\}$. Hence, each player is strictly more willing to be active when their rival is active relative to when their rival is passive. Second, without loss of generality, we will assume that either i) $s_1 > s_2$, or ii) $s_1 = s_2$ and $u_1^m \geq u_2^m$. Then, whereas some existing papers refer to the player with the higher reach as the 'stronger' player, we will only employ this language under A5 below. Indeed, under A5, it is possible that the 'stronger' player 1, with $s_1 \geq s_2$, can have a lower reach, $r_1 < r_2$.

A5) Player 1 is assigned to be the 'stronger' player (and Player 2 as the 'weaker' player) with i) $s_1 > s_2$, or ii) $s_1 = s_2$ and $u_1^m \geq u_2^m$.

Finally, to help exposition, we will sometimes focus on equilibria in 'generic' contests as defined by Definition 3. Appendix B later characterizes the full set of equilibria for all generic and non-generic contests, and shows that the equilibria in non-generic contests can involve some less interesting equilibrium multiplicities.

Definition 3. A 'generic' contest does not involve the following knife-edge cases: $r_i = u_i^m$ or $s_i = u_i^m$ for any $i = 1, 2$.

3 Equilibrium Analysis

To derive the equilibria, Section 3.1 first considers some preliminary steps before Section 3.2 provides the main characterization. Any proofs are provided in Appendix A unless stated otherwise.

3.1 Preliminaries

Lemma 1. *Any active offer, u_i, is strictly dominated for player i if a) $u_i < u_i^m$, or b) $u_i \in (u_i^m, u_j^m)$.*

This implies that player i will only consider an active offer equal to $u_i = u_i^m$ or $u_i \geq u^m \equiv max\{u_i^m, u_j^m\}$. The proof is immediate. a) Any active offer $u_i \in [0, u_i^m)$ is strictly dominated by $u_i = u_i^m$ as it would raise player i's payoffs from winning or losing (via A2), and yet never reduce her probability of winning. b) Any $u_i \in (u_i^m, u_j^m)$ is also strictly dominated by u_i^m. To see this, note from above that player j will never select any active offer $u_j < u_j^m$. Hence, moving any mass in

$u_i \in (u_i^m, u_j^m)$ to $u_i = u_i^m$ would raise player i's payoffs from winning or losing, but have no effect on her probability of winning.

Lemma 2. *Suppose only one player, player i, is active with positive probability, $\alpha_i > 0$ and $\alpha_j = 0$. Then, in equilibrium, it must be that $\underline{u}_i = \bar{u}_i = u_i^m$.*

Again, the proof is immediate. In this case, player i must set $u_i \geq u_i^m$ and player j must set $u_j = \phi$ such that player i wins with probability one. Given this, by reducing u_i to u_i^m, player i can strictly increase her payoffs via A2 and still win with certainty.

Lemma 3. *In equilibrium, player i cannot put a point mass on any active offer other than $u_i = u_i^m$. Further, if $u_1^m = u_2^m = u^m$, then at most, one player can put a point mass on u^m.*

As detailed in the proof, this just follows standard mixed-strategy results - if not, at least one player would have an incentive to redistribute their probability mass elsewhere. Now denote the size of any potential point mass at u_i^m by $\beta_i \geq 0$. As player i's probability of active participation on $u_i \geq u_i^m$ is denoted by $\alpha_i \in [0, 1]$, then it must be that $\alpha_i \geq \beta_i$.

Lemma 4. *Suppose player i selects an offer strictly above u_i^m with positive probability in equilibrium such that $\alpha_i > \beta_i \geq 0$. Then, it must be that:*
a) both players make offers above u^m with positive probability and share a common upper bound, $\bar{u} \equiv \bar{u}_1 = \bar{u}_2 > u^m$,
b) any $u \in (u^m, \bar{u}]$ is a point of increase of $F_1(u)$ and $F_2(u)$,
c) on $u \in (u^m, \bar{u}]$ for $k = 1, 2$ and $l \neq k$,

$$F_k(u) = \frac{W_l(\bar{u}) - L_l(u)}{W_l(u) - L_l(u)}. \tag{5}$$

Intuitively, if $\alpha_i > \beta_i$ in equilibrium then player i makes active offers strictly above u_i^m. If so, then the other player must be doing the same otherwise player i could optimally reduce her offers towards u_i^m. Moreover, if $u_i^m < u_j^m$, then any $u \in (u_i^m, u_j^m)$ is dominated for both players and so they must make offers strictly above u^m. As consistent with standard results (without participation costs), the two players must then continuously randomize up to a common upper bound, $\bar{u}$. By deriving expected payoffs and equilibrium payoffs for a given $\bar{u}$, one can then characterize the implied distribution in active offers for each player, (5).

3.2 Characterization

Building on Lemmas 1-4, we now characterize the full equilibria. To further aid exposition, it is convenient to denote the following two expressions:

$$\theta_i(u) = 1 - \frac{W_j(u) - L_j^\phi}{(W_j^\phi - L_j^\phi)x_j} \tag{6}$$

$$\sigma_i(u) = 1 - \frac{c_j(u)}{b_j(u)} \equiv 1 - \frac{L_j^\phi - L_j(u)}{W_j(u) - L_j(u) - x_j(W_j^\phi - L_j^\phi)} \tag{7}$$

Given the usual complexities with models of this sort, it is notable that we now demonstrate that any generic contest has a unique, tractable equilibrium and that the equilibrium can be reduced to five qualitatively distinct cases that only depend upon the relative sizes of s_1 and r_2. As the proof is lengthy, it is provided separately in Appendix B.

Theorem 1. *Given A1-5, there exists a unique equilibrium for any generic contest:*

i) When $s_1 < u_1^m$ *(and hence* $s_2 < u_2^m$*), neither player is active,* $\alpha_1 = \alpha_2 = 0$*.*
ii) When $s_1 > u_1^m$ *and* $r_2 \leq u^m$*, player* 1 *is always active at* u_1^m*,* $\alpha_1 = \beta_1 = 1$*, and player* 2 *is always passive,* $\alpha_2 = 0$*.*
iii) When $r_2 > u_2^m \geq s_1 > u_1^m$*, player* 1 *selects* u_1^m *with probability* $\beta_1 = \alpha_1 = \theta_1(u_2^m) \in (0,1)$ *and player* 2 *selects* u_2^m *with probability* $\beta_2 = \alpha_2 = \sigma_2(u_1^m) \in (0,1)$*.*
iv) When $r_2 > s_1 > u^m$*, players 1 and 2 are active with probabilities* $\alpha_1 = \theta_1(\bar{u}) \in (0,1)$ *and* $\alpha_2 = 1 - F_2(u_1^m) = \sigma_2(u_1^m) \in (0,1)$*. They both randomize on* $(u^m, \bar{u}]$ *with* $F_i(u)$ *in (5) where* $\bar{u} = s_1$*,* $\beta_1 = F_1(u^m) - (1-\alpha_1) \geq 0$ *and* $\beta_2 = F_2(u^m) - F_2(u_1^m) \geq 0$*.*
v) When $s_1 \geq r_2 > u^m$*, players 1 and 2 are active with probabilities* $\alpha_1 = 1$ *and* $\alpha_2 = 1 - F_2(u_1^m) \in (0,1)$*. They both randomize on* $(u^m, \bar{u}]$ *with* $F_i(u)$ *in (5) where* $\bar{u} = r_2$*,* $\beta_1 = F_1(u^m) \geq 0$ *and* $\beta_2 = F_2(u^m) - F_2(u_1^m) \geq 0$*.*

We now discuss the intuition of Theorem 1 in terms of reach and strength. Later sections will build on this to offer a fuller examination of how default allocation probabilities and participation costs directly affect equilibrium.

To start, it is useful to initially consider the (quasi-) symmetric case where $r_1 = r_2 = r$, $s_1 = s_2 = s$ and $u_1^m = u_2^m = u^m$. Here, as $r > s$, Theorem 1 collapses to a simple form involving only case i) and iv) depending solely on $s \lesseqgtr u^m$. First, suppose $s < u^m$ such that case i) applies. From Definition 2, this implies that

$L^{\phi} + x(W^{\phi} - L^{\phi}) > W(u^m)$. Therefore, neither player wishes to be active - given that the other player is passive, a player earns $L^{\phi} + x(W^{\phi} - L^{\phi})$ and has no incentive to be active in order to earn, at most, $W(u^m)$. This equilibrium is unique because the remaining possibility where both players are always active cannot be an equilibrium as one player would always deviate due to costly participation (A3). Next suppose $s > u^m$ such that case iv) applies. From Definition 2, this implies that $W(u^m) > L^{\phi} + x(W^{\phi} - L^{\phi})$ such that each player has an incentive to be active if the other is passive. However, both players cannot be active with probability one in equilibrium due to the assumption of costly participation. Hence, the unique equilibrium involves both players being active with interior probability, $\alpha \in (0, 1)$, and randomizing over active offers with $F(u)$, where $\bar{u} = s > u^m$ and $\beta = 0$.

We now give an initial overview of the intuition of Theorem 1 under player asymmetry. However, some simple example settings are also provided shortly in Section 4.1. First, consider case i). Here, the stronger player 1 has low strength, $s_1 < u_1^m$. Using Definition 2 and A5, this implies that both players have low strength, $s_1 = s_2 = -\infty$, such that $L_i^{\phi} + x_i(W_i^{\phi} - L_i^{\phi}) > W_i(u_i^m)$ for $i = \{1, 2\}$. Therefore, neither player wishes to be active when the other is passive and this equilibrium can be shown to be unique. Each player i earns $L_i^{\phi} + x_i(W_i^{\phi} - L_i^{\phi})$.

Next, examine case ii) where the stronger player 1 has a relatively higher strength, $s_1 > u_1^m$, but player 2 has a relatively low reach, $r_2 < u^m$. Using Definitions 1 and 2, this implies a) $W_1(u_1^m) > L_1^{\phi} + x_1(W_1^{\phi} - L_1^{\phi})$ - if player 2 is passive, player 1 has a strict incentive to be active, and b) $W_2(u^m) < L_2^{\phi}$ - if player 1 is active, player 2 strictly prefers to remain passive. Hence, this ensures a pure-strategy equilibrium where only player 1 actively competes. Moreover, given the specified values of reach and strength, this equilibrium is unique. Player 1 earns $W_1(u_1^m)$ while player 2 earns L_2^{ϕ}.

Now defer cases iii) and iv) and jump to case v). Here, $s_1 \geq r_2 > u^m$ such that player 1 has a high strength and player 2 has a sufficient reach. Via Definitions 1 and 2, this implies that both players are willing to be active with $\alpha_i > \beta_i$ - player 1 is willing to be active if player 2 is passive, but if player 1 is active then player 2 is also willing to be active. Specifically, by building on Lemma 4, the unique equilibrium involves a) both players mixing over active offers with $F(u)$ up to $\bar{u} = r_2 > u^m$, b) player 2 mixing over active participation with interior probability, $\alpha_2 \in (0, 1)$, but c) player 1 remaining strong enough to always be active, $\alpha_1 = 1$. This latter feature implies that the default allocation probabilities are never implemented within this case. As such, this form of equilibrium bears a qualitative resemblance to standard asymmetric equilibria without participation costs or default allocation probabilities (e.g. Hillman and Riley 1989 or Siegel 2010). Note that $\alpha_1 = 1$ also implies that player 2 can only guarantee an equilibrium payoff of L_2^ϕ, while player 1 earns a payoff equal to $W_2(\bar{u}) > L_1^\phi$, as equivalent to the standard model.

Next, move back to the more novel cases iii) and iv), and start with case iv). Here, $u^m < s_1 < r_2$, such that player 1's strength is sufficiently high while player 2 has a high reach. Like in case v), this implies that both players are willing to be active with $\alpha_i > \beta_i$. Hence, once again, in the unique equilibrium, both players mix over active offers with a non-degenerate distribution, $F(u)$ up to $\bar{u} > u^m$ and player 2 mixes over active participation with interior probability, $\alpha_2 \in (0, 1)$. However, unlike case v), the upper bound differs, $\bar{u} = s_1$, and player 1 also mixes over active participation with a strictly interior probability, $\alpha_1 \in (0, 1)$ as she is not strong enough to be active with probability one. Consequently, the default allocation probabilities are implemented with positive probability. Each player i earns payoffs higher than L_i^ϕ but strictly lower than if they both remained passive, $L_i^\phi + x_i(W_i^\phi - L_i^\phi)$. Hence, this case has a Prisoner's Dilemma feature that is not present in the previous literature - the players would strictly prefer everyone to remain passive, but have an individual incentive to deviate and be active.

Finally, return to case iii). Here, player 1 has a moderate strength, $s_1 \in (u_1^m, u_2^m]$ and player 2 has a relatively high reach, $r_2 > u_2^m$. Hence, this case can only occur if $u_2^m > u_1^m$. Thus it require several underlying dimensions of underlying symmetry as player 1 has the higher strength but the lower level of maximizing offer. Using Definitions 1 and 2, the conditions on reach and strength imply $W_1(u_1^m) > L_1^\phi + x_1(W_1^\phi - L_1^\phi) > W_1(u_2^m)$ and $W_2(u_2^m) > L_2^\phi$. Intuitively, player 1 is strong enough to be active at u_1^m when player 2 is passive, but player 2 would prefer to be active at u_2^m if player 1 chooses u_1^m. Further, player 1 is not strong enough to be active at u_2^m when player 2 is passive, but player 2 would be active there if player 1 is. As a result, this case produces an unusual, novel form of equilibrium that is new to the literature - both players use a binary strategy to randomize between being passive and selecting their own minimum active offer, u_i^m. In equilibrium, both players earn $L_i^\phi + x_i(W_i^\phi - L_i^\phi)(1 - \alpha_j)$ and so like case iv), the case also has a Prisoner's Dilemma feature where the players would prefer a commitment to no active offers.

