Coalitional substitution of players and the proportional Shapley value

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Abstract

We present a new axiomatization of the proportional Shapley Value. Our study is based on three axioms: efficiency, which ensures that the total worth of the grand coalition is fully distributed among the players; the disjointly productive players property, which states that removing a player who has no cooperative interactions with another player does not affect that player's payoff; and a new axiom that makes the difference to the classical Shapley value. This axiom, the coalitional substitution of players property, involves a scenario in which a player's cooperative contribution to a coalition is replaced by that of a group of new players whose combined individual worths match that of the original player. The key point is that the payoffs to the remaining players remain unaffected.

Keywords Cooperative game · Proportional Shapley value · Disjointly productive players · Coalitional substitution of players · Patronage refunds

1 Introduction

Although the Shapley value (Shapley, 1953) is widely regarded as the best-known cooperative solution concept, profits or surpluses are usually distributed proportionally among investors based on their contribution or participation in the respective cooperative coalition.

From a game-theoretical perspective, the proportional rule (Moriarity, 1975) is traditionally the most relevant method here. Recently, Zou et al. (2021) axiomatized this solution as proportional division value. One of Amer et al.'s (2007) main criticisms of the proportional rule is that it is designed in such a way that it simply ignores most marginal contributions. Instead they propose the Shapley value for "Utility Sharing in Joint Business."

A solution concept that combines advantages of both solution concepts is the proportional Shapley value (Besner, 2016; Béal et al., 2018). While the Shapley value distributes the Harsanyi dividends[1] (Harsanyi, 1959) generated by the different coalitions equally among

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[1] The Harsanyi dividend of a singleton is equal to its worth, while the dividends of all other coalitions reflect the additional contribution made by the coalition above the cumulated dividends of its subcoalitions. Therefore, dividends can be seen as "the pure contribution of cooperation" (see Billot and Thisse (2005)).

its members, the proportional Shapley value distributes them proportionally to the worths of the singletons of the individual players. For two-player games or when the marginal contributions of the players other than the singletons and the grand coalition (the coalition of all players) are reduced to the worths of the singletons, the proportional rule and the proportional Shapley value coincide.

Overall, and especially compared to the meanwhile nearly countless axiomatizations of the Shapley value (see, e.g., Lipovetsky (2020)), the number of axiomatizations of the proportional Shapley value is still quite modest (see Béal et al., 2018; Besner, 2019; Besner, 2022b).

Besner (2022a) axiomatized the Shapley value by three axioms: efficiency, the disjointly productive players property[2], and the split game property. The first two axioms are also applied here in our characterization of the proportional Shapley value. A solution concept that does not satisfy efficiency is unlikely to make much sense for the applications if, for example, investors want to collect their patronage refunds. Paying out less than the total proceeds would not be accepted, and more is simply not possible. The disjointly productive players property would also be reasonable from the investors' point of view. Why should removing a player affect my payoff if that player has contributed nothing to the investments I am involved in?

With the split game property, however, some investors may not be quite so enthusiastic about it. By this property, a player is split into two disjointly productive players, which means, in particular, that the Harsanyi dividends are zero for all coalitions containing both players. Therefore, investors who consider the standalone worth of a player to be the decisive factor in allocating a cooperation benefit might perceive this as unfair since the standalone worth of one of the new players is generally different from that of the split player in the relevant coalitions. The fact that the payoff to the other players should not change due to this axiom if the worth of all coalitions containing both new players is equal to the worth of the old coalitions with the split player does not change the fact that investors may not perceive the payoff as equitable.

Our new axiom, the coalitional substitution of players property, indeed points in the same direction. This time, a player is not split into two new players but a group of new players replaces the marginal contribution of another player within an arbitrary coalition of at least two players. Here, the sum of the standalone worths equals the standalone worth of the substituted player in the decisive coalitions, which satisfies the other investors' sense of fairness if their payoff remains unchanged as required.

The paper is organized as follows. Section 2 contains some preliminaries. In Section 3, we introduce the scenario of a coalitional substitution of players game with an additional short example of cooperative investments, the coalitional substitution of players property, our new axiomatization of the proportional Shapley value, and all the proofs. Section 4 gives a short conclusion, and the Appendix (Section 5) shows the logical independence of the axioms in the axiomatization.

