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A Complete Characterization of Equilibrium Paths in a Labour-Less Two-Sector Model of Optimal Growth*

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Abstract

In this paper, I develop a two-sector growth model in which the output of each industry is obtained only by means of capital. From a theoretical perspective, this setting reveals that the determinacy of the equilibrium path is possible even when the production technology available in the industry of consumption goods is not necessarily convex. Moreover, form a numerical point of view, I show that whenever the first-best solution of the model is tailored on the US economy, such a theoretical framework relaxes the complementarity between the propensities to consume and to save and it is also consistent with the observed countercyclical pattern of the relative price of capital goods.

Keywords: Capital accumulation; Investment; Consumption; Two-sector growth model; Relative price of capital goods.

JEL Classification: C61, E21, E22, O11.

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1 Introduction

Since its entering on the scene, the two-sectors model of optimal growth pioneered by Meade (1961) and Uzawa (1961) in which there are distinct production technologies for consumption and investment goods has been blamed for a certain opacity so that many prominent scholars tried to shed lights on some of its gray areas by writing a number of clarifying notes that started a heated debate on a number of aspects of that setting (cf. Solow 1961; Hahn 1965; Shell, 1966; Gabisch, 1973). Despite these difficulties, the importance of such a theoretical framework, however, has never been questioned; indeed, years later, the two-sector model still provides a rigorous setting to analyse how the available production factors are optimally allocated in different industries and to assess the dynamic behaviour of relative prices in multiproduct economies (cf. Pelgrin and Venditti, 2022; Senouci, 2011, 2012; Matzuo et al. 2008; Herrendorf and Velentinyi, 2006). In this scenario, where there are unceasing calls for simpler formulations, it may be interesting to find some simplifying assumptions aimed at making the model a little bit easier to handle (cf. Venditti, 2005). Since the large majority of the contributions dealing with the two-sector model consider an exogenously given labour supply, a way to simplify the underlying picture may be to take into consideration the case in which the non-reproducible factor becomes unnecessary for the production of any goods.

Following the above intuition, in this paper I develop a version of the two-sector optimal growth model in which the output of the two existing industries is obtained only by means of capital under the hypothesis of continuous full employment of the unique reproducible factor. In other words, I consider a two-product one-factor economy populated by an omniscient infinitely-lived optimizing agent that aims at maximizing its flow of instantaneous utility under the capital accumulation constraint by considering the labour input and technical progress are both absent. To the best of my knowledge, such a simplified version of the two-sector model in which the available capital has to be optimally directed in each industry has never been explored before. That setting can be taken as a possible outcome of the ongoing processes of production automation pinned down in a context of non-homogeneity of produced outputs (cf. Growiec, 2022).

From a theoretical perspective, relaying on Cobb-Douglas production functions and logarithmic preferences over consumption goods, this framework reveals that the determinacy of the equilibrium path followed by the production of consumption goods and by the total stock of capital is possible even when the production technology available in the industry of consumption goods is not necessarily convex. That finding is intriguing because in the two-sector model as well as in the one-sector one the presence of non-convexities in aggregate production technologies has been often associated to indeterminacy and multiple equilibrium paths (cf. Benhabib and Farmer, 1994; 1996). Moreover, form a numerical point of view, I show that whenever the first-best solution of the model is tailored on the US economy, such a theoretical framework dampens the strict complementarity between the marginal propensity to consume and the marginal propensity to save typical of the one-sector model with homogenous output and it is also consistent with the observed countercyclical pattern of the relative price of tradable capital goods stressed by a number of scholars (cf. Collins and Williamson, 2001; Restuccia and Urrutia, 2001; Fisher, 2006; Justiniano et al. 2011; Hergovich and Merz, 2018; Lian et al. 2020).

The paper is arranged as follows. Section 2 illustrates the model economy. Section 3 explores its quantitative implications. Finally, Section 4 concludes.

2 Theoretical framework

I consider a model economy in which time (t) is continuous − so that $t \in \mathbb{R}_+$ − and there are two productive sectors, one that produces consumption goods that contribute to the utility of the optimizing agent and another one in which are produced investment goods that boost the overall capital accumulation in the society (cf. Uzawa, 1964). Consequently, in each instant, the total stock of capital available in the economy − say $K(t)$ − is broken down between the stock of capital allocated in the sector of consumption goods and the stock of capital allocated in the sector of investment goods, respectively denoted, respectively, by $K_C(t)$ and $K_I(t)$. Therefore, continuous full employment of the available capital will always imply that

$$K(t) = K_C(t) + K_I(t) \tag{1}$$

Given the hypothesis of the absence of labour, the productive technologies available in the two industries are assumed to be described by the following Cobb-Douglas expressions:

$$Y_i(t) = S_i\left(K_i(t)\right)^{\alpha_i} \qquad i = C, I \tag{2}$$

where $Y_i(t)$ is the real output of sector i, $S_i > 0$ is an industry-specific measure of productivity, whereas $\alpha_i > 0$ is the value of the elasticity of output of the industry with respect to the stock of the employed capital in that sector.

