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Chow, Nikolai Sheung-Chi

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Nikolai Sheung-Chi CHOW *

Faculty of Finance, City University of Macau

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Abstract:

This study advances the understanding of risk measures and portfolio choice for investors exhibiting gain-loss dependent risk attitudes by integrating stochastic dominance (SD) concepts, including prospect stochastic dominance (PSD) and Markowitz stochastic dominance (MSD). We demonstrate that partial moments serve as effective risk measures, aligning with various SD criteria to capture diverse investor attitudes toward gains and losses. One contribution of this paper is the development of a decision-making criterion to identify the segment of the mean-variance efficient frontier that is efficient under different SD conditions, applicable to elliptical distributions. Leveraging partial moments, we adopt a portfolio optimization method that constructs portfolios dominating a benchmark from multiple SD perspectives, facilitating comparisons across gain-loss utility models. This approach enables a more direct comparison of alternative gain-loss utility models without relying on parameter assumptions, which often lead to differing risk-return priorities within a model.

Keywords : Gain-Loss Utility, Mean-Variance Analysis, Stochastic Dominance, Partial Moments, Prospect Theory

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1 Introduction

The Mean-Variance (MV) analysis, established by Markowitz (1952a), has fundamentally shaped the landscape of finance. Central to the MV framework is the identification of risk as the variance of returns, advocating for a portfolio that optimally balances expected return against risk. The MV framework's symmetrical view of return deviations — treating gains and losses equally — has faced criticism. In the context of risk evaluation, there is a common belief that losses should receive greater emphasis, or even exclusive focus. Consequently, this perspective has driven the development of risk measures and investment rules that focus on "downside risks", which specifically address negative deviations from a reference point.

Early suggestion by Markowitz (1959) to consider semi-variance as a more representative risk measure has evolved into a diverse set of downside risk measures. These include the Lower Partial Moment (LPM) (e.g., Bawa et al., 1985), Value-at-Risk (e.g., Jorion, 2007), Conditional Value-at-Risk (e.g., Rockafellar and Uryasev, 2000), Maximum Loss (e.g., Young, 1998), and others. In parallel, the Capital Asset Pricing Model (CAPM) proposed by Sharpe (1964) similarly focuses on risk as measured by return variance. As critiques of variance-based risk have mounted, new asset pricing frameworks incorporating downside betas or partial moments have emerged and are being explored (e.g., Hogan and Warren, 1974; Bawa and Lindenberg, 1977; Ang et al., 2006; Anthonisz, 2012; Bollerslev et al., 2022).

It is commonly believed that loss aversion in prospect theory is one of the reasons why concern about downside risk is significant (e.g., Ang et al., 2006; Farago and Tédongap, 2018; Bollerslev et al., 2022). [1] Loss aversion, a key component of the decision rule in prospect theory proposed by Kahneman and Tversky (1979) and further refined by Tversky and Kahneman (1992), suggests that losses have a greater impact on individuals than equivalent gains when evaluating risky choices. In this context, a 'gain' is defined as any positive deviation from a reference point, while a 'loss' is any negative deviation, with investment decisions often benchmarking a zero return as the reference point. Given its empirical support from numerous real-world observations (e.g., Ljungqvist and Wilhelm Jr, 2005; Fox and Poldrack, 2009; Abdellaoui et al., 2013), this concept is a strong candidate for explaining investment decisions.

A line of work related to loss aversion that provides a theoretical foundation for "downside risk" models is the mean-partial lower moment (MPLM) framework (sometimes referred to as generalized Roy's safety-first rule) proposed by Hogan and Warren (1974) and Bawa and Lindenberg (1977). The MPLM framework suggests choosing a portfolio that minimizes the first-order LPM or the second-order LPM, given an expected return level. As pointed out by Bawa (1978), the first-order MPLM criterion (second-order MPLM criterion) can be seen as maximizing the expectation of a utility function that is linear for

[1] Another example is the disappointment model proposed by Gul (1991) and the safety-first principle discussed in Roy (1952) and Bawa (1978).

returns exceeding the reference return, with a linear penalty for outcomes below the reference return (with a quadratic penalty for outcomes below the reference return). Although this line of work didn't mention anything about loss aversion, these utility models do imply constant loss aversion, a specific form of loss aversion.[2]

The relationship between loss aversion and investors' attitudes toward volatility, particularly in the context of downside risk, remains theoretically unclear. Specifically, it is uncertain whether loss aversion inherently implies that investors are averse to volatility only when it leads to losses, or how loss aversion relates to downside risk more broadly. Formally, given a continuous utility function u, loss aversion is modeled by $u'(-x) \geq u'(x)$ for all $x \geq 0$ when derivatives exist (Tversky and Kahneman, 1992). This condition is satisfied by a concave utility function, indicating that investors are also averse to variation from gains when decisions are based on comparing expected utilities. In prospect theory, risk attitude is jointly affected by loss aversion, curvature of utility, and probability weighting.[3] Without more assumptions about utility, loss aversion does not have direct implication on risk attitude (e.g., Köbberling and Wakker, 2005; Schmidt and Zank, 2008).

Given the above context, diminishing sensitivity, another important aspect of prospect theory, further complicates the analysis. Diminishing sensitivity implies that the perceived significance of a change decreases as the distance from a reference point increases. When this principle is applied to both gains and losses, it suggests that the perceived value difference between a gain (or loss) of 100 and 200 is more pronounced than that between 1,100 and 1,200. This indicates that the marginal 'utility' of gains decreases and the marginal 'disutility' of losses also decreases, implying that individuals are risk-seeking in the loss domain , and risk-averse in the gain domain, when decision maker processes probabilities linearly.[4]

The gain-loss asymmetry, which arises from loss aversion and changing sensitivity, indicates that individuals' risk preferences are not fixed but vary depending on whether they are facing potential gains, losses, or specific combinations of both (e.g., Davies and Satchell, 2007; Zakamouline and Koekebakker, 2009). Prospect theory, addressing loss aversion and changing sensitivity, employs a piecewise utility function that incorporates separate formulations for gains and losses. Similarly, it is common in other financial models to adopt piecewise utility functions defined differently across two intervals based on a specified reference point. Another notable gain-loss utility model in the finance literature was proposed by Markowitz (1952b), which features a utility function that contains a segment that is concave for values below current wealth and a segment that is convex for values above it. This model, aiming to address shortcomings in the earlier utility model by Friedman and Savage (1948), thus indicating a distinct gain-

[2] For discussion on constant loss aversion see Tversky and Kahneman (1991).

[3] We refer to the "value function" in Tversky and Kahneman (1992) as the utility function here.

[4] In the context of prospect theory, the utility function is commonly referred to as the "value function".

loss dependent risk attitude compared to that proposed by prospect theory. [5]

Under this context, Levy and Levy (2004) were among the first to explore how gain-loss dependent risk attitudes might influence the MV rule by examining utility functions that exhibit diminishing sensitivity. In a different approach, Zakamouline and Koekebakker (2009) approximates the expected utility function using mean and partial moments given a piecewise-quadratic utility function that accounts some form of loss aversion. This implies that mean and partial moments can be used as portfolio performance measure for investors with utility function algin with prospect theory. Building on this foundation, Zakamouline (2014) generalized it to a piecewise linear plus power utility function. Under a similar setting, León and Moreno (2017) derive closed-form expressions for LPM and upper partial moment (UPM) measures by assuming the Gram-Charlier density for stock returns.

Given the close association between the MV rule and portfolio choice, a natural inquiry arises: what insights can be gleaned from the gain-loss asymmetry? Levy and Levy (2004) demonstrated that a subset of the MV-efficient set remains efficient across all utility functions that exhibited diminishing sensitivity. There have been several papers that examine portfolio choice under prospect theory, such as Berkelaar et al. (2004), which analyzed the optimal portfolio choice for loss-averse investors, and Bernard and Ghossoub (2010) and He and Zhou (2011), who conducted an analytical treatment of a single-period portfolio choice model for agents with features suggested by prospect theory.

Despite these advancements, several questions remain unanswered. This study addresses the following unresolved issues:

- Can a more direct link be established between gain-loss dependent risk attitudes and risk measures? How is concern for downside risk manifested under varying gain-loss dependent risk preferences?
- In what ways does the mean-variance analysis adapt when confronted with different gain-loss dependent risk attitudes?
- What are the implications of these diverse risk preferences for portfolio selection, particularly when investors' preferences are only partially observable?

To tackle these questions, we utilize the stochastic dominance conditions proposed by Levy and Wiener (1998), Levy and Levy (2002), and Baucells and Heukamp (2006). Levy and Wiener (1998) introduced the concept of "prospect stochastic dominance" (PSD), a modified SD condition that enables the comparison of the expected utilities of investments for a class of utility functions characterized by diminishing sensitivity. Baucells and Heukamp (2006) further explored the PSD concept and introduced an SD condition that allows for the comparison of the expected utilities of investments for utility functions

[5] Some evidences on supporting such function is provided by Fishburn and Kochenberger (1979).

exhibiting both diminishing sensitivity and loss aversion. Conversely, Levy and Levy (2002) presented the concept of "Markowitz stochastic dominance" (MSD), an SD condition facilitating the comparison of the expected utilities of investments for utility functions that are concave on gains and convex on losses, akin to the function proposed by Markowitz (1952b).

Our analysis yields three principal findings. First, we show that partial moments can serve as effective risk measures for downside and upside risks under various gain-loss dependent risk attitudes. Unlike Zakamouline and Koekebakker (2009); Zakamouline (2014) and León and Moreno (2017), we avoid assuming piecewise-quadratic or linear plus power utility functions, or constant loss aversion, which may not hold in reality, as suggested by Brooks and Zank (2005).

Second, we refine MV efficiency concepts by embedding partial moments into standard portfolio analysis, identifying a "modified MV frontier" that is robust to a wide range of investor attitudes toward downside risk. Extending Levy and Levy (2004), we pinpoint the PSD-efficient MV frontier segment, creating a dual MV- and PSD-efficient frontier, valid for elliptical distributions (e.g., multivariate normal, t, Cauchy, logistic).

Third, by incorporating partial moments into an optimization method that maximizes returns subject to probabilistic constraints, we illustrate that portfolio selections accounting for diverse risk attitudes can be compared without relying on parameter assumptions. Assuming a specific utility function like Berkelaar et al. (2004), Bernard and Ghossoub (2010), and He and Zhou (2011) may not be the best way to compare portfolio selections for different gain-loss models since parameter variations can alter risk-return trade-offs. It also remains questionable whether any one utility function can adequately capture the heterogeneous preferences of a diverse investor base. This framework thus allow portfolio managers to construct portfolios that better reflect their clients' risk attitudes, particularly those influenced by loss aversion.

The remainder of this chapter is organized as follows. Section 2 discusses the concept of SD. In Section 3, we explore the relationships between SD conditions and partial moments. Section 4 demonstrates how incorporating partial moments into the MV framework can render it equivalent to analyses that employ different gain-loss utility models, under the assumption of elliptical distributions. In Section 5, we illustrate that portfolio choices can be compared for different gain-loss utility models even when we have incomplete information about investors' preferences. Finally, Section 6 provides concluding remarks. To enhance clarity, we distinguish throughout the chapter between established results, presented as properties and propositions, and our original contributions, which are formulated as lemmas and theorems. All proofs of these lemmas and theorems are provided in Section B.

