Granularity Shock: A Small Perturbation Two-Factor Model

Osadchiy, Maksim

31 March 2025

Granularity Shock: A Small Perturbation Two-Factor Model

Maksim Osadchiy[1]

The paper presents a small perturbation two-factor model designed to capture granularity risk, extending the Vasicek Asymptotic Single Risk Factor (ASRF) portfolio loss model. By applying the Lyapunov Central Limit Theorem, we demonstrate that, for small values of the Herfindahl-Hirschman Index (HHI), granularity risk, conditional on market risk, is proportional to a standard normal random variable. Instead of studying the behavior of a heterogeneous portfolio, we examine the behavior of a homogeneous portfolio subjected to a small perturbation induced by granularity risk. We introduce the Vasicek-Herfindahl portfolio loss distribution, which extends the Vasicek portfolio loss distribution for heterogeneous portfolios with low HHI values. Utilizing the Vasicek-Herfindahl distribution, we derive closed-form granularity adjustments for the probability density function and cumulative distribution function of portfolio loss, as well as for Value at Risk (VaR) and Expected Shortfall (ES). We compare the primary results of our approach with established findings and validate them through Monte Carlo simulations.

Keywords
Credit portfolio model; Granularity adjustment; Value at Risk; Expected Shortfall

1 Introduction

The Vasicek model (Vasicek O. , 1987), which is based on the Law of Large Numbers (LLN), assumes perfect granularity by considering a homogeneous portfolio with equal weights for all exposures. However, the LLN is not suitable in the case of a heterogeneous portfolio with varying weights. In this case, an additional granularity risk arises, which, due to linearity of expectation, is represented only in the residual $L - \mathbb{E}[L|Y]$, where L denotes the portfolio loss and Y represents the market shock. Gordy (2003) demonstrated that, under mild regularity conditions, as the number of positions in the heterogeneous portfolio increases, the portfolio loss converges almost surely to its conditional expectation given the common factor. Additionally, Gordy (2003) emphasized the importance of the Herfindahl-Hirschman Index (HHI) for studying granularity adjustment (GA).

The foundation for further study of GA to VaR was provided by (Gouriéroux, Laurent, & Scailett, 2000), who calculated the first and second derivatives of VaR.

(Emmer & Tasche, 2005) obtained GA to VaR for both the general case of loss distribution and for the case of the Vasicek model. The formula of GA to VaR for the case of the Vasicek model was refined by (Gordy & Lutkebohmert, 2013).

(Voropaev, 2011) then moved on to studying the behavior of the portfolio loss PDF and GA to VaR and ES using a much simpler methodology based on the method of moments.

However, despite considerable efforts, it has not yet been possible to obtain GA to VaR suitable for supervisory purposes. Currently, there is a “granularity gap” in the regulation of credit risk. On one hand, a primitive archaic approach is used that considers the sizes of loans within the

[1] Independent analyst. E-mail: maxos1us@gmail.com.

portfolio but ignores the correlations between these assets. On the other hand, the more advanced Internal Ratings-Based (IRB) approach accounts for correlations but neglects the varying sizes of loans in the portfolio. This paper aims to fill this gap concerning small values of HHI. The Vasicek-Herfindahl portfolio loss distribution introduced in our paper can be used to regulate a bank's economic capital.

This paper is organized as follows: In Section 2, we introduce the main subject of our study and provide essential information about the Vasicek model. In Section 3, we examine the behavior of the portfolio loss random variable when the HHI is close to zero. In Section 4, we present the Vasicek-Herfindahl portfolio loss distribution and investigate its properties. In Section 5, we calculate VaR and GA to VaR using our methodology and compare it with the results of (Emmer & Tasche, 2005). In Section 6, we calculate ES and GA to ES using the framework of our approach. In Section 7, we discuss the Vasicek's attempt at assessing the granularity effect. In Section 8, we review the approach of (Voropaev, 2011). In Section 9, we analyze the method proposed by (Emmer & Tasche, 2005). Finally, in Section 10, we summarize our findings.

2 A Model Framework

Consider a portfolio consisting of n loans. The weight of loan k is denoted as w_k, such that:

$$\sum_{k=1}^{n} w_k = 1 \tag{2.1}$$

where $w_k \geq 0$ for each k.

Let the portfolio loss be defined as:

$$Loss(\{X_k\}_{k=1}^{n}, Y) = \sum_{k=1}^{n} w_k \, l(X_k, Y) \tag{2.2}$$

where Y is a standard normal random variable representing market (systematic) risk, and the standard normal random variables X_k represent individual (specific, idiosyncratic) risks. The random variables $\{X_k\}_{k=1}^{n}$ and Y are independent and identically distributed (i.i.d.). The term $l(X_k, Y)$ denotes the loss associated with loan k,

$$\mathbb{E}[l(X_k, Y)] = PD \tag{2.3}$$

$$var[l(X_k, Y)] = PD(1 - PD) \tag{2.4}$$

for each k, where PD is the probability of default.

Assume that $l(X_k, y)$ is equal to 1 with probability $p(y)$ (the default) and 0 otherwise, where y is a realization of the random value Y. The random variable $l(X_k, Y)$, conditional on the market shock Y, follows a Bernoulli distribution:

$$l|Y \sim Bernoulli(p(Y)) \tag{2.5}$$

The conditional mean is given by:

$$\mathbb{E}[l|Y] = p(Y)$$ (2.6)

The conditional variance is expressed as:

$$\sigma^2(p(Y)) = p(Y)(1 - p(Y))$$ (2.7)

In the model by (Vasicek O. A., 2002) the conditional probability of loss for any loan is given by:

$$p(Y) = \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho}Y}{\sqrt{1-\rho}}\right)$$ (2.8)

where $\rho \in [0,1]$ is the asset correlation, and Φ denotes the standard normal CDF.

The conditional expectation of the portfolio loss given Y is expressed as:

$$p(Y) = \mathbb{E}[Loss|Y]$$ (2.9)

The Vasicek CDF is given by:

$$F_V(x; PD, \rho) = \Phi\left(-p^{-1}(x)\right) = \Phi\left(\frac{\sqrt{1-\rho}\Phi^{-1}(x) - \Phi^{-1}(PD)}{\sqrt{\rho}}\right)$$ (2.10)

If the weights w_i are the same, then according to the LLN, the random value $Loss|Y$ converges in probability to the conditional mean $p(Y)$:

$$Loss|Y \xrightarrow{P} p(Y)$$ (2.11)

as $n \to \infty$ (Vasicek O. A., 2002).

3 Lyapunov CLT

Suppose ξ_i is a sequence of independent random variables, each with a finite expected value μ_i and variance σ_i. Define

$$s_n{}^2 = \sum_{i=1}^{n} \sigma_i{}^2$$ (3.1)

If Lyapunov's condition

$$\lim_{n\to\infty} \frac{1}{s_n{}^{2+\delta}} \sum_{i=1}^{n} \mathbb{E}\left(|\xi_i - \mu_i|^{2+\delta}\right) = 0$$ (3.2)

is satisfied for some $\delta > 0$, then the sum

$$\frac{1}{s_n}\sum_{i=1}^{n}(\xi_i-\mu_i)\xrightarrow{d}\mathcal{N}(0,1)$$

(3.3)

converges in distribution to a standard normal variable as $n\to\infty$. Let us apply the Lyapunov CLT to our problem. We have:

$$\xi_i=w_i l(X_i,y)$$

$$\mu_i=w_i p(y)$$

$$s_n=\sqrt{h_n}\sigma(p(y))$$

where

$$h_n=\sum_{k=1}^{n}{w_k}^2$$

is the Herfindahl-Hirschman Index.
In the new variables, formula (3.3) is transformed into the following form:

$$\frac{\sum_{i=1}^{n}w_i\left(l(X_i,y)-p(y)\right)}{\sqrt{h_n}\sigma(p(y))}\xrightarrow{d}\mathcal{N}(0,1)$$

(3.4)

Taking into account equation (2.2), this formula can be expressed as:

$$\frac{Loss(\{X_k\}_{k=1}^{n},y)-p(y)}{\sqrt{h_n}\sigma(p(y))}\xrightarrow{d}\mathcal{N}(0,1)$$

(3.5)

Thus, the portfolio loss converges in distribution to the asymptotic loss L:

$$Loss(\{X_k\}_{k=1}^{n},Y)\xrightarrow{d}L(Z,Y)=p(Y)+\sqrt{h}\sigma(p(Y))Z$$

(3.6)

where $Y,Z\sim\mathcal{N}(0,1)$ are i.i.d. random variables, and

$$h=\lim_{n\to\infty}h_n$$

The heterogeneous portfolio loss risk encompasses not only market risk but also the granularity risk, which is represented by the term $\sqrt{h}\sigma(p(Y))Z$. It is important to note that granularity risk is influenced by market risk.

If $h=0$ (perfect granularity), then $L(Z,Y)=p(Y)$. Conversely, the case where $h=1$ indicates full concentration, occurring when the weight of one of the loans is 1 and the weights of all the others are 0.

The range of the function $Loss(\{X_k\}_{k=1}^{n},Y)$ is the unit interval [0,1], while the range of the function $L(Z,Y)$ is $\mathbb{R}$. However, when $h=0$, the range of the function $L(Z,Y)$ is narrowed to the unit interval [0,1].

3.1 Applicability of the Lyapunov CLT

The applicability of our approach is constrained by the limits of the Lyapunov CLT. Let $\delta = 1$. We aim to prove that:

$$\lim_{n\to\infty} \frac{1}{{s_n}^3} \sum_{i=1}^{n} \mathbb{E}[|\xi_i - \mu_i|^3] = 0 \tag{3.7}$$

is equivalent to:

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} {w_i}^3}{\left(\sum_{j=1}^{n} {w_j}^2\right)^{3/2}} = 0 \tag{3.8}$$

Proof.