As I said in the introduction, since the labour input is completely omitted from the two expressions implied by eq. (2), the model economy described in this paper considers the limiting case of full automation of production in both industries which may be driven by the ongoing expansion of the digital sphere (cf. Growiec, 2022). Moreover, aiming at analysing the role of non-convexities in the available production technologies, for the moment I do not fix any upper bound for the parameters that convey the values of the elasticities of output respect to the stock of capital employed in each sector by simply assuming that − in each industry − output is an increasing function of the unique factor of production.

Furthermore, assuming that the employed capital equally depreciates in both industries, the capital accumulation law can be written as

$$\dot{K}(t) = Y_I(t) - \delta K(t) \tag{3}$$

where $\delta > 0$ is the common value of the depreciation rate of capital.

According to the expression in eq. (3), $Y_I(t)$ is the output of capital goods available for net investment to add to the stock of malleable machines available in the economy (cf. Meade, 1961).

2.1 The social planner problem

In the model economy described above, the decision to produce and consume a certain amount of resources straightforwardly implies to allocate a certain share of capital in the sector of consumption goods by leaving the remaining fraction to the sector of investment goods (Cai, 2006). Consequently, given the expressions in (1) − (3), a benevolent and well-informed social planner that lives forever and endowed with logarithmic preferences over consumption will allocate the available capital in the two sectors by solving the following problem:

$$\begin{gathered} \max_{Y_C(\cdot)\in\mathcal{A}_0(\overline{K})} \int_{t=0}^{\infty} \exp(-\rho t)(\ln Y_C(t))\,dt \\ \text{s.to} \\ \dot{K}(t) = S_I\left(K(t) - \left(\frac{Y_C(t)}{S_C}\right)^{\frac{1}{\alpha_C}}\right)^{\alpha_I} - \delta K(t) \\ K(0) = \overline{K} \end{gathered} \tag{4}$$

where $Y_C(\cdot)$ is the set of control functions, $\mathcal{A}_0\left(\overline{K}\right)$ is the set of admissible control strategies, $\rho > 0$ is the discount rate, whereas $\overline{K} > 0$ is the initial value of the total capital stock (cf. Herrendorf and Velentinyi, 2006).

In order to have economically meaningful trajectories, the set of admissible control strategies $Y_C(\cdot)$ starting from the initial couple $\left\{0, \overline{K}\right\}$ is defined as

$$\mathcal{A}_0\left(\overline{K}\right) := \left\{Y_C(\cdot) \in \mathbb{L}^1_{\mathbf{loc}}(\mathbb{R}_+;\mathbb{R}_+) : K(t) \in \mathbb{R}_+ \quad \forall t \in \mathbb{R}_+\right\} \tag{5}$$

According to the definition in (5), $Y_C(\cdot)$ has to belong to the set of locally integrable (or summable) functions such that the level of consumption goods which are produced and the stock of capital are positive all over the relevant time horizon.

The first-order conditions (FOCs) for the social planner problem in (4) that convey the first-best solution are given by

$$1 - \frac{\alpha_I S_I q(t)(Y_C(t))^{\frac{1}{\alpha_C}}}{\alpha_C S_C^{\frac{1}{\alpha_C}}\left(K(t) - \left(\frac{Y_C(t)}{S_C}\right)^{\frac{1}{\alpha_C}}\right)^{1-\alpha_I}} = 0 \tag{6}$$

$$\dot{q}(t) = q(t)\left(\rho + \delta - \frac{\alpha_I S_I}{\left(K(t) - \left(\frac{Y_C(t)}{S_C}\right)^{\frac{1}{\alpha_C}}\right)^{1-\alpha_I}}\right) \tag{7}$$

$$\lim_{t\to\infty} \exp(-\rho t)\, q(t)\, K(t) = 0 \tag{8}$$

where $q(t)$ is the costate variable associated to the capital accumulation constraint.

Eq. (6) is the FOC with respect to $Y_C(t)$ that has to hold in each instant. Moreover, the intertemporal relationship in eq. (7) conveys the optimal trajectory of the costate variable. Furthermore, the endpoint limit on the value of capital in (8) is the required transversality condition.