2 Stochastic Dominance conditions

Consider two risky investments, characterized by the random variables X and Y, representing their random returns. The cumulative distribution functions (CDFs) of X and Y are denoted as F and G, respectively. An agent with utility function u is considered to prefer X over Y if the following inequality is satisfied:

$$\mathbb{E}[u(X)] - \mathbb{E}[u(Y)] = \int u(x)dF(x) - \int u(x)dG(x) \geq 0.$$

That is, between two feasible investments, an agent will chose the one with higher expected utility. This is also equivalent to assume a linear processing of probabilities for investors.[6] We begin by defining several classes of utility functions as follows.

Definition 2.1 (Sets of utility functions). *Given CDFs F and G. Let* $\mathcal{U}^A$,$\mathcal{U}^D$, $\mathcal{U}^S$, $\mathcal{U}^R$, $\mathcal{U}^S(F,G)$, $\mathcal{U}^R(F,G)$, $\mathcal{U}^{SL}(F,G)$, *and* $\mathcal{U}^{RL}(F,G)$ *be the sets of the utility functions u such that:*

$$\begin{aligned}
\mathcal{U}^A\left[\mathcal{U}^D\right] &:= \left\{ u : u \text{ is non-decreasing and concave}\left[\text{convex}\right] \right\},\\
\mathcal{U}^S\left[\mathcal{U}^R\right] &:= \left\{ u :\ u^+ \in \mathcal{U}^A[\mathcal{U}^D] \text{ and } u^- \in \mathcal{U}^D[\mathcal{U}^A] \right\},\\
\mathcal{U}^L &:= \left\{ u :\ u'(-x) \geq u'(x) \text{ for all } x \geq 0, \text{ if derivatives exit} \right\},\\
\mathcal{U}^S(F,G)\left[\mathcal{U}^R(F,G)\right] &:= \left\{ u :\ u \in \mathcal{U}^S[\mathcal{U}^R] \text{ and } \int u\, dF \,\&\, \int u\, dG \text{ are finite} \right\},\\
\mathcal{U}^{SL}(F,G) &:= \left\{ u :\ u \in \mathcal{U}^S(F,G) \cap \mathcal{U}^L \text{ and } u''(-x) \geq -u''(x) \text{ for all } x \geq 0, \text{ if derivatives exit} \right\},\\
\mathcal{U}^{RL}(F,G) &:= \left\{ u :\ u \in \mathcal{U}^R(F,G) \cap \mathcal{U}^L \text{ and } -u''(-x) \geq u''(x) \text{ for all } x \geq 0, \text{ if derivatives exit} \right\}.
\end{aligned}$$

where u is a continuous utility function such that $u : \mathbb{R} \to \mathbb{R}$, $u^+ = u$ *restricted for* $x \geq 0$, *and* $u^- = u$ *restricted for* $x \leq 0$.

$\mathcal{U}^A$ and $\mathcal{U}^D$ are the set of all non-decreasing concave functions and non-decreasing convex functions. $\mathcal{U}^S$ is the set of S-shaped functions that are concave on gains and convex on losses, highlighting a risk-averse attitude towards pure gains and risk-seeking for pure losses; and $\mathcal{U}^R$ for reverse S-shaped functions, convex on gains and concave on losses. The inclusion of $\mathcal{U}^S(F,G)$ and $\mathcal{U}^R(F,G)$ ensures the expected utilities under given distributions are finite. $\mathcal{U}^S(F,G)$ is the class of all S-shaped utility functions including those in Tversky and Kahneman (1992). [7]$\mathcal{U}^R(F,G)$ is the class of all Markowitz utility functions analyzed in Levy and Levy (2002).

[6] See chapter 5 in Wakker (2010) for related discussion.

[7] Empirical studies, including Kahneman and Tversky (1979), Fiegenbaum (1990), Sinha (1994), and Duxbury and Summers (2004), have shown that people often prefer higher variance when facing pure losses, which is a consequence of S-shaped utility function.

In studies related to loss aversion, a common assumption is constant loss aversion, typically modeled by introducing a scaling factor (often denoted by λ), which adjusts the utility magnitude in the loss domain.[8] This assumption could be relaxed by considering a more general version of loss aversion, that is $u(-y) - u(-x) \geq u(x) - u(y)$ for all $0 \leq y \leq x$ in $\mathcal{U}^{SL}(F,G)$ and $U^{RL}(F,G)$. In addition, $\mathcal{U}^{SL}(F,G)$ and $U^{RL}(F,G)$ also impose the condition that absolute changes in curvature when moving from losses to more significant losses are at least as great as the absolute changes in curvature from gains to more significant gains.[9]

Traditional stochastic dominance (SD) conditions include first degree stochastic dominance (FSD) and second degree stochastic dominance (SSD). FSD requires that one distribution dominate another for all non-decreasing utility functions, while SSD extends this to all non-decreasing and concave utility functions, reflecting risk aversion everywhere. The optimal selection rules, formulated as SD conditions, apply to the utility function classes discussed above are defined in Definition 2.2. Most of these rules are examined in Levy and Wiener (1998), Levy and Levy (2002), and Baucells and Heukamp (2006).

Definition 2.2 (Stochastic Dominance Conditions)**.** *Given CDFs F and G, Stochastic Dominance concepts are defined as follows:*

(i) $F \succeq_P G$ if $\int_{x_1}^{0} [G(z) - F(z)]\, dz \geq 0$ and $\int_{0}^{x_2} [G(z) - F(z)]\, dz \geq 0$, $\forall\, x_1 \leq 0 \leq x_2$.

(ii) $F \succeq_M G$ if $\int_{-\infty}^{x_1} [G(z) - F(z)]\, dz \geq 0$ and $\int_{x_2}^{\infty} [G(z) - F(z)]\, dz \geq 0$, $\forall\, x_1 \leq 0 \leq x_2$.

(iii) $F \succeq_p G$ if $\int_{-x}^{0} [G(z) - F(z)] dz \geq \max\left\{0, \int_0^x [F(z) - G(z)] dz\right\}$, $\forall\, x \geq 0$.

(iv) $F \succeq_m G$ if $\int_{-\infty}^{-x} [G(z) - F(z)] dz \geq \max\left\{0, \int_x^{\infty} [F(z) - G(z)] dz\right\}$, $\forall\, x \geq 0$.

The stochastic order $F \succeq_P G$ represents the prospect stochastic dominance condition (PSD) proposed in Levy and Wiener (1998), which denotes that F dominates G by PSD. Similarly, $F \succeq_M G$ is the Markowitz stochastic dominance condition (MSD) proposed in Levy and Levy (2002), indicating that F dominates G by MSD. The condition $F \succeq_p G$, proposed in Baucells and Heukamp (2006), denotes that F dominates G by the modified PSD that takes loss aversion into account (PSD-L). Lastly, $F \succeq_m G$ is the new SD condition that signifies F dominates G by the modified MSD that also takes loss aversion into account (MSD-L). In addition, $F \succeq_{FSD} G$ ($F \succeq_{SSD} G$) denotes that F dominates G by FSD (SSD). All SD conditions defined here are partial orders; thus, $F \succ G$ indicates $F \succeq G$ and $G \not\succeq F$. Note also that in this paper, $X \succeq Y$ is equivalent to $F \succeq G$ when F is the CDF of X and G is the CDF of Y.

[8] For more details see Tversky and Kahneman (1991).

[9] In alignment with existing literature, we adopt the status quo as the reference point. Nonetheless, it is feasible to substitute this baseline of zero return with alternative return levels.

The relationship between some of the above SD conditions and different classes of utility functions is explored in Levy and Wiener (1998), Levy and Levy (2002), and Baucells and Heukamp (2006). We build upon and extend their work to more comprehensively cover cases with unbounded variables as follows.

Theorem 2.1. *Let F and G be CDFs of random variables X and Y respectively, with $\mathbb{E}[|X|] < \infty$ and $\mathbb{E}[|Y|] < \infty$, then the following results hold:*

(i) $F \succeq_P G$ if and only if $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$ for all $u \in \mathcal{U}^S(F,G)$.

(ii) $F \succeq_M G$ if and only if $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$ for all $u \in \mathcal{U}^R(F,G)$.

(iii) $F \succeq_p G$ if and only if $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$ for all $u \in \mathcal{U}^{SL}(F,G)$.

(iv) $F \succeq_m G$ if and only if $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$ for all $u \in \mathcal{U}^{RL}(F,G)$.

Theorem 2.1 establishes a nuanced relationship between stochastic dominance and utility models, delineating how investors' preferences for risky investments are shaped by their risk attitudes as encoded in their utility functions. These conditions allow us to compare investments even when we have incomplete information about investors' preferences. [10]

Additionally, we are interested in the class of non-decreasing, continuous utility functions that imply the MLPM criterion or generalized Roy's safety-first rule. The MLPM criterion states that, for $n = 1,2$, given the same mean return and reference outcome level d, decisions should be based on minimizing the n-order LPM. As pointed out by Bawa (1978), the mean-lower partial moment criterion is equivalent to the maximization of the expectation of $v(x;n,d)$:

$$v(x;n,d) = \begin{cases} \alpha + \beta x & \text{for } x \geq d, \\ \alpha + \beta x - \lambda(d-x)^n & \text{for } x < d. \end{cases} \tag{2.1}$$

for $n = 1,2$, where α, β, and λ are positive constants. If we consider the reference outcome d as the status quo, then we can view the above utility function from the perspective of changing sensitivity

[10] Levy and Wiener (1998) addresses scenarios involving unbounded random variables. In Theorem 4 of Levy and Wiener (1998), U_p is the set of all S-shaped utility functions as suggested by Kahneman and Tversky (1979). $u \in U_p$ is equivalent to $u'(x) \geq 0$ for all x, $u''(x) \leq 0$ for all $x > 0$, and $u''(x) \geq 0$ for all $x < 0$. Consider u_0 as follows:

$$u_0(x) = \begin{cases} x^{0.88} & \text{if } x \geq 0 \\ -2.25(-x)^{0.88} & \text{if } x < 0 \end{cases}$$

which is the one estimated by Tversky and Kahneman (1992) and $u_0 \in U_p$. Now take $F(x)$ and $G(x)$ as follows:

$$F(x) = G(x) = \begin{cases} 1 - \frac{1}{x^{0.38}} & \text{if } x \geq 1 \\ 0 & \text{if } x < 1 \end{cases}$$

Then $\int_{x_1}^{x_2} [G(z) - F(z)]dz = 0$ for all $x_1 \leq 0 \leq x_2$, that is, $F \succeq_P G$. However, the difference between $\mathbb{E}[u(X)]$ and $\mathbb{E}[u(Y)]$ in this case is not well defined since both of these are equal to $\int_1^\infty \frac{0.38}{\sqrt{x}}dx$ which is infinite.

and loss aversion. The utility function v belongs to the class $\mathscr{U}^{RL}(F,G)$ when we want to compare the expectation of $v(X;n,d)$ and $v(Y;n,d)$.