4 Examples

This section offers some simple examples of the equilibrium. Aside from highlighting some features of Theorem 1, it also further illustrates the concepts of reach and strength, and forms a useful base for other later parts of the paper. Specifically, in an otherwise standard all-pay contest, Section 4.1 first offers an example that allows for direct participation costs, as consistent with entry fees, set-up costs or foregone outside options, while Section 4.2 expands this to show how our framework can also allow for indirect participation costs in the form of a minimum bid or reservation offer.

4.1 An Example with Direct Participation Costs

Suppose each player i values the contest's prize at $V_i > 0$. The total cost of player i submitting an active offer of $u_i \geq 0$ equals $k_i u_i^a + A_i$ where i) $k_i u_i^a$ is the associated effort cost (with $k_i > 0$ and $a > 0$ as parameters), and ii) $A_i \in (0, V_i]$ is a direct participation cost. Each player i's default allocation probability continues to be denoted by $x_i \in (0,1)$. Using our framework, we can then write player i's payoff functions as $W_i(u_i) = V_i - k_i u_i^a - A_i$ and $L_i(u_i) = -k_i u_i^a - A_i$ which further imply $u_1^m = u_2^m = 0$, $W_i^\phi = V_i$ and $L_i^\phi = 0$.

Following Definitions 1 and 2, one can compute reach and strength, (8) and (9). Intuitively, player i's reach and strength are both increasing in the prize value, V_i, but decreasing in the costs of making a given active bid, k_i, A_i and a. Further, player i's strength is also decreasing in her default allocation probability, x_i.

$$r_i = \begin{cases} \left(\frac{V_i - A_i}{k_i}\right)^{1/a} & \text{if } r_i \geq u_i^m = 0 \Leftrightarrow V_i \geq A_i \\ -\infty & \text{if } r_i < u_i^m = 0 \Leftrightarrow V_i < A_i \end{cases} \tag{8}$$

$$s_i = \begin{cases} \left[\frac{1}{k_i}\left(V_i - \frac{A_i}{(1-x_i)}\right)\right]^{1/a} & \text{if } s_i \geq u_i^m = 0 \Leftrightarrow V_i(1-x_i) \geq A_i \\ -\infty & \text{if } s_i < u_i^m = 0 \Leftrightarrow V_i(1-x_i) < A_i \end{cases} \tag{9}$$

For ease of exposition, let $k_1 = k_2 = 1$ and $a = 1$. Then, when no lower than $u_i^m = 0$, reach and strength reduce to $r_i = (V_i - A_i)$ and $s_i = V_i - \frac{A_i}{(1-x_i)}$, where $s_1 \geq s_2$ requires $V_1 - \frac{A_1}{(1-x_1)} \geq V_2 - \frac{A_2}{(1-x_2)}$. Notice that this does not require player 1 to have the higher prize value - it can be offset by a lower participation cost or default allocation probability. The equilibrium cases in Theorem 1 then follow straightforwardly (albeit without case iii which cannot exist given $u_1^m = u_2^m = 0$): case i) applies with no active participation if $V_1 < \frac{A_1}{(1-x_1)}$ (such that $s_1 < u^m$), case ii) applies where only player 1 is active if $V_1 - \frac{A_1}{(1-x_1)} > 0 > V_2 - A_2$ (such that $s_1 > 0 > r_2$), case iv) applies where both players mix over active offers with $\alpha_1, \alpha_2 \in (0,1)$ if $V_2 - A_2 > V_1 - \frac{A_1}{(1-x_1)} > 0$ (such that $r_2 > s_1 > 0$), and case v) applies where both players mix over active offers with $\alpha_1 = 1$ and $\alpha_2 \in (0,1)$ if $V_1 - \frac{A_1}{(1-x_1)} \geq V_2 - A_2 > 0$ (such that $s_1 \geq r_2 > 0$).[14]

4.2 An Example with Indirect Participation Costs

In contrast to direct participation costs, other 'indirect' forms of participation costs can derive from a minimum required bid or 'reservation offer'. In such settings, any valid active offer must be weakly larger than some level, $u^R \geq 0$. From A2 and Lemma 1, such a reservation offer will only have an effect on equilibrium if $u^R > u_i^m$ for at least one player; any affected player i then has to choose between $u_i = \phi$ and $u_i \geq u^R > u_i^m$. Hence, like in the seminal analysis (under symmetry with linear effort costs) by Hillman and Samet (1987), a reservation offer can create indirect participation costs by prompting player i to submit an active offer at a higher level than she might have done otherwise.

This can be easily captured within our more general framework by slightly modifying the game. First, without loss, we can modify each player i's win and loss functions to equal $W_i(u^R)$ and $L_i(u^R)$ for $u_i \in [0, u^R)$ but to remain otherwise unchanged. Denote these as $\tilde{W}_i(\cdot)$ and $\tilde{L}_i(\cdot)$, respectively. From above, we then know that player i will only consider any active offer $u_i \geq max\{u_i^m, u^R\}$. Hence, player i's new maximizer becomes $\tilde{u}_i^m \equiv max\{u_i^m, u^R\}$. Second, we can use these to calculate a revised level of strength, $\tilde{s}_i$, from (4), and apply Theorem 1 to the modified game. For instance, consider the introduction of a binding reservation offer, $u^R > u_1^m = u_2^m = 0$, into our previous example with $k_1 = k_2 = 1$, $a = 1$ and $A_1 = A_2 = 0$. The reservation offer acts now as the only participation cost. It reduces player i's willingness to compete when their rival is passive by decreasing their strength, $\tilde{s}_i = V_i - \frac{x_i u^R}{(1-x_i)}$ when $\tilde{s}_i \geq u^R$, and makes it less likely that the equilibrium falls into a case involving more active participation.

[14]Specifically, case iv) has $\bar{u} = s_1 = V_1 - \frac{A_1}{(1-x_1)}$, $\Pi_1^* = W_1(\bar{u}) = A_1 \frac{x_1}{(1-x_1)}$, $\Pi_2^* = W_2(\bar{u}) = V_2 - A_2 - V_1 + \frac{A_1}{(1-x_1)}$, $\alpha_1 = \theta_1(\bar{u}) = 1 - \left(\frac{\Pi_2^*}{x_2 V_2}\right)$, $\alpha_2 = 1 - F_2(0) = \frac{A_1}{(1-x_1)V_1}$, and $F_i(u_i) = \frac{\Pi_j^* + u_i + A_j}{V_j}$ for $i = \{1,2\}$, and case v) has $\bar{u} = r_2 = V_2 - A_2$, $\Pi_1^* = W_1(\bar{u}) = (V_1 - V_2) + (A_2 - A_1)$, $\Pi_2^* = W_2(\bar{u}) = 0$, $\alpha_1 = 1$, $\alpha_2 = 1 - F_2(0) = \frac{V_2 - A_2}{V_1}$, $F_1(u_1) = \frac{u_1 + A_2}{V_2}$ and $F_2(u_2) = 1 - \left(\frac{V_2 - A_2}{V_1}\right) + \left(\frac{u_1}{V_1}\right)$.

5 Contest Design

To continue demonstrating the benefits of our framework, this section analyzes how a contest organizer would optimally design the novel features of our model: i) the participation costs, and ii) the default allocation probabilities. As later detailed, these contest design tools have received little or no attention within the existing literature (e.g. Chowdhury et al. 2023 and Fu and Wu 2019).

We will assume that the organizer wishes to maximize offers. Specifically, a contest design will be referred to as *'offer-maximizing'* if it maximizes any weighted combination of the sum of total expected offers, $E(u_1) + E(u_2)$, and the expected winning offer, $E(u_{max})$: $\lambda[E(u_1) + E(u_2)] + (1 - \lambda)E(u_{max})$ $\forall\lambda \in [0, 1]$. Aside from capturing familiar contest objectives related to the associated level of effort or bids, this objective can also correspond to consumer surplus in our later clearinghouse sales context. Throughout, whenever the players are mixing over the interval $(u^m, \bar{u}]$, we refer to an 'improvement' (or 'reduction') in player i's offers in the sense of first-order stochastic dominance (FOSD). Holding constant player j's strategy, such an improvement (or reduction) ensures both an increase (or decrease) in player i's expected offer, $E(u_i)$, and the contest's expected winning offer, $E(u_{max})$.[15]

We first consider the optimal design of participation costs under the assumption that the contest organizer can individually manipulate each player's (direct) costs of being active, $A_1 \geq 0$ and $A_2 \geq 0$, at the start of the game. To focus on the optimal design of participation costs per se, we assume that $\boldsymbol{A} = \{A_1, A_2\}$ comprises of players' set-up costs rather than entry fees which could be otherwise used to enhance the prize fund.

While the spirit of our results can be shown more broadly, we focus on the following setting. First, apart from $\boldsymbol{A}$, we assume the players are otherwise symmetric. This implies that the players will vary in both reach and strength whenever $A_1 \neq A_2$. Second, for each player i and for all active offers, we re-define the payoff functions as follows: $L(u) \equiv l(u) - A_i$ and $W(u) \equiv w(u) - A_i$ where $w(u)$ and $l(u)$ satisfy versions of A1-A3 and where $u^m \geq 0$ is their maximizer. In particular, as consistent with A3, to ensure that costly participation remains even if $A_i = 0$,

[15]Technically, it is sufficient to define a FOSD improvement (or reduction) to occur when a) the player's new offer distribution $\hat{F}_i(u)$ is weakly less (greater) than their original offer distribution $F_i(u)$ for all active offers, $u \in [0, \infty)$, b) the player's new probability of being passive, $(1-\hat{\alpha}_i)$, is weakly less (greater) than their original probability of being passive, $(1 - \alpha_i)$, and c) either i) $\hat{F}_i(u) < (>) F_i(u)$ for at least some $u \in [0, \infty)$, and/or ii) $(1 - \hat{\alpha}_i) < (>) (1 - \alpha_i)$.

we assume that some other form of exogenous, baseline participation cost always applies such that $L^{\phi} > l(u^m)$ and $W^{\phi} > w(u^m)$. Finally, we maintain some basic potential for the players to be active by letting $w(u^m) > L^{\phi} + x(W^{\phi} + L^{\phi})$ such that $s_i > u^m$ when $A_i = 0$ for any player i.

As a preliminary step, consider the effects of a marginal increase in $A_{j \neq i}$ while holding A_i constant. From (3)-(4), this will make active participation more costly for player j and so it strictly reduces her reach and strength (when they exceed u^m), but leaves player i's reach and strength unchanged. Hence, we know that player 1 will have the higher strength and reach, $s_1 > s_2$ and $r_1 > r_2$, whenever $A_1 < A_2$. We can now state the following for any generic or non-generic contest:

Proposition 1. *In any contest under our assumptions, it is always strictly offer-maximizing to set* $A_1 = 0$ *and* $A_2 = \bar{A} \equiv \frac{x(W^{\phi}-L^{\phi})(L^{\phi}-l(u^m))}{w(u^m)-l(u^m)-x(W^{\phi}-L^{\phi})} > 0$ *such that* $s_1 = r_2 > u^m$.

One may have predicted that the offer-maximizing design would involve zero participation costs for both players. Indeed, this is easy to show if participation costs are forced to be symmetric, $A_1 = A_2$. However, Proposition 1 demonstrates that such logic is incorrect if the contest designer can employ an asymmetric contest design. Indeed, despite the players being a priori symmetric, an offer-maximizing organizer will optimally set $A_1 = 0$ and $A_2 = \bar{A} > 0$. The fact that asymmetric participation costs may arise endogenously in this way underlines the importance of our framework. As example applications, this implies that a contest organizer may wish to use asymmetric participation costs to stimulate higher bids in otherwise symmetric tendering contexts. Further, in our later clearinghouse setting, it suggests that a digital platform's use of asymmetric, discriminatory advertising fees to businesses wishing to list on its website may be pro-competitive by raising offers to consumers.

To consider the intuition, suppose 1 is the stronger player with $s_1 \geq s_2$ (such that $A_1 \leq A_2$). Then, as detailed in the proof, it must be offer-maximizing to set A_1 low enough such that $s_1 > u^m$. If not, the players would be too inactive. Then, given our setting, we know $s_1 > u^m$ can occur in either equilibrium case iv) or v). However, much of the key intuition can be understood via case iv). Here, first consider a decrease in the stronger player's participation cost, A_1, while holding A_2 constant. This increases 1's strength and prompts her to be more aggressive via an increase in $\bar{u} = s_1$. In turn, via a form of strategic complementarity, player 2 responds by also becoming more aggressive. Hence, equilibrium active offers improve, as consistent with the instinctive logic of lowering participation costs.

However, now consider an *increase* in the weaker player's participation cost, A_2, while holding A_1 constant. Ceteris paribus, this encourages 2 to be less aggressive and without any change in 1's strategy, this would induce 2 to be active with a lower probability. However, to maintain equilibrium, 1 has to respond by relocating some passive participation probability towards her point mass at u^m. This then reduces 2's payoff from being passive (and increases her payoff from being active above u^m) and thus restores 2's willingness to randomize in the same fashion as before the change in A_2.

