On the Equivalence of Strategy-proofness and Directed Local Strategy-proofness under Preference Extensions

Kumar, Ujjwal and Roy, Souvik

January 2025

On the Equivalence of Strategy-proofness and Directed Local Strategy-proofness under Preference Extensions*

Ujjwal Kumar[1] and Souvik Roy[2]

[1]Hausdorff Center for Mathematics and Institute for Microeconomics, University of Bonn

[2]Applied Statistics Unit, Indian Statistical Institute, Kolkata

April, 2025

Abstract

We consider a model in which outcomes are bundles of alternatives, each of size at most a fixed (but arbitrary) number. Each agent's type is a strict preference over individual alternatives, which is then lexicographically extended to induce a strict preference over outcomes. A *social choice function* assigns an outcome to each type profile of agents. A social choice function is said to be *locally strategy-proof* if no agent can benefit by misreporting her type to another type that the designer considers plausible. The main departure from existing literature lies in the asymmetry of type misreports, which is captured using a *directed* graph that encodes the designer's beliefs about feasible misreports. An environment is said to satisfy *Directed-Local-Global Equivalence (DLGE)* property if every locally strategy-proof social choice function defined on it is, in fact, (globally) strategy-proof. In this paper, we

*The authors would like to thank the audience of BGSE Workshop, University of Bonn for their helpful comments. Ujjwal Kumar acknowledges that his work was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy - GZ 2047/1, Projekt-ID 390685813.

provide a complete characterization of DLGE environments via a property we refer to as *Property Strong DL*. Additionally, we derive necessary and sufficient conditions for DLGE under several specific notions of locality, such as *adjacent*, *k-push-up*, *k-push-down*, and k_1*-push-up* $\cup$ k_2*-push-down* (some of which were studied in Altuntaş et al. (2023)) both in the setting where outcomes are individual alternatives and where any subset of alternatives may constitute a feasible outcome. Our analysis also extends to single-peaked domains as well. The main result in Cho and Park (2023) and several main results in Altuntaş et al. (2023) follow as corollaries of our framework.

JEL CLASSIFICATION: D71

KEYWORDS: Local strategy-proofness, (global) strategy-proofness, directed-local-global-equivalence, lexicographic preference extension function.

1. Introduction

The concept of *strategy-proofness* in social choice theory stipulates that no agent should be able to benefit by misrepresenting her preferences. In contrast, *local strategy-proofness* imposes a weaker, yet meaningful, requirement: agents cannot benefit from certain deviations that are of particular relevance to the designer. Understanding the structure of strategy-proof social choice functions remains a fundamental and challenging question in the literature. This difficulty is especially pronounced when the domain of admissible preferences lacks restrictive structural properties—such as being unrestricted, single-peaked, single-dipped, or single-crossing. In such general domains, characterizing strategy-proof rules becomes a formidable task. This raises a natural question: can one identify more tractable conditions or methods for verifying strategy-proofness? Local strategy-proofness offers a promising direction in this regard. The principal objective of this paper is to explore the connection between (global) strategy-proofness and its local counterpart, thereby shedding light on the extent to which local conditions can serve as proxies for global strategy-proofness.

The motivation for studying *local strategy-proofness* is well-established in the literature (Carroll, 2012; Sato, 2013; Kumar et al., 2021; Mishra et al., 2016). As noted earlier, a central motivation lies in the fact that verifying strategy-proofness of a social choice rule can be considerably simplified if the designer knows that it suffices to check for certain specific deviations or misreports, as prescribed by local strategy-proofness. Moreover, the concept of local strategy-proofness captures the behavior of *behavioral agents*—agents who may refrain from misreporting in certain situations due to intrinsic costs associated with lying, such as ethical or moral considerations. In such settings, even if a misreport could yield a more preferred outcome, the agent may not find it acceptable. Hence, the designer only needs to ensure robustness against the subset of deviations that remain feasible for these agents. This behavioral interpretation further underscores the practical relevance of local strategy-proofness.

We consider a model with a single agent (without loss of generality) and a finite set A of m alternatives, where $m \geq 2$. The agent's type is represented by a strict preference (ranking) P over the elements of A. The collection of all admissible strict preferences is denoted by the *domain* $\mathcal{D}$.

The set of possible outcomes is given by $\tilde{A} \subseteq 2^A$, that is, outcomes are subsets of alternatives. Each preference $P \in \mathcal{D}$ is associated with a unique and *consistent* strict preference $\tilde{P}$ over $\tilde{A}$, via a *preference extension function* $\eta : \mathcal{D} \to \tilde{\mathcal{D}}$, where $\eta(P) := \tilde{P}$ and $\tilde{\mathcal{D}}$ is the set of all strict preferences over $\tilde{A}$.