To gain additional insights, we also consider more restricted classes of utility functions that include $v(x;n,d)$ for $n=1,2$. For instance, one class comprises utility functions in both $\mathscr{U}^R$ and $\mathscr{U}^{SL}(F,G)$, where the utility function is linear on both gains and losses and incorporates a loss aversion feature—this class includes $v(x;1,d)$. In the case of $v(x;2,d)$, we distinguish two scenarios. First, we consider utility functions in both $\mathscr{U}^A$ and $\mathscr{U}^R(F,G)$, where the function is linear on gains and concave on losses without requiring loss aversion. Second, we consider those in both $\mathscr{U}^A$ and $\mathscr{U}^{RL}(F,G)$, where the function is linear on gains, concave on losses, and must satisfy the loss aversion criterion. The optimal selection rules for these classes are detailed in Section A.

SD conditions corresponding to the above different investor preferences are denoted as follows. The condition $F \succeq_L G$ hold for F to be preferred to G by all investors with non-decreasing, continuous utility functions that are linear in both gains and losses and exhibit loss aversion. The condition $F \succeq_R G$ applies to investors whose utility functions are non-decreasing and continuous, concave in the loss domain, and linear in the gain domain—implying risk aversion for losses and risk neutrality for gains. Finally, the condition $F \succeq_r G$ is imposed when F is preferred to G by all investors whose utility functions are non-decreasing, continuous, risk-averse for losses, and exhibit loss aversion while remaining linear in the gain domain.

3 Implications on Security and Potential

The distinction between managing "good" and "bad" outcomes is a recurring theme in investment decision-making studies. Markowitz's portfolio theory frames mean returns as "good" and variance as "bad", viewing the latter as a risk measure to be minimized. Roy's "safety-first" principle further emphasized the importance of avoiding outcomes below a predetermined threshold, framing underperformance as undesirable. Arzac and Bawa (1977) extended this idea by developing portfolio optimization methods and asset pricing models based on partial moments, enabling a more nuanced assessment of downside risk. Building on these foundations, Shefrin and Statman (2000) introduced behavioral portfolio theory, drawing on Lopes (1987)'s SP/A theory, which highlights two core motivations in decision-making: security (S), which focuses on loss, and potential (P), which emphasizes gain. Bernardo and Ledoit (2000) proposed a model based on ranking investments according to the gain-loss ratio. In this section, we explore how gain-loss utility models are aligned with this framework, illustrating how these models encapsulate the dual motivations of security and potential.

3.1 Necessary conditions

Let μ_X and μ_Y represent the means of X and Y, respectively. Similarly, their variances are denoted as σ_X^2 and σ_Y^2. We begin by discussing an established result concerning implications of PSD, MSD, PSD-L, and MSD-L on expected returns, as detailed in the following corollary:

Corollary 3.1. *If $F \succeq_P G$, $F \succeq_M G$, $F \succeq_p G$ or $F \succeq_m G$, then $\mu_X \geq \mu_Y$.*

Corollary 3.1 is a direct result of Theorem 2.1 since the identity function is contained in $\mathcal{U}^S(F,G)$, $\mathcal{U}^R(F,G)$, and $\mathcal{U}^{SL}(F,G)$ given that expected returns are finite. If a CDF dominates the other CDF in a PSD, MSD, PSD-L, or MSD-L sense, then the expected return of the first risky asset will be greater than that of the second. Furthermore, consider that the expected return can be divided into its two first partial moments. For any random variable Z with $\mathbb{E}[|Z|] < \infty$ and CDF H, we define the following:

$$\mathbb{E}[Z] = \mu_Z^+ - \mu_Z^- \quad \text{such that} \quad \mu_Z^+ := \int_0^\infty x dH(x) \quad \text{and} \quad \mu_Z^- := \int_{-\infty}^0 -x dH(x), \tag{3.1}$$

where μ_Z^+ is the first-order UPM with respect to a reference point 0 (or measure of expected gain) and μ_Z^- is the first-order LPM with respect to a reference point 0 (or measure of expected loss). These formulations allow us to derive the relationship between PSD, MSD, PSD-L, or MSD-L and the first partial moment. Under this setting, necessary conditions for PSD, MSD, PSD-L, and MSD-L that involve first-order partial moments are given in Theorem 3.1.

Theorem 3.1. *If $\mathbb{E}[|X|] < \infty$ and $\mathbb{E}[|Y|] < \infty$, then the following statements are true:*

(i) If $F \succeq_P G$, then $\mu_X^+ \geq \mu_Y^+$ and $\mu_X^- \leq \mu_Y^-$.

(ii) If $F \succeq_M G$, then $\mu_X^+ \geq \mu_Y^+$ and $\mu_X^- \leq \mu_Y^-$.

(iii) If $F \succeq_p G$ or $F \succeq_m G$, then $\mu_Y^- - \mu_X^- \geq \max\left\{0, \mu_Y^+ - \mu_X^+\right\}$.

Theorem 3.1 demonstrates that PSD, MSD, PSD-L, and MSD-L influence not only expected returns but also expected gains and losses, aligning with the security-potential framework of Lopes (1987)'s SP/A theory. Parts (i) and (ii) show that, regardless of whether an investor's utility exhibits diminishing sensitivity or follows Markowitz-type functions (Levy and Levy, 2002), dominance of investment X over Y under PSD or MSD requires higher expected gains ($\mu_X^+ \geq \mu_Y^+$), reflecting greater potential (P), and lower expected losses ($\mu_X^- \leq \mu_Y^-$), enhancing security (S), thus balancing upside opportunities with downside protection. **Part (iii)** highlights that, for loss-averse investors under PSD-L or MSD-L, security is prioritized: any sacrifice in expected gain ($\mu_Y^+ - \mu_X^+$) must be compensated by a corresponding reduction in expected loss ($\mu_Y^- - \mu_X^-$), emphasizing safety when gains are comparable. These conditions illustrate how gain-loss utility models encapsulate the dual motivations of security and potential driving investment decisions.

Theorem 3.1 is also useful for checking whether a decision rule based on risk measures or performance measures that are related to first-order UPM or first-order LPM is consistent with PSD, MSD, or PSD-L. We illustrate this by the following example.

Example 3.1 (Omega ratio)**.** *The Omega ratio was first introduced by Keating and Shadwick (2002). For any random variable Z with $\mathbb{E}[|Z|] < \infty$, the Omega ratio that use zero as the return threshold can be expressed as:*[11]

$$\Omega_Z(0) = \frac{\mu_X^+}{\mu_X^-}.$$

Thus, Omega measures are consistent with PSD, and MSD according to Theorem 3.1. This consistency of the Omega ratio with PSD and MSD highlights its practical relevance to the security-potential framework. As $\Omega_Z(0) = \mu_X^+/\mu_X^-$ increases, an investment offers greater potential (higher μ_X^+) relative to its security risk (lower μ_X^-), resonating with Lopes (1987)'s SP/A theory.

The second moment a random variable could also be partitioned into two second-order partial moments (or semivariance). For any random variable Z with finite second moment $\mathbb{E}[Z^2]$ and CDF H We define the following:

$$\mathbb{E}[Z^2] = \tau_Z^+ + \tau_Z^- \quad \text{such that} \quad \tau_Z^+ := \int_0^\infty x^2 dH(x) \quad \text{and} \quad \tau_Z^- := \int_{-\infty}^0 x^2 dH(x), \tag{3.2}$$

where τ_Z^+ is the second-order UPM with respect to reference point 0 and τ_Z^- is the second-order LPM with respect to reference point 0. Under this setting, necessary conditions for PSD, MSD, and PSD-L, involving these second-order partial moments, are established in Theorem 3.2.

Theorem 3.2. *If $\mathbb{E}[X^2] < \infty$ and $\mathbb{E}[Y^2] < \infty$, then the following statements are true:*

(i) *Given that $F \succeq_P G$:*

  (a) *$\mu_X^+ = \mu_Y^+$ implies $\tau_X^+ \le \tau_Y^+$.*

  (b) *$\mu_X^- = \mu_Y^-$ implies $\tau_X^- \ge \tau_Y^-$.*

(ii) *If $F \succeq_M G$, then $\tau_X^+ \ge \tau_Y^+$ and $\tau_X^- \le \tau_Y^-$.*

(iii) *Given that $\mu_X^+ = \mu_Y^+$ and $\mu_X^- = \mu_Y^-$, if $F \succeq_p G$, then $\tau_X^- - \tau_Y^- \ge \max\left\{0, \tau_X^+ - \tau_Y^+\right\}$.*

(iv) *If $F \succeq_m G$, then $\tau_Y^- - \tau_X^- \ge \max\left\{0, \tau_Y^+ - \tau_X^+\right\}$.*

Theorem 3.2 extends the security-potential duality to second partial moments, offering insights into risk-reward relationship akin to moment implications for risk averters and seekers (e.g., Fishburn, 1980; Chan et al., 2022). **Part (i)** shows that under PSD, equal expected gains ($\mu_X^+ = \mu_Y^+$) require lower squared

[11] See Farinelli and Tibiletti (2008).

positive deviations ($\tau_X^+ \le \tau_Y^+$) and equal expected losses ($\mu_X^- = \mu_Y^-$) allow higher squared negative deviations ($\tau_X^- \ge \tau_Y^-$), reflecting a trade-off between potential and security, analogous to SSD ($\mu_X = \mu_Y$ implies $\mathbb{E}[X^2] \le \mathbb{E}[Y^2]$) and risk-seeking dominance ($\mu_X = \mu_Y$ implies $\mathbb{E}[X^2] \ge \mathbb{E}[Y^2]$). **Part (ii)** indicates that under MSD, X dominates Y with higher squared gains ($\tau_X^+ \ge \tau_Y^+$), amplifying potential (P), and lower squared losses ($\tau_X^- \le \tau_Y^-$), enhancing security (S), mirroring the SP/A framework's balance of upside and downside variability. Parts (iii) and (iv) further address PSD-L and MSD-L, where dominance hinges on differences in squared gains and losses, underscoring how gain-loss models align with behavioral preferences for balancing opportunity and safety.

3.1.1 An application of necessary conditions

Comparing distributions with SD conditions normally involves comparing integrals of distributions for a large amount of outcomes (e.g., Linton et al., 2005). Assume that you have 25 investments to consider. Then, fully understanding the relationship between these investments requires $25 \times 24 = 600$ pairs of comparisons, with each comparison involving the comparisons of integrals of distributions for a huge amount of outcomes. This is time-consuming in general. In this part, we make use of the results in Section 3.1 and illustrate that the number of comparisons can be reduced significantly by using a simple method. To do so, we first consider the following immediate consequence of both Theorems 3.1 and 3.2.

Corollary 3.2. *For random variables X and Y, the following statements are true:*

(i) *If $\mu_X^+ < \mu_Y^+$, then $F \not\succeq_P G$.*

(ii) *If $\mu_X^- > \mu_Y^-$, then $F \not\succeq_P G$.*

(iii) *Given that $\mu_X^+ = \mu_Y^+$, if $\tau_X^+ > \tau_Y^+$, then $F \not\succeq_P G$.*

(iv) *Given that $\mu_X^- = \mu_Y^-$, if $\tau_X^- < \tau_Y^-$, then $F \not\succeq_P G$.*

We utilize the aforementioned results to formulate the following hypotheses: $H_0^+ : \mu_X^+ \ge \mu_Y^+$, $H_0^- : \mu_X^- \le \mu_Y^-$, $H_0^{2+} : \tau_X^+ \le \tau_G^+$, and $H_0^{2-} : \tau_X^- \ge \tau_G^-$. Rejecting H_0^+ or H_0^- is equivalent to reject $F \succeq_P G$. When we compare the two distributions, if both H_0^+ and H_0^- are not rejected, then we can further check whether H_0^{2+} or H_0^{2-} is rejected, in this situation, again, it is equivalent to check whether $F \succeq_P G$ is rejected. By using this procedure, we can reduce the number of comparisons before we go to compare the integral of distributions for a huge amount of outcomes. To demonstrate this, we present an empirical example utilizing data in Kenneth French's online data library. The data consists of the excess returns of 25 portfolios formed based on size and Book-to-Market ratio (BE/ME).[12]

[12] See https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html for more details about the portfolios.