The net result involves an overall improvement in offers: i is active with a higher probability (at u^m), while 2's behavior is unchanged. Hence, it is this effect that ensures $A_1 = A_2 = 0$ is not optimal. In essence, the stronger player has to act more aggressively in order to encourage the further weakened player to contest. Specifically, within case iv), the designer faces incentives to decrease A_1 while increasing A_2 such that s_1 rises and r_2 falls until $s_1 = r_2$. At this point, it is offer-maximizing to set the lowest values of $\{A_1, A_2\}$ possible while still ensuring set $s_1 = r_2$, which gives $A_1 = 0$ and $A_2 = \bar{A} > 0$. Notice that $\bar{A}$ is increasing in $x > 0$ and our measure of baseline participation costs, $L^\phi - l(u^m) > 0$.

The existing literature on all-pay auctions has suggested that when two players become more asymmetric, they are likely to compete less aggressively via the 'discouragement effect' (e.g. Baye et al. 1993). Results on contest design build on this to show how competition can be increased by handicapping the ex ante stronger player and favoring the ex ante weaker player in order to 'level the playing field'.[16] Somewhat similarly, we demonstrate how an organizer can induce more fierce competition by using asymmetric participation costs. However, our results differ in two important ways. First, in contrast to much of the literature, we show how an asymmetric contest design can stimulate competition even when the players are otherwise symmetric.[17] Second, and more unusually, rather than leveling the playing field, our results suggest that offer-maximizing organizers should use participation costs to create or enhance any difference between the two players' strengths. Indeed, the weaker (stronger) player with the relatively higher (lower) participation costs should optimally be made even weaker (stronger) by increasing (reducing) their participation costs. Throughout, it is important to note that it

[16]For instance, Szech (2015) and Franke et al. (2018) show this in relation to the use of tie-break rules or headstarts/multiplicative biases, respectively.

[17]As noted in the introduction, a small stream of literature has found a similar principle can also apply but these focus on different settings and different mechanisms (e.g. Drugov and Ryvkin 2017, Barbieri and Serena 2022, and Pérez-Castrillo and Wettstein 2016).

is the *combined* presence of (baseline) participation costs and default allocations that is key in our result as $\bar{A}$ would equal zero if either $x = 0$ or $L^{\phi} - l(u^m) = 0$.

Finally, consider how a contest organizer would optimally design the default allocation probabilities, $\boldsymbol{x} \equiv \{x_1, x_2\}$. The use of $\boldsymbol{x}$ offers a practical, low-cost form of contest design that has remained unstudied within the previous literature. By using our framework, one can show that an organizer will optimally 'favor' one of the players by increasing one player's default allocation probability and decreasing the other's. Again, an asymmetric contest design is optimal even when the players are otherwise symmetric. In the extreme, the designer will actually make one of the players the 'default winner' by setting $x_j \to 1$ and $x_i \to 0$. Hence, in the context of our motivating examples from the introduction, this is consistent with contract-tender settings where some firm is automatically selected in the event of no bids - our result suggests that such a contest design may be optimal for an organizer because it can stimulate more competitive bidding. The result can also have interesting implications outside the context of contest design. For instance, it suggests that electoral campaigning activity may be greater in settings where there would otherwise be a clear-cut favorite to win the vote in the absence of any campaigning. As the explanation and intuition of the default allocation probability result is very similar to Proposition 1, we defer further details to the Supplementary Online Appendix.

6 Clearinghouse Models under Full Asymmetry

This section now shows how our framework can be used to characterize the full clearinghouse equilibrium under arbitrary asymmetry for the first time. Clearinghouse models (e.g. Baye and Morgan 2001, Baye et al. 2004, Baye et al. 2006) are commonly used within industrial organization and marketing to study the role of price comparison platforms (or 'clearinghouses') on pricing and advertising. Previously, the literature had only been able to consider firm asymmetries in simplified or special settings.[18] However, by using the tools of contest theory, we can freely allow for a full set of asymmetries. As a consequence, our paper should open up new theoretical and empirical research where asymmetry is important, such as platform design or the regulation of platform fees.

To begin, consider a relatively general version of a duopoly clearinghouse

[18]For instance, see Narasimhan (1988), Baye et al. (1992), Wildenbeest (2011), Arnold et al. (2011), Shelegia and Wilson (2021) and Myatt and Ronayne (2024).

model. Suppose there are two firms, $i = \{1, 2\}$, that each sell a single good. All consumers have the same product preferences and so each consumer has an identical demand function for firm i's good, $D_i(p_i)$, given firm i's price, p_i. However, note that this demand function can differ across the two firms. Let firm i have a constant marginal cost, $k_i \geq 0$, so that firm i's potential profits *per-consumer* equal $\pi_i(p_i) = (p_i - k_i)D_i(p_i)$. We assume these profits are strictly quasi-concave in p_i with a unique maximizer at firm i's monopoly price, p_i^m.

Consumers are split into two types. Each firm i has a base of 'non-shopper' consumers with mass $\lambda_i > 0$ who only consider purchasing from their designated firm i. In addition, there is a group of 'shopper' consumers with mass $S > 0$ who are initially allocated to the firms in respective proportions, x_1 and x_2. However, any shoppers allocated to firm i become aware of firm $j \neq i$ iff firm j advertises at a digital platform (or 'clearinghouse'). Hence, if firm j does not advertise, the shoppers allocated to firm i only consider firm i but if firm j advertises, then the shoppers assigned to firm i trade with the firm offering the best deal (using any tie-breaking rule in the event of a tie). Within a one-shot game, each firm i simultaneously selects its price, p_i, and whether to advertise for a fixed cost, $A_i > 0$.

We now translate the model into our framework. First, given firm i's price, it is straightforward to calculate firm i's implied utility offer, u_i, its monopoly utility offer, u_i^m, and its per-consumer profits in terms of u_i, $\pi_i(u_i)$.[19] One can then construct firm i's payoffs from winning and losing as follows. Suppose firm i opts to be 'active' by advertising. If it has the highest offer, it wins all the shoppers to receive $W_i(u_i) = (S + \lambda_i)\pi_i(u_i) - A_i$, but otherwise, it earns $L_i(u_i) = \lambda_i \pi_i(u_i) - A_i$. Alternatively, if firm i opts to be passive by not advertising, then it will optimally offer u_i^m. If firm j advertises, then firm i will then only trade with its share of non-shoppers to obtain $L_i^\phi = \lambda_i \pi_i(u_i^m)$. However, if firm j is also passive, firm i will also retain its share of shoppers to receive $(x_i S + \lambda_i)\pi_i(u_i^m)$. Equivalently, in the language of the framework, when both firms are passive firm i will earn $L_i^\phi + x_i(W_i^\phi - L_i^\phi)$ where $W_i^\phi = (S + \lambda_i)\pi_i(u_i^m)$. Finally, one can verify that A1-A4 apply given $\lambda_i > 0$, $S > 0$, $x_i > 0$, and $A_i > 0$ $\forall i$. The measures of strength and reach can then be calculated, and Theorem 1 can be stated to fully derive the

[19] Specifically, as the consumers have identical product preferences, all consumers value firm i's offer with the associated consumer surplus, $u_i = CS_i(p_i) = \int_{p_i}^{\infty} D_i(x)dx$, where $u_i^m = CS_i(p_i^m)$. To then calculate firm i's per-consumer profit function in terms of u_i, one can denote $p_i(u_i) = CS_i^{-1}(u_i)$ and $d_i(u_i) = D_i(p_i(u_i))$ to obtain $\pi_i(u_i) = d_i(u_i)(p_i(u_i) - k_i)$.

market equilibrium.

7 Equilibrium with $n > 2$ Symmetric Players

Within the all-pay contest literature, there is a well-known equilibrium multiplicity problem when there are $n > 2$ symmetric players. Specifically, in addition to the symmetric equilibrium, there also exists a continuum of asymmetric equilibria (Baye et al. 1996). In what follows, Section 7.1, first uses our framework to demonstrate how the combination of participation costs and default allocation probabilities, even if arbitrarily small, can resolve this issue to ensure that only the symmetric equilibrium remains. To do so, we build on the clearinghouse literature where a parallel problem exists; the symmetric clearinghouse setting with zero advertising costs (à la Varian 1980) also has an infinite number of equilibria when there are more than two firms. There, however, Arnold and Zhang (2014) show how the presence of positive advertising costs à la Baye and Morgan (2001) can ensure equilibrium uniqueness. We demonstrate that this equilibrium uniqueness also holds in our more general contest framework, and highlight how it requires the simultaneous presence of participation costs and default allocations. Given this unique equilibrium, Section 7.2 then provides some comparative statics related to the 'competitiveness' of a contest.

7.1 Unique Equilibrium

In their seminal paper, Baye et al. (1996) demonstrate how a standard, symmetric, single-prize all-pay contest (without participation costs or default allocation probabilities) has an infinite number of equilibria when there are more than two players. This has remained a long-standing problem within the literature, providing uncertainty over players' predicted behavior. Specifically, such multiple equilibria have the following features (translated into our set-up and notation). Due to the absence of participation costs, each player is always active with probability one, $\alpha = 1$. There then exists i) a unique symmetric equilibrium where all players mix over $[u^m, \bar{u}]$ with no mass at u^m, and ii) a continuum of asymmetric equilibria where at least two players mix over $[u^m, \bar{u}]$, while others mix over $[\underline{u}_i, \bar{u}]$ with a positive mass point at u^m, where $\underline{u}_i > u^m$ is a free individual parameter (and where the relevant player i bids u^m with probability one if $\underline{u}_i \geq \bar{u}$).

However, we now show how the combination of participation costs and default allocation probabilities can guarantee equilibrium uniqueness for all parameter values, even in non-generic contests where $r = u^m$ or $s = u^m$. To proceed under $n \geq 2$ symmetric players, we maintain assumptions A1-A3 and an n-player version of A4 such that $x_i = (X/n) > 0$ for all i:

Proposition 2. *Suppose there are $n \geq 2$ symmetric players. Given A1-A3 and $x_i = (X/n) > 0$ for all i, the unique equilibrium of any generic or non-generic contest is symmetric:*
i) When $s \leq u^m$, all players are passive, $\alpha_i = 0$ $\forall i$.
ii) When $s > u^m$, all players are active with probability $\alpha_i = \alpha \in (0,1)$ in (10). They all randomize on $[u^m, \bar{u}]$ with $F_i(u) = F(u)$ in (11) where $\bar{u} = s$ and $\beta_i = 0$ $\forall i$.

$$\alpha = 1 - \left(\frac{c(u^m)}{b(u^m)}\right)^{\frac{1}{n-1}} \tag{10}$$

$$F(u) = \left(\frac{W(\bar{u}) - L(u)}{W(u) - L(u)}\right)^{\frac{1}{n-1}} \tag{11}$$

Proposition 2 shows how all asymmetric equilibria disappear and that only the symmetric equilibrium remains once *both* participation costs and default allocation probabilities become positive. Provided participation is not too costly, the remaining unique equilibrium involves the players mixing between passive and active participation with the same interior probability, and randomizing symmetrically over $[u^m, \bar{u}]$ without mass at u^m. This applies for all parameters in both generic and non-generic contests. Hence, our result could be used as a equilibrium refinement in the standard model to select the symmetric equilibrium.

Although lengthy to prove, the intuition of Proposition 2 can be understood as follows. All the potential asymmetric equilibria involve at least one player using mass at the lowest possible active offer, u^m. Such an offer at u^m is relatively uncompetitive because at least two other players always mix over $[u^m, \bar{u}]$. Within our framework, the use of such a mass point at u^m becomes dominated and cannot be part of equilibrium. Thus, only symmetric equilibria can remain.Then, one can use a logic akin to Theorem 1, to show that only a *single* symmetric equilibrium exists. Again, like at other points in our paper, the *combination* of positive participation costs and default allocation probabilities is key in driving this result.[20]

[20]Indeed, when $X = 0$ a continuum of asymmetric equilibria exists even when participation costs are positive. For instance, when $X = 0$, there exists a continuum of equilibria where i) all $n > 2$ firms randomize on $(u^m, r]$ with $F(u) = \left(\frac{L^\phi - L(u)}{W(u) - L(u)}\right)^{\frac{1}{n-1}}$, ii) $n-1$ firms remain passive with probability

7.2 The Effects of Competition

Finally, we now discuss some features of the unique equilibrium with respect to the 'competitiveness' of a contest. Interest in the effects of competition within full information all-pay contests, such as ours, has been rejuvenated due to Fang et al.'s (2020) recent analysis. While allowing for multiple prizes, they show a variety of insightful results under the assumption that players face convex effort costs. In contrast, within our single-prize setting, we show how the introduction of participation costs and default allocations can reverse some previous findings in relation to i) an increase in the number of players, and ii) 'scaling', where the number of players and size of prize are both increased proportionately. To see these features most easily, we place some more structure on our symmetric $n \geq 2$ model. Specifically, we focus on a symmetric version of the linear example from Section 4.2 with reservation offer, $u^R > u^m = 0$.

7.2.1 Increase in the Number of Players

First, consider an increase in the number of players, n. In a standard, symmetric all-pay contest with linear effort costs (and no participation costs or default allocations), it is well-known that expected individual offers, $E(u)$, are decreasing in n (e.g. Hillman and Samet 1987). This is further confirmed in a more general setting with convex effort costs by Fang et al. (2020). Similarly, in the parallel clearinghouse literature with zero advertising costs à la Varian (1980), it is also well-known that the expected price, $E(p)$, is increasing in the number of firms (e.g. Morgan et al. 2006). Intuitively, a higher number of players reduces the expected reward per player by diminishing the chance of any given player winning the contest and so players are discouraged from competing aggressively. We refer to this as the 'active competition effect'.