2 Preliminaries

We denote the real numbers with $\mathbb{R}$, the rational numbers with $\mathbb{Q}$, and the natural numbers with $\mathbb{N}$. Let the countably infinite set $\mathfrak{U}$ be the universe of players. We denote by $\mathcal{N}$

[2]In Casajus and Tido Takeng (2024), this axiom is called the second-order null player property.

the set of all non-empty and finite subsets of $\mathfrak{U}$. A cooperative game with transferable utility (TU-game) is a pair (N, v) such that $N \in \mathcal{N}$ and v is a **coalition function**, i.e., $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. The subsets $S \subseteq N$ are called **coalitions** and $v(S)$ is called the **worth** of the coalition S. If S contains only one player $i \in N$, S is called a **singleton**. $|S|$ denotes the cardinality of the coalition S and Ω^S denotes the set of all nonempty subsets of S, (S, v) is the **restriction** of (N, v) to the player set $S \in \Omega^N$.

The set of all games (N, v) is denoted by $\mathbb{V}(N)$, the set of all TU-games (N, v) such that $v(\{i\}) > 0$ for all $i \in N$ or $v(\{i\}) < 0$ for all $i \in N$ is denoted by $\mathbb{V}_0(N)$, and if the worths of the singletons must all be positive rational numbers or must all be negative rational numbers, we denote this set of TU-games by $\mathbb{V}_{0_{\mathbb{Q}}}(N)$.

The **Harsanyi dividends** $\Delta_v(T)$ (Harsanyi, 1959), are defined inductively by

$$\Delta_v(T) := \begin{cases} v(T) - \sum_{S \subsetneq T} \Delta_v(S), & \text{if } T \in \Omega^N, \text{ and} \\ 0, & \text{if } T = \emptyset. \end{cases} \tag{1}$$

Let $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$. We call a coalition $S \subseteq N$ **essential** in (N, v) if $\Delta_v(S) \neq 0$. The **marginal contribution** $MC_i^v(S)$ of a player $i \in N$ to a coalition $S \subseteq N \backslash \{i\}$ is given by $MC_i^v(S) := v(S \cup \{i\}) - v(S)$. We call two players $i, j \in N$, $i \neq j$, **symmetric** in (N, v) if for all $S \subseteq N \backslash \{i, j\}$, we have $v(S \cup \{i\}) = v(S \cup \{j\})$, they are called **disjointly productive** in (N, v) if, for all $S \subseteq N \backslash \{i, j\}$, we have $v(S \cup \{i, j\}) - v(S \cup \{j\}) = v(S \cup \{i\}) - v(S)$, which is equivalent, by Lemma 3.3 in Besner (2022a), to

$$\Delta_v(S \cup \{i, j\}) = 0. \tag{2}$$

For all $N \in \mathcal{N}$, a **TU-value** or **solution** φ is an operator that assigns to any $(N, v) \in \mathbb{V}(N)$ a payoff vector $\varphi(N, v) \in \mathbb{R}^N$. The **Shapley value** Sh (Shapley, 1953) is given by

$$Sh_i(N, v) := \sum_{S \subseteq N,\, S \ni i} \frac{\Delta_v(S)}{|S|} \quad \text{for all } i \in N.$$

While the Shapley value distributes the Harsanyi dividends evenly among all members of a coalition, the following value distributes the dividends proportionally to the singleton worths among the respective players. For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}_0(N)$, the **proportional Shapley value** Sh^p (Besner, 2016; Béal et al., 2018) is given by

$$Sh_i^p(v) = \sum_{S \subseteq N, S \ni i} \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \Delta_v(S) \quad \text{for all } i \in N. \tag{3}$$

We make use of the following axioms for TU-values.

Efficiency, E. For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, we have $\sum_{i \in N} \varphi_i(N, v) = v(N)$.

This axiom means that the worth of the grand coalition must be completely shared between the players.

Symmetry, S. For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and $i, j \in N$ such that i and j are symmetric in (N, v), we have $\varphi_i(N, v) = \varphi_j(N, v)$.

The symmetry axiom assures that equals should also be treated equally.

Disjointly productive players, DP (Besner, 2022a). For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and $i, j \in N$ such that i and j are disjointly productive players in (N, v), we have $\varphi_i(N, v) = \varphi_i(N \backslash \{j\}, v)$.

By this axiom, a player's payoff does not change when another player, disjointly productive in relation to that player, leaves the game.

In the case of using a subdomain in the following part, an axiom is required to hold whenever a game belongs to the subdomain.

3 Coalitional substitution of players

The following definition concerns games that result from the substitution of a player's marginal contribution to one coalition S (and all subcoalitions of $S, |S| \geq 2,$) by the impact of a set of new players who are disjointly productive to the substituted player.