We start with the expression:

$$\mathbb{E}[|\xi_i - \mu_i|^3] = {w_i}^3 \mathbb{E}[|l(X_k, y) - p(y)|^3] = {w_i}^3 p(y)(1 - p(y))\left((1 - p(y))^2 + p^2(y)\right) \tag{3.9}$$

Thus, we have:

$$\lim_{n\to\infty} \frac{1}{{s_n}^3} \sum_{i=1}^{n} \mathbb{E}\,[|\xi_i - \mu_i|^3] = \frac{\mathbb{E}[|l(X_k, y) - p(y)|^3]}{\sigma^3(p(y))} \lim_{n\to\infty} \frac{\sum_{i=1}^{n} {w_i}^3}{(\sum_{i=1}^{n} {w_i}^2)^{3/2}}$$
$$= \frac{(1 - p(y))^2 + p^2(y)}{\sqrt{p(y)\left(1 - p(y)\right)}} \lim_{n\to\infty} \frac{\sum_{i=1}^{n} {w_i}^3}{\left(\sum_{j=1}^{n} {w_j}^2\right)^{3/2}} \tag{3.10}$$

Q.E.D.

To simulate the random variable $Loss$, it is necessary to use a set $w_k \geq 0$, $k = 1, \dots, n$, such that:

$$\sum_{k=1}^{n} w_k = 1$$
$$\sum_{k=1}^{n} {w_k}^2 \ll 1$$
$$\frac{\sum_{k=1}^{n} {w_k}^3}{\left(\sum_{j=1}^{n} {w_j}^2\right)^{3/2}} \ll 1 \tag{3.11}$$

We use the geometric progression defined as follows:

$$w_k = \frac{s^{k-1}}{\sum_{j=1}^{\infty} s^j} = (1 - s)s^{k-2} \tag{3.12}$$

where $0 < s < 1$. Let

$$h = \sum_{k=1}^{\infty} {w_k}^2 = \frac{1-s}{1+s} \Rightarrow s = \frac{1-h}{1+h} \tag{3.13}$$

Now we can evaluate:

$$\frac{\sum_{k=1}^{\infty} {w_k}^3}{\left(\sum_{j=1}^{\infty} {w_j}^2\right)^{3/2}} = \frac{\sum_{k=1}^{\infty} s^{3k}}{\left(\sum_{j=1}^{\infty} s^{2j}\right)^{3/2}} = \frac{(1-s^2)^{3/2}}{1-s^3} \tag{3.14}$$

If s is chosen in the left neighborhood of 1, then $\frac{\sum_{k=1}^{\infty} {w_k}^3}{\left(\sum_{j=1}^{\infty} {w_j}^2\right)^{3/2}}$ is close to 0.

4 Vasicek-Herfindahl Distribution

Let random variable

$$V = p(Y) \Rightarrow V \sim Vasicek(PD, \rho) \tag{4.1}$$

where $Vasicek(PD, \rho)$ represents the Vasicek loan loss distribution. Let

$$H = Z\sigma(V)\sqrt{h} \tag{4.2}$$

where

$$\sigma(x) = \sqrt{x(1-x)} \tag{4.3}$$

The asymptotic portfolio loss risk L is the sum of the "classical" Vasicek portfolio loss risk V and the granularity risk H:

$$L = V + H = V + Z\sigma(V)\sqrt{h} \tag{4.4}$$

$h\sigma^2(V)$ represents the conditional variance of portfolio loss:

$$h\sigma^2(V) = var[L|V] \tag{4.5}$$

If $h \ll 1$, then the granularity risk, $H = Z\sigma(V)\sqrt{h}$, is considered a small perturbation. This is GA to the portfolio loss risk:

$$GA^L = Z\sigma(V)\sqrt{h} \tag{4.6}$$

Note that

$$H = L - \mathbb{E}[L|Y] \tag{4.7}$$

Equation (4.4) illustrates that it is incorrect to assert that $\mathbb{E}[L|Y]$ represents systematic risk while $L - \mathbb{E}[L|Y]$ denotes specific risk. This is because the residual $L - \mathbb{E}[L|Y] = Z\sigma(p(Y))\sqrt{h}$ is influenced by the market risk Y.

The PDF of the portfolio loss L is given by:

$$f_L(x) = f_{V+H}(x) = \int_0^1 f_{H|V}(x-v|v) f_V(v)dv = \int_0^1 f_{H|V}(x-v|v) dF_V(v)$$
$$= \int_0^1 \frac{\varphi\left(\frac{x-v}{\sqrt{h}\sigma(v)}\right)}{\sqrt{h}\sigma(v)} dF_V(v)$$

(4.8)

where $\varphi(x)$ is the standard normal PDF. Similarly, the CDF of the portfolio loss L is:

$$F_L(x) = F_{V+H}(x) = \mathbb{P}[V+H<x] = \int_0^1 \mathbb{P}[H<x-v|V=v] f_V(v)dv$$
$$= \int_0^1 F_{H|V}(x-v|v) f_V(v)dv = \int_0^1 \Phi\left(\frac{x-v}{\sqrt{h}\sigma(v)}\right) dF_V(v)$$

(4.9)

The PDF of the random variable H, conditional on the random variable V, is given by:

$$f_{H|V}(x|v) = \frac{\varphi\left(\frac{x}{\sqrt{h}\sigma(v)}\right)}{\sqrt{h}\sigma(v)}$$

(4.10)

The CDF of the random variable H, conditional on the random variable V, is given by:

$$F_{H|V}(x|v) = \Phi\left(\frac{x}{\sqrt{h}\sigma(v)}\right)$$

(4.11)

Now we introduce the Vasicek-Herfindahl PDF of the portfolio loss:

$$f_{VH}(x; PD, \rho, h) = \int_0^1 \frac{\varphi\left(\frac{x-v}{\sqrt{h}\sigma(v)}\right)}{\sqrt{h}\sigma(v)} dF_V(v; PD, \rho)$$

(4.12)

and the Vasicek-Herfindahl CDF:

$$F_{VH}(x; PD, \rho, h) = \int_0^1 \Phi\left(\frac{x-v}{\sqrt{h}\sigma(v)}\right) dF_V(v; PD, \rho)$$

(4.13)

Using integration by parts, the function can be transformed into the following form:

$$F_{VH}(x; PD, \rho, h) = \int_0^1 F_V(v; PD, \rho) d\Phi\left(\frac{v - x}{\sqrt{h}\sigma(v)}\right)$$

(4.14)

4.1 Additivity of Granular Shocks

Suppose that the "large" portfolio consists of N "small" portfolios. The weight of the "small" portfolio i is denoted as u_i, and its HHI is represented by h_i. The loss of the "small" portfolio i is given by:

$$L_i = V + \sigma(V) Z_i \sqrt{h_i}$$

where $\{Z_k\}_{k=1}^{n}$ are i.i.d. random variables, and $Z_i \sim N(0,1)$. The loss of the "large" portfolio can be calculated as:

$$L = \sum_{i=1}^{N} u_i L_i = V + \sigma(V) \sum_{i=1}^{N} Z_i u_i \sqrt{h_i} = V + \sigma(V) Z \sqrt{\tilde{h}}$$

where

$$\tilde{h} = \sum_{i=1}^{N} {u_i}^2 h_i$$

4.2 Model Validation Using Monte Carlo Simulation

The simulated CDF is given by:

$$F_s(x) = \sum_{k=1}^{N} I(Loss_k \le x)$$

(4.15)

where $I(x)$ is the indicator function, $Loss_k$ is the k^{th} Monte Carlo simulation of the random variable $Loss$, and N is the total number of simulations.

The simulated PDF is:

$$f_s(x_j) = \frac{1}{N} \sum_{k=1}^{N} I(x_j \le Loss_k < x_{j+1})$$

(4.16)

where

$$x_j = x_0 + j\Delta x$$

(4.17)

and Δx is the bin width.

The theoretical Vasicek-Herfindahl ΔCDF is given by:

$$\Delta F_{VH}(x; PD, \rho, h) = F_{VH}(x; PD, \rho, h) - F_{VH}(x; PD, \rho, 0)$$

(4.18)

The theoretical Vasicek-Herfindahl ΔPDF is defined as:

$$\Delta f_{VH}(x; PD, \rho, h) = f_{VH}(x; PD, \rho, h) - f_{VH}(x; PD, \rho, 0) \tag{4.19}$$

The simulated ΔCDF is expressed as:

$$\Delta F_s(x; h) = F_s(x; h) - F_s(x; 0) \tag{4.20}$$

The simulated ΔPDF is represented by:

$$\Delta f_s(x; h) = f_s(x; h) - f_s(x; 0) \tag{4.21}$$

The difference between the theoretical Vasicek-Herfindahl ΔCDF with the simulated ΔCDF is given by the equation:

$$\Delta\Delta F(x) = \Delta F_{VH}(x) - \Delta F_s(x) \tag{4.22}$$

Similarly, the difference between the theoretical Vasicek-Herfindahl ΔPDF and the simulated ΔPDF is represented as:

$$\Delta\Delta f(x) = \Delta f_{VH}(x) - \Delta f_s(x) \tag{4.23}$$

Figure 1 (with $h = 0.01$) and Figure 2 (with $h = 0.1$) illustrate the differences between the theoretical Vasicek-Herfindahl ΔCDF and ΔPDF and their corresponding simulated functions. Both figures demonstrate a decline in model quality as HHI values increase. Nevertheless, even with the relatively large value of $h = 0.1$, the model still accurately represents the shapes of both the PDF and CDF.

