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Abstract

The Hicks or direct elasticity of substitution is traditionally derived from the production function. This paper exploits duality theory to present a more general derivation from the input distance function, which is exactly dual to the Shadow Elasticity of Substitution. The new elasticity is more general than the traditional one as it can handle situations of technical inefficiency, non-separability between inputs and outputs, and multiple outputs, but is equal to the traditional elasticity under the classical conditions. The new derivation is related to the Morishima and Antonelli Elasticities of Complementarity in the same way that the Shadow Elasticity of Substitution is related to the Morishima and Allen-Uzawa Elasticities of Substitution. Furthermore, distance (technical efficiency) is not constant for the Morishima and Antonelli Elasticities of Complementarity

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1. Introduction

The elasticity of substitution introduced by Hicks (1932) was reformulated by Lerner (1933) as the degree to which the marginal rate of substitution between two inputs varies as the ratio of the quantity of those inputs varies while output is held constant. Assuming integrability (Hicks and Allen, 1934b), Hicks and Allen (1934a) proposed the following formula:

$$\sigma_{ij} = \left.\frac{\partial \ln(f_i / f_j)}{\partial \ln(x_j / x_i)}\right|_{y} \tag{1}$$

where output, y, which is held constant, is produced according to the production function $y = f(\mathbf{x})$ whose derivatives are indicated by subscripts. $\mathbf{x}$ is the vector of inputs. This is now known as the Hicks or Direct elasticity of substitution, though it is usually defined as the reciprocal of (1).

Traditionally, this elasticity of substitution is derived from the production function, which is natural given (1). This paper exploits duality theory to present an alternative, more general derivation from the input distance function. The new derivation is related to the Morishima Elasticity of Complementarity (Kim, 2000) in the same way that the Shadow Elasticity of Substitution (MacFadden, 1963) is related to the Morishima Elasticity of Substitution (Morishima, 1967; Blackorby and Russell, 1975). I also estimate a system of cost share equations derived from the input distance function using the classic Berndt and Wood (1975) data set and compute the elasticity for each factor combination.

2. Relationships between Elasticities of Substitution and Complementarity

Pigou (1934) pointed out that, in general, unless further restrictions are made, the value of the two factor, two price class (see Mundlak, 1968) of elasticity of substitution:

$$\sigma_{ij} = \frac{\partial \ln(f_i / f_j)}{\partial \ln(x_j / x_i)} \tag{2}$$

or its inverse is not symmetric to which price or quantity in the denominator changes. In the case of the Hicks Elasticity of Substitution (HES), Pigou (1934) suggested holding the output quantity constant, as in (1). Given certain conditions, McFadden (1963) gave a formula for the reciprocal of (1):

$$HES_{ij} = \frac{\frac{1}{f_i x_i} + \frac{1}{f_j x_j}}{-\frac{f_{ii}}{f_i f_i} + 2\frac{f_{ij}}{f_i f_j} - \frac{f_{jj}}{f_j f_j}} \tag{3}$$

Mundlak (1968), Morishima (1967), and Blackorby and Russell (1975) introduced the asymmetric Morishima Elasticity of Substitution:

$$MES_{ij} = \frac{\partial \ln(C_i(y,\mathbf{p}) / C_j(y,\mathbf{p}))}{\partial \ln(p_j / p_i)} \tag{4}$$

where $C(y, p_1, ..., p_m)$ is the cost function in output quantity and input prices whose derivatives are indicated by subscripts. The rationale behind this formula is that according to Shephard's Lemma the derivative of the cost function is equal to the optimal factor input. This elasticity is exactly Robinson's (1933) notion of the elasticity of the change in the input ratio with respect to the price ratio holding output constant and letting all other inputs adjust optimally while holding their prices constant. Equation (2), on the other hand, holds all other inputs constant and lets the shadow prices of the inputs adjust. Blackorby and Russell (1989) show that for a change in p_i (4) can be found as:

$$MES_{ij} = \frac{\partial \ln x_j(y,\mathbf{p})}{\partial \ln p_i} - \frac{\partial \ln x_i(y,\mathbf{p})}{\partial \ln p_i} \tag{5}$$

The two terms on the RHS of (5) are the cross-price and own-price demand elasticities. The MES is, therefore, asymmetric, so that $MES_{ij} \neq MES_{ji}$. Mundlak (1968) indicates that we can make this elasticity symmetric by imposing a condition of constant cost yielding McFadden's (1963) shadow elasticity of substitution:

$$SES_{ij} = \frac{-\dfrac{C_{ii}}{C_i C_i} + 2\dfrac{C_{ij}}{C_i C_j} - \dfrac{C_{jj}}{C_j C_j}}{\dfrac{1}{C_i p_i} + \dfrac{1}{C_j p_j}} \tag{6}$$

Equations (3) and (6) look very similar but there is an asymmetry between them. While the HES holds all other inputs and output constant, the SES holds all other input prices, output, and also cost constant.