We denote the portfolios by using their names in Kenneth French's online data library. The one-month US Treasury bill rate serves as risk-free rate for calculating excess returns. This paper focuses on the monthly excess return from July 1963 to December 2021. All data are directly obtained from Kenneth French's online data library. To test H_0^+, H_0^-, H_0^{2+}, and H_0^{2-}, we use the bootstrap method in Efron and Tibshirani (1994) which will be further elaborated in Appendix C. We perform these tests to the empirical distribution for each of the 600 pairs of two portfolios and exhibit the results in Table 1.

Table 1: Redundant comparisons suggested by the corresponding hypotheses

Size	**BE/ME**	**Hypotheses**				
		H_0^+	H_0^-	H_0^{2+}	H_0^{2-}	**Total**
1	1	0	24	21	0	24
1	2	1	21	15	2	23
1	3	5	16	11	3	20
1	4	6	8	9	5	16
1	5	1	10	9	2	13
2	1	0	23	18	1	24
2	2	3	16	11	3	19
2	3	10	8	6	7	18
2	4	10	4	5	9	17
2	5	3	15	12	1	18
3	1	3	21	13	2	24
3	2	10	10	5	5	20
3	3	15	2	2	11	17
3	4	15	1	3	11	18
3	5	5	8	9	2	14
4	1	7	15	4	4	21
4	2	15	7	2	7	22
4	3	17	2	2	10	19
4	4	15	1	3	9	18
4	5	6	10	8	2	16
5	1	21	1	0	15	22
5	2	21	0	0	18	21
5	3	22	0	0	20	22
5	4	21	1	0	5	22
5	5	11	9	4	2	20

Note: We use 5% as the significance level in this table.

Table 1 provides a comprehensive overview of the results. The first column shows the sizes of the companies we considered in the portfolio; the second column shows the BE/ME of the companies; the numbers in columns 3, 4, 5, and 6 correspond to the number of times we rejected H_0^+, H_0^-, H_0^{2+}, and H_0^{2-}. These rejections occur when comparing the portfolio indicated in Columns 1 and 2 (denoted as F) with one of the other 24 portfolios (denoted as G). The last column displays the number of rejections for $F \succeq_P G$. On average, the outlined approach reduces 19.5 out of the 24 comparisons, resulting in the requirement to perform only 18.75% of the original comparisons. This reduction in the number of comparisons improves efficiency and streamlines the analysis process.[13]

[13]There are other potential applications for this approach. One such example could be employing a PSD concept to ana-

3.2 Partial Moment Framework

In this subsection, we connect stochastic dominance conditions to the partial moment framework through necessary and sufficient relationships. We first define the first LPM $\mathscr{L}_Z(x)$ and the first UPM $\mathscr{U}_Z(x)$, for any random variable Z with corresponding CDF H as shown in the following:

$$\mathscr{L}_Z(x) := \mathbb{E}[(x-Z)_+] = \int_{-\infty}^{x} (x-z)dH(z) \quad \text{and} \quad \mathscr{U}_Z(x) := \mathbb{E}[(Z-x)_+] = \int_{x}^{\infty} (z-x)dH(z) \,,$$

where $(x)_+ = \max\{x, 0\}$. Noted that $\mathscr{L}_Z(x)$ could be considered as the expected loss below the target return x, and $\mathscr{U}_Z(x)$ as the expected gain above the target return x. Thus, $\mathscr{U}_Z(0) = \mu_Z^+$ and $\mathscr{L}_Z(0) = \mu_Z^-$ represent the expected gains and losses when the target return is set at zero.

LPM are commonly used to measure asymmetric risk by focusing on deviations from a target return, for capturing losses. Similarly, UPM is utilized to assess upside potential, enabling a more complete understanding of performance in risk-return optimization.It has also been used in forming trading strategy (e.g., Gao et al., 2022), volatility forecast (e.g., Liu and O'Neill, 2018) and studied asset price (e.g., Hogan and Warren, 1974; Bawa and Lindenberg, 1977; Bernardo and Ledoit, 2000; Anthonisz, 2012). For asset pricing, Hogan and Warren (1974) and Bawa and Lindenberg (1977) proposed the MPLM framework which assume the representative agent will maximize mean and minimize first order lower partial moment or second order lower partial moment with respect to a reference return level. This approach yields the so-called "downside beta" which measure the relationship between market return and security's return given that the market retun fall below a reference return level.

Although the MPLM framework, as proposed by Hogan and Warren (1974) and Bawa and Lindenberg (1977), provides a foundation for downside risk models, its behavioral relevance remains debated. Bawa (1978) links it to Roy (1952)'s safety-first principle, minimizing disaster risk, and shows it maximizes the expected value of a utility function (2.1).[14] Under this perspective, MPLM selection rules can also be viewed as tractable mechanisms for identifying potentially efficient choices that could be incorporated within the computationally intractable SD admissible sets. [15]

For gain-loss dependent objectives, first-order MPLM is particularly relevant since the utility function (2.1) is consistent with the classes $\mathcal{U}^S$, $\mathcal{U}^R$, $\mathcal{U}^{SL}$, and $\mathcal{U}^{RL}$, and thus supports identifying potentially efficient portfolios under various SD conditions. Formally, let $\mathscr{X}^*$ denote the set of optimal choices within the feasible set $\mathscr{X}$ under the first-order MPLM framework, with $\mathscr{X}^* \subseteq \mathscr{X}$. Then, every $X^* \in \mathscr{X}^*$ is efficient, in the sense that no non-optimal choice $Y \in \mathscr{X} \setminus \mathscr{X}^*$ exists such that $Y \succ_P X^*$, $Y \succ_M X^*$, $Y \succ_p X^*$,

lyze market efficiency like Cho et al. (2007) and Chui et al. (2020) utilized an SSD concept in their respective market efficiency analyses.

[14] Fishburn (1977) first analyzed such utilities with non-negative real exponents (which implies considering LPMs that are not limited to integer orders), while Holthausen (1981) extended this to weighted UPM and LPM sums.

[15] See Bawa (1978) for details.

or $Y \succ_m X^*$. Since the expectation of utility function (2.1) represents an affine transformation of the difference between weighted mean return and weighted first-order LPM, the first-order MPLM framework is equivalent to maximizing UPM and minimizing LPM relative to the same target return, with appropriate priority. We now delve into the theorems related to LPM and UPM.

Theorem 3.3. *For any pair of random variables X and Y with CDFs F and G, the followings are true:*

(i) $F \succeq_P G$ *if and only if*

$$\min\Big\{\big[\mathscr{L}_Y(0)-\mathscr{L}_Y(x_1)\big]-\big[\mathscr{L}_X(0)-\mathscr{L}_X(x_1)\big],\big[\mathscr{U}_X(0)-\mathscr{U}_X(x_2)\big]-\big[\mathscr{U}_Y(0)-\mathscr{U}_Y(x_2)\big]\Big\} \geq 0,$$
$$\forall\, x_1 \leq 0 \leq x_2.$$

(ii) $F \succeq_M G$ *if and only if* $\min\Big\{\mathscr{L}_Y(x_1)-\mathscr{L}_X(x_1),\mathscr{U}_X(x_2)-\mathscr{U}_Y(x_2)\Big\} \geq 0, \quad \forall\, x_1 \leq 0 \leq x_2.$

(iii) $F \succeq_p G$ *if and only if*

$$\big[\mathscr{L}_Y(0)-\mathscr{L}_Y(-x)\big]-\big[\mathscr{L}_X(0)-\mathscr{L}_X(-x)\big] \geq \max\Big\{0,\big[\mathscr{U}_Y(0)-\mathscr{U}_Y(x)\big]-\big[\mathscr{U}_X(0)-\mathscr{U}_X(x)\big]\Big\},$$
$$\forall\, x \geq 0.$$

(iv) $F \succeq_m G$ *if and only if* $\mathscr{L}_Y(-x)-\mathscr{L}_X(-x) \geq \max\Big\{0,\mathscr{U}_Y(x)-\mathscr{U}_X(x)\Big\}, \quad \forall\, x \geq 0.$

The theorem states that $F \succeq_P G$ if and only if the minimum of two differences is greater than or equal to 0 for all $x_1 \leq 0 \leq x_2$. The first difference is the difference between the first LPMs evaluated at 0 and x_1, i.e., $\big[\mathscr{L}_Y(0)-\mathscr{L}_Y(x_1)\big]-\big[\mathscr{L}_X(0)-\mathscr{L}_X(x_1)\big]$. The second difference is the difference between the first UPMs evaluated at 0 and x_2 for random variable X and Y, i.e., $\big[\mathscr{U}_X(0)-\mathscr{U}_X(x_2)\big]-\big[\mathscr{U}_Y(0)-\mathscr{U}_Y(x_2)\big]$. Theorem 3.3 gives us a representation equivalent to $F \succeq_P G$. This result will be useful whenwe construct portfolio optimization method in Section 5. Similarly, results related to $F \succeq_R G$, $F \succeq_L G$, and $F \succeq_r G$ are listed as follows.

Corollary 3.3. *Let F and G be CDFs of random variables X and Y respectively, with $\mathbb{E}[|X|] < \infty$ and $\mathbb{E}[|Y|] < \infty$, then the following results hold:*

(i) $F \succeq_L G$ *if and only if* $\mathscr{L}_Y(0)-\mathscr{L}_X(0) \geq \max\Big\{0,\mathscr{U}_Y(0)-\mathscr{U}_X(0)\Big\}.$

(ii) $F \succeq_R G$ *if and only if* $\mathscr{L}_Y(x)-\mathscr{L}_X(x) \geq 0, \quad \forall\, x \leq 0$ *and* $\mathscr{U}_X(0)-\mathscr{U}_Y(0) \geq 0.$

(iii) $F \succeq_r G$ *if and only if* $\mathscr{L}_Y(x)-\mathscr{L}_X(x) \geq 0, \quad \forall\, x \leq 0$ *and* $\mathscr{L}_Y(0)-\mathscr{L}_X(0) \geq \mathscr{U}_Y(0)-\mathscr{U}_X(0).$

Theorem 3.3 and Corollary 3.3 establish convenient characterizations for stochastic dominance conditions in terms of first-order lower and upper partial moments. As Lopes (1987) notes, both the desire for security and the pursuit of potential coexist in all investors, with some individuals favoring security over potential and others preferring the reverse. Loss aversion–related results indicate that potential becomes secondary if security is high enough, which appears to be more consistent with empirical work ((e.g., Ang et al., 2006; Bollerslev et al., 2022)). This interplay underscores the relevance of the partial moment framework in capturing behavioral investment dynamics.

4 Mean-Variance Analysis

This section is devoted to the relationship between PSD, PSD-L, MSD,MSD-L and MV Analysis under the premise of elliptically distributed asset returns. We will first define the elliptical distribution, followed by a discussion of results related to our analysis.. In the final part of this section, we will extend the findings of Levy and Levy (2004), investigating the connections between MV-efficient portfolios and those that are PSD-efficient, MSD-efficient, PSD-L-efficient, or MSD-L-efficient.