Differentiating eq. (6) with respect to time and using its implied expression for $q(t)$ allows us to find the Euler equation for the production of consumption goods. Straightforward algebra reveals that such an Euler equation can be written as

$$\frac{\dot{Y}_C(t)}{Y_C(t)} = \frac{\alpha_C\left((1-\alpha_I)K(t)\frac{\dot{K}(t)}{K(t)} - \left(K(t) - \left(\frac{Y_C(t)}{S_C}\right)^{\frac{1}{\alpha_C}}\right)\frac{\dot{q}(t)}{q(t)}\right)}{K(t) - \alpha_I\left(\frac{Y_C(t)}{S_C}\right)^{\frac{1}{\alpha_C}}} \tag{9}$$

The expression in eq. (9) shows that the optimal growth rate of the output of consumption goods is given by a non-linear combination between the growth rate of capital – which can be easily retrieved from eq. (3) – and the optimal growth rate of the costate variable implied by eq. (7). Obviously, eq. (9) together with eq. (3) supplemented with the technological constraints implied by the expressions in eq.s (1) and (2) provide the dynamic system that describes the motion of the model economy.

2.2 Steady state and golden rule

In the theoretical framework under scrutiny, steady-state allocations are defined as the set of pairs $\mathcal{S} := \{Y_C^*, K^*\} \in \mathbb{R}_{++}^2$ such that $\dot{Y}_C(Y_C^*, K^*) = \dot{K}(Y_C^*, K^*) = 0$ whose point values will be also associated to the equilibrium level of the production of investment goods ($Y_I^* > 0$) and the corresponding steady-state value of the amounts of capital allocated, respectively, in the sector of consumption goods ($K_C^* > 0$) and in the sector of investment goods ($K_I^* > 0$). In case of asymptotic stability, some elements of $\mathcal{S}$ will be also characterized by the fact that $\lim_{t\to\infty} Y_C(t) = Y_C^* \wedge \lim_{t\to\infty} K(t) = K^*$, a feature that may lead to the convergence of all the remaining endogenous variables.

The long-run value of each endogenous variable conveyed as a function of the model's parameters can be easily retrieved as follows. First, setting $\dot{q}(t) = 0$ in eq. (7) allows us to find the steady-state level of the stock of capital allocated in the sector of investment goods, namely

$$K_I^* = \left(\frac{\alpha_I S_I}{\rho + \delta}\right)^{\frac{1}{1-\alpha_I}} \tag{10}$$

Second, eq. (10) together with the expressions in eq. (2) pins down the steady-state level of the output in the sector of investment goods, that is

$$Y_I^* = S_I \left(\frac{\alpha_I S_I}{\rho + \delta}\right)^{\frac{\alpha_I}{1-\alpha_I}} \tag{11}$$

Third, setting $\dot{K}(t) = 0$ in eq. (3) and using the result in eq. (11) allows us to obtain the steady-state value of the total stock of capital, namely

$$K^* = \frac{S_I}{\delta} \left(\frac{\alpha_I S_I}{\rho + \delta}\right)^{\frac{\alpha_I}{1-\alpha_I}} \tag{12}$$

Forth, consistently with eq. (1), subtracting the expression in eq. (10) from the expression in eq. (12) allows us to retrieve the steady-state level of the stock of capital allocated in the sector of consumption goods, that is

$$K_C^* = \frac{S_I (\rho + \delta (1 - \alpha_I))}{\delta (\rho + \delta)} \left(\frac{\alpha_I S_I}{\rho + \delta}\right)^{\frac{\alpha_I}{1-\alpha_I}} \tag{13}$$

Eq. (13) straightforwardly reveals that the equilibrium stock of capital allocated in the production of consumption goods is positive whenever α_I is lower that $1 + \rho/\delta$. Moreover, eq. (13) together with the expressions in eq. (2) allows us to derive the steady-state level of the output in the sector of consumption goods, that is

$$Y_C^* = S_C \left(\frac{S_I (\rho + \delta (1 - \alpha_I))}{\delta (\rho + \delta)}\right)^{\alpha_C} \left(\frac{\alpha_I S_I}{\rho + \delta}\right)^{\frac{\alpha_C \alpha_I}{1-\alpha_I}} \tag{14}$$

Eq.s (12) and (13) imply that $(\rho + \delta (1 - \alpha_I)) / (\rho + \delta)$ $(\alpha_I \delta / (\rho + \delta))$ is the equilibrium share of capital allocated in the sector of consumption (investment) goods. It is worth noticing that its magnitude – as well as the ones of K_I^* and K_C^* – is completely unrelated to the conditions of production prevailing in the sector of consumption goods which matter only for the determination of Y_C^* through eq. (14) (cf. Senouci, 2011).

Furthermore, setting $\dot{K}(t) = 0$ in eq. (3) by taking into account the results in eq.s (1) and (2), allows us to find the golden-rule level of the stock of capital that in the present context amounts to the equilibrium level of K that maximizes the steady-state production of Y_C. Straightforward algebra reveals that such a critical level of the capital stock is given by the following expression:

$$K_{\text{GR}} = \frac{S_I}{\delta}\left(\frac{\alpha_I S_I}{\delta}\right)^{\frac{\alpha_I}{1-\alpha_I}} \tag{15}$$

As illustrated in the diagram of Figure 1, the expressions in eq.s (15) and (12) imply that as long as the social planner discounts future utility streams at a positive rate, the golden-rule level of the capital stock is higher than its long-run value and it falls short of its maximum level ($K_{\max} \equiv (S_I/\delta)^{1/(1-\alpha_I)}$) that would prevail if the equilibrium value of the production of consumption goods were set to zero. Once again, it is worth noticing that both K_{GR} and $K_{\max}$ are not related to the production technology prevailing in the sector of consumption goods.