Blackorby and Russell (1981) and Kim (2000) proposed the Morishima elasticity of complementarity based on the input distance function as the dual of the MES. The formula is:

$$MEC_{ij} = \frac{\partial \ln\left(D_i(y,\mathbf{x})/D_j(y,\mathbf{x})\right)}{\partial \ln(x_j/x_i)} = \frac{\partial \ln p_i(y,\mathbf{x})}{\partial \ln x_j} - \frac{\partial \ln p_j(y,\mathbf{x})}{\partial \ln x_j} \tag{7}$$

where $D(y,\mathbf{x})$ is the input distance function (Färe and Primont, 1995). The MEC is, therefore, the difference between two elasticities of price with respect to input quantities based on the inverse demand functions derived from the input distance function. Like the MES, the MEC is asymmetric.

The elasticity shows how the price ratio changes as one of the inputs in the input ratio changes under the assumption of cost-minimisation and holding all other inputs and output constant. But it is impossible for only one input to change and for output to be held constant, unless distance is changing. And this is exactly what must be the case for the MEC and the related Antonelli Elasticity of Complementarity (Kim, 2000). In other words, the free disposal assumption is violated.

We can impose the restriction of constant distance on the MEC (7) in the same way that constant cost was imposed on the MES (4) to yield the SES (6). As an input is changed, another input will then be forced to change so that constant distance is maintained. The elasticity will then measure substitution along a curve of constant distance in the same way as the SES measures substitution along an isocost curve. Because output is an argument of the input distance function, output will also be held constant in the proposed elasticity and substitution also occurs along an isoquant. It stands to reason that this restricted elasticity will be symmetric in the same way that the SES is symmetric.

3. New Derivation of the Hicks Elasticity of Substitution

The input distance function for a single output is given by (for more details see Färe and Primont, 1995):

$$D(y, \mathbf{x}) \geq 1 \tag{8}$$

Based on Shephard's Lemma, the derivatives with respect to inputs i and j are given by (Färe and Primont, 1995):

$$\frac{p_i}{C} = D_i \text{ and } \frac{p_j}{C} = D_j \tag{9}$$

where the prices p can be either market or shadow prices. We wish to evaluate the elasticity of complementarity – i.e. where prices respond to changes in inputs as in (7), which is given by:

$$\sigma_{ij} = \frac{d(p_i(y,\mathbf{x}) / p_j(y,\mathbf{x}))}{d(x_j / x_i)} \frac{p_j x_j}{p_i x_i} \tag{10}$$

subject to the quantities of all other inputs and output being held constant:

$$dx_k = dy = 0 \quad \forall k \neq i, j \tag{11}$$

and, additionally, distance is held constant:

$$\sum_{k=1}^{m} \frac{\partial D}{\partial x_k} dx_k = dD = 0 \tag{12}$$

Substituting (9) into (12) conditions (11) and (12) imply that:

$$dx_i = -(p_j / p_i) dx_j \tag{13}$$

We evaluate the two total differentials in (10) by applying the quotient rule and substituting in the expression in (13) in place of dx_i:

$$d(p_i(\mathbf{x},y)/p_j(\mathbf{x},y)) = -\frac{D_{ii}D_j - D_{ji}D_i}{D_jD_j}\frac{p_j}{p_i}dx_j + \frac{D_{ij}D_j - D_{jj}D_i}{D_jD_j}dx_j \tag{14}$$

$$d(x_j/x_i) = \frac{p_jx_j}{x_i^2p_i}dx_j + \frac{1}{x_i}dx_j \tag{15}$$

Then we take the ratio of (14) and (15) and multiply by $\left(p_jx_j\right)/\left(p_ix_i\right)$ to obtain the elasticity:

$$\sigma_{ij} = \frac{-D_{ii}D_jD_j + 2D_{ij}D_iD_j - D_{jj}D_iD_i}{D_ix_i + D_jx_j}\frac{x_jx_i}{D_iD_j} \tag{16}$$

which can be simplified to (17):

$$\sigma_{ij} = \frac{-\frac{D_{ii}}{D_iD_i} + 2\frac{D_{ij}}{D_iD_j} - \frac{D_{jj}}{D_jD_j}}{\frac{1}{D_ix_i} + \frac{1}{D_jx_j}} \tag{17}$$

This expression is identical to the reciprocal of (3) except that each derivative of the production function is replaced with a derivative of the input distance function. It is also clear that the elasticity is symmetric.