4.1 Elliptical Distribution

Elliptical distributions, a class of probability distributions that includes the multivariate normal distribution and the multivariate t-distribution, are critical in finance due to their adaptability and capability to handle heavy-tailed data, which are common in financial returns. We define an elliptically distributed random vector as follows.

Definition 4.1. *Let $\boldsymbol{Z}$ be any N-dimensional random vector. $\boldsymbol{Z}$ is "elliptically distributed" with mean vector $\boldsymbol{\mu_Z} \in \mathbb{R}^N$, dispersion matrix $\boldsymbol{\Sigma_Z} \in \mathbb{R}^{N\times N}$, and characteristic generator ϕ, denoted as $\boldsymbol{Z} \sim \mathcal{E}_N(\boldsymbol{\mu_Z}, \boldsymbol{\Sigma_Z}, \phi)$, if and only if*

$$\boldsymbol{Z} \overset{d}{=} \boldsymbol{\mu_Z} + \boldsymbol{\Lambda}\mathcal{R}\boldsymbol{U^{(k)}}$$

where $\boldsymbol{U^{(k)}}$ is a k-dimensional random vector uniformly distributed on $S^{k-1} := \{\boldsymbol{z} \in \mathbb{R}^k : ||\boldsymbol{z}||_2 = 1\}$ and $||\cdot||_2$ is the Euclidean norm. $\mathcal{R}$ is a nonnegative random variable that is stochastically independent of $\boldsymbol{U^{(k)}}$, $\boldsymbol{\Lambda} \in \mathbb{R}^{N\times k}$ with $\operatorname{rank}(\boldsymbol{\Lambda}) = k$ such that $\boldsymbol{\Sigma_Z} = \boldsymbol{\Lambda}\boldsymbol{\Lambda}'$ and characteristic generator $\phi : \mathbb{R}_+ \to \mathbb{R}$ is a function such that the characteristic function $\boldsymbol{t} \mapsto \varphi_{\boldsymbol{Z}-\boldsymbol{\mu_Z}}(\boldsymbol{t})$ of $\boldsymbol{Z} - \boldsymbol{\mu_Z}$ corresponds to $\boldsymbol{t} \mapsto \phi(\boldsymbol{t}'\boldsymbol{\Sigma_Z}\boldsymbol{t})$, $\boldsymbol{t} \in \mathbb{R}^N$.

Before delving further into our analysis, it's important to acknowledge the relevant properties of elliptical distributions. [16]

Property 4.1. *Let $\boldsymbol{Z}$ be an N-dimensional random vector such that $\boldsymbol{Z} \sim \mathcal{E}_N(\boldsymbol{\mu_Z}, \boldsymbol{\Sigma_Z}, \phi)$ with $\operatorname{rank}(\boldsymbol{\Sigma_Z}) = k$, then the following statements are true:*

(i) $\mathbb{E}[\boldsymbol{Z}] = \boldsymbol{\mu_Z}$.

(ii) $Cov(\boldsymbol{Z}) = \frac{E(\mathcal{R}^2)}{k}\boldsymbol{\Sigma_Z}$.

[16] See p.11-12 of Frahm (2004), and p.17 of Gupta et al. (2013).

(iii) $\boldsymbol{\Lambda}^{-1}(\boldsymbol{Z}-\boldsymbol{\mu_Z}) \sim \mathscr{E}_k(\boldsymbol{0}, \boldsymbol{I_k}, \phi)$.

where $\boldsymbol{\Lambda}^{-1}$ is the generalized inverse of $\boldsymbol{\Lambda}$, and $\boldsymbol{I_k}$ is the k-dimensional identity matrix.

The random vector $\boldsymbol{Z}$ possesses a scale-location-parameter distribution, with its mean vector $\boldsymbol{\mu_Z}$, and its covariance matrix $Cov(\boldsymbol{Z})$ equal to the second moment of random variable $\mathscr{R}$ divided by rank($\boldsymbol{\Sigma_Z}$) times the dispersion matrix $\boldsymbol{\Sigma_Z}$. By applying a transformation using the generalized inverse (sometimes referred to as the Moore–Penrose inverse) of matrix $\boldsymbol{\Lambda}$ (denoted as $\boldsymbol{\Lambda}^{-1}$) to the vector equal to $\boldsymbol{Z}$ subtracting the mean vector $\boldsymbol{\mu_Z}$, the random vector follows an elliptically symmetric distribution $\mathscr{E}_k(\boldsymbol{0}, \boldsymbol{I}_k, \phi)$, with a zero mean vector and k-dimensional identity matrix as the dispersion matrix, standardizing $\boldsymbol{Z}$.

In this section, we consider elliptically distributed random vectors such that their probability density function exists and their support is $\mathbb{R}^N$. Examples of distributions that satisfy these conditions include multivariate normal distribution, multivariate t-distribution, multivariate Cauchy, multivariate logistic distributions, etc. We will denote such kind of random vector as $\boldsymbol{Z} \sim \mathscr{E}_N^*(\boldsymbol{\mu_Z}, \boldsymbol{\Sigma_Z}, \phi)$. In this section, we will first discuss random variables X and Y such that $X \sim \mathscr{E}_1^*(\mu_X, \Sigma_X, \phi)$ and $Y \sim \mathscr{E}_1^*(\mu_Y, \Sigma_Y, \phi)$ with corresponding CDFs F and G. In the one-dimensional case, matrix $\boldsymbol{\Lambda}$ becomes the square root of dispersion parameter, that is $\sqrt{\Sigma_Z}$, and the standardized random variables will become $(Z-\mu_Z)/\sqrt{\Sigma_Z}$ for $Z = X$ and Y. Some of our results will be related to the intersection of the cumulative distribution functions of X and Y. We derive a related Lemma as follows.

Lemma 4.1 (Single-crossing Lemma)**.** *Let $X \sim \mathscr{E}_1^*(\mu_X, \Sigma_X, \phi)$ and $Y \sim \mathscr{E}_1^*(\mu_Y, \Sigma_Y, \phi)$, then the following statements are true:*

(i) *F and G will cross if and only if $\sigma_Y^2 \neq \sigma_X^2$.*

(ii) *F and G will cross exactly one time and G will cross F from above if $\sigma_Y^2 > \sigma_X^2$.*

For CDFs that cross only once, this characteristic is often described as the "single-crossing property" in various works. Manski (1988) posits that if function G intersects function F from above, under specific conditions, it can be inferred that G embodies greater risk than F. In addition, de Castro and Galvao (2019) demonstrate an application of the single-crossing property in a dynamic context. We demonstrate one of it's implication in the following example.

Example 4.1 (Safety First)**.** *For a "disaster" level d, Levy and Levy (2009) consider a safety first criterion such that F is preferred to G if and only if $G(d) \leq F(d)$. For $X \sim \mathscr{E}_1^*(\mu_X, \Sigma_X, \phi)$ and $Y \sim \mathscr{E}_1^*(\mu_Y, \Sigma_Y, \phi)$, G and F will cross at most one time by Lemma 4.1, thus $F \succeq_P G$ implies that F is preferred to G with this safety first principle for all $d \leq 0$.*

4.2 Modified Mean-Variance Analysis

Consider normally distributed returns, one of the common assumptions people will use when applying the MV analysis. One unexpected finding regarding PSD on comparing assets is the lack of any dominance between two normal distributions, say F and G, when both have the same expected return. For MSD, even a higher expected return cannot establish dominance between two normal distributions if their variances differ. This contrasts with the traditional theory of stochastic dominance for risk-averse and risk-seeking individuals, where dominance between F and G can be established despite equal expected returns. We state this formally as the following theorem.

Theorem 4.1 (Impossibility Theorem). *Let $X \sim \mathcal{E}_1^*(\mu_X, \Sigma_X, \phi)$ and $Y \sim \mathcal{E}_1^*(\mu_Y, \Sigma_Y, \phi)$ such that $\Sigma_X \neq \Sigma_Y$. Then, the followings hold:*

(i) *Given that $\mu_X = \mu_Y$, $F \not\succeq_P G$ and $G \not\succeq_P F$.*

(ii) *$F \not\succeq_M G$ and $G \not\succeq_M F$.*

Theorem 4.1 provides additional justification for the potential need to adjust MV analysis when catering to investors whose preferences align with prospect theory. The MV rule posits that a risky investment should be preferred if it exhibits a higher expected return and lower return variance than another, provided it has either a strictly higher expected return or a strictly lower return variance. However, Theorem 4.1 demonstrates that, for investors with S-shaped utility functions, equal expected returns preclude any PSD dominance between two investments with differing variances. For investors with reverse S-shaped utility functions, the situation is more restrictive: even unequal expected returns do not guarantee MSD dominance when variances differ.

In the context of normal distributions, as discussed in Levy and Levy (2004), it is essential to understand the crossover point of CDFs. This crossover point turns out can be characterized in terms of the first partial moments and the variance of elliptically distributed random variables. We establish the following lemma that formally describes the conditions under which two CDFs intersect.

Lemma 4.2. *Let $X \sim \mathcal{E}_1^*(\mu_X, \Sigma_X, \phi)$ and $Y \sim \mathcal{E}_1^*(\mu_Y, \Sigma_Y, \phi)$ where $\sigma_X^2 < \sigma_Y^2$. Then the following statements are true:*

(i) *F will cross G exactly for one time from below at x_0 such that $x_0 > 0$ if $\mu_X^+ \geq \mu_Y^+$.*

(ii) *F will cross G exactly for one time from below at x_0 such that $x_0 < 0$ if $\mu_X^- \geq \mu_Y^-$.*

The above lemma shows that the cross-over point depends on the first partial moments and variance of the two distributions. This result is particularly important when we discuss the concept of PSD. This lemma plays a vital role in understanding and simplifying the conditions for PSD. In the following, we illustrate how it allows us to revisit and reinterpret the result from Levy and Levy (2004) in a more compact way.

Example 4.2. *With Lemmas B.4 and 4.2, the important lemma on p.1035 of Levy and Levy (2004) can be restated as: Assume that both F and G are normal CDFs such that* $\mu_X < \mu_Y$ *and* $\sigma_X < \sigma_Y$*. Then, G dominates F by PSD if and only if* $\mu_X^- \geq \mu_Y^-$*.*

In this example, two normal CDFs, F and G, are compared. Despite X having a lower expected return and less return standard deviation than Y, the dominance in terms of PSD isn't determined solely by these factors. Instead, G dominates F by PSD if and only if the expected loss for X is at least as large as that for Y. This implies that, all else being equal, for all decision makers with the S-shaped utility function, they would prefer Y over X due to its lower expected loss. We generalized the above results as shown in the following theorem.

Theorem 4.2. *Let* $X \sim \mathcal{E}_1^*(\mu_X, \Sigma_X, \phi)$ *and* $Y \sim \mathcal{E}_1^*(\mu_Y, \Sigma_Y, \phi)$ *such that* $\sigma_X^2 \leq \sigma_Y^2$*. Then, the following statements are true:*

(i) $F \succeq_P G$ *if and only if* $\mu_X \geq \mu_Y$ *and* $\mu_X^+ \geq \mu_Y^+$.