Definition 3.1. *Let $N \in \mathcal{N}, (N,v) \in \mathbb{V}(N), S \subseteq N, |S| \geq 2, i \in S$, $Q \in \mathcal{N}$, $Q \cap N = \emptyset$, and $N^Q := N \cup Q$. A TU-game $(N^Q, v_i^{S,Q}) \in \mathbb{V}(N^Q)$ is called a **coalitional substitution of players game** corresponding to (N,v) if $v_i^{S,Q}$ is such that*

$$\begin{aligned}
\sum\nolimits_{j\in Q} v_i^{S,Q}(\{j\}) &= v(\{i\}),\\
v_i^{S,Q}(T\cup Q) &= v(T\cup\{i\}),\ \textit{for all } T\subseteq S\backslash\{i\},\\
v_i^{S,Q}(T\cup R) &= v(T)+\sum\nolimits_{j\in R} v_i^{S,Q}(\{j\}),\ \textit{for all } T\subseteq N\backslash\{i\} \textit{ and } R\subsetneq Q,\\
&\qquad \textit{or } R=Q \textit{ and additional } T\not\subseteq S\backslash\{i\},\ \textit{and}\\
v_i^{S,Q}(T\cup\{i\}\cup R) &= v(T\cup\{i\})-v\big((T\cap S)\cup\{i\}\big)+v(T\cap S)+v(\{i\})+\sum\nolimits_{j\in R} v_i^{S,Q}(\{j\}),\\
&\qquad \textit{for all } T\subseteq N\backslash\{i\} \textit{ and } R\subseteq Q.
\end{aligned}$$

Remark 3.2. *Under the same preconditions, Definition 3.1 is equivalent to the following statements:*

$$\begin{aligned}
\sum\nolimits_{j\in Q} v_i^{S,Q}(\{j\}) &= v(\{i\}),\\
\Delta_{v_i^{S,Q}}(R\cup\{i\}) &= 0,\ \textit{for all } \emptyset\neq R\subseteq S\backslash\{i\},\\
\Delta_{v_i^{S,Q}}(R\cup Q) &= \Delta_v(R\cup\{i\}),\ \textit{for all } R\subseteq S\backslash\{i\},\\
\Delta_{v_i^{S,Q}}(T) &= \Delta_v(T),\ \textit{for all } T\subseteq N \textit{ such that } T\neq R\cup\{i\} \textit{ with } \emptyset\neq R\subseteq S\backslash\{i\},\\
\Delta_{v_i^{S,Q}}(P) &= 0,\ \textit{for all } P\subseteq N^Q, P\cap Q\neq\emptyset, |P|\neq 1,\\
&\qquad \textit{and such that } P\neq(R\cup Q) \textit{ with } R\subseteq S\backslash\{i\}.
\end{aligned}$$

It is easy to see, by (1), that we have for a player $i \in N$ and a coalition $S \subseteq N\backslash\{i\}$, $MC_i^v(S) = \sum_{R\subseteq S, R\neq\emptyset} \Delta_v(R)$. Therefore, the marginal contribution to the coalition S in the coalitional substitution of players game reduces for the player i to the singleton worth $v(\{i\})$. The missing part is taken over by the new players from coalition Q which as singletons have together the same impact as the player i. All the other coalitions which are subsets of N have the same Harsanyi dividends in both games and all other new coalitions have a dividend of zero.

To illustrate our model, we give an example: Let us look to a group of investors and companies (our players) which collaborate to fund and develop various renewable energy projects. They form multiple coalitions to invest in different projects, such as wind farms, solar power plants, and hydroelectric dams. Each investor contributes capital, influencing the success of these projects. These are the worths of the singletons of the different players.

At some point, one investor decides to withdraw from investment into one project/coalition for some reasons. This could be, e.g., a policy shift of the company.

Now, to maintain the viability of this project, new investors must step in to replace the old investor while ensuring the same level of impact, meaning that the new investors contribute equivalent capital as the exiting investor. This says in our model that the singleton worths of the new investors should together sum up to the singleton worth of the withdrawing investor and the worth of the coalition which represents the respective investment proceeds should be the same as before. On the structure of the complete setup there should be no other change.

Decisive for the old investors would be that their payoff would be in the new setup the same as in the old one. This property is captured by the following new axiom.