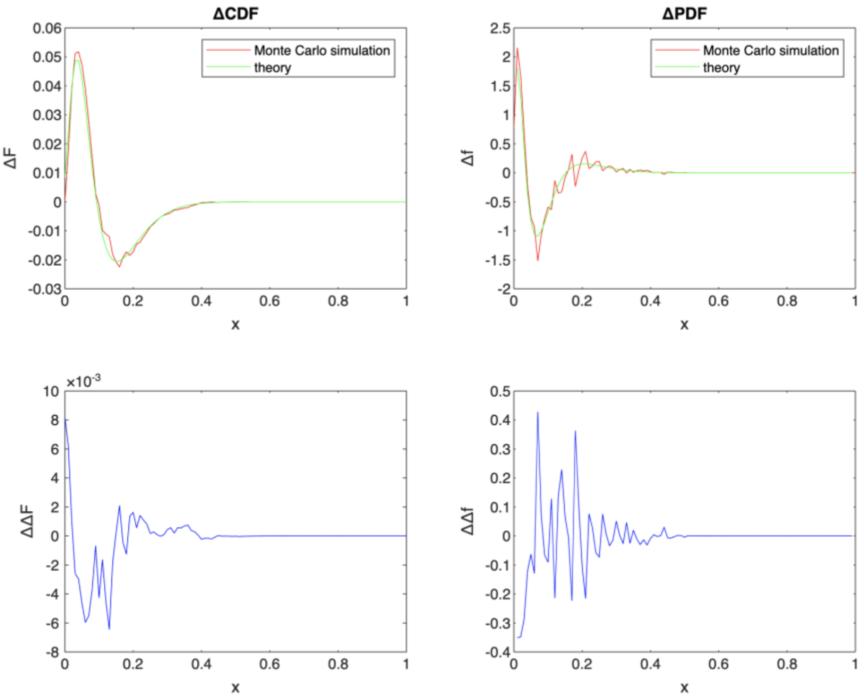

Figure 1. Theoretical Vasicek-Herfindahl ΔCDF and ΔPDF vs simulated functions (top row). Below the plots of the functions are the corresponding plots of the differences between theoretical and simulated functions. Number of Monte Carlo simulations: 20 000. Parameters used: PD=0.1, ρ=0.1, h=0.01, n=20 000.

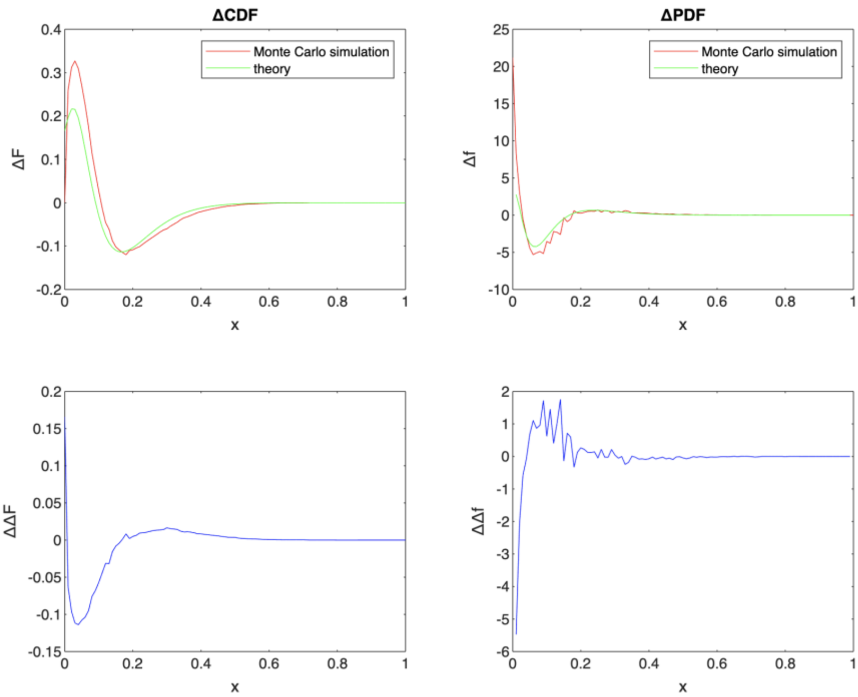

Figure 2. Theoretical Vasicek-Herfindahl ΔCDF and ΔPDF vs simulated functions (top row). Below the plots of the functions are the corresponding plots of the differences between theoretical and simulated functions. Number of Monte Carlo simulations: 20 000. Parameters used: PD=0.1, ρ=0.1, h=0.1, n=20 000.

4.3 Properties of the Vasicek-Herfindahl Distribution

4.3.1 Normalization Property of the PDF

The total area under the PDF curve is equal to 1:

$$\int_{-\infty}^{\infty} f_{VH}(x; PD, \rho, h)dx = 1$$

(4.24)

Proof.

$$\int_{-\infty}^{\infty} f_{VH}(x; PD, \rho, h)dx = \int_{0}^{1} \frac{dF_V(v; PD, \rho)}{\sqrt{h}\sigma(v)} \int_{-\infty}^{\infty} \varphi\left(\frac{x - v}{\sqrt{h}\sigma(v)}\right) dv = \int_{0}^{1} dF_V(v; PD, \rho)$$
$$= F_V(v; PD, \rho)|_0^1 = 1$$

(4.25)

Q.E.D.

4.3.2 Expected Loss

The unconditional mean of the asymptotic loss L is equal to the unconditional probability of default:

$$\mathbb{E}[L] = PD$$

(4.26)

Proof.

$$\mathbb{E}[L] = \mathbb{E}\left[V + Z\sqrt{V(1-V)h}\right] = \mathbb{E}[V] + \sqrt{h}\mathbb{E}[Z]\mathbb{E}\left[\sqrt{V(1-V)}\right] = \mathbb{E}[V] = PD \tag{4.27}$$

Q.E.D.

4.3.3 Variance

The unconditional variance of the asymptotic loss L is a linear function of h:

$$var[L] = PD(1-PD)h + (1-h)var[p(Y)] \tag{4.28}$$

where

$$var[p(Y)] = var\big[\mathbb{E}[L|Y]\big] = \Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); \rho) - PD^2 \tag{4.29}$$

Proof.

$$\begin{aligned}\mathbb{E}[L^2] = \mathbb{E}\left[p^2(Y) + 2p(Y)Z\sqrt{p(Y)\big(1-p(Y)\big)h} + Z^2 p(Y)\big(1-p(Y)\big)h\right] \\ = \mathbb{E}[p^2(Y)] + 2\sqrt{h}\mathbb{E}[Z]\mathbb{E}\left[p(Y)\sqrt{p(Y)\big(1-p(Y)\big)}\right] \\ + h\mathbb{E}[Z^2]\mathbb{E}\big[p(Y)\big(1-p(Y)\big)\big] = \mathbb{E}[p^2(Y)] + h\mathbb{E}\big[p(Y)\big(1-p(Y)\big)\big] \\ = (1-h)\mathbb{E}[p^2(Y)] + h\mathbb{E}[p(Y)] = (1-h)\Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); \rho) + hPD\end{aligned} \tag{4.30}$$

$$\begin{aligned}var[L] = \mathbb{E}[L^2] - \mathbb{E}^2[L] = (1-h)\Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); \rho) + hPD - PD^2 = \\ = (1-h)(\Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); \rho) - PD^2) + (1-h)PD^2 + hPD - PD^2 \\ = (1-h)var[p(Y)] + PD(1-PD)h\end{aligned} \tag{4.31}$$

Q.E.D.

It is straightforward to verify that the Law of Total Variance:

$$var[L] = \mathbb{E}\big[var[L|Y]\big] + var\big[\mathbb{E}[L|Y]\big] \tag{4.32}$$

is valid. Indeed,

$$\mathbb{E}[L|Y] = p(Y) \tag{4.33}$$

$$var\big[\mathbb{E}[L|Y]\big] = var[p(Y)] = \mathbb{E}[p^2(Y)] - (\mathbb{E}[p(Y)])^2 = \mathbb{E}[p^2(Y)] - PD^2 \tag{4.34}$$

$$var[L|Y] = h\sigma^2(Y) = hp(Y)\big(1-p(Y)\big) \tag{4.35}$$

$$\mathbb{E}\big[var[L|Y]\big] = h\mathbb{E}[p(Y)] - h\mathbb{E}[p^2(Y)] = hPD - h\mathbb{E}[p^2(Y)] \tag{4.36}$$

$$\begin{aligned}\mathbb{E}\big[var[L|Y]\big] + var\big[\mathbb{E}[L|Y]\big] = hPD - h\mathbb{E}[p^2(Y)] + \mathbb{E}[p^2(Y)] - PD^2 \\ = (1-h)var[p(Y)] + (1-h)PD^2 + hPD - PD^2 \\ = (1-h)var[p(Y)] + hPD(1-PD) = var[L]\end{aligned}$$

(4.37)

4.3.4 Symmetry Property

The distribution exhibits a symmetry property:

$$F_{VH}(x;PD,\rho,h) = 1 - F_{VH}(1-x;1-PD,\rho,h)$$

(4.38)

This is similar to the symmetry property presented by (Vasicek O. A., 2002) (p.4):

$$F_V(x;PD,\rho) = 1 - F_V(1-x;1-PD,\rho)$$

(4.39)

Proof.

$$\begin{aligned}
F_{VH}(x;PD,\rho,h) &= \int_0^1 \Phi\left(\frac{x-v}{\sqrt{h}\sigma(v)}\right) dF_V(v;PD,\rho) \\
&= -\int_0^1 \Phi\left(\frac{x-v}{\sqrt{v(1-v)h}}\right) dF_V(1-v;1-PD,\rho) \\
&= \int_0^1 \Phi\left(\frac{x-(1-u)}{\sqrt{(1-u)uh}}\right) dF_V(u;1-PD,\rho) = \\
&= 1 - \int_0^1 \Phi\left(\frac{(1-x)-u}{\sqrt{h}\sigma(u)}\right) dF_V(u;1-PD,\rho) = 1 - F_{VH}(1-x;1-PD,\rho)
\end{aligned}$$

(4.40)

Q.E.D.