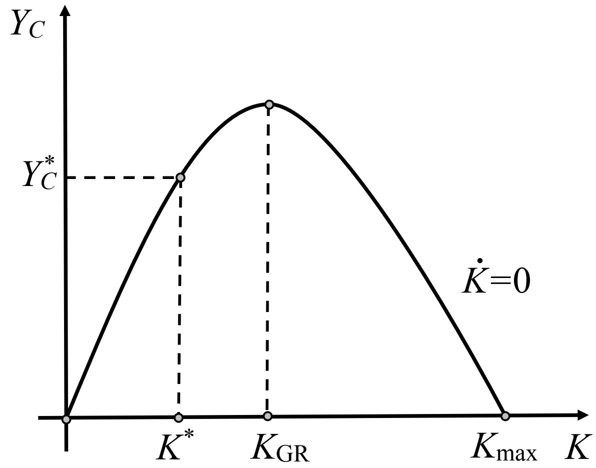

Figure 1: The golden rule

2.3 Local dynamics

Given the expressions in eq.s (1) – (3), (9), (12) and (14), the local dynamics of Y_C and K around the unique element of $\mathcal{S}$ is given by the following linear system:

$$\begin{pmatrix} \dot{Y}_C(t) \\ \dot{K}(t) \end{pmatrix} = \begin{bmatrix} 0 & -\frac{\alpha_C(1-\alpha_I)(\rho+\delta)\delta Y_C^*}{(\rho+\delta-\alpha_I(\rho+\delta(1-\alpha_I)))K^*} \\ -\frac{\rho+\delta}{\alpha_C S_C}\left(\frac{(\rho+\delta(1-\alpha_I))K^*}{\rho+\delta}\right)^{1-\alpha_C} & \rho \end{bmatrix} \begin{pmatrix} Y_C(t) - Y_C^* \\ K(t) - K^* \end{pmatrix} \tag{16}$$

The trace of the Jacobian matrix in (16) – say $\boldsymbol{J} \in \mathbb{R}^{2\times 2}$ – is equal to $\rho > 0$ whereas its determinants is given by the ratio

$$-\frac{(\rho+\delta)(1-\alpha_I)(\rho+\delta(1-\alpha_I))\delta}{\alpha_I\delta + (1-\alpha_I)(\rho+\delta(1-\alpha_I))} \tag{17}$$

As long as the technology prevailing in the sector of investment goods is convex, i.e., as long as $\alpha_I \in (0,1)$, the expression in eq. (17) is negative. In this case, no matter the shape of the technology prevailing in the sector of consumption goods and the level capital intensity in the two industries, $\boldsymbol{J}$ will have two real eigenvalues with opposite sign, say $\lambda_1 < 0$ and $\lambda_2 > 0$. Therefore, as it would happen in a companion one-sector model with logarithmic preferences and Cobb-Douglas technology with decreasing returns to scale, the unique element of S will be a saddle point and the slope of the saddle path will be positive (cf. Cass, 1965). In other words, given the value of $\overline{K}$ there will be a unique value of $Y_C(0)$ that place the system in (16) on the stable branch of the saddle path whereas all the others will tend to diverge by violating the resource constraint implied by eq.s $(1)-(3)$ or the transversality condition in (8). Moreover, as long as $\alpha_I \in (0,1)$, whenever the selected value of $\overline{K}$ undershoots – or overshoots – its steady-state reference, then the same must hold even for $Y_C(0)$.

Interestingly, in the model economy under scrutiny, the determinacy of the equilibrium path followed by Y_C and K – and thorough them by all the other endogenous variables – is verified even when the technology prevailing in the consumption goods industry – which may even be the one that produces the largest share of total output – is not convex, i.e., whenever α_C is higher than 1. Obviously, this feature of the model follows from the fact that Y_C – differently from Y_I – does not contribute to capital accumulation.