In contrast, as we demonstrate shortly, when participation costs and default allocation probabilities are both strictly positive, the expected offer, $E(u)$, can *rise* in response to more players. Hence, if participation costs vary across different real-world settings, this result could help reconcile the mixed empirical findings regarding how contest offers are affected by the number of players (see Dechenaux et al. 2015) or the empirical literature on how the number of firms affects market prices in clearinghouse settings (e.g. Allen et al. 2014, Lach and Moraga-González

$1-\alpha = \left(\frac{L^{\phi}-L(u^m)}{W(u^m)-L(u^m)}\right)^{\frac{1}{n-1}}$, but iii) the remaining firm arbitrarily splits the same probability mass between being passive and active with an offer u^m.

2017).

To see this in more detail, re-consider Proposition 2 within our linear example setting from Section 4.2. Here, we know that the equilibrium involving active players exists if $s > u^R$ or equivalently, $u^R < (1-x)V$. After setting passive offers to zero for ease, $\phi \equiv 0$, one can show that a player's expected offer, $E(u) = \int_{u^R}^{s} uf(u)du$, equals $\frac{V}{n} - \left(\frac{u^R}{(n-X)}\right)T$ where $T = X + (1-X)\left(\frac{nu^R}{V(n-X)}\right)^{\frac{1}{n-1}}$. For any $n \geq 2$ and any $X > 0$, it then follows that $E(u)$ is strictly *increasing* in n if u^R is sufficiently close to the boundary for active participation, $(1-x)V$. Intuitively, an increase in n now generates a second, opposing effect which we refer to as the 'passive competition effect'. In contrast to the active competition effect, this effect prompts the players to make *higher* offers by reducing each player's chance of winning when passive via their default allocation probability, $x = X/n$. The passive competition effect is strongest when participation costs are relatively large. Indeed, when participation costs are close to the boundary for active participation, $(1-x)V$, this effect can dominate the active competition effect, such that $E(u)$ increases in n. This result is novel to the literature because it requires the combination of participation costs and default allocations, $u^R > 0$ and $X > 0$. If instead, $u^R = 0$ or $X = 0$, then $E(u)$ is decreasing in n.

7.2.2 (Single-Prize) Scaling

Now consider a different change in competitiveness in the form of (single-prize) 'scaling' where an initial contest with n players and a single prize worth V is scaled up by a factor, $m \geq 1$, where m is an integer. This creates a new, scaled-up contest with $\hat{n} = mn$ players and a single prize worth $\hat{V} = mV$. While such scaling keeps the average reward per player constant at V/n, it still increases competitiveness because players have to beat a larger number of rivals to win.[21]

In a standard, symmetric, single-prize all-pay contest with linear effort costs (and no participation costs or default allocations) by Hillman and Samet (1987), one can easily verify that scaling does not change the level of expected offers,

[21]This (single-prize) version of scaling coincides with other concepts used elsewhere in the contest literature outside all-pay contests, such as 'replicating' (e.g. Wärneryd 2001), or its converse, 'divisioning' (e.g. Brookins and Jindapon 2022). However, it differs somewhat to Fang et al.'s (2020) (multi-prize) definition of scaling where the scaled-up contest involves an increased *number* of prizes. In their setting, even if the initial contest only has one prize of size V, the scaled-up contest involves m prizes each worth V, rather than a single prize worth mV. We focus on our (single-prize) definition because our framework is not equipped to study multiple prizes.

$E(u) = V/n \equiv \hat{V}/\hat{n}$. Intuitively, each player optimally responds to the enhanced prize and higher number of rivals by using an offer distribution with a larger variance, but an equal expected offer. Now consider Fang et al.'s (2020) analysis involving a version of Hillman and Samet (1987) with multiple prizes and a general cost function. They show that (multi-prize) scaling has no effect if costs are linear, but that such scaling *reduces* expected offers if costs are convex (Theorem 3, Fang et al. 2020).

In contrast, by using Proposition 2, we can show that (single-prize) scaling can produce strictly *higher* expected offers even with linear costs. To see this simply, consider the example from Section 7.2.1 again but set the total default allocation probability equal to one, $X = 1$. After scaling the initial contest, each player's default allocation probability reduces from $x = 1/n$ to $\hat{x} = 1/mn$. In the scaled-up contest, strength, $\hat{s} = \hat{V} - \frac{\hat{x}u^R}{(1-\hat{x})} = mV - \frac{1}{nm-1}u^R$, is increasing in m,and active participation requires $\hat{s} > u^R$ or $u^R < \frac{nm-1}{n}V$. Hence, to ensure that active participation can occur for any $m \geq 1$,we require $u^R < \frac{n-1}{n}V$. It then follows that the expected offer within the scaled-up contest, $E(\hat{u})$, equals $\frac{\hat{V}}{nm} - \frac{u^R}{(nm-1)} = \frac{V}{n}\left[1 - \frac{n}{nm-1}\frac{u^R}{V}\right]$. As $E(\hat{u})$ is increasing in m, (single-prize) scaling produces higher expected offers. This implies that consolidating m identical contests into a grand contest can raise the level of individual and aggregate bids. While this result contrasts with previous results in all-pay settings, it is in line with much of the scaling literature under different types of contests (e.g. Wärneryd 2001, Fu and Lu 2009).

8 Conclusion

Due to the presence of fixed participation costs, players often face non-trivial decisions of whether to actively participate in contests. In such cases, as consistent with some common situations, the outcome of the contest can depend upon the 'default allocation' - how the prize is awarded if no player actively competes. To start understanding this issue, our paper makes three main contributions. First, it provides a general, tractable framework that can explicitly characterize all potential equilibria in all-pay contests under endogenous participation with arbitrary asymmetries, while allowing for flexible forms of both participation costs and default allocations. Second, it analyzes how the novel features of our model, participation costs and default allocations, can be used as new, practical tools in contest design. We show how *asymmetric* participation costs or default allocation probabilities can optimally stimulate competition even in an otherwise symmetric setting. Finally, the paper uses the framework to solve the broad family of 'clearinghouse' models (e.g. Baye and Morgan 2001, Baye et al. 2004, Baye et al. 2006) under arbitrary asymmetry for the first time. By using methods from contest theory, we are able to allow for a full set of player asymmetries in a way that should open up new theoretical and empirical research where asymmetry is important, such as platform design or the regulation of platform steering and fees. In doing so, we are also able to offer a new equilibrium refinement for multi-player symmetric all-pay contests, and provide some comparative statics related to the 'competitiveness' of a contest that differ markedly to the existing literature. Throughout the paper, we show how the *combination* of participation costs and default allocations is key.

Appendix A: Main Proofs

Proof of Lemma 3. By adapting well-known results in the contest or clearinghouse literatures (e.g. Narasimhan 1988, Hillman and Riley 1989, Baye et al. 1992, Baye et al. 1996), we know that in equilibrium: i) no player will ever use a point mass at $u > u^m$, and ii) if one player has a point mass at u, then the other player will not. Hence, when combined with Lemma 1, player i can only possibly use a point mass at u_i^m or u_j^m. We now prove that player i will never put a point mass at $u_j^m \neq u_i^m$. First, suppose $u_i^m > u_j^m$. A point mass at u_j^m cannot be optimal as $u_i = u_j^m$ is dominated via Lemma 1. Second, suppose $u_i^m < u_j^m$. By reversing the previous argument together with Lemma 1, we know that player j will never select $u_j \in [u_i^m, u_j^m)$. Thus, if player i had a mass point at u_j^m, then she would optimally deviate by moving the mass from u_j^m to u_i^m in order to increase her payoffs from winning (or losing) without affecting her probability of winning. □

Proof of Lemma 4. Suppose $\alpha_i > \beta_i \geq 0$. a) From Lemma 1, we know no player will set an active offer in the interval $(\min(u_i^m, u_j^m), u^m)$. Hence, given Lemmas 2 and 3, player i must make offers above u^m with positive probability. For this to be optimal, it must be that player j also makes offers above u^m with positive probability, otherwise i would deviate. Hence, $\bar{u}_1, \bar{u}_2 > u^m$. By adapting standard well-known results (e.g. Narasimhan 1988, Hillman and Riley 1989, Baye et al. 1992, Baye et al. 1996), one can then demonstrate $\bar{u}_1 = \bar{u}_2$ as well as b). For c), we know that any player $l = \{1, 2\}$ has an expected payoff from any $u \in (u^m, \bar{u}]$ equal to $L_l(u) + F_k(u)[W_l(u) - L_l(u)]$ given $k \neq l$. For player l to mix over $u \in (u^m, \bar{u}]$, she must earn the same equilibrium payoffs, Π_l^*, over this interval. At $u_l = \bar{u}$, she can guarantee to win (as there are no mass points at $\bar{u}$). Hence, it must be that $\Pi_l^* \equiv W_l(\bar{u})$ which implies the unique active offer distribution $F_k(u)$ in (5). □

The proof of Theorem 1 is contained separately in Appendix B.

Proof of Proposition 1. We proceed through a number of steps. Given our assumptions, it is useful to firstly summarize which equilibrium cases are relevant (across both generic and non-generic contests in Theorems 1 and 2).[22] Specifically, from the text, we know $u_1^m = u_2^m = u^m$, and for any player i we also know that $r_i > s_i$ for any $A_i \geq 0$, and $s_i > u^m$ when $A_i = 0$. Therefore, cases T1iii and T2c can never apply. However, all other cases remain possible. This leaves T1i and T2b where $\alpha_1 = \alpha_2 = 0$, T2a where $\alpha_2 = 0$ and $\alpha_1 = \beta_1 \in [0, 1) \cap [\sigma_1(u_2^m), \theta_1(u^m)]$, T1ii and T2d where $\alpha_2 = 0$ and $\alpha_1 = \beta_1 = 1$, T1iv where $1 > \alpha_i > \beta_i \geq 0$ for $i = \{1, 2\}$, and T1v where $1 = \alpha_1 > \beta_1 \geq 0$ and $1 > \alpha_2 > \beta_2 \geq 0$.

From these, it is immediate that any $\boldsymbol{A}$ consistent with T1i and T2b can never be offer maximizing as both players would always be passive. Further, any $\boldsymbol{A}$ consistent with T1ii, T2a, or T2d (which all have $\alpha_1 = \beta_1 \in (0, 1]$ and $\alpha_2 = 0$) can never be offer maximizing either because it would be dominated by some $\boldsymbol{A}$ consistent with T1v. Intuitively, in T1v (where $1 = \alpha_1 > \beta_1 \geq 0$ and $1 > \alpha_2 > \beta_2 \geq 0$), we know that i) player 2 is active with positive, rather than zero, probability, and ii) player 1 is active with a weakly higher probability and has an average active offer strictly above u^m. Hence, the offer-maximizing $\boldsymbol{A}$ must lie somewhere within the remaining cases, T1iv and T1v. To understand more, the next two lemmas detail the comparative statics within these two cases.

Lemma 5. *Let $u^m < r_2 \leq s_1$ such that T1v applies. Then a) a marginal increase in A_1 leaves both players' offers unchanged, while b) a marginal increase in A_2 reduces both players' offers in the sense of FOSD.*

Proof. In T1v, given $u_1^m = u_2^m = u^m$, we know $\bar{u} = r_2$, $F_1(u) = F_2(u) = \frac{w(r_2) - l(u)}{w(u) - l(u)}$, $\alpha_1 = 1$, $1 - \alpha_2 = F_2(u_1^m)$, $\beta_1 = F_1(u^m)$, $\beta_2 = 0$, $\Pi_1^* = w(r_2) - A_1$ and $\Pi_2^* = w(r_2) - A_2 \equiv L^{\phi}$. a) From the text we know $\partial r_2 / \partial A_1 = 0$. Hence, a marginal increase in A_1 has no impact on the players' offers because $F_1(u)$, $F_2(u)$, $(1 - \alpha_1)$, and $(1 - \alpha_2)$ all remain unchanged. b) From the text, also recall $\frac{\partial r_2}{\partial A_2} = \frac{1}{w'(r_2)} < 0$.

[22]For ease of exposition, we refer to the cases of Theorems 1 and 2 in abbreviated form, e.g. T1i refers to case i of Theorem 1, and T2a refers to case a of Theorem 2.

Hence, both player's offers reduce because $F_1(u)$, $F_2(u)$ and $(1-\alpha_2)$ are all strictly increasing in A_2 via r_2, while α_1 is independent of A_2. □

Lemma 6. *Let $u^m < s_1 < r_2$ such that case T1iv applies. Then a) a marginal increase in A_1 reduces both players' offers in the sense of FOSD, and b) a marginal increase in A_2 improves player 1's offers in the sense of FOSD but leaves player 2's offers unchanged.*

Proof. In T1iv, given $u_1^m = u_2^m = u^m$, we know $\bar{u} = s_1$, $F_1(u) = F_2(u) = \frac{w(s_1)-l(u)}{w(u)-l(u)}$, $1-\alpha_1 = \frac{w(s_1)-A_2-L^\phi}{(W^\phi-L^\phi)x} \in (0,1)$, $\beta_1 = F_1(u^m) - (1-\alpha_1)$, $1-\alpha_2 = F_2(u_1^m)$, $\beta_2 = 0$ and $\Pi_i^* = w(s_1) - A_i$ for $i = 1,2$. a) From the text we know $\partial s_1/\partial A_1 < 0$. Hence, both players' offers reduce in the sense of FOSD as $F_1(u)$, $F_2(u)$, $(1-\alpha_1)$ and $(1-\alpha_2)$ are all increasing in A_1. b) From the text, also recall $\partial s_1/\partial A_2 = 0$. Thus, in terms of offers, the only changes that occur involve a decrease in $1-\alpha_1$ and an associated increase in β_1, $\frac{\partial(1-\alpha_1)}{\partial A_2} = -\frac{\partial \beta_1}{\partial A_2} < 0$. Hence, player 1's offers improve in the sense of FOSD, but player 2's offers remain unchanged. □

To complete the proof of Proposition 1, suppose $A_1 \leq A_2$ such that $s_1 \geq s_2$ and $r_1 \geq r_2$. First, consider case T1iv where $u^m < s_1 < r_2$. Here, we know that a marginal reduction in A_1 (and associated increase in s_1) will improve both players' offers, and a marginal increase in A_2 (and associated reduction in r_2) will improve player 1's offers, but leave 2's unchanged. Hence, within this case, it is strictly offer maximizing to reduce A_1 and increase A_2 until the boundary point where we approach $s_1 = r_2$. At this point, we enter case T1v where $u^m < r_2 \leq s_1$. In this case, we know that a marginal reduction in A_2 (and associated increase in r_2) can improve both player's offers. Therefore, we know that reducing A_2 until the point where $r_2 = s_1 > u^m$ must be strictly offer-maximizing.