(ii) $G \succeq_P F$ *if and only if* $\mu_X \leq \mu_Y$ *and* $\mu_X^- \geq \mu_Y^-$.

(iii) $F \succeq_M G$ $(G \succeq_M F)$ *if and only if* $\mu_X \geq \mu_Y$ $(\mu_Y \geq \mu_X)$ *and* $\sigma_X^2 = \sigma_Y^2$.

(iv) *Given* $\mu_X \geq 0$, $F \succeq_p G$ *if and only if* $\mu_Y^- - \mu_X^- \geq \max\left\{0, \mu_Y^+ - \mu_X^+\right\}$.

(v) $G \succeq_p F$ $(G \succeq_m F)$ *if and only if* $G \succeq_P F$ $(G \succeq_M F)$.

(vi) $F \succeq_m G$ *if and only if* $\mu_X \geq \mu_Y$.

Theorem 4.2 provides conditions for dominance under PSD, MSD, and their loss-averse variants (PSD-L and MSD-L) for elliptical distributions. For **part (i)**, $F \succeq_P G$ requires X have an expected return and expected gain at least as high as those of Y, suggesting that investors prioritize both overall return and upside potential. For **part (ii)**, $G \succeq_P F$ holds when Y has a higher expected return and lower expected loss. This indicates that, despite Y's potentially higher volatility (since $\sigma_X^2 \leq \sigma_Y^2$), its lower downside risk can make it preferable. **Part (iii)** states that $F \succeq_M G$ $(G \succeq_M F)$ requires equal variances alongside a higher expected return.

For **part (iv)**, under PSD-L with $\mu_X \geq 0$, $F \succeq_p G$ if the reduction in expected loss from Y to X exceeds any loss in expected gain. This prioritizes security over potential gains for loss-averse investors with S-shaped utility functions, indicating that they weigh downside risk reductions more heavily. **Part (v)** suggesting that loss aversion does not contribute in this case. Finally, **part (vi)** shows that $F \succeq_m G$ depends solely on higher expected returns.

These results suggest that traditional MV analysis, focusing solely on mean and variance, is insufficient for investors with gain-loss dependent risk attitudes. Incorporating partial moments—expected gains (μ^+) and losses (μ^-)—enables a modified MV framework that better captures prospect theory preferences. For PSD and PSD-L, portfolio selection should balance expected returns with upside potential and downside protection. For MSD, equal variance constraints may limit practical applicability for elliptical distributions, while MSD-L's focus on returns alone offers a streamlined approach.

The second part of the Theorem 4.2 is also relevant to Levy and Levy (2004) because they show that a segment of the MV frontier may be PSD-inefficient. With Theorem 4.2, one could just compare the portfolios in that segment with other portfolios on the MV frontier and find out which portfolios are PSD-efficient, meaning that there is no other portfolio that dominates it in the PSD sense. We will illustrate this in Section 4.3.

Using Theorem 4.2, we can get a rule to compare elliptical distributions in the sense of PSD, PSD-L, MSD, and MSD-L by using moments and partial moments. We call it the modified MV rule which can be stated formally in the following corollary.[17]

Corollary 4.1 (Modified MV rule). *Let $X \sim \mathcal{E}_1^*(\mu_X, \Sigma_X, \phi)$ and $Y \sim \mathcal{E}_1^*(\mu_Y, \Sigma_Y, \phi)$. Then, the following statements are true:*

(i) *If $\mu_X \geq \mu_Y$, $\mu_X^+ \geq \mu_Y^+$, and $\sigma_X^2 \leq \sigma_Y^2$, then $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$ for all $u \in \mathcal{U}^S(F,G)$.*

(ii) *If $\mu_X \geq \mu_Y$, $\mu_X^- \leq \mu_Y^-$, and $\sigma_X^2 \geq \sigma_Y^2$, then $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$ for all $u \in \mathcal{U}^S(F,G)$.*

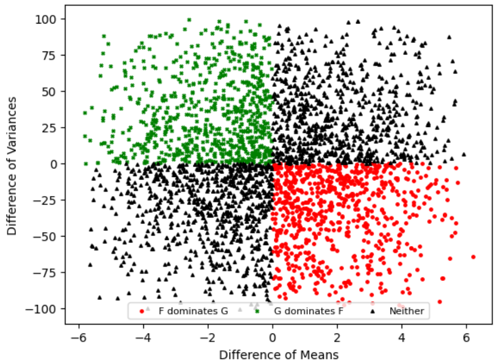
(iii) *If $\mu_X \geq 0$, $\mu_Y^- - \mu_X^- \geq \max\left\{0, \mu_Y^+ - \mu_X^+\right\}$, and $\sigma_X^2 \leq \sigma_Y^2$, then $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$ for all $u \in \mathcal{U}^{SL}(F,G)$.*

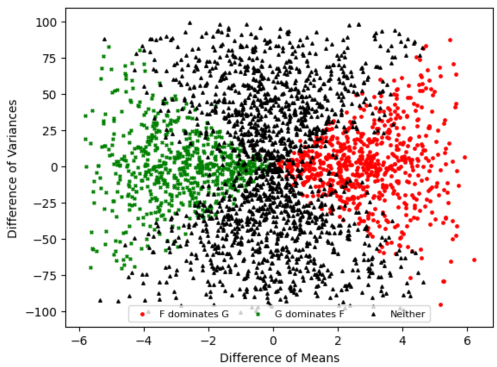
(iv) *$\mu_X \geq \mu_Y$ and $\sigma_X^2 \leq \sigma_Y^2$ if and only if $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$ for all $u \in \mathcal{U}^{RL}(F,G)$.*

With Theorem 4.2 and Corollary 4.1, one can see that the traditional MV rule has a very different implication compared to PSD and PSD-L. To gain a better understanding of the relationship between the traditional MV rule, PSD and PSD-L, we generate 3,000 pairs of normally distributed series with random mean and random variance and use them to compare the MV rule with the modified MV rule, as stated in Corollary 4.1. The comparison is shown in Figure 1.

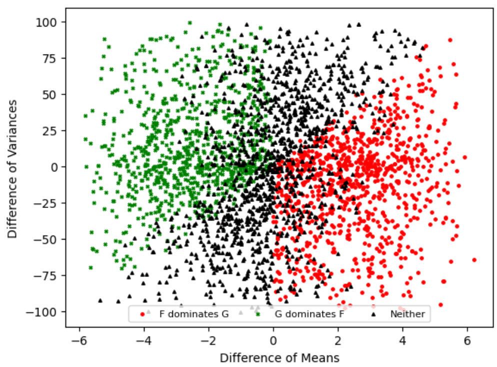
[17] Implications from parts (iii) and (v) will be ignored, since $\mu_X \geq \mu_Y$ and $\sigma_X^2 = \sigma_Y^2$ is equivalent to FSD dominance, and $G \succeq_p F$ is equivalent to $G \succeq_P F$.

(a) MV rule





(b) Modified MV rule for $\mathcal{U}^S(F,G)$



(c) Modified MV rule for $\mathcal{U}^{SL}(F,G)$

Figure 1: Comparison between MV rule and Modified MV rules

In Figure 1, we refer to the first distribution in a pair as F and the second distribution as G. We calculate the means and variances for all pairs of distributions and plot the differences $\mu_X - \mu_Y$ and $\sigma_X^2 - \sigma_Y^2$ of each pair in Figure 1a, 1b and 1c. In these figures, we denote F dominates G by red and G dominates F by green.[18]

Figure 1 shows that we will never get a "wrong" result when we use the MV rule. When a distribution dominates another distribution by the MV rule (e.g., $F \succeq_{MV} G$), it will never be the case that the distribution is dominated by another distribution by the PSD or PSD-L (e.g., $G \succeq_P F$ or $G \succeq_P F$). This implies that the discussion of whether concave utility functions or S-shaped utility functions better describe human behavior might not be as important when comparing elliptical distributions. One can simply use the MV rule to compare elliptical distributions, and the dominated distribution will never be the dominant one in the PSD or PSD-L sense.

[18]We draw the mean μ_i for distribution i from a uniform distribution such that $\mu_i \in [-3,3]$. The graph does not change significantly when We change it to $\mu_i \in [0,3]$.

4.3 Mean-Variance Portfolio Choice

Levy and Levy (2004) is one of the very first studies to examine how prospect theory may affect the MV rule by using the PSD rule. Under the assumptions that returns follow a multivariate normal distribution, portfolios can be formed without restrictions, and no two assets are perfectly correlated, Levy and Levy (2004) find out that, with diversification, the elements of an MV-efficient set which is excluded from the PSD-efficient set is at most one segment on the MV frontier.[19] A portfolio is said to be MV-efficient if no other portfolio dominates it under the mean–variance rule. In a similar vein, a portfolio is PSD-efficient (or MSD-efficient, PSD-L-efficient, MSD-L-efficient) if no other portfolio dominates it in the respective sense. In what follows, we leverage the results of Section 4.2 to illustrate which portfolios are excluded from each of these efficiency sets and which are not.

A nice property of multivariate normally distributed random variables is that their linear combinations are also normally distributed. This suggests that conclusions drawn from analyzing assets with normally distributed returns can easily extend to portfolios composed of assets with multivariate normally distributed returns. This holds true even when we examine an elliptical distribution say $\boldsymbol{Z} \sim \mathscr{E}_N(\boldsymbol{\mu_Z}, \boldsymbol{\Sigma_Z}, \phi)$ with $\text{rank}(\boldsymbol{\Sigma_Z}) = k$, and $\theta \in \mathbb{R}^N$, then $\theta'\boldsymbol{Z} \sim \mathscr{E}_1(\theta'\boldsymbol{\mu_X}, \theta'\boldsymbol{\Sigma}\theta, \phi)$, where $\theta_1, \ldots, \theta_N$ as portfolio weights on assets $1, \ldots, N$ with corresponding random returns $Z_1, \ldots, Z_N$ and $\boldsymbol{Z} = (Z_1, \ldots, Z_N)$. In this section, we assume that only a finite quantity of each asset can be shorted, thus the set of possible asset allocations is defined as:

$$\Theta = \left\{\theta \in \mathbb{R}^N : \sum_{i=1}^{N} \theta_i = 1, \theta_n \geq M \text{ for } n = 1, \ldots, N\right\}.$$

where M is any non positive number that takes a finite value, that is $M \in \mathbb{R}_{\leq 0}$. Our analysis will primarily concentrate on portfolios' return $\theta'\boldsymbol{Z}$ such that $\theta \in \Theta$. Furthermore, we assume that $\boldsymbol{Z} \sim \mathscr{E}^*_N(\boldsymbol{\mu_Z}, \boldsymbol{\Sigma_Z}, \phi)$ and that no pair of asset returns are perfectly correlated.

In the context of portfolio distributions, Chamberlain (1983) posits that if asset returns are jointly elliptically distributed, the mean and variance alone can describe these distributions.[20] Equipped with this understanding, we're now poised to examine the connection between MV-efficient portfolios and the various SD-efficient portfolios, thereby expanding on the insights presented by Levy and Levy (2004).

Theorem 4.3 (Levy & Levy Revised)**.** *Consider N assets with corresponding return vector $\boldsymbol{Z} \sim \mathscr{E}^*_N(\boldsymbol{\mu_Z}, \boldsymbol{\Sigma_Z}, \phi)$ and no pair of asset returns being perfectly correlated. For portfolios generated by these N assets with different weight vector $\theta \in \Theta$, then the following statements are true:*

(i) The PSD-efficient set is a subset of the MV-efficient set.