The analogy with the textbook Ramsey model recalled above can be further extended; indeed, it is worth noticing that the two elements on the main diagonal of $\boldsymbol{J}$ are exactly the same that would prevail in a companion one-sector model with logarithmic preferences and decreasing-returns-to-scale Cobb-Douglas technology (cf. Cass, 1965). By contrast, the values taken by the remaining two elements of $\boldsymbol{J}$ will be not necessarily equal to the corresponding ones of the companion one-sector model and their actual magnitude will depend on the adopted calibration. We all know, however, that in the one-sector model consumption is assumed to reduce capital accumulation one-to-one so that the first element on the second row of $\boldsymbol{J}$ – say $j_{2,1}$ – is always equal to -1, because in that model the decision to consume an additional unit of Y_C reduces savings and investment at the same pace. Considering the expressions in eq. (12), in the theoretical framework under scrutiny this holds when the parameters of the model fulfil a particular relationship which is described by the following equality:

$$K_C^* = \left(\frac{\alpha_C S_C}{\rho + \delta}\right)^{\frac{1}{1-\alpha_C}} \tag{18}$$

The equality in eq. (18) is the one that prevails when the marginal propensity to consume of the optimizing agent is strictly complementary to its marginal propensity to save and – recalling the result in eq. (10) – it corresponds to the case in which K_C^* is determined just in the same way of K_I^*, i.e., by equating the marginal productivity of capital employed in the sector of consumption goods to the opportunity cost of capital which is given by the sum between the discount and the depreciation rates.

Suppose now that $\mathbb{V}_1(\lambda_1) \in \mathbb{R}^{2\times 1}$ is the column-eigenvector associated to the convergent eigenvalue. Thereafter, the dynamics of the output in the sector of consumption goods and the one of capital are conveyed by

$$\begin{pmatrix} \dot{Y}_C(t) \\ \dot{K}(t) \end{pmatrix} = \begin{pmatrix} Y_C^* \\ K^* \end{pmatrix} + \begin{bmatrix} \frac{v_{1,1}}{v_{2,1}} \\ 1 \end{bmatrix} \exp(\lambda_1 t) \left(\overline{K} - K^*\right) \tag{19}$$

where $v_{i,j}$ is the i-th element of $\mathbb{V}_j(\lambda_j)$.

Relying on a suitable discretization and an ad hoc calibration, the expression in (19) will be used to analyse the numerical properties of the two-sector model economy.

3 Numerical properties

In order to assess its numerical properties, the theoretical framework developed above is calibrated on an annual basis by taking as reference the US economy. Specifically, following Christiano and Fisher (2003), the values of the two sectorial elasticities of output with respect to capital – respectively, α_C and α_I – are taken from the estimates of Hornstein and Praschnick (1997). The common value of the depreciation rate of capital (δ) is fixed as in Kydland and Prescott (1982). The value of the discount rate (ρ) is set at the point value suggested by Itskhoki and Moll (2019). Moreover, after having normalized to 1 the value of S_I, the one of S_C is fixed in order to replicate the observed long-run level of the investment-output ratio whose actual magnitude – according to historical data – is given by 17.33%.[1] The values of each parameter together with their description is summarized in Table 1.

Parameter	Description	Value
S_C	Productivity in the sector of consumption goods	2.14
α_C	Elasticity of capital in the sector of consumption goods	0.45
S_I	Productivity in the sector of investment goods	1.00
α_I	Elasticity of capital in the sector of investment goods	0.26
δ	Capital depreciation rate	0.10
ρ	Discount rate	0.03

Table 1: Calibration

The figures in Table 1 imply that in the steady-state equilibrium the share of capital employed in the sector of consumption goods (K_C^*/K^*) amount to 80% so that the remaining 20% is allocated in the sector of investment goods. Unsurprisingly, in the calibrated labourless economy the consumption goods industry is more capital-intensive than the investment goods one. In addition, under the baseline calibration the equality in eq. (18) is not verified.

[1] Macroeconomic US data on national account can be retrieved at https://fred.stlouisfed.org.

Specifically, the adopted parameter values imply that the first element on the second row of $\boldsymbol{J}$ is less than one in absolute value. To be precise, $j_{2,1} = -0.4844$. This means that according to the parameter values in Table 1 the decision to produce and consume an additional unit of Y_C slows down capital accumulation but less then one-to-one as it happens in the conventional one-sector model of optimal growth (cd. Cass, 1965). In other words, given the values of the involved parameters, in the two-sector model developed above the marginal productivity of the stock of capital employed in the sector of consumption goods is so high that the production and the consumption of an additional unit of Y_C is possible by slowing capital accumulation by a lower amount.[2] Therefore, in the calibrated model, the sum between the marginal propensity to consume and the marginal propensity to save – and invest – is larger than 1 (cf. Senouci, 2011).