Implementing the point $r_2 = s_1 > u^m$ by manipulating $\{A_1, A_2\}$ is always possible given our assumption that $r_1 > s_1 > u^m$ when $A_1 = 0$. Indeed, there are an infinite number of pairs of $\{A_1, A_2\}$ for which $r_2 = s_1 > u^m$. With use of (3) and (4), any such pair must satisfy $A_1 + \frac{x(W^\phi-L^\phi)(L^\phi-(l(u^m)-A_i))}{w(u^m)-l(u^m)-x(W^\phi-L^\phi)} = A_2$ or equivalently, $A_2 = \frac{A_1(w(u^m)-l(u^m))+x(W^\phi-L^\phi)(L^\phi-l(u^m))}{w(u^m)-l(u^m)-x(W^\phi-L^\phi)}$. From Lemma 5, the offer maximizing pair must be the one with the lowest value of A_2. Hence, it is offer maximizing to set $A_1 = 0$ and $A_2 = \frac{x(W^\phi-L^\phi)(L^\phi-l(u^m))}{w(u^m)-l(u^m)-x(W^\phi-L^\phi)} \equiv \bar{A}$ where $\bar{A} > 0$ due to our assumptions $L^\phi > l(u^m)$ and $w(u^m) > L^\phi + x(W^\phi + L^\phi)$.

Proof of Proposition 2. The proof proceeds in a series of steps that build on those in Arnold and Zhang (2014):

STEP 1: To begin, it is trivial to reproduce versions of Lemmas 1-3 for the case of $n > 2$ symmetric players. Further, by following standard results in the literature (e.g. Baye et al., 1992), it is also straightforward to note a few additional results (without full proof) that apply when at least two players are active with positive probability. i) At least one player must have a lower bound of their support, $\underline{u}_i$, equal to u^m. To show this, suppose one or more players share the lowest lower bound, $min\{\underline{u}_j\} > u^m$. Then at least one such player would optimally deviate by relocating probability mass from just above the lower bound to u^m because this would strictly increase their payoffs via A2 and yet leave their probability of winning (nearly) unchanged. ii) There can be no interval of active offers, $u \in (u', u'')$ with $u' < u''$, that is only in the support of one player. If so, that player's expected payoffs would be decreasing across the interval and so they would optimally reallocate the probability mass in the interval to the lower end of the interval. iii) These two results then imply that if at least two players are active with positive probability, then the lower bound for at least two players, $\underline{u}_i$ and $\underline{u}_j$, must equal u^m.

STEP 2: In equilibrium, no player i can have $\alpha_i = 1$. We prove this by contradiction across three exhaustive cases. First, suppose there are at least two players, i and j, with $\alpha_i = \alpha_j = 1$. In this case, it must be that $\underline{u}_i = \underline{u}_j = u^m$. If not, with $\underline{u}_i < \underline{u}_j$, then player i would always wish to deviate. Further, from Lemma 3, we also know there can be no ties in active offers within any equilibrium. Hence, players i and j must lose whenever they select u^m and so earn an equilibrium payoff of $L(u^m)$. They would then wish to deviate to being passive to earn L^ϕ via A3. Hence, at most, only one player can have $\alpha_i = 1$. Second, suppose $\alpha_i = 1$ and $\alpha_j = 0$ for all $j \neq i$. For this to be an equilibrium, i cannot wish to deviate to being passive, and any player j cannot wish to deviate to $u^m + \varepsilon$ for sufficiently small ε. This requires $W(u^m) \geq L^\phi + x(W^\phi - L^\phi)$ and $L^\phi \geq W(u^m + \varepsilon)$ respectively, which provides a contradiction for small enough ε given $x(W^\phi - L^\phi) > 0$. Finally, suppose only one player i has $\alpha_i = 1$, and at least one player j has $\alpha_j \in (0, 1)$. From Step 1, we know at least two players k and l must have $\underline{u}_k = \underline{u}_l = u^m$. This cannot be an equilibrium if player i is neither k or l, as then players k and l will definitely lose at u^m and so would prefer to deviate. Hence, suppose player i equals k. If so, then i has to put a point mass on u^m or else l would lose for sure at u^m and so would wish to deviate. Therefore, given $\alpha_i = 1$ and $\alpha_h < 1 \; \forall h \neq i$, all players other than i must have an equilibrium payoff $\Pi^*_h = L^\phi$, while player i earns $\Pi^*_i = L^\phi + x(W^\phi - L^\phi)\Pi_{h \neq i}(1 - \alpha_h) > L^\phi$. From $\Pi^*_h = L^\phi$ and $\alpha_j \in (0, 1)$, it must be that $\bar{u}_j = r$ otherwise j would deviate to above $\bar{u}$ if $\bar{u} < r$. However, from $\Pi_i^* > L^\phi$, it must be that $\bar{u}_i < \bar{u}_j = r$. Yet this leads to a contradiction: at $u_i = \bar{u}_i$, i's payoff is lower than Π_i^*. At $\bar{u}_i$, player j who randomizes just above $\bar{u}_i$ has an expected payoff equal to $\Pi_j^* = L(\bar{u}_i) + (W(\bar{u}_i) - L(\bar{u}_i))\Pi_{h \neq j} F_h(\bar{u}_i)$ and this must equal $\Pi_j^* = L^\phi$. However, at $\bar{u}_i$, player i earns $L(\bar{u}_i) + (W(\bar{u}_i) - L(\bar{u}_i))\Pi_{h \neq i} F_h(\bar{u}_i)$ and this must be less than L^ϕ because $F_j(\bar{u}_i) < 1$ and $F_i(\bar{u}_i) = 1$.

STEP 3: In equilibrium, any player who is active with positive probability has the same upper bound, $\bar{u}$. Suppose not. Specifically, suppose there are two active players, i and j, with $\bar{u}_i \equiv \bar{u} > \bar{u}_j > u^m$. From Step 1, we know it must be that $F_i(\bar{u}) = 1$ and $F_i(\bar{u} - \varepsilon) < 1$ for any $\varepsilon > 0$, such that $F_i(\bar{u}_j) \in (0,1)$. As there can be no ties, we also know that $\Pi_i^* = W(\bar{u})$. For this to be equilibrium, we require i) player i's expected payoff at $\bar{u}$ to be weakly larger than her expected payoff at $\bar{u}_j$: $W(\bar{u}) \geq L(\bar{u}_j) + (W(\bar{u}_j) - L(\bar{u}_j))\Pi_{h \neq i} F_h(\bar{u}_j)$, and ii) player j's expected payoff at $\bar{u}_j$ to be weakly larger than her expected payoff at $\bar{u}$: $L(\bar{u}_j) + (W(\bar{u}_j) - L(\bar{u}_j))\Pi_{h \neq j} F_h(\bar{u}_j) \geq W(\bar{u})$. However, this leads to a contradiction because both inequalities cannot hold simultaneously as $\Pi_{h \neq i} F_h(\bar{u}_j) > \Pi_{h \neq j} F_h(\bar{u}_j)$ given $F_i(\bar{u}_j) \in (0,1)$ and $F_j(\bar{u}_j) = 1$. Therefore, all active players must set a common upper bound, $\bar{u}$, and so achieve an equilibrium payoff, $W(\bar{u})$.

STEP 4: In equilibrium, all players must have $\alpha_i = \alpha \in [0,1)$. From above, we know that each player must be passive with positive probability as no player can be active with probability one. Thus, any player i must earn equilibrium payoffs of $\Pi_i^* = L^\phi + x(W^\phi - L^\phi)\Pi_{h \neq i}(1 - \alpha_h)$. If any player i is active with positive probability, $\alpha_i \in (0,1)$, then we further know that $\Pi_i^* = W(\bar{u})$ such that $W(\bar{u}) = L^\phi + x(W^\phi - L^\phi)\Pi_{h \neq i}(1 - \alpha_h)$. Clearly, it then follows that any player i with $\alpha_i \in (0,1)$ must share the same value of $\alpha_i = \alpha \in (0,1)$. Further it cannot be that one or more players have $\alpha_i = \alpha \in (0,1)$ while one or more players have $\alpha_k = 0$ as player k would then earn strictly lower expected payoffs than an active player i and so wish to deviate to $\bar{u}$, a contradiction; $\Pi_k^* = L^\phi + x(W^\phi - L^\phi)\Pi_{h \neq k}(1 - \alpha_h) < \Pi_i^* = L^\phi + x(W^\phi - L^\phi)\Pi_{h \neq i}(1 - \alpha_h) = W(\bar{u})$.

STEP 5: If $s \leq u^m$, then the equilibrium is unique and symmetric with $\alpha = 0$. From Step 4, we know $\alpha_i = \alpha \in [0,1)$ for all i. Given $s \leq u^m$, we know $L^\phi + x(W^\phi - L^\phi) \geq W(u^m)$. In this case, there is always an equilibrium at $\alpha = 0$ as no player has a strict incentive to deviate to $\alpha_i > 0$ as they earn $L^\phi + x(W^\phi - L^\phi)$ by being passive and $W(u^m)$ at most from being active. Moreover, there is never an equilibrium with $\alpha \in (0,1)$ as this would require any individual player to be indifferent between being passive and being active at u^m, such that $L^\phi + x(W^\phi - L^\phi)(1-\alpha)^{n-1} = L(u^m) + (W(u^m) - L(u^m))(1-\alpha)^{n-1}$. However, given

$s \leq u^m$, this can never hold for $\alpha \in (0, 1)$. Hence, the only possible equilibrium involves $\alpha = 0$.

STEP 6: If $s > u^m$, then the equilibrium is unique and symmetric with $\alpha \in (0, 1)$ and $F_i(u) = F(u)$ for all i and for all $u \in [u^m, \bar{u}]$. If $s > u^m$, then $L^\phi + x(W^\phi - L^\phi) < W(u^m)$. Hence, there can be no equilibrium with $\alpha = 0$ as a player would wish to deviate to u^m instead. Therefore, it must be that $\alpha \in (0, 1)$. Next, we show that for any two players, i and j, it cannot be that $F_i(u') > F_j(u')$ for some offer $u' > u^m$. First, suppose that both players select u' with positive probability. $F_i(u') > F_j(u')$ then implies that player j has a higher probability of winning at u' and so the two players cannot have the same equilibrium payoffs, contrary to an earlier result, $\Pi_i^* = W(\bar{u})\ \forall i$. Second, suppose u' is only selected with positive probability by player i and not j. From Step 3, we know $\bar{u}_i = \bar{u}_j = \bar{u} > u^m$, and so there would have to be some $\hat{u} \in (u', \bar{u})$ in the support of both players, with $F_i(\hat{u}) > F_j(\hat{u})$ (unless player j has a mass point at $\hat{u}$ but this is ruled out as we know mass points can only arise at u^m for one player, via a n-player version of Lemma 3). Hence, like above, this leads to a contradiction as the two players cannot have the same equilibrium payoffs. Third, suppose u' is only selected with positive probability by player j and not i. At u', we know player j must earn $L(u') + (W(u') - L(u'))(1 - F_i(u'))\Pi_{k \neq i,j}(1 - F_k(u')) = \Pi_j^*$, while player i would earn $\Pi_i(u') = L(u') + (W(u') - L(u'))(1 - F_j(u'))\Pi_{k \neq i,j}(1 - F_k(u'))$. Given $F_i(u') > F_j(u')$, this leads to $\Pi_i(u') > \Pi_j^* = W(\bar{u})$, which again gives a contradiction. Fourth, suppose u' is not selected by either player with positive probability. In this case, consider the highest offer below u' which is selected by at least one player with positive probability, u''. Such an offer $u'' > u^m$ has to exist, otherwise $\alpha = 0$. As there are no point masses above u^m, we must have $F_i(u'') = F_i(u')$ and $F_j(u') = F_j(u'')$. Hence, if $F_j(u') > F_j(u')$ then $F_j(u'') > F_j(u'')$ and so one can then apply the previous deductions again to show a contradiction with u'' instead of u'.

Lastly, for any two players, i and j, it also cannot be that $F_i(u^m) > F_j(u^m)$. From above, we know for all i: $\alpha_i = \alpha$ and $F_i(u) = F(u)$ for $u > u^m$. Hence, it must also be that $F_i(u^m) = F(u^m)\ \forall i$. Therefore, when $s > u^m$, the equilibrium is unique and symmetric.