Coalitional substitution of players, CSP. For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and a coalitional substitution of players game $(N^Q, v_i^{S,Q}) \in \mathbb{V}(N^Q)$ corresponding to (N, v), we have

$$\varphi_j(N^Q, v_i^{S,Q}) = \varphi_j(N, v) \text{ for all } j \in N\backslash\{i\}.$$

Our focus is also on the payoffs for the other players. If the TU-value satisfies Efficiency **E**, we have, by Definition 3.1,

$$v_i^{S,Q}(N^Q) = v(N) + v(\{i\}). \tag{4}$$

Therefore, we get a result which is easy to verify.

Remark 3.3. *Let $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and $(N^Q, v_i^{S,Q}) \in \mathbb{V}(N^Q)$ be a coalitional substitution of players game corresponding to (N, v). If φ is a TU-value that satisfies **E** and **CSP**, we have,*

$$\varphi_i(N, v) = \varphi_i(N^Q, v^{S,Q}) + \left[\sum_{j \in Q} \varphi_j(N^Q, v_i^{S,Q})\right] - v(\{i\}). \tag{5}$$

The statement from this remark can be extended to multiple consecutive substitutions of the same player for more then one coalition.

Remark 3.4. *Let $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, $|N| \geq 2, i \in N$, and $\mathcal{T}_i := \{T \subseteq N : T \ni i, |T| \geq 2\}$ be the set which contains all subsets of N that contain the player i, except the singleton. We denote by $(N^{Q(\mathcal{T}_i)}, v_i^{\mathcal{T}_i,Q(\mathcal{T}_i)})$ the TU-game derived from (N, v) by a successively coalitional substitution of the player i in all coalitions $T \in \mathcal{T}_i$ accordingly to Definition 3.1, starting with the coalitions with two players, where $Q(\mathcal{T}_i)$ denotes the set containing all players that substitute the player i in the successively derived coalitional substitutions in the coalitions $T \in \mathcal{T}_i$. If φ satisfies **CSP**, we have*

$$\varphi_j(N^{Q(\mathcal{T}_i)}, v_i^{\mathcal{T}_i,Q(\mathcal{T}_i)}) = \varphi_j(N, v) \text{ for all } j \in N\backslash\{i\}.$$

We have $|\mathcal{T}_i| = 2^{|N|-1} - 1$. It follows, by Definition 3.1,

$$v_i^{\mathcal{T}_i,Q(\mathcal{T}_i)}(N^{Q(\mathcal{T}_i)}) = v(N) + (2^{|N|-1} - 1)v(\{i\}).$$

*If φ is a TU-value that satisfies **E** and **CSP**, we have,*

$$\varphi_i(N,v) = \varphi_i(N^{Q(\mathcal{T}_i)}, v_i^{\mathcal{T}_i,Q(\mathcal{T}_i)}) + \left[\sum_{j\in Q(\mathcal{T}_i)} \varphi_j(N^{Q(\mathcal{T}_i)}, v_i^{\mathcal{T}_i,Q(\mathcal{T}_i)})\right] - (2^{|N|-1}-1)v(\{i\}). \tag{6}$$

*The singleton $\{i\}$ is the only essential coalition containing player i. Therefore, if φ also satisfies **DP**, we obtain*

$$\varphi_i(N^{Q(\mathcal{T}_i)}, v_i^{\mathcal{T}_i,Q(\mathcal{T}_i)}) = v(\{i\}). \tag{7}$$

and, by (6),

$$\varphi_i(N,v) = \left[\sum_{j\in Q(\mathcal{T}_i)} \varphi_j(N^{Q(\mathcal{T}_i)}, v_i^{\mathcal{T}_i,Q(\mathcal{T}_i)})\right] - (2^{|N|-1}-1)v(\{i\}). \tag{8}$$

Now, as an intermediate step, we can formulate a Lemma.

Lemma 3.5. *If a TU-value φ satisfies **E**, **DP**, and **CSP**, then φ also satisfies **S**.*

Proof. Let $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, and $i,j \in N$ be two symmetric players in (N,v), and φ be a TU-value that satisfies **E**, **DP**, and **CSP**.

Accordingly to Definition 3.1, we chose $S = N, Q_1 = \{k_1\}, k_1 \in \mathfrak{U}, k_1 \notin N$, and substitute the player i in the coalitional substitution of players game $(N^{Q_1}, v_i^{N,Q_1}) \in \mathbb{V}(N^{Q_1})$ corresponding to (N,v).