On a visual perspective, assuming that initially there is less capital than the level desired in the long run, the parameters values in Table 1 lead to the trajectories of Y_C and K illustrated in the diagram of Figure 2.[3]

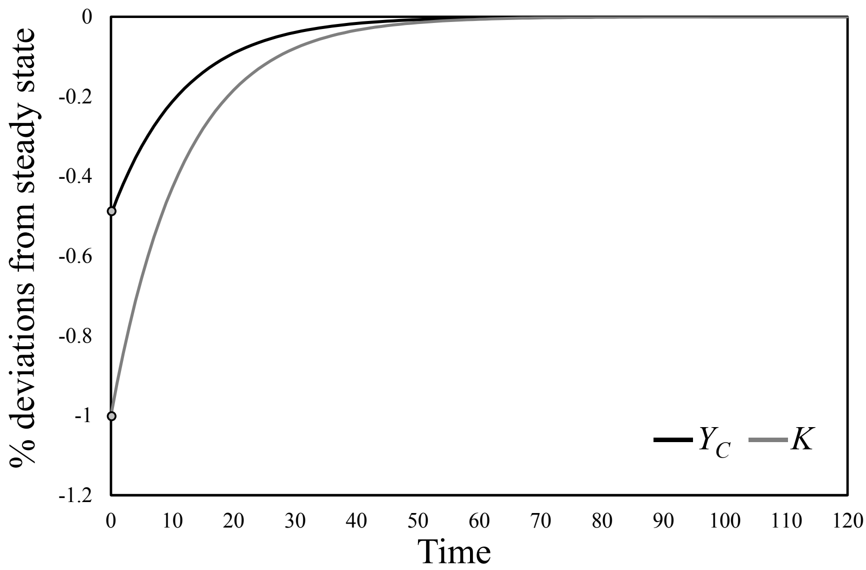

Figure 2: Saddle-path dynamics

The plot in Figure 2 shows that when the stock of capital undershoots its steady-state value by 1% the production of consumption goods does the same but to a lower extent (about 0.5%). Obviously, the smaller deviations of Y_C from Y_C^* during the adjustment process are due to the risk-aversion implied by the logarithmic preferences of the social planner that solves the intertemporal problem in (4). Thereafter, as it happens in the standard one sector model, Y_C

[2] Obviously, this follows from the fact that K_C^* is lower than the amount appointed by eq. (18).

[3] The time step of the simulations is set to 1. MAT LAB codes are available from the author upon reasonable request.

and K monotonically converge towards their long-run references on the saddle path mentioned in Section 2.

From a numerical point of view, another interesting aspect of the model economy under scrutiny is certainly the way in which capital is optimally allocated over time in the two sectors of production. The implied adjustments of K_C and K_I obtained when K initially undershoots its steady-state value by 1% as it does in Figure 2 are plotted in Figure 3.

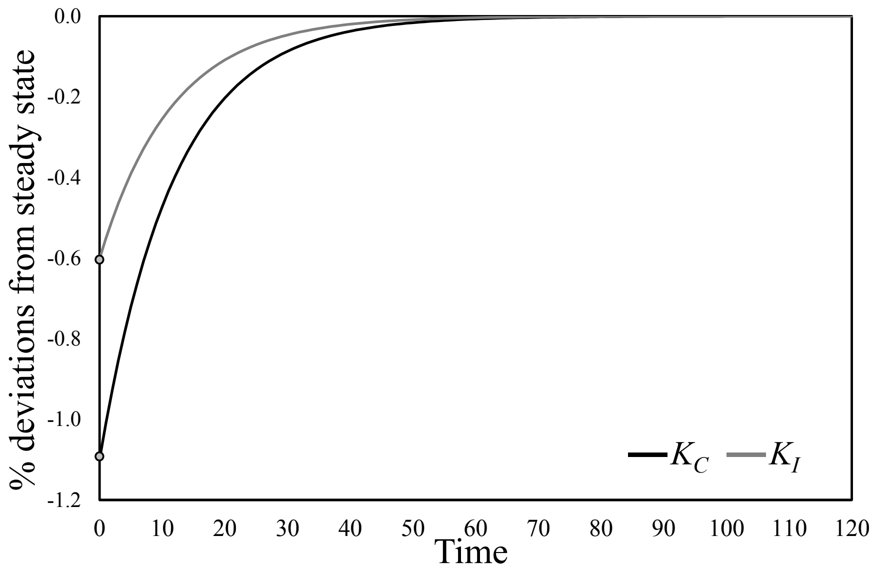

Figure 3: Capital adjustments in the two sectors

The plot in Figure 3 shows that the amount of capital allocated in the sector of consumption (investment) goods undershoots its long-run reference to a higher (lower) extent with respect to the aggregate stock of capital and then they monotonically converge towards their steady-state values with no factor reversal.

As far as the sectoral allocation of capital is concerned, it is worth recalling that in a decentralized setting in which the two goods are produced by atomistic profit-maximizing firms that take the prices of the two goods as given, the value of the marginal product of capital must be equalized across industries in order to verify the condition for the allocative efficiency of the unique production factor (cf. Huzawa, 1961, 1964; Senouci, 2011). Consequently, in a competitive setting, the ratio between the marginal productivities of capital employed in the two sectors must return the implicit path of the relative price of the two goods that implements the first-best allocation described by the solution of the problem in (4). Formally speaking, according to the expressions implied by eq. (2), such a relative price should be equal to

$$\frac{p_I(t)}{p_C(t)} = \frac{\alpha_C S_C (K_I(t))^{1-\alpha_I}}{\alpha_I S_I (K_C(t))^{1-\alpha_C}} \tag{20}$$

where $p_I(t)$ ($p_C(t)$) is the price of investment (consumption) goods.