STEP 7: Finally, when $s > u^m$, we derive the equilibrium values of α and $F(u)$ in (10) and (11), together with $\bar{u} = s$ and $\beta = 0$. First, given $F_i(u^m) = F(u^m)$, all the players could, in principle, use an identical mass point at u^m. However, this is ruled out by a n-player version of Lemma 3 which says that only one player at most can use such a mass point. Hence, $\beta = 0$. Second, given $\alpha \in (0, 1)$, each player must be indifferent between i) being passive and selecting u^m, such that $\Pi_i^* = L^\phi + x(W^\phi - L^\phi)(1-\alpha)^{n-1} = L(u^m) + (W(u^m) - L(u^m))(1-\alpha)^{n-1}$, and ii) selecting any $u \in (u^m, \bar{u}]$, such that $\Pi_i^* = W(\bar{u}) = L(u) + F(u)^{n-1}(W(u) - L(u))$. By rearranging, these provide (10) and (11). Finally by setting $\Pi_i^* = W(\bar{u}) = L^\phi + x(W^\phi - L^\phi)(1-\alpha)^{n-1}$ and inserting (10) for the value of α, one can show that $\bar{u} = s$. One can then verify that $1 - \alpha = F(u^m)$ such that $\beta = 0$ as required. □

Appendix B: Proof of Theorem 1

This appendix provides the proof of Theorem 1 by deriving a more general result, Theorem 2, which characterizes the set of equilibria for *both* generic *and* non-generic contests. For convenience, it is useful to define

$$\delta_i(u) = 1 - \frac{W_i(u) - L_i(u_i^m)}{W_i(u_i^m) - L_i(u_i^m)}. \tag{12}$$

Theorem 2. *Given A1-5, the equilibrium in any generic or non-generic contest follows Theorem 1 unless any of the following knife-edge cases apply. If so, the equilibrium is potentially non-unique:*
a) When $s_1 = u_1^m$, player 2 is always passive, $\alpha_2 = 0$, but player 1 selects u_1^m with any probability $\alpha_1 = \beta_1 \in [0,1) \cap [\sigma_1(u_2^m), \theta_1(u^m)]$.
b) When $r_2 = u_2^m$ and $s_1 < u_1^m$, then neither player is active, $\alpha_1 = \alpha_2 = 0$.
c) When $r_2 = u_2^m$, $s_1 > u_1^m$ and $u_1^m < u_2^m$, player 1 is always active with u_1^m, $\alpha_1 = \beta_1 = 1$, and player 2 selects u_2^m with any probability $\alpha_2 = \beta_2 \in [0, \min\{\sigma_1(u_1^m), \delta_1(u_2^m)\}]$.
d) When $r_2 = u_2^m$, $s_1 > u_1^m$ and $u_1^m \geq u_2^m$, player 1 is always active with u_1^m, $\alpha_1 = \beta_1 = 1$, and player is always passive, $\alpha_2 = 0$.

The proof of Theorem 2 proceeds as follows. Step 1 provides an exhaustive list of possible equilibrium forms. Step 2 defines some further features for each equilibrium form, and characterizes some necessary parameter conditions for each form to exist. Finally, Step 3 shows how these parameter conditions are enough to characterize the equilibria for the entire parameter space in a way that is consistent with Theorem 2.

Step 1: Possible Equilibrium Forms

Lemmas 1-4 in Section 3 offer a start in thinking about possible equilibrium forms. However, to narrow this down further, Lemma 7 shows that both players cannot be active in equilibrium with probability one.

Lemma 7. *Suppose both players are active with positive probability in equilibrium such that $\alpha_k > 0$ for $k = 1, 2$. Then it cannot be that $\alpha_1 = \alpha_2 = 1$. Instead, either i) $\alpha_k = \beta_k > 0$ for $k = 1, 2$, in which case it must be that $u_j^m > u_i^m$ and $\alpha_j = \beta_j \in (0, 1)$ for some j, or ii) $\alpha_k > \beta_k \geq 0$ for $k = 1, 2$, in which case it must be that $\alpha_2 \in (0, 1)$.*

Proof of Lemma 7. Suppose both players are active with positive probability in equilibrium such that $\alpha_k > 0$ for $k = 1, 2$. Then, we know it must be that either i) $\alpha_k = \beta_k > 0$ for $k = 1, 2$, or ii) $\alpha_k > \beta_k \geq 0$ for $k = 1, 2$ because Lemma 4 rules out the possibility that $\alpha_i > \beta_i \geq 0$ but $\alpha_j = \beta_j > 0$. First consider i). Here, two initial conditions must hold. First, from Lemma 3, it must be that $u_1^m \neq u_2^m$. Hence, without loss, let $u_j^m > u_i^m$. Then it must be that $\alpha_j = \beta_j < 1$. If instead, $\alpha_j = \beta_j = 1$, then player i would lose with certainty by selecting u_i^m, and so would deviate to $\alpha_i = \beta_i = 0$ to earn $L_i^\phi > L_i(u_i^m)$ via A3. Now consider ii). Here, it cannot be that $\alpha_2 = 1$. We prove this by contradiction under two exhaustive cases. First, suppose $\alpha_2 = 1$ with $\alpha_1 = 1$ and let $u_i^m = u^m \geq u_j^m$. From Lemma 4, we know both players must mix on $(u^m, \bar{u}]$ and that player i must select u_i (arbitrarily close to) u^m with positive probability, with no ties at such a point. Given $\alpha_j = 1$, i must always lose when making such an offer and so earn an equilibrium payoff (arbitrarily close to) $L_i(u_i^m = u^m)$. However, i would then strictly prefer to deviate by setting $u_i = \phi$ as $L_i^\phi - L_i(u_i^m) > 0$ via A3. Second, suppose $\alpha_2 = 1$ with $\alpha_1 \in (0, 1)$. Given $\alpha_2 = 1$, player 1 will earn L_1^ϕ when passive. From Lemma 4, we know that player 1 must mix up to $\bar{u}$ and that player 1 will win with certainty at $\bar{u}$, earning $W_1(\bar{u})$. Hence, for player 1 to mix between ϕ and $\bar{u}$, we require $L_1^\phi = W_1(\bar{u})$. This implies $\bar{u} = r_1$. For $\alpha_2 = 1$, we need to rule out any deviations to ϕ and so we require $W_2(\bar{u}) \geq L_2^\phi + (1 - \alpha_1)x_2(W_2^\phi - L_2^\phi)$. From the definition of strength, the RHS is equivalent to $W_2(s_2)$. Hence, we require $W_2(\bar{u}) \geq W_2(s_2)$ which implies $\bar{u} \leq s_2$. Therefore, when combined with $\bar{u} = r_1$ and $s_1 < r_1$, we require $s_1 < r_1 = \bar{u} \leq s_2$. This implies $s_1 < s_2$; a contradiction via A5. $\square$

Using this with the results from Lemmas 1-4, we can now list the possible equilibrium forms as follows.

Lemma 8. *Given Lemmas 1-4 and 7, the only possible equilibrium forms are:*
1. Neither player is active, $\alpha_1 = \alpha_2 = 0$*.*
2. Player i is always active with u_i^m*,* $\alpha_i = \beta_i = 1$*, and player j is always passive,* $\alpha_j = 0$*.*
3. Player i randomizes between being active at u_i^m *and being passive,* $\alpha_i = \beta_i \in (0, 1)$*, while player j is always passive,* $\alpha_j = 0$*.*
4. Each player k randomizes between being active at u_k^m *and being passive,* $\alpha_k = \beta_k \in (0, 1)$*, for* $k = 1, 2$*, where* $u_j^m > u_i^m$*.*
5. Player j randomizes between being active at u_j^m *and being passive,* $\alpha_j = \beta_j \in (0, 1)$*, but player i is always active with* u_i^m*,* $\alpha_i = \beta_i = 1$*, where* $u_j^m > u_i^m$*.*
6. Both players are active above u^m *and active with interior probability,* $1 > \alpha_i > \beta_i \geq 0$ *for* $i = 1, 2$*.*
7. Both players are active above u^m *where one player, player 2, is active with interior probability,* $1 > \alpha_2 > \beta_2 \geq 0$*, and one player, Player 1, is active with probability one,* $1 = \alpha_1 > \beta_1 \geq 0$*.*

Proof of Lemma 8. By definition, any equilibrium must have $\alpha_i \geq \beta_i \geq 0$ for $i = \{1, 2\}$. Hence, the only possible outcomes can be exhaustively listed by a) $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0$, b) $\alpha_i = \beta_i > 0$ and $\alpha_j = \beta_j \geq 0$, and c) $\alpha_i > \beta_i \geq 0$ for at least one player i. We now show how these possible outcomes are fully covered by equilibrium forms 1-7 in the Lemma. First, a) corresponds directly to form 1. Second, we can split b) into four sub-cases that directly correspond to forms 2, 3, 4, and 5 respectively. Note that Lemma 7 rules out $\alpha_i = \beta_i = 1$ for both players, and also ensures that $u_j^m > u_i^m$ must hold in forms 4 and 5. Finally, if c) applies then we know from Lemma 4 that $\alpha_j > \beta_j \geq 0$ must also apply, with both players being active above u^m. Hence, c) can be split into two sub-cases that correspond directly to forms 6 and 7. In form 7, note it cannot be that $\alpha_2 = 1$ due to Lemma 7. □

Step 2: Further Results on each Equilibrium Form

We now detail some further features of the equilibrium forms and define some necessary parameter conditions for the existence of each form. These results apply for both generic and non-generic contests.

Lemma 9. *Equilibrium Form 1:* $\alpha_1 = \alpha_2 = 0$ *is an equilibrium iff* $s_i \leq u_i^m$ *for* $i = \{1, 2\}$*.*

Proof of Lemma 9. If $\alpha_1 = \alpha_2 = 0$, then each player i expects to earn $L_i^\phi + x_i(W_i^\phi - L_i^\phi)$. For $\alpha_1 = \alpha_2 = 0$, we require no player i to have an incentive to deviate by submitting an active offer, even if they were to win at u_i^m with probability one. This requires $W_i(u_i^m) \leq L_i^\phi + x_i(W_i^\phi - L_i^\phi)$ for $i = \{1, 2\}$. From the definition of strength this is equivalent to $s_i \leq u_i^m$ for $i = \{1, 2\}$. □

Lemma 10. *Equilibrium Form 2: $\alpha_i = \beta_i = 1$ and $\alpha_j = 0$ is an equilibrium iff $i = 1$, $j = 2$, $s_1 \geq u_1^m$ and $r_2 \leq u^m$.*

Proof of Lemma 10. If $\alpha_i = \beta_i = 1$ and $\alpha_j = 0$ then from Lemma 2, player i earns $W_i(u_i^m)$ and player j earns L_j^ϕ. For this to be an equilibrium, it is first necessary that player i has no incentive to deviate to $u_i = \phi$ to earn $L_i^\phi + x_i(W_i^\phi - L_i^\phi)$. Hence, we require $W_i(u_i^m) \geq L_i^\phi + x_i(W_i^\phi - L_i^\phi)$, which from the definition of strength, gives $s_i \geq u_i^m$. Second, it is necessary that player j has no incentive to deviate. If j deviated, she would optimally deviate to either i) just above u^m to earn slightly below $W_j(u^m)$ if $u_i^m = u^m \geq u_j^m$, or ii) u^m to earn $W_j(u^m)$ if $u_i^m < u_j^m = u^m$. Hence, as a necessary condition, we require $L_j^\phi \geq W_j(u^m)$, which by using the definition of reach requires $r_j \leq u^m$. Thus, this equilibrium requires $s_i \geq u_i^m$ and $r_j \leq u^m$. As we now prove, these two conditions cannot both hold unless $i = 1$ and $j = 2$. We proceed by contradiction. Suppose $j = 1$ such that $r_1 \leq u^m$. As $x_1 > 0$, this gives $s_1 < r_1 \leq u^m$. First suppose $u^m = u_2^m$. This then implies $s_1 < u_2^m$ which when combined with A5, gives $s_2 \leq s_1 < u_2^m$ such that $s_2 \geq u_2^m$ can never apply. Finally, suppose $u^m = u_1^m$. Then $s_1 < r_1 \leq u^m$ gives $s_1 < u_1^m$ which implies $s_1 = -\infty$ from Definition 2. From A5, this further implies $s_2 \leq s_1 = -\infty$ such that $s_2 \geq u_2^m \geq 0$ can also never apply. Hence, it must be that $i = 1$ and $j = 2$. □

Lemma 11. *Equilibrium Form 3: $\alpha_i = \beta_i \in (0, 1)$ and $\alpha_j = 0$ is an equilibrium iff $i = 1$, $j = 2$, $s_1 = u_1^m$ and $\alpha_1 = \beta_1 \in (0, 1) \cap [\sigma_1(u_2^m), \theta_1(u^m)]$.*

Proof of Lemma 11. Suppose $\alpha_i = \beta_i \in (0, 1)$ and $\alpha_j = 0$. First, in order for player i to be willing to mix between u_i^m and ϕ, we require $W_i(u_i^m) = L_i^\phi + x_i(W_i^\phi - L_i^\phi)$ given $\alpha_j = 0$. This implies $s_i = u_i^m$ from the definition of strength. Further, as player j is passive, she must earn $\Pi_j^* = L_j^\phi + x_j(W_j^\phi - L_j^\phi)(1 - \alpha_i)$. To be an equilibrium, we require neither player to have an incentive to deviate. For i, this is trivial because she has no other profitable deviations. For j, we proceed to consider two exhaustive situations: $u_j^m \geq u_i^m$ and $u_j^m < u_i^m$.