By (2) and Remark 3.2, the player i is disjointly productive to all other players in the new game. Therefore, by (4) and (5), we have

$$\varphi_i(N,v) = \varphi_{k_1}(N^{Q_1}, v_i^{N,Q_1})$$

For the game (N^{Q_1}, v_i^{N,Q_1}), we chose $S = N^{Q_1}, Q_2 = \{k_2\}, k_2 \in \mathfrak{U}, k_2 \notin N^{Q_1}$, and substitute the player j in the coalitional substitution of players game $(N^{Q_1 Q_2}, v_j^{N^{Q_1},Q_2}) \in \mathbb{V}(N^{Q_1 Q_2})$ corresponding to (N^{Q_1}, v_i^{N,Q_1}). It follows, by **CSP**,

$$\varphi_i(N,v) = \varphi_{k_1}(N^{Q_1 Q_2}, v_j^{N^{Q_1},Q_2}),$$

and, analogously.

$$\varphi_j(N,v) = \varphi_{k_2}(N^{Q_1 Q_2}, v_j^{N^{Q_1},Q_2}),$$

Now, we perform the same procedure again for the game (N,v), but this time with the player sets $Q_3 = \{k_2\}$ and $Q_4 = \{k_1\}$. We obtain

$$\varphi_i(N,v) = \varphi_{k_2}(N^{Q_3 Q_4}, v_j^{N^{Q_3},Q_4}),$$

and

$$\varphi_j(N,v) = \varphi_{k_1}(N^{Q_3 Q_4}, v_j^{N^{Q_3},Q_4}).$$

Since the players i and j are symmetric in (N,v), the games $(N^{Q_1 Q_2}, v_j^{N^{Q_1},Q_2})$ and $(N^{Q_3 Q_4}, v_j^{N^{Q_3},Q_4})$ are identical and **S** is shown. □

Now, we are ready to present our main result.

Theorem 3.6. *Let $N \in \mathcal{N}$ and $(N, v) \in \mathbb{V}_{0_{\mathbb{Q}}}(N)$. Sh^p is the unique TU-value that satisfies* ***E****,* ***DP****, and* ***CSP****.*

Proof. Let $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}_{0_{\mathbb{Q}}}(N)$.
I. Existence: It is well-known that Sh^p satisfies **E**. By (3) and (2), it is obvious that Sh^P satisfies **DP**, and, by (3) and Remark 3.2, that Sh^P satisfies **CSP**.
II. Uniqueness: Let φ be a TU-value that satisfies all axioms of Theorem 3.6 and, therefore, by Lemma 3.5, also **S**.

If $|N| = 1$, uniqueness is given by **E**.

Let now $|N| \geq 2$ and, w.l.o.g., be $v(\{i\}) > 0$ for all $i \in N$. The following part of the proof is constructive. We have $2^{|N|} - 1 - |N|$ coalitions $S_k, |S_k| \geq 2$, where, for each player $i \in N$, $2^{|N|-1} - 1$ coalitions contain the player i. For each coalition S_k, the worth of the singleton of each player $\ell \in S_k$ can be written as a fraction. We have

$$v(\{\ell\}) = \frac{p_\ell^{S_k}}{q_\ell^{S_k}} \text{ with } p_\ell^{S_k}, q_\ell^{S_k} \in \mathbb{N}.$$

We choose a main denominator q^{S_k} of the fractions of all players $\ell \in S_k$ by $q^{S_k} := \prod_{\ell \in S_k} q_\ell^{S_k}$. With $z_\ell^{S_k} := p_\ell^{S_k} \cdot \prod_{i \in S_k \setminus \{\ell\}} q_i^{S_k}$, we have

$$v(\{\ell\}) = \frac{z_\ell^{S_k}}{q^{S_k}}.$$

We substitute each player $\ell \in S_k$ in each coalition $S_k \subseteq N$, according to Definition 3.1, by adding $z_\ell^{S_k}$ many new players, starting by the coalitions S_k with $|S_k| = 2$ for all players $\ell \in N$ and go then to the coalitions with three players and so on. The procedure is the same as in Remark 3.4 but for all players $\ell \in N$ in parallel with the same final result. Denote by $(N^{\overline{Q}}, v^{\overline{Q}}) \in \mathbb{V}(N^{\overline{Q}})$ the final resulting coalitional substitution of players game and by $\overline{Q}_\ell^{S_k}$ the set containing all players that substitute the original player $\ell \in N$ in the coalition $S_k \subseteq N$ in the final player set. By (7), we have

$$\varphi_\ell(N^{\overline{Q}}, v^{\overline{Q}}) = v(\{\ell\}) \text{ for all } \ell \in N,$$

and, by (8),

$$\varphi_\ell(N, v) = \left[\sum_{\substack{S_k \subseteq N, \\ S_k \ni \ell}} \sum_{j \in \overline{Q}_\ell^{S_k}} \varphi_j(N^{\overline{Q}}, v^{\overline{Q}}) \right] - (2^{|N|-1} - 1) v(\{\ell\}) \text{ for all } \ell \in N.$$