Eq. (20) shows that the relative price of investment goods is a decreasing function of the total factor productivity (TFP) in that sector relative to the TFP in the consumption good sector whereas its path depends on the allocation of capital in the two industries over time (cf. Ferreira et al. 2014). The frictionless path of the ratio p_I/p_C implied by the adjustments of K_C and K_I plotted in Figure 3 is illustrated in Figure 4.

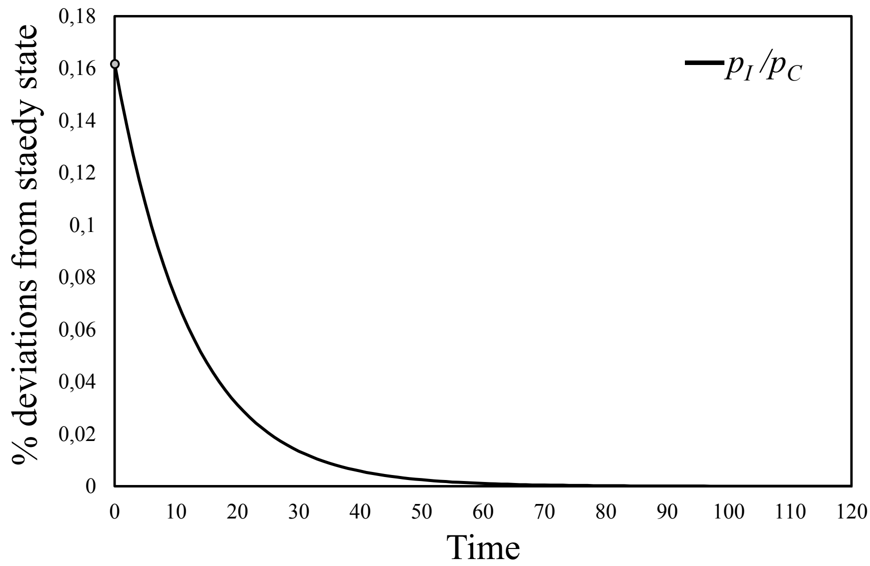

Figure 4: The implied path of the relative price of investment goods

The plot in Figure 4 reveals the price of investment goods measured in units of the price of consumption goods which is consistent with solution of the central planner problem follows a decreasing pattern towards its steady-state value. Since the quantity adjustments plotted in Figures 2 and 3 imply that the production of the two goods – Y_C and Y_I – follows an increasing pattern towards the respective steady-state values, this means that whenever the supply functions of the two goods are upward sloped the growth rate of the price of investment goods has to fall short of the growth rate of the price of consumption goods all over the adjustment process towards the stationary solution.[4]

The declining path of the ratio p_I/p_C tracked in Figure 4 is a feature of the adopted calibration that follows (i) from the convexity of both production technologies, i.e., from the fact that $\alpha_i \in (0,1)$ for $i = C, I$, and (ii) from the fact that the production of consumption goods always exceeds the one of investment goods because of the adopted values of S_j, $j = C, I$. From an empirical point of view, this kind of dynamic behaviour of prices is consistent with the

[4]In a one-factor economy, denoting with $i(t)$ the rental rate paid by firms for each unit of employed capital, the supply function of each goods would be given by $p_j(t) = i(t)(Y_j(t))^{1/(1-\alpha_j)}/\alpha_j S_j^{1/\alpha_j}$, $j = C, I$.

countercyclical pattern displayed by actual data on the relative price of tradable capital goods stressed by many authors and often attributed to technical progress and/or market integration (cf. Collins and Williamson, 2001; Restuccia and Urrutia, 2001; Fisher, 2006; Justiniano et al. 2011; Hergovich and Merz, 2018; Lian et al. 2020).[5] Relaying on a competitive setting, the labour-less two-sector model developed above shows that even in the absence of these factors a decreasing path of the relative price of capital goods is still possible (cf. Guerrazzi and Candido, 2025).

To close the numerical analysis of the model it is worth noticing that as long as $\alpha_I \in (0, 1)$, assuming that in the sector of consumption goods there is a non-convex technology, i.e., taking a value of α_C is higher than 1, would not alter the qualitative picture of the quantity adjustments tracked in Figures 2 and 3. Nevertheless, no matter the actual share of produced outputs, it would produce an increasing pattern for the relative price of capital goods which is against both to competitive pricing in the sector of consumption goods and the available empirical evidence.