4.4 Taylor Series of CDF and PDF

Let $z = h\sigma^2(v)$. Taking into account that $0 < z \ll 1$, we will expand $\Phi\left(\frac{u}{\sqrt{z}}\right)$ into a Taylor series at $z = 0$:

$$\Phi\left(\frac{u}{\sqrt{z}}\right) = \theta(u) + \sum_{k=1}^{\infty} \left(\frac{z}{2}\right)^k \frac{\delta^{(2k-1)}(u)}{k!} = \theta(u) + \frac{z}{2}\delta'(u) + o(z)$$

(4.41)

(see proof in Appendix). Hence, the Vasicek-Herfindahl CDF is given by:

$$\begin{aligned}
F_{VH}(x;PD,\rho,h) &= \int_0^1 \Phi\left(\frac{x-v}{\sqrt{h}\sigma(v)}\right) dF_V(v;PD,\rho) \\
&= F_V(x;PD,\rho) \\
&\quad - \sum_{k=1}^{\infty} \frac{(h/2)^k}{k!} \int_0^1 \delta^{(2k-1)}(x-v)\sigma^{2k}(v) f_V(v;PD,\rho) dv = F_V(x;PD,\rho) \\
&\quad + \sum_{k=1}^{\infty} \frac{(h/2)^k}{k!} \frac{\partial^{2k-1}}{\partial x^{2k-1}} \left(\sigma^{2k}(x) f_V(x;PD,\rho)\right) \\
&= F_V(x;PD,\rho) + \frac{h}{2}\frac{\partial}{\partial x}\left(\sigma^2(x) f_V(x;PD,\rho)\right) + o(h)
\end{aligned}$$

(4.42)

We used the Delta function property:

$$\int_{-\infty}^{+\infty} \delta^{(k)}(x) f(x) dx = (-1)^k f^{(k)}(0)$$

(4.43)

The Vasicek-Herfindahl PDF is given by:

$$f_{VH}(x; PD, \rho, h) = \frac{\partial}{\partial x} F_{VH}(x; PD, \rho, h) = \sum\nolimits_{k=0}^{\infty} \frac{(h/2)^k}{k!} \frac{\partial^{2k}}{\partial x^{2k}} \left(\sigma^{2k}(x) f_V(x; PD, \rho)\right)$$
$$= f_V(x; PD, \rho) + \frac{h}{2} \frac{\partial^2}{\partial x^2} \left(\sigma^2(x) f_V(x; PD, \rho)\right) + o(h)$$

(4.44)

Hence, the GA to CDF is given by:

$$GA^{CDF} = \frac{h}{2} \frac{\partial}{\partial x} \left(\sigma^2(x) f_V(x; PD, \rho)\right)$$

(4.45)

and the GA to PDF is expressed as:

$$GA^{PDF} = \frac{h}{2} \frac{\partial^2}{\partial x^2} \left(\sigma^2(x) f_V(x; PD, \rho)\right)$$

(4.46)

5 Value at Risk

The Value at Risk $VaR_\alpha(L(h)) = x(\alpha, h)$ is the root of the integral equation

$$1 - \alpha = \int_0^1 \Phi\left(\frac{x(\alpha, h) - v}{\sqrt{h}\sigma(v)}\right) dF_V(v; PD, \rho)$$

(5.1)

where α is a confidence level. The numerical value of this root can be easily determined using known values of the parameters PD, ρ, h and α, for example, by utilizing MATLAB.

$x(\alpha, h)$ represents the α-quantile of $L(h)$:

$$x(\alpha, h) = q_\alpha(L(h))$$

where, for any random variable X,

$$q_\alpha(X) = inf\{x \in \mathbb{R}: \mathbb{P}[X \leq x] \geq \alpha\}$$

5.1 VaR Approximation

Since a closed-form solution for $x(\alpha, h)$ of the integral equation (5.1) is not available, we will consider expanding $x(\alpha, h)$ into a Taylor series at $h = 0$.

5.1.1 First-Order Derivative of VaR with respect to h

Differentiating the integral equation (5.1) with respect to h yields:

$$0 = \frac{\partial x(\alpha,h)}{\partial h}\int_0^1 \frac{1}{\sqrt{h}\sigma(v)}\varphi\left(\frac{v-x(\alpha,h)}{\sqrt{h}\sigma(v)}\right)dF_V(v;PD,\rho) + \frac{1}{2}\int_0^1 \frac{v-x(\alpha,h)}{\left(\sqrt{h}\sigma(v)\right)^3}\varphi\left(\frac{v-x(\alpha,h)}{\sqrt{h}\sigma(v)}\right)\sigma^2(v)dF_V(v;PD,\rho) \tag{5.2}$$

From equations (4.12) and (5.2), it follows that

$$\frac{\partial x(\alpha,h)}{\partial h} = -\frac{1}{2f_{VH}(x(\alpha,h);PD,\rho,h)}\int_0^1 \frac{v-x(\alpha,h)}{\left(\sqrt{h}\sigma(v)\right)^3}\varphi\left(\frac{v-x(\alpha,h)}{\sqrt{h}\sigma(v)}\right)\sigma^2(v)dF_V(v;PD,\rho) \tag{5.3}$$

It follows from equations (4.4) and (5.3) that

$$\frac{\partial x(\alpha,h)}{\partial \varepsilon} = \frac{1}{\varepsilon}\mathbb{E}[H|L = x(\alpha,h)] \tag{5.4}$$

where $\varepsilon = \sqrt{h}$. This result is consistent with Lemma 1 (Gouriéroux, Laurent, & Scailett, 2000), with notation differences.

Let $z = h\sigma^2(v)$. We expand $-\frac{u\varphi(u/\sqrt{z})}{(\sqrt{z})^3}$ into a Taylor series at $z = 0$:

$$-\frac{u\varphi\left(u/\sqrt{z}\right)}{z^{3/2}} = \sum_{k=0}^{\infty}\frac{\delta^{(2k+1)}(u)}{k!}(z/2)^k = \delta'(u) + \frac{z}{2}\delta'''(u) + o(z) \tag{5.5}$$

It follows from equations (5.3), (5.5), as well as the property of the Delta function

$$\int_{-\infty}^{+\infty}\delta^{(k)}(x)f(x)dx = (-1)^k f^{(k)}(0) \tag{5.6}$$

that:

$$\begin{aligned} -\int_0^1 \frac{v-x(\alpha,h)}{\left(\sqrt{h}\sigma(v)\right)^3}\varphi\left(\frac{v-x(\alpha,h)}{\sqrt{h}\sigma(v)}\right)\sigma^2(v)dF_V(v;PD,\rho) \\ = \sum_{k=0}^{\infty}\frac{(h/2)^k}{k!}\int_0^1 \delta^{(2k+1)}\left(v-x(\alpha,h)\right)\sigma^{2(k+1)}(v)dF_V(v;PD,\rho) \\ = -\sum_{k=0}^{\infty}\frac{(h/2)^k}{k!}\frac{\partial^{2k+1}}{\partial x^{2k+1}}\left(\sigma^{2(k+1)}(x)f_V(x;PD,\rho)\right)\Big|_{x=x(\alpha,h)} \end{aligned} \tag{5.7}$$

It follows from equations (4.44) and (5.7) that

$$\frac{\partial x(\alpha,h)}{\partial h} = -\frac{\sum_{k=0}^{\infty}\frac{(h/2)^k}{k!}\frac{\partial^{2k+1}}{\partial x^{2k+1}}\left(\sigma^{2(k+1)}(x) f_V(x;PD,\rho)\right)}{2\sum_{k=0}^{\infty}\frac{(h/2)^k}{k!}\frac{\partial^{2k}}{\partial x^{2k}}\left(\sigma^{2k}(x) f_V(x;PD,\rho)\right)}\Bigg|_{x=x(\alpha,h)} \tag{5.8}$$

which leads to

$$\left.\frac{\partial x(\alpha,h)}{\partial h}\right|_{h=0} = -\frac{\frac{\partial}{\partial x}\left(\sigma^2(x) f_V(x;PD,\rho)\right)}{2 f_V(x;PD,\rho)}\Bigg|_{x=x(\alpha)} \tag{5.9}$$

where the α-quantile of $L(h=0)$

$$x(\alpha) = q_\alpha(L(h=0)) = VaR_\alpha\big(L(h=0)\big) = \Phi\left(\frac{\sqrt{\rho}\Phi^{-1}(1-\alpha) + \Phi^{-1}(PD)}{\sqrt{1-\rho}}\right) \tag{5.10}$$

is the root of the equation

$$1-\alpha = F_V(x(\alpha);PD,\rho) \tag{5.11}$$

It follows from equations (5.9) and $\sigma^2(x) = x(1-x)$ that:

$$\left.\frac{\partial x(\alpha,h)}{\partial h}\right|_{h=0} = -\frac{1}{2}\left(1-2x+x(1-x)\frac{\partial}{\partial x} ln\big(f_V(x;PD,\rho)\big)\right|_{x=x(\alpha)} \tag{5.12}$$