For each $S_k \subseteq N$, to all players $j \in \bigcup_{m \in S_k} \overline{Q}_m^{S_k}$ all other players outset this set are disjointly productive and, by **DP**, removing all other players does not change the payoff for the players inside this set which are all symmetric players. Denote the restriction of the game where all players outside $\bigcup_{m \in S_k} \overline{Q}_m^{S_k}$ are removed by $(\overline{Q}^{S_k}, v^{\overline{Q}})$. We obtain

$$\varphi_\ell(N, v) = \left[\sum_{\substack{S_k \subseteq N, \\ S_k \ni \ell}} \sum_{j \in \overline{Q}_\ell^{S_k}} \varphi_j(\overline{Q}^{S_k}, v^{\overline{Q}}) \right] - (2^{|N|-1} - 1) v(\{\ell\}) \text{ for all } \ell \in N. \tag{9}$$

Since all players $j \in \overline{Q}^{S_k}$ are symmetric in $(\overline{Q}^{S_k}, v^{\overline{Q}})$ and we have $\Delta_{v^{\overline{Q}}}(\overline{Q}^{S_k}) = \Delta_v(S_k)$ and $\sum_{j \in \overline{Q}_\ell^{S_k}} v^{\overline{Q}}(\{j\}) = v(\{\ell\})$, it follows, by **S** and **E**,

$$\sum_{j \in \overline{Q}_\ell^{S_k}} \varphi_j(\overline{Q}^{S_k}, v^{\overline{Q}}) = \frac{v(\{j\})}{\sum_{i \in S_k} v(\{i\})} \Delta_v(S_k) + v(\{\ell\}),$$

and, therefore, by (9),

$$\varphi_\ell(N, v) = \sum_{\substack{S_k \subseteq N, \\ S_k \ni \ell}} \frac{v(\{j\})}{\sum_{i \in S_k} v(\{i\})} \Delta_v(S_k) \underset{(3)}{=} Sh_\ell^p(N, v).$$

□

Remark 3.7. *Theorem 3.6 remains valid on $(N, v) \in \mathbb{V}_0(N)$ if we require continuity of the TU-value in $v(\{i\})$ for all $(N, v) \in \mathbb{V}_0(N)$ and all $i \in N$ in an additional axiom.*

4 Conclusion

In this paper, we have introduced a new axiomatization of the proportional Shapley Value by incorporating the novel coalitional substitution of players property. This new axiom allows us to model scenarios in which a player's cooperative contribution to a coalition is replaced by a group of new players whose combined individual worths are equal to that of the original player. Importantly, this substitution ensures that the payoffs to the remaining players remain unchanged, thereby maintaining fairness in the allocation process and ensuring consistency.

Our results provide a theoretical foundation for applications, for example, in the areas of cooperative investment, patronage refunds, and coalition-based decision-making where proportional fairness is required. The marginal contributions of players to all coalitions are taken into account. Future research could explore further generalizations of this model, including its implications for dynamic coalition formation and real-world economic settings.

5 Appendix

Remark 5.1. *The axioms in Theorem 3.6 are logically independent:*

- ***E**: The TU-value $\varphi := 2Sh^p$ satisfies **DP** and **CSP** but not **E**.*
- ***DP**: We define $\mathcal{E}^{(N,v)}$ as the set of all coalitions $S \subseteq N, |S| \geq 2$, which are essential in (N, v). Let*

$$\gamma := \begin{cases} \dfrac{v(N) - \sum_{i \in N} v(\{i\})}{|\mathcal{E}^{(N,v)}|}, & \text{if } |\mathcal{E}^{(N,v)}| \geq 1, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

We use the convention that an empty sum evaluates to zero. Then, the TU-value φ, defined for all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}^N$, by

$$\varphi_i(N, v) := v(i) + \sum_{S \in \mathcal{E}^{(N,v)}, S \ni i} \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \gamma \text{ for all } i \in N,$$

*satisfies **E** and **CSP** but not **DP**.*

- ***CSP**: The Shapley value Sh satisfy **E** and **DP** but not **CSP**.*

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