[19] These assumptions need to be checked when dealing with real word data. For example, the assumptions that returns follow a multivariate normal distribution could be checked by using the method proposed in Richardson and Smith (1993).

[20] Schuhmacher et al. (2021) gives some justification that MV analysis is valid even for a specific skew-elliptical distribution.

(ii) The portfolio in the MV-efficient set is PSD-efficient if and only if no other portfolio in the MV-efficient set offers a higher expected return and lower expected loss.

(iii) The MSD-efficient set is equivalent to the MV-efficient set.

(iv) The PSD-efficient (MSD-efficient) set is equivalent to the PSD-L-efficient (MSD-L-efficient) set.

This revision maintains the original MV frontier while excluding the segment of portfolios that are not part of the PSD-efficient set. Theorem 4.3 refined the results of Levy and Levy (2004) in significant ways. First, rather than relying on the multivariate normal distribution, our analysis employs the more general framework of elliptically distributed random vector, which captures a broader range of return distributions. Second, while Levy and Levy (2004) suggests the PSD-efficient set might exclude one segment from the MV-efficient set, our study establishes clear criteria for determining when MV-efficient portfolios align with PSD-efficiency. Additionally, this result also provide some support on research like Barberis et al. (2001) that why using a single risky asset make sense in a prospect theory context. We illustrate the modified MV frontier using real data as shown in Figure 2. [21]

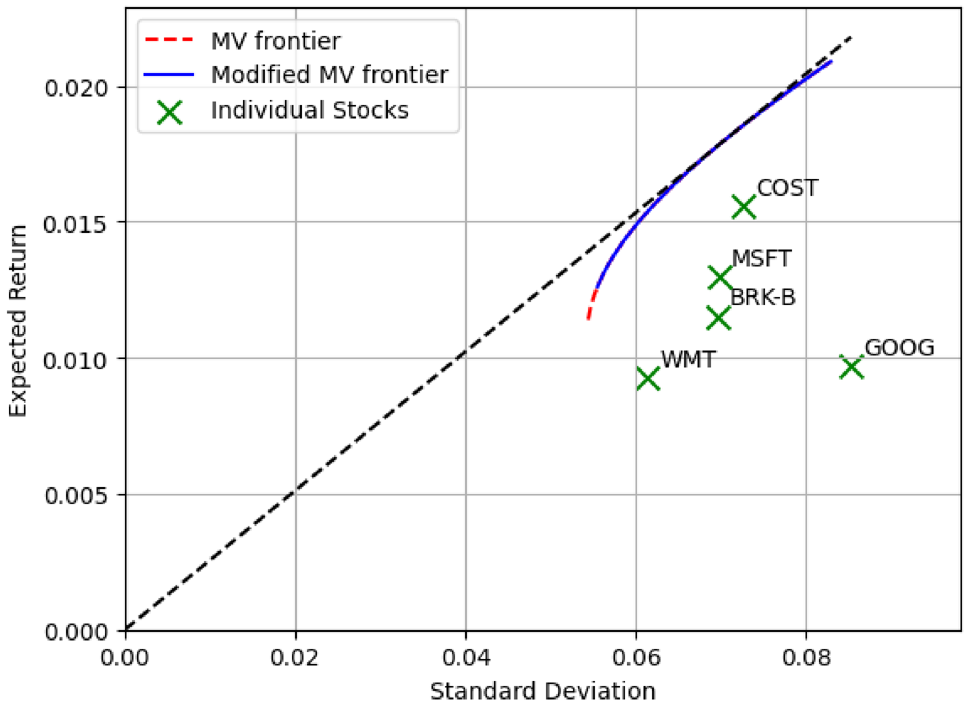


Figure 2: Standard Deviation-Return Trade-off

[21] We use method in Diamond et al. (2014) to get the MV frontier in here.

5 Portfolio Choices in General

In the preceding section, we delved into the subject of SD-efficient portfolios, assuming these portfolios were formulated from base assets with elliptically distributed returns. Without specific assumptions regarding the distribution, identifying PSD-efficient portfolios can be a complex task. This complexity arises due to SD order being a partial order, which typically hinders the use of a real value function for ranking the available choices. Consequently, we cannot leverage traditional optimization techniques to pinpoint portfolios bearing the highest "SD rank". In this section, our focus will shift towards a distinct portfolio optimization problem that can integrate the concept of SD.

5.1 General Case

Let $R_1, ..., R_N$ be random returns for assets $1, ..., N$, with joint cumulative distribution function $\mathscr{F}(\mathbf{r})$ where $\mathbf{r} \in R_1 \times ... \times R_N$. Assume that $\mathbb{E}[R_n{}^2] < \infty$ for all $n = 1, ..., N$, and further let $w_1, ..., w_N$ as portfolio weights on assets $1, ..., N$ such that $w = (w_1, ..., w_N)'$ and $w \in \Theta$. The marginal CDF $\mathscr{F}_w(\eta)$ for asset allocation w is defined as:

$$\mathscr{F}_w(\eta) = \int_{\{w'\mathbf{r} \leq \eta\}} d\mathscr{F}(\mathbf{r})$$

where the corresponding portfolio return is denoted as $R_w = R_1 w_1 + ... + R_N w_N$. Letting Y be a target random return with finite expected value and corresponding CDF G. Consider the distribution G as a benchmark. A compelling inquiry arises: Given a group agents with preferences shaped by an S-shaped utility function, how might we identify a portfolio that outperforms this benchmark? Under these settings, we can form the following problem:

$$\begin{aligned} &\max \mathbb{E}[R_w] \\ \text{subject to } &\mathscr{F}_w \succeq_P G, \\ &w \in \Theta. \end{aligned} \tag{5.1}$$

where $\mathbb{E}[R_w]$ is the expected portfolio return. A similar problem for risk averters (based on SSD) is first considered in Dentcheva and Ruszczyński (2003) and further explored in Post and Kopa (2017), Post et al. (2018), and Fang and Post (2022). The problem here can be explained as follows: given the expected utility of choosing portfolio random return R_w is higher than the target random return Y for all utility functions $u \in \mathcal{U}^S(\mathscr{F}_w, G)$, the objective function $\mathbb{E}[R_w]$ is maximized. Since $\succeq_P$ is a partial order, this denotes the absence of a single objective function to represent it in general. Typically, optimization under these conditions presents substantial challenges. For the above problem, it is not clear how the assets allocations change will affect the cumulative distribution function of the corresponding portfolio return R_w. In order to solve the above problem, we use Corollary 3.3 and transform problem (5.1) to an equivalent problem as follows:

$$
\begin{aligned}
&\max\ \mathbb{E}[R_w] \\
\text{subject to }\ &\left[\mathscr{L}_Y(0)-\mathscr{L}_Y(\eta_1)\right]-\left[\mathscr{L}_{R_w}(0)-\mathscr{L}_{R_w}(\eta_1)\right] \ \geq\ 0,\quad \forall \eta_1 \leq 0, && (5.2)\\
&\left[\mathscr{U}_{R_w}(0)-\mathscr{U}_{R_w}(\eta_2)\right]-\left[\mathscr{U}_Y(0)-\mathscr{U}_Y(\eta_2)\right] \ \geq\ 0,\quad \forall \eta_2 \geq 0, && (5.3)\\
&w \in \Theta.
\end{aligned}
$$

Consider that we have finitely many states $t = 1,\dots,T$, let r_{nt} denote the return of asset n in period t, p_t as probabilities of these realizations where $p_t = 1/T$. Assume random variables under concerns are uniform-bounded, then $\mathscr{L}_Z(\eta)$ could be express as follows:

$$
\mathscr{L}_Z(\eta) = \sum_{t=1}^{T} p_t S_t(\eta)
$$

with variables $S_t(\eta)$ and corresponding binary variables b_t that satisfy the following inequalities:

$$
\begin{aligned}
&S_t(\eta) \geq \eta - \sum_n^N w_n r_{nt}, && t = 1,\dots,T,\\
&S_t(\eta) \leq \eta - \sum_n^N w_n r_{nt} - K_L b_t(\eta), && t = 1,\dots,T,\\
&S_t(\eta) \geq 0, && t = 1,\dots,T,\\
&S_t(\eta) \leq K_U(1-b_t(\eta)), && t = 1,\dots,T,\\
&b_t(\eta) \in \{0,1\}, && t = 1,\dots,T.
\end{aligned}
\tag{5.4}
$$

where $K_L = \min\{y_t, r_{nt}\} - \max\{y_t, r_{nt}\}$ and $K_U = \max\{y_t, r_{nt}\} - \min\{y_t, r_{nt}\}$. The upper bound of $\mathscr{L}_Z(\eta)$ could be expressed as

$$
\mathscr{L}_Z(\eta) \leq \sum_{t=1}^{T} p_t S_t^*(\eta)
$$

with variables $S_t^*(\eta)$ that satisfies the following inequalities:

$$
\begin{aligned}
&S_t^*(\eta) \geq \eta - \sum_n^N w_n r_{nt}, && t = 1,\dots,T,\\
&S_t^*(\eta) \geq 0, && t = 1,\dots,T.
\end{aligned}
\tag{5.5}
$$

For $\mathscr{U}_Z(\eta)$, it and it's upper bound could be defined in a similar way by replacing $\eta - \sum_n^N w_n r_{nt}$ with $\sum_n^N w_n r_{nt} - \eta$. For this case, we replace $S_t(\eta)$ with $D_t(\eta)$, $S_t^*(\eta)$ with $D_t^*(\eta)$ and $b_t(\eta)$ with $c_t(\eta)$ to denote new decision variables. The above formation makes use of the standard technique on handling max operator in mixed integer linear programming and linear programming. It is common in the literature to use ideas related to (5.5) to solve portfolio optimization problems with second or higher order stochastic dominance constraints, including Dentcheva and Ruszczyński (2003), Dentcheva and Ruszczyński (2006), Post et al. (2018), and Fang and Post (2022). Using these concepts, one can replace (5.2) and (5.3) with the following inequalities.

$$
\begin{aligned}
&\left[\mathscr{L}_Y(0)-\mathscr{L}_Y(\eta_1)\right] \geq \sum_{t=1}^{T} p_t\left[S_t^*(0)-S_t(\eta_1)\right], && \forall \eta_1<0,\\
&\left[\mathscr{U}_Y(0)-\mathscr{U}_Y(\eta_2)\right] \leq \sum_{t=1}^{T} p_t\left[D_t(0)-D_t^*(\eta_2)\right], && \forall \eta_2>0,\\
&S_t^*(0) \geq -\sum_{n=1}^{N} w_n r_{nt}, && t=1,\ldots,T,\\
&S_t(\eta_1) \geq \eta_1-\sum_{n=1}^{N} w_n r_{nt}, && \forall \eta_1<0, t=1,\ldots,T,\\
&S_t(\eta_1) \leq \eta_1-\sum_{n=1}^{N} w_n r_{nt}-K_L b_t(\eta_1), && \forall \eta_1<0, t=1,\ldots,T,\\
&S_t(\eta_1) \leq K_U(1-b_t(\eta_1)), && \forall \eta_1<0, t=1,\ldots,T,\\
&D_t(0) \geq \sum_{n=1}^{N} w_n r_{nt}, && t=1,\ldots,T,\\
&D_t(0) \leq \sum_{n=1}^{N} w_n r_{nt}-K_L c_t(0), && t=1,\ldots,T,\\
&D_t(0) \leq K_U(1-c_t(0)), && t=1,\ldots,T,\\
&D_t^*(\eta_2) \geq \sum_{n=1}^{N} w_n r_{nt}-\eta_2, && \forall \eta_2>0, t=1,\ldots,T,\\
&S_t^*(0), S_t(\eta_1), D_t(0), D_t^*(\eta_2) \geq 0, && \forall \eta_1<0<\eta_2, t=1,\ldots,T,\\
&b_t(\eta_1), c_t(0) \in \{0,1\}, && \forall \eta_1<0, t=1,\ldots,T.
\end{aligned}
\tag{5.6}
$$

More details of the above problem will be discussed in Appendix D. The above problem is also analyzed for partial order $\succeq_p$, $\succeq_M$, $\succeq_m$, $\succeq_L$, $\succeq_R$, and $\succeq_r$ in Appendix D.