4 Concluding remarks

In this paper, I developed a parsimonious two-sector optimal growth model with concave utility in which the output of the two industries is produced only by means of capital. From a theoretical perspective, this setting revealed that the determinacy of the equilibrium path followed by the optimal production of consumption goods and the total stock of capital is possible even when the production technology available in the industry of consumption goods does not necessarily displays diminishing marginal returns and that sector – as it usually happens in real-world economies – is the one that produces the largest share of total output. By contrast, a convex technology prevailing in the sector of investment goods is instead necessary for such a determinacy and it is also responsible for the determination of the share of capital employed in each industry as well as of the magnitude of the golden-rule level of capital (cf. Senouci, 2011). Moreover, from a numerical point of view, I showed that a calibrated version of the model supplemented with competitive pricing is able to dampen the complementarity between the marginal propensities to consume and save of the optimizing agent and to replicate the observed long-lasting decline of the relative price of tradable capital goods even without technical progress (cf. Collins and Williamson, 2001; Restuccia and Urrutia, 2001; Fisher, 2006; Justiniano et al. 2011; Hergovich and Merz, 2018; Lian et al. 2020).

The analysis carried out in this paper can be developed in many different directions. A natural extension could be the re-introduction of labour and the derivation of a version of the model in which labour supply is endogenous, a feature never addressed by the literature on two-sector models of optimal growth with the exception of settings characterized by very

[5]Further evidence of the dynamic behaviour of the relative price of capital goods in the US is given in Appendix.

peculiar assumptions about the preferences of the optimizing agent and/or about the factor elasticities displayed by the technologies available in the two industries (cf. Zhang, 2005; Harrison and Weder, 2002; Benhabib and Nishimura, 1998; Benhabib and Farmer, 1996). The non-reproducible factor, however, can hardly be considered as malleable as capital so it would be worth to consider not only the situation in which labour is employed in both industries, but also the two distinct cases in which labour is somehow immobile and it enters only one of the production technologies prevailing in the two sectors (cf. Hashimoto and Sakuragawa, 1999).

Furthermore, while the one-sector model of optimal growth deals with the case of output homogeneity, it is also worth recalling that the two-sector model deals with the opposite polar case in which there is full specialization of production in each industry. In real-world economy, however, it often happens that a commodity is tagged as a consumption or as an investment good depending on the identity of the agent involved in the purchase of the commodity itself. This feature suggests that whenever the national output is not a homogenous good, a decentralized perspective – bypassed in this paper – in which consumers and firms take their own decisions deserves to be taken in serious consideration to understand the actual drivers of capital accumulation and the determinants of households' utility.

Appendix: The observed path of the relative price of capital goods

An empirical appraisal of the path followed by the relative price of capital goods in the US can be grasped by taking the ratio between the estimated values of the producer price index for capital equipment (PPICPE) and price index for personal consumption expenditure (PCEPI). The pattern of such a ratio over the period 1959-2014 together with its linear trend is illustrated in Figure A1.

Consistently with the calibrated version of the model presented in the main text, the graph in Figure A1 shows that the actual relative price of capital goods followed a decreasing path. In real-world data, the main deviations from this monotonic pattern occurred in correspondence of the two oil shocks of the Seventies (1973 and 1979) and – to a lower extent – during the Great Recession of 2008-2009. Those divergences may signal that during episodes of that type the values of the productivity parameters in eq. (2), namely, S_C and S_I, can hardly be taken as time-invariant (cf. Ferreira et al. 2014).

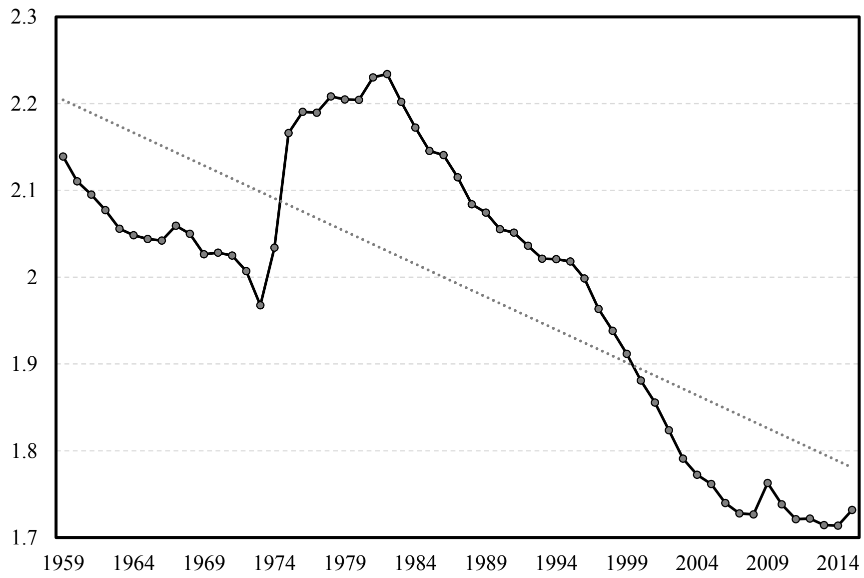

Figure A1: The relative price of capital goods in the US (1959-2014)

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