Since the Vasicek PDF is given by:

$$f_V(x;PD,\rho) = \sqrt{\frac{1-\rho}{\rho}} exp\left\{-\frac{1}{2\rho}\left(\sqrt{1-\rho}\Phi^{-1}(x) - \Phi^{-1}(PD)\right)^2 + \frac{1}{2}\left(\Phi^{-1}(x)\right)^2\right\} \tag{5.13}$$

it follows that:

$$\frac{\partial ln f_V(x;PD,\rho)}{\partial x} = \frac{(2\rho-1)\Phi^{-1}(x) + \sqrt{1-\rho}\Phi^{-1}(PD)}{\rho\varphi\big(\Phi^{-1}(x)\big)} \tag{5.14}$$

From equations (2.10), (5.11) and (5.14), we have:

$$\frac{\partial ln f_V(x;PD,\rho)}{\partial x} = \frac{\Phi^{-1}\big(x(\alpha)\big) - \sqrt{\frac{1-\rho}{\rho}}\Phi^{-1}(1-\alpha)}{\varphi\big(\Phi^{-1}(x)\big)} \tag{5.15}$$

It follows from equations (5.12) and (5.15) that:

$$\left.\frac{\partial x(\alpha,h)}{\partial h}\right|_{h=0} = -\frac{1}{2}\left(1-2x(\alpha)+\frac{x(\alpha)\big(1-x(\alpha)\big)}{\varphi\left(\Phi^{-1}\big(x(\alpha)\big)\right)}\left(\Phi^{-1}\big(x(\alpha)\big)-\sqrt{\frac{1-\rho}{\rho}}\Phi^{-1}(1-\alpha)\right)\right) \tag{5.16}$$

Hence, the GA to VaR is:

$$GA^{VaR} = -\frac{h}{2}\left(1-2x(\alpha)+\frac{x(\alpha)\big(1-x(\alpha)\big)}{\varphi\left(\Phi^{-1}\big(x(\alpha)\big)\right)}\left(\Phi^{-1}\big(x(\alpha)\big)-\sqrt{\frac{1-\rho}{\rho}}\Phi^{-1}(1-\alpha)\right)\right) \tag{5.17}$$

Based on the approach outlined by (Gouriéroux, Laurent, & Scailett, 2000), (Emmer & Tasche, 2005) (Remark 2.3) with a correction (Gordy & Marrone, 2012) (minus before $\sqrt{\frac{1-\rho}{\rho}}$ instead of plus) derived the same formula with precision up to notation ($q_{1-\alpha}(X) = -\Phi^{-1}(1-\alpha)$, $\Phi\left(\frac{c-\sqrt{\rho}q_{1-\alpha}(X)}{\sqrt{1-\rho}}\right) = x(\alpha)$).

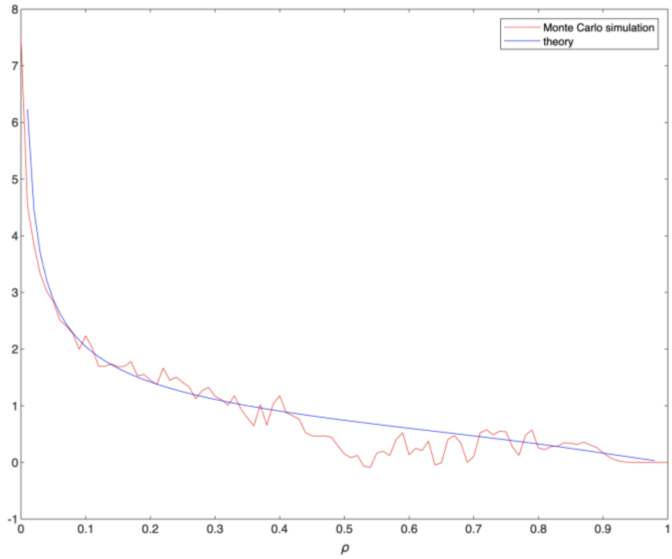

Figure 3. Comparison of the dependence on ρ of the simulated function $\frac{x(\alpha,h)-x(\alpha)}{h}$ (red line) and of the theoretical function $\left.\frac{\partial x(\alpha,h)}{\partial h}\right|_{h=0}$ (blue line). Number of Monte Carlo simulations: 20 000. The parameters used: $PD = 0.1$, $n = 15\ 000$, $h = 0.01$, $\alpha = 0.01$.

6 Expected Shortfall

The Expected Shortfall is defined as:

$$ES_\alpha(L) = \mathbb{E}[L|L > VaR_\alpha(L)] = \frac{1}{\alpha}\int_0^\alpha VaR_\gamma\,(L)d\gamma \tag{6.1}$$

Let us expand $VaR_\gamma(L(h))$ into a Taylor series at $h = 0$:

$$VaR_\gamma(L(h)) = VaR_\gamma(L(h=0)) + h\frac{\partial VaR_\gamma(L(h))}{\partial h}\bigg|_{h=0} + o(h)$$

(6.2)

where $VaR_\gamma(L(h=0)) = x(\gamma)$ is the root of equation:

$$1-\gamma = F_V(x(\gamma); PD, \rho)$$

(6.3)

After differentiating (6.3) with respect to γ, we obtained:

$$-1 = \frac{dx(\gamma)}{d\gamma}\frac{\partial F_V(x; PD, \rho)}{\partial x}\bigg|_{x=x(\gamma)} = \frac{dx(\gamma)}{d\gamma} f_V(x(\gamma); PD, \rho)$$

(6.4)

Hence,

$$\frac{dx(\gamma)}{d\gamma} = -\frac{1}{f_V(x(\gamma); PD, \rho)}$$

(6.5)

By the chain rule, the formula (5.9)

$$\frac{\partial x(\gamma,h)}{\partial h}\bigg|_{h=0} = -\frac{\frac{\partial}{\partial x}\left(\sigma^2(x) f_V(x; PD, \rho)\right)}{2 f_V(x; PD, \rho)}\Bigg|_{x=x(\gamma)}$$

(6.6)

is transformed into the following form:

$$\frac{\partial x(\gamma,h)}{\partial h}\bigg|_{h=0} = \frac{1}{2}\frac{dx(\gamma)}{d\gamma}\left(\frac{\partial}{\partial x}\left(\sigma^2(x) f_V(x; PD, \rho)\right)\right)\bigg|_{x=x(\gamma)} = \frac{1}{2}\frac{\partial}{\partial \gamma}\left(\sigma^2\big(x(\gamma)\big) f_V(x(\gamma); PD, \rho)\right)$$

(6.7)

Hence,

$$\int_0^\alpha \frac{\partial x(\gamma,h)}{\partial h}\bigg|_{h=0} d\gamma = \frac{1}{2}\sigma^2\big(x(\gamma)\big) f_V(x(\gamma); PD, \rho)\bigg|_{\gamma=0}^{\gamma=\alpha} = \frac{1}{2}\sigma^2\big(x(\alpha)\big) f_V(x(\alpha); PD, \rho)$$

(6.8)

It follows from equations (6.1) and (6.8) that

$$\frac{\partial ES_\alpha(L(h))}{\partial h}\bigg|_{h=0} = \frac{1}{2\alpha}\sigma^2\big(x(\alpha)\big) f_V(x(\alpha); PD, \rho)$$

(6.9)

$$ES_\alpha\big(L(h)\big) = ES_\alpha\big(L(0)\big) + \frac{h}{2\alpha}\sigma^2\big(x(\alpha)\big) f_V(x(\alpha); PD, \rho) + o(h)$$

(6.10)

$$\frac{ES_\alpha\big(L(h)\big) - ES_\alpha\big(L(0)\big)}{h} \approx \frac{1}{2\alpha}\sigma^2\big(x(\alpha)\big) f_V(x(\alpha); PD, \rho)$$

(6.11)

Hence, the GA to ES is

$$GA^{ES} = \frac{h}{2\alpha}\sigma^2\big(x(\alpha)\big)f_V(x(\alpha); PD, \rho) \tag{6.12}$$

For Monte Carlo simulations of conditional expectation, we use the formula

$$ES_\alpha(L) = \mathbb{E}[L|L > VaR_\alpha(L)] = \frac{\sum_{k=1}^{N} L_k I(L_k > VaR_\alpha(L))}{\sum_{k=1}^{N} I(L_k > VaR_\alpha(L))} \tag{6.13}$$

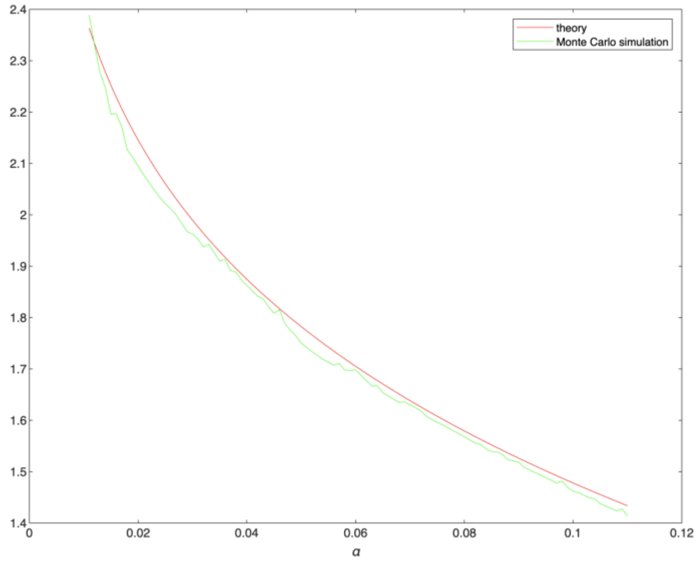

Figure 4. Comparison of the dependence on α of the simulated function $\frac{ES_\alpha(L(h)) - ES_\alpha(L(0))}{h}$ *(green line) and of the theoretical function* $\frac{\partial ES_\alpha(L(h))}{\partial h}\Big|_{h=0}$ *(red line). Number of Monte Carlo simulations: 25 000. The parameters used: PD=0.1,* $\rho = 0.1$*, n=25 000, h=0.01.*

7 Vasicek's Attempt

From formula (10) (Vasicek O. A., 2002) (p. 8), after obvious transformations, the formula

$$var[L] = h\Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); 1) + (1-h)\Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); \rho) - PD^2 \tag{7.1}$$

follows for the unconditional variance, taking into account:

$$a = 1$$
$$b = \Phi^{-1}(PD)$$
$$H = T$$
$$\Phi_2(x, x; 1) = \Phi(x)$$

However, the following formula from (Vasicek O. A., 2002) is erroneous:

$$var[L] \approx (\rho + (1-\rho)h)\varphi^2\big(\Phi^{-1}(PD)\big) \tag{7.2}$$

Let’s demonstrate how this error occurred.