5.2 Illustration

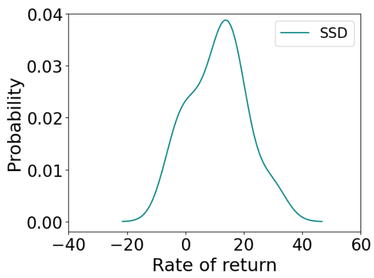
For practical purposes, we set up $M_1 + M_2 + 2$ return levels such that $\eta_1^{M_1} < \eta_1^{M-1} < \ldots < \eta_1^1 < \eta_1^0 = 0 = \eta_2^0 < \eta_2^1 < \ldots < \eta_2^{M-1} < \eta_2^{M_2}$ to replace the condition $\forall \eta_1 \leq 0 \leq \eta_2$ in order to get an approximate solution. The above problem can be solved by a standard mixed integer linear programming solver. We illustrate this by using the data from Dentcheva and Ruszczyński (2003). The optimal portfolio w_p which is based on PSD is $w_p = (0,0,0,0.311,0,0.060,0.369,0.261)$. While the optimal portfolio w_s based on SSD is $w_s = (0,0,0.068,0.188,0,0.391,0.231,0.122)$. Optimal portfolio based on PSD-L, MSD, MSD-L, Loss aversion ($\succeq_L$), Risk averse on losses ($\succeq_R$), or RL-L ($\succeq_r$) can also be solved by a standard mixed integer linear programming solver.

Figure 3 illustrates the return distributions of these optimal portfolios alongside a simple average portfolio, while Table 2 reports their descriptive statistics, including expected return (μ), variance (σ^2), mean positive return (μ^+), positive tail measure ($\tau^+ - (\mu^+)^2$), mean negative return (μ^-), and negative tail measure ($\tau^- - (\mu^-)^2$). Three key observations arise from this analysis, detailed below.

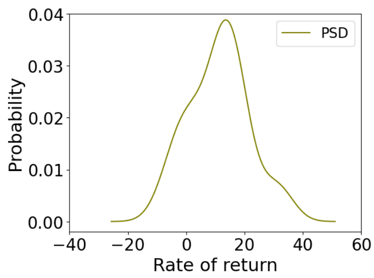
Higher Returns and Volatility: An optimal portfolio based on PSD yields higher returns and greater volatility compared to one based on SSD. This outcome is intuitive, as PSD does not impose risk aversion uniformly across all scenarios. By not accounting for certain forms of "inappropriate" risk, PSD allows for higher returns. For market participants, this flexibility means they can achieve greater rewards by accepting higher volatility. This finding aligns with the equity premium puzzle noted by Mehra and Prescott (1985), where average equity returns exceed what traditional asset pricing models predict. In the context of prospect theory, Benartzi and Thaler (1995), Barberis et al. (2001) and Barberis et al. (2021) argue that investors demand higher expected returns, with Barberis et al. (2001) demonstrating investors' tolerance for increased volatility relative to traditional frameworks.

Lower Diversification: The optimal portfolio under PSD appears less diversified than its SSD counterpart. This discrepancy suggests that PSD favors a more concentrated asset selection and weighting, challenging the conventional wisdom that diversification is essential for risk mitigation. This observation is consistent with empirical evidence from Goetzmann and Kumar (2008), who find that U.S. individual investors often hold under-diversified portfolios relative to the market portfolio. By not enforcing risk aversion across all scenarios, PSD may prioritize assets offering higher returns, potentially increasing concentration risk but also enhancing return potential.

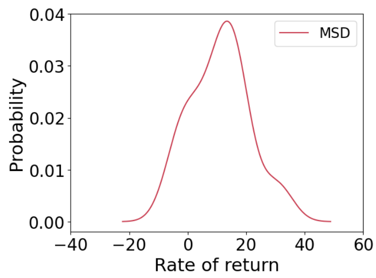
Higher Returns Under Loss Aversion: For PSD, MSD, and their loss-averse variants (PSD-L and MSD-L), the loss-averse versions achieve higher return levels. This result is logical, as these models impose fewer restrictions on upside variation while still addressing downside risk through loss aversion. By allowing greater flexibility to capture potential gains, these approaches optimize for higher returns, aligning with prospect theory's insight that investors are more loss-averse than gain-averse, yet willing to pursue higher rewards when upside potential is less constrained.



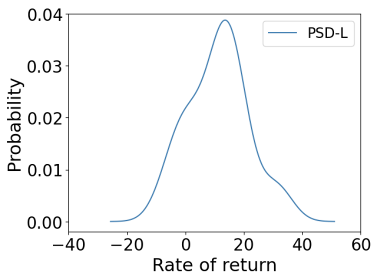
(a) SSD



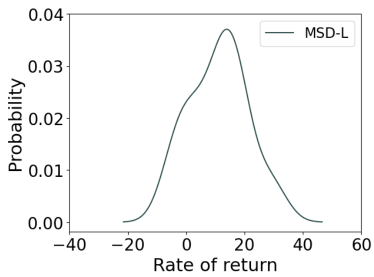
(b) PSD



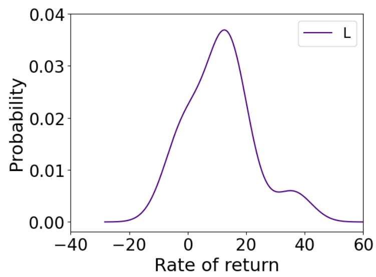
(c) MSD



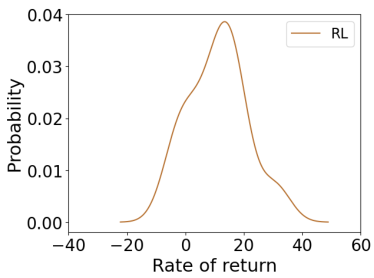
(d) PSD-L



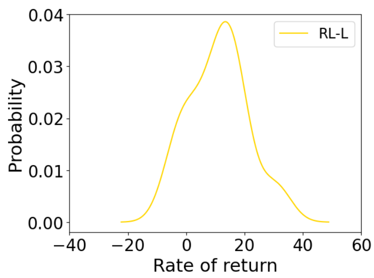
(e) MSD-L



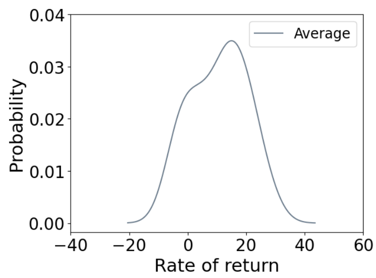
(f) Loss aversion



(g) Risk averse on losses



(h) RL-L



(i) Simple Average

Figure 3: Distributions of different Optimal Portfolio Choices

Table 2: Comparison of Optimal Portfolio Choices

Method	μ	σ^2	μ^+	$\tau^+ - (\mu^+)^2$	μ^-	$\tau^- - (\mu^-)^2$
SSD	11.008%	87.824	11.492%	75.141	0.484%	1.570
PSD	11.173%	95.961	11.726%	80.553	0.552%	2.456
MSD	11.013%	91.602	11.497%	78.914	0.484%	1.570
PSD-L	11.175%	96.082	11.727%	80.698	0.552%	2.431
MSD-L	11.044%	91.874	11.595%	77.173	0.551%	1.934
Loss Aversion	11.206%	116.424	11.758%	101.007	0.552%	2.430
Risk Averse on Losses	11.013%	91.602	11.497%	78.914	0.484%	1.570
RL-L	11.013%	91.602	11.497%	78.914	0.484%	1.570
Simple Average	10.653%	85.522	11.206%	71.594	0.552%	1.551

6 Concluding Remarks

In this paper, we have investigated the relationship between risk measures, portfolio choices, and gain-loss dependent objectives, integrating stochastic dominance conditions consistent with prospect theory and related gain-loss frameworks. By systematically exploring Prospect Stochastic Dominance (PSD), Markowitz Stochastic Dominance (MSD), and their extensions incorporating loss aversion (PSD-L and MSD-L), we establish rigorous linkages between these dominance criteria and partial moment-based risk measures. These measures effectively capture investor preferences reflecting differing sensitivities to gains and losses.

Our study makes three principal contributions. First, we establish that partial moments are consistent with different stochastic dominance conditions, providing a robust framework for measuring downside and upside risks under varying investor attitudes. As Lopes (1987) pointed out, investors are motivated by both security (playing it safe) and potential (seeking growth), with these motivations present in varying strengths, suggesting our framework captures this duality.

Second, we introduce a modified MV efficiency criterion to isolate segments of the MV-efficient frontier that are robust under various gain-loss utility assumptions. This refined frontier addresses the limitations inherent in conventional MV analysis by explicitly accommodating investors' asymmetric risk attitudes and loss aversion. Third, we present a portfolio optimization method that leverages partial moments to construct portfolios that dominate a given benchmark from multiple stochastic dominance perspectives, facilitating direct comparisons across alternative gain-loss utility models without relying on restrictive parameter assumptions.

These findings have significant implications for both theory and practice in finance. Theoretically, our work bridges the gap between traditional mean-variance analysis and behavioral finance by incorporating asymmetric risk preferences into portfolio selection. Practically, the methods developed offer investors and portfolio managers tools to construct portfolios that better reflect their risk attitudes, particularly concerning loss aversion and the differential treatment of gains and losses.

While our study provides valuable insights, it is important to note certain limitations. For instance, the assumption of elliptical distributions in some parts of the analysis may not hold in all market conditions, and future research could explore the robustness of our findings under alternative distributional assumptions. Additionally, empirical validation using diverse real-world datasets would further substantiate the practical applicability of our methods. It might also be interesting to explore linkages between result in this paper to different area of empirical works. Market efficiency studies, as in Post (2003), Post and Levy (2005), Cho et al. (2007), and Chui et al. (2020), could assess how gain-loss preferences affect pricing anomalies. Similarly, computational studies, such as Hodder et al. (2015), suggest advanced optimization methods to handle computational complexity, thereby ensuring scalability.

Future theoretical research could extend our framework to dynamic settings, incorporating multi-period portfolio optimization and accounting for transaction costs. Moreover, investigating how our methods perform across different asset classes or during various market regimes would be beneficial. Another promising avenue is to integrate probability weighting functions, as in cumulative prospect theory, capturing the full spectrum of prospect theory' implication in investment decisions.

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