Begin with the situation with $u_j^m \geq u_i^m$. Here, player j could deviate from ϕ to u_j^m (or just above u_j^m if $u_i^m = u_j^m$). To rule this out, we need $\Pi_j^* = L_j^\phi + x_j(W_j^\phi -$

$L_j^\phi)(1-\alpha_i) \geq W_j(u_j^m)$ which is equivalent to $\alpha_i \leq \theta_i(u_j^m)$. For later, it is useful to note that this condition binds, that is $\theta_i(u_j^m) < 1$, when $W_j(u_j^m) > L_j^\phi \leftrightarrow u_j^m < r_j$. But more immediately, note that to allow $\alpha_i > 0$ as required, we need $\theta_i(u_j^m) > 0$ or equivalently, $W_j(u^m) < L_j^\phi + x_j(W_j^\phi - L_j^\phi)$. Via the definition of strength, this implies $s_j < u_j^m$, which in turn implies $s_j = -\infty$. Hence, as $s_i = u_i^m \geq 0$, it must be that $i = 1$ and $j = 2$ from A5. Given these player identities, we require $W_2(u^m) < L_2^\phi + x_2(W_2^\phi - L_2^\phi)$ which implies from (7) that $\sigma_1(u_2^m) < 0$.

Now consider the other situation with $u_j^m < u_i^m$. Here, we need to consider two possible deviations by player j to a) u_j^m, or b) just above u_i^m. First consider deviation a). This will not be optimal if

$$\Pi_j^* = L_j^\phi + x_j(W_j^\phi - L_j^\phi)(1-\alpha_i) \geq L_j(u_j^m) + (W_j(u_j^m) - L_j(u_j^m))(1-\alpha_i) \quad (13)$$

Given $L_j^\phi > L_j(u_j^m)$ from A3, this condition holds for any α_i if $W_j(u_j^m) \leq L_j(u_j^m) + x_j(W_j^\phi - L_j^\phi)$. The condition also continues to hold for higher $W_j(u_j^m)$ if $W_j(u_j^m) \leq L_j^\phi + x_j(W_j^\phi - L_j^\phi)$ (such that $s_j \leq u_j^m$) because there, even at $\alpha_i = 0$, (13) holds. For $W_j(u_j^m) > L_j^\phi + x_j(W_j^\phi - L_j^\phi)$ (such that $s_j > u_j^m$), then one can rearrange (13) to require $\alpha_i \geq \sigma_i(u_j^m) = 1 - \frac{L_j^\phi - L_j(u_j^m)}{W_j(u_j^m) - L_j(u_j^m) - x_j(W_j^\phi - L_j^\phi)}$. Now consider deviation b) to just above $u_i^m > u_j^m$. To rule this out we require $\Pi_j^* = L_j^\phi + x_j(W_j^\phi - L_j^\phi)(1-\alpha_i) \geq W_j(u_i^m)$ or equivalently, $\alpha_i \leq \theta_i(u_i^m) = 1 - \frac{W_j(u_i^m) - L_j^\phi}{(W_j^\phi - L_j^\phi)x_j}$.

We now explore deviations a) and b) in three exhaustive parameter regions and show that in each case, it must be that $\alpha_i = \beta_i \in (0,1)$ must lie within the interval, $[\sigma_i(u_j^m), \theta_i(u^m)]$, as required. Finally, we then prove that in each case, it must be that $i = 1$ and $j = 2$.

First, suppose $s_j \leq u_j^m$. Here, deviation a) is never profitable using earlier results, but to rule out deviation b) we need $\alpha_i \leq \theta_i(u_i^m)$. To then allow $\alpha_i > 0$ as required, we need $\theta_i(u_i^m) > 0$ or equivalently, $W_j(u_i^m) < L_j^\phi + x_j(W_j^\phi - L_j^\phi)$. Given $u_i^m > u_j^m$, this always holds because $W_j(u_i^m) < W_j(u_j^m) \leq L_j^\phi + x_j(W_j^\phi - L_j^\phi)$ where the last part follows from $s_j \leq u_j^m$. So, if $s_j \leq u_j^m$ then any $\alpha_i = \beta_i \in (0,1)$ can be an equilibrium provided $\alpha_i \leq \theta_i(u_i^m)$.

Second, suppose $s_j > u_j^m$ and $r_j \leq u_i^m$. Deviation b): the latter condition on r_j implies $\theta_i(u_i^m) \geq 1$ such that any $\alpha_i = \beta_i \in (0,1)$ will automatically satisfy $\alpha_i \leq \theta_i(u_i^m)$. However, to rule out deviation a), given $s_j > u_j^m$, we know from earlier results that we require $\alpha_i \geq \sigma_i(u_j^m)$. Hence, to allow $\alpha_i = \beta_i \in (0,1)$ we need $\sigma_i(u_j^m) < 1$. This is assured by $s_j > u_j^m$ and A3: $L_j^\phi - L_j(u_j^m) > 0$ and (which in turn gives $W_j(u_j^m) - L_j(u_j^m) - x_j(W_j^\phi - L_j^\phi) > 0$). So if $s_j > u_j^m$ and $r_j \leq u_i^m$ then any $\alpha_i = \beta_i \in (0,1)$ can be an equilibrium provided $\alpha_i \geq \sigma_i(u_j^m)$.

Third, suppose $s_j > u_j^m$ and $r_j > u_i^m$. From earlier results we need both $\alpha_i \geq \sigma_i(u_j^m)$ and $\alpha_i \leq \theta_i(u_i^m)$ to rule out deviations b) and a), respectively. Whilst $s_j > u_j^m$ and $r_j > u_i^m$ ensure that $\sigma_i(u_j^m) < 1$ and $\theta_i(u_i^m) > 0$ as required, we still need to ensure $\theta_i(u_i^m) \geq \sigma_i(u_j^m)$ in order for some $\alpha_i = \beta_i \in (0,1)$ to be possible. This condition can be rewritten as

$$1 - \frac{W_j(u_i^m) - L_j^\phi}{(W_j^\phi - L_j^\phi)x_j} \geq 1 - \frac{L_j^\phi - L_j(u_j^m)}{W_j(u_j^m) - L_j(u_j^m) - x_j(W_j^\phi - L_j^\phi)}$$

or $W_j(u_i^m) \leq W_j(s_j)$, which is equivalent to $s_j \leq u_i^m$. So if $s_j > u_j^m$ and $r_j > u_i^m \geq s_j$, then any $\alpha_i = \beta_i \in (0,1)$ can be an equilibrium provided $\alpha_i = \beta_i \in [\sigma_i(u_j^m), \theta_i(u_i^m)]$.

Finally, notice that in all three regions, it follows that $i = 1$ and $j = 2$ from A5. In the first region, we require $s_j \leq u_j^m < u_i^m = s_i$ and so $i = 1$. In the second region, we require $s_j > u_j^m$ and $r_j \leq u_i^m = s_i$, and so $s_i > s_j$ follows from $s_j < r_j$ for $s_j \geq u_j^m$. In the third region, we require $s_j > u_j^m$ and $r_j > s_i = u_i^m \geq s_j$, and so even if $s_i = s_j$ the fact that $u_i^m > u_j^m$ implies $i = 1$ from A5.

To summarize, when $u_i^m > u_j^m$, an equilibrium with $\alpha_i = \beta_i \in (0,1)$ and $\alpha_j = 0$ can arise iff $i = 1$, $j = 2$ and either i) $s_2 \leq u_2^m < u_1^m = s_1$ and $\alpha_1 \leq \theta_1(u_1^m)$ (where $\sigma_1(u_2^m) \leq 0$); ii) $s_2 > u_2^m$, $r_2 \leq u_1^m = s_1$, and $\alpha_1 \geq \sigma_1(u_2^m)$ (where $\theta_1(u_1^m) \geq 1$), or iii) $s_2 > u_2^m$, $r_2 > u_1^m = s_1$, and $\alpha_1 \in [\sigma_1(u_2^m), \theta_1(u_1^m)]$ (where $0 < \sigma_1(u_2^m) \leq \theta_1(u_1^m) < 1$). Therefore, in all three cases $i = 1$, $j = 2$, $\alpha_1 \in (0,1) \cap [\sigma_1(u_2^m), \theta_1(u^m)]$. Furthermore, when $u_j^m = u^m \geq u_i^m$, we also found an equilibrium with $\alpha_i = \beta_i \in (0,1)$ and $\alpha_j = 0$ can arise iff $i = 1$, $j = 2$, $s_1 = u_1^m \leq u_2^m$, $s_2 < u_2^m$, $\alpha_1 \leq \theta_1(u^m)$, and $\sigma_1(u_2^m) \leq 0$. Thus, overall, an equilibrium with $\alpha_i = \beta_i \in (0,1)$ and $\alpha_j = 0$ arises iff $i = 1$, $j = 2$, $s_1 = u_1^m$ and $\alpha_1 = \beta_1 \in (0,1) \cap [\sigma_1(u_2^m), \theta_1(u^m)]$. □

Lemma 12. *Equilibrium Form 4: $\alpha_i = \beta_i \in (0,1)$, $\alpha_j = \beta_j \in (0,1)$ with $u_j^m > u_i^m$ is an equilibrium iff $i = 1$, $j = 2$, $u_1^m < s_1 \leq u_2^m < r_2$, $\alpha_1 = \beta_1 = \theta_1(u_2^m)$, and $\alpha_2 = \beta_2 = \sigma_2(u_1^m)$.*

Proof of Lemma 12. Suppose $\alpha_i = \beta_i \in (0,1)$ and $\alpha_j = \beta_j \in (0,1)$ with $u_j^m > u_i^m$. Given this, for player j to mix over ϕ and u_j^m, we require $L_j^\phi + x_j(W_j^\phi - L_j^\phi)(1-\alpha_i) = W_j(u_j^m)$. This implies that player i must have $\alpha_i = \beta_i = \theta_i(u_j^m)$. For $\theta_i(u_j^m) \in (0,1)$ as required, one then needs $s_j < u_j^m < r_j$. Given $\alpha_i = \beta_i = \theta_i(u_j^m)$ and $u^m = u_j^m$, there are no possible profitable deviations for player j. Now for player i to mix with $\alpha_i = \beta_i \in (0,1)$ we require $L_i^\phi + x_i(W_i^\phi - L_i^\phi)(1-\alpha_j) = L_i(u_i^m) + (1-\alpha_j)(W_i(u_i^m) - L_i(u_i^m))$. This implies player j must have $\alpha_j = \sigma_j(u_i^m)$ and that player i has an associated equilibrium payoff equal to $W_i(s_i)$. For $\sigma_j(u_i^m) \in (0,1)$, we require $s_i > u_i^m$. Further, to ensure player i does not wish to deviate to just above u_j^m we also require $W_i(s_i) \geq W_i(u_j^m)$ or $s_i \leq u_j^m$. Thus, we need $u_i^m < s_i \leq u_j^m$ together with $s_j < u_j^m < r_j$. From Definition 2, note that $s_j < u_j^m$ implies $s_j = -\infty$. Hence, it must be that $i = 1$ and $j = 2$ via A5 as $s_j = -\infty < 0 \leq u_i^m < s_i$. □

Lemma 13. *Equilibrium Form 5: $\alpha_i = \beta_i = 1$, $\beta_j = \alpha_j \in (0,1)$ with $u_j^m > u_i^m$ is an equilibrium iff $i = 1$, $j = 2$, $r_2 = u_2^m > u_1^m$, $s_1 > u_1^m$, and $\alpha_2 = \beta_2 \in (0,1) \cap (0, \min\{\delta_2(u_2^m), \sigma_2(u_1^m)\}]$.*

Proof of Lemma 13. Given $\alpha_i = \beta_i = 1$, player j can only earn L_j^ϕ when passive. However, given $u_j^m > u_i^m$, player j will earn $W_j(u_j^m)$ when active. Hence, for player j to mix with $\beta_j = \alpha_j \in (0,1)$, we require $L_j^\phi = W_j(u_j^m)$ such that $u_j^m = r_j$. As $x_j > 0$, this implies $u_j^m = r_j > s_j$ and so it must be that $s_j = -\infty$. We also require player i to have no incentive to deviate from u_i^m to i) ϕ or ii) just above u_j^m. To rule out i), we require $L_i(u_i^m) + (W_i(u_i^m) - L_i(u_i^m))(1-\alpha_j) \geq L_i^\phi + x_i(W_i^\phi - L_i^\phi)(1-\alpha_j)$ or $\alpha_j = \beta_j \leq \sigma_j(u_i^m)$. To rule out ii), we require $L_i(u_i^m) + (W_i(u_i^m) - L_i(u_i^m))(1-\alpha_j) \geq W_i(u_j^m)$ or $\alpha_j = \beta_j \leq 1 - \frac{W_i(u_j^m) - L_i(u_i^m)}{W_i(u_i^m) - L_i(u_i^m)} = \delta_j(u_j^m)$. Thus, we require $\alpha_j = \beta_j \leq \min\{\delta_j(u_j^m), \sigma_j(u_i^m)\}$. Hence, to allow for $\alpha_j = \beta_j > 0$, we require $\min\{\delta_j(u_j^m), \sigma_j(u_i^m)\} > 0$. Given $u_j^m > u_i^m$, this is satisfied if $s_i > u_i^m$. When combined with $s_j < u_j^m = r_j$ such that $s_j = -\infty$, it must be that $i = 1$ and $j = 2$ via A5 as $s_j = -\infty < 0 \leq u_i^m < s_i$. □

Lemma 14. *Equilibrium Form 6: $1 > \alpha_i > \beta_i \geq 0$ for $i = \{1,2\}$ is an equilibrium iff $r_2 > s_1 > u^m$, $\bar{u} = s_1$, $\alpha_1 = \theta_1(\bar{u})$, $\alpha_2 = \sigma_2(u_1^m)$, $\beta_1 = F_1(u^m) - (1-\alpha_1) \geq 0$, and $\beta_2 = F_2(u^m) - F_2(u_1^m) \geq 0$.*

Proof of Lemma 14. Suppose $1 > \alpha_i > \beta_i \geq 0$ for $i = \{1,2\}$. From Lemma 4, each player k must mix over $u_k \in \{\phi\} \bigcup (u^m, \bar{u}]$. For this to be part of equilibrium, each player k must earn their equilibrium payoff, Π_k^*, from any such u_k. Thus, to be indifferent between ϕ and $\bar{u}$ specifically, requires $\Pi_k^* = L_k^\phi + x_k(1-\alpha_l)(W_k^\phi - L_k^\phi) = W_k(\bar{u})$ such that $\alpha_l = 1 - \frac{W_k(\bar{u}) - L_k^\phi}{x_k(W_k^\phi - L_k^\phi)} \equiv \theta_l(\bar{u})$ for any $k,\ l \neq k \in \{1,2\}$.