Vasicek used the tetrachoric expansion of the bivariate normal CDF:

$$\Phi_2(x, x; \rho) \approx \Phi^2(x) + \rho\varphi^2(x) \tag{7.3}$$

$$\Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); \rho) \approx PD^2 + \rho\varphi^2\big(\Phi^{-1}(PD)\big) \tag{7.4}$$

Applying this expansion to the case $\rho = 1$ yields the incorrect result:

$$\Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); 1) \approx PD^2 + \varphi^2\big(\Phi^{-1}(PD)\big) \tag{7.5}$$

whereas, in fact,

$$\Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); \rho) \xrightarrow[\rho \to 1^-]{} PD \tag{7.6}$$

As a result, Vasicek arrived at the incorrect approximation:

$$\begin{aligned} var[L] &\approx h\left(PD^2 + \varphi^2\big(\Phi^{-1}(PD)\big)\right) + (1-h)\left(PD^2 + \rho\varphi^2\big(\Phi^{-1}(PD)\big)\right) - PD^2 \\ &= (\rho + (1-\rho)h)\varphi^2\big(\Phi^{-1}(PD)\big) \end{aligned} \tag{7.7}$$

instead of the correct approximation:

$$\begin{aligned} var[L] &\approx hPD + (1-h)\left(PD^2 + \rho\varphi^2\big(\Phi^{-1}(PD)\big)\right) - PD^2 \\ &= hPD(1-PD) + (1-h)\rho\varphi^2\big(\Phi^{-1}(PD)\big) \end{aligned} \tag{7.8}$$

Furthermore, on page 8, Vasicek presented equation (12):

$$\mathbb{P}[L \le x] = F_V(x; p, \rho + h(1-\rho)) \tag{7.9}$$

without proper justification. The fallacy of this formula is demonstrated in Figure 5, where the function

$$dF_V(x) = \frac{F_V(x; p, \rho + h(1-\rho)) - F_V(x; p, \rho)}{h} \tag{7.10}$$

is compared to the corresponding simulated function

$$dF_s(x) = \frac{f_s(x; p, \rho, h) - f_s(x; p, \rho, 0)}{h} \tag{7.11}$$

as well as the function

$$dF_{VH}(x) = \frac{F_{VH}(x; p, \rho, h) - F_{VH}(x; p, \rho, 0)}{h} = \frac{F_{VH}(x; p, \rho, h) - F_V(x; p, \rho)}{h} \tag{7.12}$$

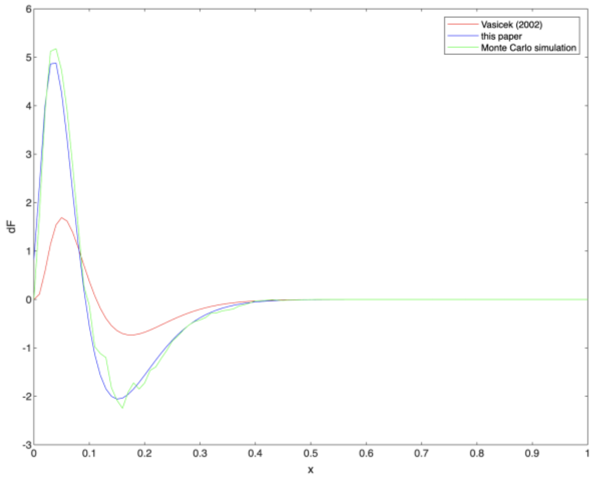

Figure 5. Comparison of $dF_V(x)$ (red line) with the corresponding simulated function (green line) and with $dF_{VH}(x)$ (blue line). Number of Monte Carlo simulations: 20 000. The parameters used: $PD = 0.1$, $\rho = 0.1$, $n = 20\,000$, $h = 0.01$.

The poor quality of Vasicek's attempt to assess the granularity effect is evident in Figure 5.

8 Approach of (Voropaev, 2011)

Based on the information obtained in our paper about the behavior of the portfolio loss distribution near zero of the HHI, let us consider the approach of (Voropaev, 2011).

Transformation of variables $v = x - u$ allows us to write the Vasicek-Herfindahl function PDF (4.12) in the form of formula (3.1) of (Voropaev, 2011):

$$f^*(x) = \int_{-\infty}^{\infty} g(u|x-u) f(x-u)\, du \tag{8.1}$$

where

$$g(u|x) = \frac{\varphi\left(\frac{u}{\sqrt{h}\sigma(x)}\right)}{\sqrt{h}\sigma(x)} \theta(x)\theta(1-x) \tag{8.2}$$

$$\begin{gathered} f^*(x) = f_{VH}(x; PD, \rho, h) \\ f(x) = f_V(x; PD, \rho) \end{gathered} \tag{8.3}$$

By expanding the integrand into a Taylor series at $u = x$, we obtain

$$g(u|x-u)f(x-u) = \sum_{k=0}^{\infty} (-1)^k \frac{u^k}{k!} \frac{\partial^k}{\partial x^k}\big(g(u|x)f(x)\big) \tag{8.4}$$

$$f^*(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!} \frac{d^k}{dx^k}\big(f(x) m_k(x)\big) \tag{8.5}$$

where $m_k(x)$ is the k^{th} moment of distribution H conditional on V:

$$m_k(x) = \int_{-\infty}^{\infty} u^k g(u|x)\, du \tag{8.6}$$

Since $g(u|x)$ was unknown to Voropaev, he was unable to clarify the meaning of these moments, which renders his work unsuitable for practical applications. We will go further and show that his approach leads to the same results as our method.

It follows from equations (8.2) and (8.6) that

$$m_k(x) = \theta(x)\theta(1-x) \int_{-\infty}^{\infty} u^k \frac{\varphi\left(\frac{u}{\sqrt{h}\sigma(x)}\right)}{\sqrt{h}\sigma(x)} du \tag{8.7}$$

Since $\frac{\varphi\left(\frac{u}{\sqrt{h}\sigma(x)}\right)}{\sqrt{h}\sigma(x)}$ is the even function of u, odd moments are equal to zero. Hence,

$$f^*(x) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} \frac{d^{2k}}{dx^{2k}} \left(f(x) m_{2k}(x)\right) \tag{8.8}$$

Since

$$\int_{-\infty}^{\infty} u^{2k} exp(-au^2)\, du = \sqrt{\frac{\pi}{a}} \frac{(2k-1)!!}{(2a)^k} \tag{8.9}$$

then

$$m_{2k}(x) = \theta(x)\theta(1-x) \int_{-\infty}^{\infty} u^{2k} \frac{\varphi\left(\frac{u}{\sqrt{h}\sigma(x)}\right)}{\sqrt{h}\sigma(x)} du = \theta(x)\theta(1-x)\left(h\sigma^2(x)\right)^k (2k-1)!! \tag{8.10}$$

Hence, by employing Voropaev's approach, we derive the equation:

$$f^*(x) = \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!} h^k \frac{\partial^{2k}}{\partial x^{2k}} \left(\sigma^{2k}(x) f_V(x; PD, \rho)\right) \tag{8.11}$$

In contrast, our approach leads to the formula (4.44). Since

$$\frac{(2k-1)!!}{(2k)!} = \frac{1}{k!}\left(\frac{1}{2}\right)^k \tag{8.12}$$

both equalities coincide, leading to the result:

$$f_{VH}(x; PD, \rho, h) = f^*(x) \tag{8.13}$$

9 Approach of (Emmer & Tasche, 2005)

The approach proposed by (Emmer & Tasche, 2005) is based on the decomposition:

$$L = \mathbb{E}[L|Y] + c(L - \mathbb{E}[L|Y])$$

where $c = 1$, along with the expansion of $VaR_\alpha(L) = x(\alpha, h, c)$ into a Taylor series at $c = 0$. However, the convergence of this series when $c = 1$ has not been proven.