In the single output case, when $D(y,\mathbf{x}) = 1$ and inputs and outputs are separable, evaluating (10) subject to (11) should lead to the same result as evaluating (10) subject to (11) and (12). In this classical case, (17) should equal the reciprocal of (3) and be the original concept proposed by Lerner (1933) and Hicks and Allen (1934a). However, (17), which is the exact dual of (6), is a more general indicator as it can also handle situations of technical inefficiency, non-separability between inputs and outputs, and multiple outputs.

4. Numerical Example

This section provides a numerical illustration by estimating a series of cost share equations derived from the input distance function and deriving the elasticities of substitution from the results. Following Kim (2000), I use the classic Berndt and Wood (1975) data set and the translog function to allow comparability with previous studies.

Linear, shares based regression is used, despite the existence of better nonlinear approaches (e.g. Kim, 1992). I estimate the model using the instrumental variables in the Berndt and Wood (1975) dataset. Technical change is modeled using the standard linear trends biased technical change approach despite its potential shortcomings (Lim and Shumway, 1997). Among studies using the same data set, Kim (2000) models technological change using linear time trends, while Berndt and Wood (1975) do not include trends in their cost share equations.

The translog input distance function for a single output, y, in period t, is given by:

$$\begin{aligned}
&\alpha_0 + \alpha_y \ln y_t + \sum_{i=1}^{m} \alpha_i \ln x_{it} + \alpha_\tau \tau_t + \frac{1}{2}\beta_\tau \tau_t^2 + \frac{1}{2}\beta_{yy} \ln y_t^2 + \sum_{i=1}^{m} \beta_{iy} \ln x_{it} \ln y_t \\
&+ \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} \beta_{ij} \ln x_{it} \ln x_{jt} + \beta_{y\tau} \ln y_t \tau_t + \sum_{i=1}^{m} \beta_{i\tau} \ln x_{it} \tau_t \geq 1
\end{aligned} \tag{18}$$

where τ is a linear time trend and the α's and β's are coefficients to be estimated. The input distance function is homogenous of degree one, increasing, and concave with respect to the inputs and decreasing with respect to the output (Kim, 2000). Using (9) and imposing the condition that $D = 1$ (technical efficiency) results in the following cost share equations:

$$S_{it} = \alpha_i + \beta_{iy} \ln y_t + \sum_{j=1}^{m} \beta_{ij} \ln x_{jt} + \beta_{i\tau}\tau \tag{19}$$

where S_{it} is the cost share of input i in period t. I estimate Equation (19) for the capital, labor, and energy cost shares imposing homogeneity of degree zero in the inputs by normalizing on the quantity of materials and symmetry of the coefficients and deducting the means of the explanatory variables. I then compute the elasticities at the mean where all the variables are zero. Under these assumptions, the elasticity of substitution (18) is estimated as:

$$\sigma_{ij} = \frac{\dfrac{\alpha_i - \beta_{ii}}{\alpha_i^2} + 2\dfrac{\beta_{ij}}{\alpha_i \alpha_j} + \dfrac{\alpha_j - \beta_{jj}}{\alpha_j^2}}{\dfrac{1}{\alpha_i} + \dfrac{1}{\alpha_j}} \qquad i \neq j \tag{20}$$

which is reported in Table 1. As this elasticity is the inverse of the traditional one, greater values indicate less substitutability. The poorest substitutes are capital and energy and the best substitutes are labor and materials. Capital has restricted substitutability with all the other inputs.

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Table 1. Hicks Elasticities of Substitution in Elasticity of Complementarity Form

	Capital	Labor	Energy	Materials
Capital				
Labor	1.36565			
Energy	1.47382	0.72817		
Materials	1.16291	0.34248	0.77318	