Without loss let $u_i^m \leq u_j^m$. Initially consider a first possibility where $\beta_i > 0$. Then, player i must earn its equilibrium payoff, Π_i^*, from setting u_i^m. Hence,

$\Pi_i^* = L_i(u_i^m) + (W_i(u_i^m) - L_i(u_i^m))(1-\alpha_j)$. By setting this equal to the previous expression, $\Pi_i^* = L_i^\phi + x_i(1-\alpha_j)(W_i^\phi - L_i^\phi)$, one obtains an alternative expression for $\alpha_j = \sigma_j(u_i^m)$. Hence, by setting $\alpha_j = \sigma_j(u_i^m) = \theta_j(\bar{u})$, we find $\Pi_i^* = W_i(s_i)$ such that $\bar{u} = s_i$. Now consider player j. She must earn her equilibrium profit when selecting select u_j (arbitrarily close to) u^m. Thus, $\Pi_j^* = L_j(u^m) + (W_j(u^m) - L_j(u^m))(1-\alpha_i+\beta_i)$. By setting this equal to $\Pi_j^* = W_j(\bar{u})$, it gives $\beta_i = \frac{W_j(\bar{u})-L_j(u^m)}{W_j(u^m)-L_j(u^m)} - \theta_i(\bar{u}) \equiv F_i(u^m) - (1-\alpha_i)$. By rearranging the expression for β_i, we then require $s_i > s_j$ to ensure $\beta_i > 0$ as assumed. Hence it must be that $i = 1$ and $j = 2$. By definition it follows that $\beta_2 = F_2(u^m) - (1-\alpha_2)$. Then, using the definition of strength, one can show $(1-\alpha_2) = F_2(u_1^m)$ such that $\beta_2 = F_2(u^m) - F_2(u_1^m) \geq 0$ given $u_1^m \leq u_2^m = u^m$. Lastly, given $s_1 > s_2$, to ensure $\alpha_k \in (\beta_k, 1)$ for $k \in \{1,2\}$, we require $r_2 > s_1 > u^m$.

Now continue to assume $u_i^m \leq u_j^m$, but consider the remaining possibility with $\beta_i = 0$. For player j to be indifferent between setting $u_j^m = u^m$ and being passive, we require $L_j(u_j^m) + (W_j(u_j^m) - L_j(u_j^m))(1-\alpha_i) = L_j^\phi + x_j(W_j^\phi - L_j^\phi)(1-\alpha_i)$. Hence, one obtains an alternative expression for $\alpha_i = \sigma_i(u_j^m)$. Then by setting $\alpha_i = \sigma_i(u_j^m) = \theta_i(\bar{u})$, we find $\Pi_j^* = W_j(s_j)$ such that $\bar{u} = s_j$. Given $\beta_i = 0$, player i should not want to deviate to u_i^m. Hence, we require $\Pi_i^* = W_i(\bar{u}) \geq L_i(u_i^m) + (W_i(u_i^m) - L_i(u_i^m))(1-\alpha_j)$. After rearranging, this gives $W_i(s_j) \leq L_i^\phi + \frac{c_i(u_i^m)}{b_i(u_i^m)}(W_i^\phi - L_i^\phi)x_i = W_i(s_i)$ or $s_i \leq s_j$. Given $\beta_j \geq 0$, player i should also earn Π_i^* by setting u_i just above u_j^m such that $\Pi_i^* = W_i(\bar{u}) = L_i(u_j^m) + (W_i(u_j^m) - L_i(u_j^m))(1-\alpha_j+\beta_j)$. This confirms that $\beta_j = F_j(u_j^m) - (1-\alpha_j)$. This equals zero if $s_i = s_j$ but is otherwise positive. Hence, when $s_1 = s_2$, player i can either be 1 or 2, but when $s_i < s_j$ then it must be that $j = 1$ and $i = 2$ from A5. Either way, this again confirms that $\bar{u} = s_1$ and $\beta_1 = F_1(u^m) - (1-\alpha_1) \geq 0$ and $\beta_2 = F_2(u^m) - (1-\alpha_2) = F_2(u^m) - F_2(u_1^m) = 0$. Further, again, we require the same conditions, $r_2 > s_1 > u^m$, to ensure $\alpha_k \in (0,1)$ and $\alpha_k > \beta_k$ for $k \in \{1,2\}$.

Finally, we also need to verify that $F_k(u)$ in (5) is well-behaved for both $k = \{1,2\}$ with i) $F_k(\bar{u}) = 1$, and ii) $F_k'(u) > 0$ for all $u \in (u^m, \bar{u}]$. i) is satisfied automatically. For ii), note $F_k'(u)$ has the same sign as $-L_l'(u)[W_l(u) - W_l(\bar{u})] - W_l'(u)[W_l(\bar{u}) - L_l(u)]$, and that this is guaranteed to be positive for all $u \in (u^m, \bar{u}]$ if $W_l(\bar{u}) > L_l(u^m)$ for $l = \{1,2\}$. As $L_l^\phi > L_l(u^m)$, this condition would be satisfied if $W_l(\bar{u}) \geq L_l^\phi$. Using the definition of reach, this requires $r_l \geq \bar{u}$ for $l = \{1,2\}$ which follows given $r_2 > \bar{u} = s_1 > u^m$, and $r_1 > s_1$. □

Lemma 15. *Equilibrium Form 7:* $1 > \alpha_2 > \beta_2 \geq 0$ *and* $1 = \alpha_1 > \beta_1 \geq 0$ *is an equilibrium iff* $s_1 \geq r_2 > u^m$, $\bar{u} = r_2$, $\alpha_2 = 1 - F_2(u_1^m)$, $\beta_1 = F_1(u^m) > 0$ *and*

$\beta_2 = F_2(u^m) - F_2(u_1^m) \geq 0$.

Proof of Lemma 15. Suppose $1 > \alpha_2 > \beta_2 \geq 0$ and $1 = \alpha_1 > \beta_1 \geq 0$. From Lemma 4, each player k must mix over $u_k \in \{\phi\} \bigcup (u^m, \bar{u}]$. First, it must be true that $\beta_1 > 0$ in equilibrium. If not, with $\beta_1 = 0$, then player 2 would always lose when choosing u_2 arbitrarily close to u^m and so she would prefer to deviate to ϕ instead as $L_2^\phi - L_2(u_2^m) > 0$ via A3. Second, it then follows that $\beta_2 = 0$ when $u_1^m \geq u_2^m$. To understand this, note that given $\beta_1 > 0$, β_2 must equal zero if $u_1^m = u_2^m$ from Lemma 3. Further, if $u_1^m > u_2^m$ then player 2 will always lose at u_2^m given $\alpha_1 = 1$. Therefore, player 2 would optimally set $\beta_2 = 0$ and instead, deviate to ϕ as $L_2^\phi - L_2(u_2^m) > 0$ via A3. Third, for $\alpha_2 \in (0, 1)$, player 2 must earn Π_2^* from any $u_2 \in \{\phi\} \bigcup (u^m, \bar{u}]$. Hence, she must be indifferent between setting u i) equal to ϕ, ii) just above u^m, and iii) equal to $\bar{u}$. Given $\alpha_1 = 1$, this implies $\Pi_2^* = L_2^\phi = L_2(u^m) + (W_2(u^m) - L_2(u^m))\beta_1 = W_2(\bar{u})$ such that $\bar{u} = r_2$ and $\beta_1 = \frac{W_2(\bar{u}) - L_2(u^m)}{W_2(u^m) - L_2(u^m)} \equiv F_1(u^m)$. Fourth, to ensure $\bar{u} > u^m$, we require $r_2 > u^m$. (This also ensures $F_1(u^m) > 0$ given $L_2^\phi > L_2(u^m)$ via A3.) Fifth, given $\alpha_1 > \beta_1 > 0$, player 1 must earn Π_1^* from any $u_1 \in [u^m, \bar{u}]$. Given $\beta_2 = 0$ when $u_1^m \geq u_2^m$, player 1 must earn $\Pi_1^* = L_1(u_1^m) + (1 - \alpha_2)(W_1(u_1^m) - L_1(u_1^m))$ by selecting $u_1 = u_1^m$. By setting this equal to $\Pi_1^* = W_1(\bar{u})$, one obtains $\alpha_2 = 1 - \frac{W_1(\bar{u}) - L_1(u_1^m)}{W_1(u_1^m) - L_1(u_1^m)} \equiv 1 - F_2(u_1^m)$. Given $\bar{u} > u^m$, our previous condition, $r_2 > u^m$, ensures $\alpha_2 \in (0, 1)$ as required. It then follows that player 2 has a mass point at u_2^m of size $\beta_2 = F_2(u^m) - (1 - \alpha_2) = F_2(u^m) - F_2(u_1^m)$. As consistent with our earlier claim, this is positive if $u_1^m < u_2^m$, and zero if $u_1^m \geq u_2^m$. Further, player 1 should not want to deviate to being passive, so we require $W_1(\bar{u}) \geq L_1^\phi + x_1(W_1^\phi - L_1^\phi)(1 - \alpha_2) \equiv W_1(s_1)$. This implies $s_1 \geq r_2$. So overall, we require $s_1 \geq r_2 > u^m$. (Finally, we need to verify that $F_k(u)$ in (5) is well-behaved for both $k = \{1, 2\}$ with i) $F_k(\bar{u}) = 1$, and ii) $F_k'(u) > 0$ for all $u \in (u^m, \bar{u}]$. Using the details from the proof for Lemma 14, this requires $r_k \geq \bar{u}$ for $k = \{1, 2\}$. Here, this follows given $r_1 > s_1 \geq \bar{u} = r_2 > u^m$.) □

Step 3: Characterizing the Parameter Space

To complete the derivation, Step 3 uses the results from Step 2 to identify the possible equilibria in each region and show how the equilibrium results are consistent with Theorems 1 and 2.

First, by using Step 2, it is tedious but straightforward to show that the equilibria detailed in Lemmas 9-15 cover all valid parameter cases under our assumptions and definitions. Hence, at least one equilibrium form exists in each possible parameter constellation.

Second, we need to show that Theorems 1 and 2 cover all possible equilibria, that each equilibrium is correctly detailed within the Theorems, and that the equilibria within Theorem 1 are unique. To proceed, we work through Lemmas 9-15 in reverse order. The necessary and sufficient conditions regarding the levels of reach and strength in Lemma 15, $s_1 \geq r_2 > u^m$, are not compatible with any other Lemma from Step 2 and are fully captured by case v) in Theorem 1. Similarly, the necessary and sufficient conditions in Lemma 14 are not compatible with any other Lemmas from Step 2 and are fully covered by case iv) Theorem 1. The necessary and sufficient conditions in Lemma 13 are fully covered by case c) of Theorem 2. However, at these conditions, Lemma 10 can also apply and so case c) of Theorem 2 permits $\alpha_2 = 0$ as well as $\alpha_2 \in (0, 1)$. The necessary and sufficient conditions in Lemma 12 are not compatible with any other Lemmas from Step 2 and are fully covered by cases iii) Theorem 1. The necessary and sufficient conditions in Lemma 11 are fully covered by case a) of Theorem 2. However, at these conditions, the necessary and sufficient conditions for Lemmas 10 and 9 can also apply if $r_2 \leq u^m$ or $s_2 \leq u_2^m$, respectively. Nevertheless, these equilibrium possibilities are still consistent with Theorem 2a because $\theta_1(u^m) \geq 1$ if $r_2 \leq u^m$, and $\sigma_1(u_2^m)$ and $\theta_1(u^m)$ equal zero if $s_2 \leq u_2^m$. The necessary and sufficient conditions in Lemma 10 are not compatible with any other Lemma from Step 2 and are fully covered in Theorem 1 case ii), apart from the overlap situations that we have already covered, and apart from the situation where $s_1 > u_1^m$ and $r_2 = u_2^m \leq u_1^m$ but this is covered by case d) of Theorem 2. Finally, the necessary and sufficient conditions in Lemma 9 are not compatible with any other Lemma from Step 2 and are fully covered in Theorem 1i) apart from the overlap situations that we have already covered, and apart from the situation where $s_1 < u_1^m$ and $r_2 = u_2^m$ but this is covered by case b) of Theorem 2.

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