The right-hand side of equation (5.9)

$$\left.\frac{\partial x(\alpha,h)}{\partial h}\right|_{h=0} = -\left.\frac{\frac{\partial}{\partial x}\left(\sigma^2(x) f_V(x;PD,\rho)\right)}{2 f_V(x;PD,\rho)}\right|_{x=x(\alpha)} = -\frac{1}{h}\left.\frac{\frac{\partial}{\partial x}\left(h\sigma^2(x) f_V(x;PD,\rho)\right)}{2 f_V(x;PD,\rho)}\right|_{x=x(\alpha)}$$
$$= -\frac{1}{h}\left.\frac{\frac{\partial}{\partial x}\left(var[L|X=x] f_V(x;PD,\rho)\right)}{2 f_V(x;PD,\rho)}\right|_{x=x(\alpha)}$$

coincides, up to a factor of $h/2$, with the right-hand side of equation (7) from (Emmer & Tasche, 2005):

$$\left.\frac{\partial^2 x(\alpha,h,c)}{\partial c^2}\right|_{c=0} = -\left.\frac{\frac{\partial}{\partial x}\left(var[L|X=x] f_V(x;PD,\rho)\right)}{f_V(x;PD,\rho)}\right|_{x=x(\alpha)} \tag{9.1}$$

Hence

$$\left.\frac{\partial x(\alpha,h,1)}{\partial h}\right|_{h=0} = \frac{h}{2}\left.\frac{\partial^2 x(\alpha,h,c)}{\partial c^2}\right|_{c=0} \tag{9.2}$$

Let us prove that this equality holds true. Since

$$L = \mathbb{E}[L|Y] + c(L - \mathbb{E}[L|Y])$$

and

$$L - \mathbb{E}[L|Y] = Z\sigma(p(Y))\sqrt{h} \tag{9.3}$$

we introduce the random variable

$$\tilde{L} = \mathbb{E}[\tilde{L}|Y] + bZ\sigma\big(p(Y)\big) \tag{9.4}$$

where $b = c\sqrt{h}$. If $c = 1$, $\tilde{L} = L$. Given $b = c\sqrt{h}$, we need to prove that

$$\left.\frac{\partial x(\alpha,c^2h)}{\partial h}\right|_{h=0} = \frac{1}{2}\left.\frac{\partial^2 x(\alpha,b^2)}{\partial b^2}\right|_{b=0} = \frac{h}{2}\left.\frac{\partial^2 x(\alpha,c^2h)}{\partial c^2}\right|_{c=0} \tag{9.5}$$

For the random variable $\tilde{L}$

$$1-\alpha=\int_0^1 \Phi\left(\frac{x(\alpha,b^2)-v}{b\sigma(v)}\right)dF_V(v;PD,\rho)$$

(9.6)

Let us differentiate this equation with respect to the variable b:

$$0=\frac{\partial x(\alpha,b^2)}{\partial b}\int_0^1 \frac{1}{b\sigma(v)}\varphi\left(\frac{v-x(\alpha,b^2)}{b\sigma(v)}\right)dF_V(v;PD,\rho)$$
$$+\int_0^1\left(\frac{v-x(\alpha,b^2)}{b^2\sigma(v)}\right)\varphi\left(\frac{v-x(\alpha,b^2)}{b\sigma(v)}\right)dF_V(v;PD,\rho)$$

(9.7)

Hence,

$$\frac{1}{b}\frac{\partial x(\alpha,b^2)}{\partial b}=\frac{\int_0^1\left(-\frac{v-x(\alpha,b^2)}{\left(b\sigma(v)\right)^3}\right)\varphi\left(\frac{v-x(\alpha,b^2)}{b\sigma(v)}\right)\sigma^2(v)dF_V(v;PD,\rho)}{\int_0^1\frac{1}{b\sigma(v)}\varphi\left(\frac{v-x(\alpha,b^2)}{b\sigma(v)}\right)dF_V(v;PD,\rho)}$$

(9.8)

It follows from formulas (12.3) and (12.4) that:

$$\frac{1}{b}\frac{\partial x(\alpha,b^2)}{\partial b}\xrightarrow[b\to 0^+]{}\frac{\int_0^1\delta'\left(v-x(\alpha,b^2)\right)\sigma^2(v)dF_V(v;PD,\rho)}{\int_0^1\delta\left(v-x(\alpha,b^2)\right)dF_V(v;PD,\rho)}=-\left.\frac{\frac{\partial\left(\sigma^2(x)f_V(x;PD,\rho)\right)}{\partial x}}{f_V(x;PD,\rho)}\right|_{x=x(\alpha)}$$

(9.9)

Hence,

$$\left.\frac{\partial x(\alpha,b^2)}{\partial b}\right|_{b=0}=0$$

(9.10)

which, for a fixed value of h, is equivalent to

$$\left.\frac{\partial x(\alpha,c^2h)}{\partial c}\right|_{c=0}=0$$

(9.11)

This corresponds to equation (5) in (Emmer & Tasche, 2005). Denote

$$w(\alpha)=-\left.\frac{\frac{\partial\left(\sigma^2(x)f_V(x;PD,\rho)\right)}{\partial x}}{f_V(x;PD,\rho)}\right|_{x=x(\alpha)}$$

(9.12)

Let us differentiate equation (9.7) with respect to the variable b:

$$\frac{\partial^2 x(\alpha,b^2)}{\partial b^2}\int_0^1 \frac{1}{b\sigma(v)}\varphi\left(\frac{v-x(\alpha,b^2)}{b\sigma(v)}\right)dF_V(v;PD,\rho)$$
$$+\frac{\partial x(\alpha,b^2)}{\partial b}\int_0^1\left(-\frac{1}{b^2\sigma(v)}\right)\varphi\left(\frac{v-x(\alpha,b^2)}{b\sigma(v)}\right)dF_V(v;PD,\rho)$$
$$-\left(\frac{\partial x(\alpha,b^2)}{\partial b}\right)^2\int_0^1\left(\frac{1}{b\sigma(v)}\right)^2\left(-\frac{v-x(\alpha,b^2)}{b\sigma(v)}\right)\varphi\left(\frac{v-x(\alpha,b^2)}{b\sigma(v)}\right)dF_V(v;PD,\rho)$$
$$+\frac{\partial x(\alpha,b^2)}{\partial b}\int_0^1\frac{1}{b\sigma(v)}\left(\frac{v-x(\alpha,b^2)}{b^2\sigma(v)}\right)\left(\frac{v-x(\alpha,b^2)}{b\sigma(v)}\right)\varphi\left(\frac{v-x(\alpha,b^2)}{b\sigma(v)}\right)dF_V(v;PD,\rho)$$
$$-\frac{\partial x(\alpha,b^2)}{\partial b}\int_0^1\left(-\frac{v-x(\alpha,b^2)}{b^2\sigma(v)}\right)\left(\frac{v-x(\alpha,b^2)}{b\sigma(v)}\right)\varphi\left(\frac{v-x(\alpha,b^2)}{b\sigma(v)}\right)\left(\frac{v-x(\alpha,b^2)}{b^2\sigma(v)}\right)dF_V(v;PD,\rho)$$
$$+\frac{\partial x(\alpha,b^2)}{\partial b}\int_0^1\varphi\left(\frac{v-x(\alpha,b^2)}{b\sigma(v)}\right)\left(-2\frac{v-x(\alpha,b^2)}{b^3\sigma(v)}\right)dF_V(v;PD,\rho)$$
$$-\int_0^1\left(-\frac{v-x(\alpha,b^2)}{b\sigma(v)}\right)\varphi\left(\frac{v-x(\alpha,b^2)}{b\sigma(v)}\right)\left(\frac{v-x(\alpha,b^2)}{b^2\sigma(v)}\right)^2 dF_V(v;PD,\rho)$$
$$+\int_0^1\varphi\left(\frac{v-x(\alpha,b^2)}{b\sigma(v)}\right)\left(-2\frac{v-x(\alpha,b^2)}{b^3\sigma(v)}\right)dF_V(v;PD,\rho)=0$$

(9.13)

To use formulas (12.3) - (12.6), we transform (9.13) to the equality

$$
\begin{aligned}
&\frac{\partial^2 x(\alpha,b^2)}{\partial b^2}\int_0^1 \frac{1}{b\sigma(v)}\,\varphi\left(\frac{v-x(\alpha,b^2)}{b\sigma(v)}\right)dF_V(v;PD,\rho)\\
&\qquad - w(\alpha)\int_0^1 \frac{1}{b\sigma(v)}\,\varphi\left(\frac{v-x(\alpha,b^2)}{b\sigma(v)}\right)dF_V(v;PD,\rho)\\
&\qquad - \big(bw(\alpha)\big)^2\int_0^1 \left(-\frac{v-x(\alpha,b^2)}{\big(b\sigma(v)\big)^3}\right)\varphi\left(\frac{v-x(\alpha,b^2)}{b\sigma(v)}\right)dF_V(v;PD,\rho)\\
&\qquad + b^2 w(\alpha)\int_0^1 \left(\frac{\big(v-x(\alpha,b^2)\big)^2}{\big(b\sigma(v)\big)^5}\right)\varphi\left(\frac{v-x(\alpha,b^2)}{b\sigma(v)}\right)\sigma^2(v)dF_V(v;PD,\rho)\\
&\qquad - b^3 w(\alpha)\int_0^1 \left(-\frac{\big(v-x(\alpha,b^2)\big)^3}{\big(b\sigma(v)\big)^7}\right)\varphi\left(\frac{v-x(\alpha,b^2)}{b\sigma(v)}\right)\sigma^2(v)dF_V(v;PD,\rho)\\
&\qquad + 2bw(\alpha)\int_0^1 \left(-\frac{v-x(\alpha,b^2)}{\big(b\sigma(v)\big)^3}\right)\varphi\left(\frac{v-x(\alpha,b^2)}{b\sigma(v)}\right)\sigma^2(v)dF_V(v;PD,\rho)\\
&\qquad - b^2\int_0^1 \left(-\frac{\big(v-x(\alpha,b^2)\big)^3}{\big(b\sigma(v)\big)^7}\right)\varphi\left(\frac{v-x(\alpha,b^2)}{b\sigma(v)}\right)\sigma^4(v)dF_V(v;PD,\rho)\\
&\qquad + 2\int_0^1 \left(-\frac{v-x(\alpha,b^2)}{\big(b\sigma(v)\big)^3}\right)\varphi\left(\frac{v-x(\alpha,b^2)}{b\sigma(v)}\right)\sigma^2(v)dF_V(v;PD,\rho) = 0
\end{aligned}
\tag{9.14}
$$

Hence,

$$
\begin{aligned}
&\frac{\partial^2 x(\alpha,b^2)}{\partial b^2} f_V\big(x(\alpha,b^2);PD,\rho\big) + \big(bw(\alpha)\big)^2 \left.\frac{\partial f_V(x;PD,\rho)}{\partial x}\right|_{x=x(\alpha)}\\
&\qquad + b^2 w(\alpha)\left.\frac{\partial^2\big(\sigma^2(x) f_V(x;PD,\rho)\big)}{\partial x^2}\right|_{x=x(\alpha)}\\
&\qquad + b^3 w(\alpha)\left.\frac{\partial^3\big(\sigma^2(x) f_V(x;PD,\rho)\big)}{\partial x^3}\right|_{x=x(\alpha)} + 3bw(\alpha)\left.\frac{\partial f_V(x;PD,\rho)}{\partial x}\right|_{x=x(\alpha)}\\
&\qquad - 2bw(\alpha)\left.\frac{\partial\big(\sigma^2(x) f_V(x;PD,\rho)\big)}{\partial x}\right|_{x=x(\alpha)} + b^2\left.\frac{\partial^3\big(\sigma^4(x) f_V(x;PD,\rho)\big)}{\partial x^3}\right|_{x=x(\alpha)}\\
&\qquad + \left.\frac{\partial\big(\sigma^2(x) f_V(x;PD,\rho)\big)}{\partial x}\right|_{x=x(\alpha)} = 0
\end{aligned}
\tag{9.15}
$$

Finally,

$$
\left.\frac{\partial^2 x(\alpha,b^2)}{\partial b^2}\right|_{b=0} = -\frac{1}{f_V(x;PD,\rho)}\left.\frac{\partial\big(\sigma^2(x) f_V(x;PD,\rho)\big)}{\partial x}\right|_{x=x(\alpha)}
\tag{9.16}
$$

Q.E.D.

10 Conclusion

The approach by (Emmer & Tasche, 2005) is based on the decomposition:

$$Loss = \mathbb{E}[Loss|Y] + c(Loss - \mathbb{E}[Loss|Y])$$

where $c = 1$, and is built upon the method proposed by (Gouriéroux, Laurent, & Scailett, 2000) to calculate the derivatives of VaR. As a result, (Emmer & Tasche, 2005) obtained GA for VaR.

Our approach exploits the fact that, for small values of HHI, granularity risk is merely a small perturbation. We use the approximation:

$$Loss - \mathbb{E}[Loss|Y] \approx Z\sigma(p(Y))\sqrt{h}$$

which is obtained using the Lyapunov CLT. The random value $Z\sigma(p(Y))\sqrt{h}$ is the GA to the random value $Loss$. Consequently, in addition to the first factor, the market shock Y, a second factor, the granularity shock $Z\sigma(p(Y))\sqrt{h}$, is added to the model. Rather than studying the behavior of a heterogeneous portfolio, we focus on the behavior of a homogeneous portfolio subjected to a granularity shock.

The expansion of VaR into a Taylor series up to the first order of HHI yields the same result using both approaches. However, the approach proposed by (Emmer & Tasche, 2005) is specifically designed to expand VaR and ES, while our approach is more general and, in particular, allows us to obtain PDF and CDF of portfolio loss, taking into account the granularity effect.

We see that the HHI is a crucial factor for modeling the granularity effect and can be utilized to regulate a bank's economic capital.

The GA to VaR is given by:

$$GA^{VaR} = -\frac{h}{2}\left(1 - 2x(\alpha) + \frac{x(\alpha)\big(1 - x(\alpha)\big)}{\varphi\left(\Phi^{-1}\big(x(\alpha)\big)\right)}\left(\Phi^{-1}\big(x(\alpha)\big) - \sqrt{\frac{1-\rho}{\rho}}\Phi^{-1}(1-\alpha)\right)\right)$$

where

$$x(\alpha) = VaR_\alpha\big(Loss(h=0)\big) = \Phi\left(\frac{\sqrt{\rho}\Phi^{-1}(1-\alpha) + \Phi^{-1}(PD)}{\sqrt{1-\rho}}\right)$$

The formula for GA^{VaR} is quite complex and lacks clarity. The IRB approach uses the formula

$$VaR_\alpha\big(Loss(h=0)\big) = F_V{}^{-1}(1-\alpha; PD, \rho)$$

We recommend adopting a more transparent and intuitively clear formula for supervisory applications:

$$VaR_\alpha\big(Loss(h)\big) = F_{VH}{}^{-1}(1-\alpha; PD, \rho, h)$$

where

$$F_{VH}(x;PD,\rho,h)=\int_0^1 F_V(v;PD,\rho)d\Phi\left(\frac{v-x}{\sqrt{h}\sigma(v)}\right)$$

11 Appendix 1

For the function $\Phi(u/\sqrt{z})$, the heat conduction equation

$$\Phi_z(u/\sqrt{z})=\frac{1}{2}\Phi_{uu}(u/\sqrt{z})$$

is valid. The generalized heat equation

$$\frac{\partial^k}{\partial z^k}\Phi(u/\sqrt{z})=\frac{1}{2^k}\frac{\partial^{2k}}{\partial u^{2k}}\Phi(u/\sqrt{z})$$

can be proved by induction:

$$\frac{\partial^{k+1}}{\partial z^{k+1}}\Phi(u/\sqrt{z})=\frac{1}{2^k}\frac{\partial}{\partial z}\frac{\partial^{2k}}{\partial u^{2k}}\Phi(u/\sqrt{z})=\frac{1}{2^k}\frac{\partial^{2k}}{\partial u^{2k}}\frac{\partial}{\partial z}\Phi(u/\sqrt{z})=\frac{1}{2^{k+1}}\frac{\partial^{2(k+1)}}{\partial u^{2(k+1)}}\Phi(u/\sqrt{z})$$

Given the formulas (12.1) and (12.2), we get

$$\lim_{z\to 0^+}\frac{\partial^k}{\partial z^k}\Phi(u/\sqrt{z})=\lim_{h\to 0^+}\frac{1}{2^k}\frac{\partial^{2k}}{\partial u^{2k}}\Phi(u/\sqrt{z})=\frac{\delta^{(2k-1)}(u)}{2^k}$$

Now, we can expand function $\Phi(u/\sqrt{z})$ into a Taylor series at $z=0$:

$$\Phi(u/\sqrt{z})=\theta(u)+\sum_{k=1}^{\infty}\frac{(z/2)^k}{k!}\delta^{(2k-1)}(u)=\theta(u)+\frac{z}{2}\delta'(u)+o(z)$$

12 Appendix 2

The Dirac delta function is the weak derivative of the Heaviside step function:

$$\delta(x)=\theta'(x) \tag{12.1}$$

Consecutive differentiations of the limit representation of the Heaviside step function

$$\Phi(x/\varepsilon)\xrightarrow[\varepsilon\to 0^+]{}\theta(x) \tag{12.2}$$

with respect to x yields:

$$\frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)\xrightarrow[\varepsilon\to 0^+]{}\delta(x) \tag{12.3}$$

$$-\frac{x}{\varepsilon^3}\varphi\left(\frac{x}{\varepsilon}\right)\xrightarrow[\varepsilon\to 0^+]{}\delta'(x)$$

(12.4)

$$\frac{x^2}{\varepsilon^5}\varphi\left(\frac{x}{\varepsilon}\right)\xrightarrow[\varepsilon\to 0^+]{}\delta''(x)+\frac{1}{\varepsilon^2}\delta(x)$$

(12.5)

$$-\frac{x^3}{\varepsilon^7}\varphi\left(\frac{x}{\varepsilon}\right)\xrightarrow[\varepsilon\to 0^+]{}\delta'''(x)+\frac{3}{\varepsilon^2}\delta'(x)$$

(12.6)

References

Emmer, S., & Tasche, D. (2005). Calculating credit risk capital charges with the one-factor model. The Journal of Risk, 7(2), 85-101.

Gordy, M. (2004). Granularity adjustment in portfolio credit risk measurement. In G. P. Szegö, Risk Measures for the 21st Century. Wiley.

Gordy, M., & Lutkebohmert, E. (2013). Granularity Adjustment for Regulatory Capital Assessment. International Journal of Central Banking, 38-77.

Gordy, M., & Marrone, J. (2012). Granularity Adjustment for Mark-to-Market Credit Risk Models. Journal of Banking and Finance, 36(7), 1896-1910.

Gouriéroux, C., Laurent, J. P., & Scailett, O. (2000). Sensitivity analysis of values at risk. Journal of Empirical Finance, 7, 225-245.

Merton, R. (1974). On the pricing of corporate debt: the risk structure of interest rates. J. Finance, 29, 449-470.

Vasicek, O. (1987). Probability of Loss on Loan Portfolio. KMV Corporation.

Vasicek, O. A. (2002, December). Loan Portfolio Value. Risk Magazine, 15(12), 160-162. https://www.bankofgreece.gr/MediaAttachments/Vasicek.pdf

Voropaev, M. (2011). An Analytical Framework for Credit Portfolio. Risk Measures. Risk, 24(5), 72-78.