For any strict preference P (or $\tilde{P}$), we denote its weak counterpart by R (respectively, $\tilde{R}$).

An *environment* $\mathcal{E}$ is defined as a tuple $\langle A, \mathcal{D}, \tilde{A}, \eta, G \rangle$, where A is the set of alternatives, $\mathcal{D}$ is the domain of admissible strict preferences over A, $\tilde{A} \subseteq 2^A$ is the set of outcomes, $\eta : \mathcal{D} \to \tilde{\mathcal{D}}$ is a preference extension function mapping preferences over A to preferences over $\tilde{A}$, and G is a (fixed but arbitrary) *directed graph* whose nodes are the elements of $\mathcal{D}$. The graph G encodes the set of misreports that are of concern to the designer. Specifically, the presence of a directed edge $(P, P') \in E(G)$ indicates that the designer wishes to guard against the agent misreporting her true preference P as P'.

In this paper, we restrict attention to outcomes consisting of at most κ alternatives and assume that preferences over sets are extended via the *lexicographic extension*.[1,2]

[1]Formally, $\tilde{A} = \{S \subseteq A \mid |S| \leq \kappa\}$ for some fixed $\kappa \in \{1, \ldots, m\}$.

[2]A preference $\tilde{P}$ on $\tilde{A}$ is said to be a lexicographic extension of a preference P on A if for any two sets $S, S' \in \tilde{A}$, we have $S\tilde{P}S'$ if and only if there exists k such that $r_j(S, P) = r_j(S', P)$ for all $j \leq k$, and either $|S'| = k$ or $r_{k+1}(S, P)\, P\, r_{k+1}(S', P)$. Here, $r_j(S, P)$ denotes the j-th most preferred alternative in S according to P.

A *social choice function* (SCF) f maps each preference $P \in \mathcal{D}$ to an outcome in $\tilde{A}$, i.e., $f : \mathcal{D} \to \tilde{A}$. The function f is said to be *strategy-proof* on a pair of preferences $(P, P') \in \mathcal{D}^2$ if the agent does not strictly benefit by misreporting her true preference P as P', that is,

$$f(P)\tilde{R}f(P') \quad \text{where } \tilde{R} \text{ is the weak preference induced by } \eta(P).$$

The function f is *locally strategy-proof* (with respect to G) if it is strategy-proof on every pair $(P, P') \in E(G)$, where $E(G)$ denotes the set of directed edges in the graph G. In contrast, f is said to be *(globally) strategy-proof* if it is strategy-proof on every pair $(P, P') \in \mathcal{D}^2$.

An environment $\mathcal{E}$ is said to satisfy *directed-local-global equivalence* (DLGE) if every social choice function that is locally strategy-proof (with respect to G) is also globally strategy-proof.[3]

The notion of manipulability is inherently asymmetric and is best understood as an *ordered* relation. There exists no compelling technological or psychological rationale to justify the assumption that the feasibility of a manipulation from a preference type P to another type P' is equivalent to that of a manipulation from P' to P. Consequently, it is more natural to model such concerns using a *directed graph*, which captures the directional asymmetry of potential manipulations. We pose a fundamental question: *Which environments satisfy directed-local-global equivalence (DLGE)?* In this paper, we provide a complete characterization of such environments.

For two disjoint outcomes $S, S' \in \tilde{A}$ and a preference $P \in \mathcal{D}$, we write SPS' if S is strictly preferred to S' according to the lexicographic extension of P—that is, there exists an element $s \in S$ such that sPs' for all $s' \in S'$. We introduce a

[3]The converse holds trivially by definition.

structural condition on the graph G of the environment, termed *Property Strong DL* (abbreviated as *Property SDL*). This condition imposes a requirement on the existence of certain types of directed paths within G. Formally, the graph G is said to satisfy *Property SDL* if, for all pairs of preferences $P, P' \in \mathcal{D}$ and for all $S \in \tilde{A}$, there exists a directed path $\pi = (P = P^1, P^2, \ldots, P^t = P')$ in G such that the following condition holds: for every alternative $b \in A \setminus S$ with SPb, there do not exist (not necessarily distinct) elements $s, s' \in S$ and indices $1 < q < r \leq t$ such that

$$\{s, s'\}Pb, \quad bP^q\{s, s'\}, \quad \text{and} \quad \{s, s'\}P^r b.$$

Theorem 3.1 establishes that an environment satisfies directed-local-global equivalence (DLGE) if and only if it satisfies *Property SDL*. In the special case where the outcome set consists solely of singleton alternatives, Property SDL simplifies to a weaker condition, which we refer to as *Property DL*. This yields Corollary 4.1, which shows that Property DL fully characterizes all DLGE environments in the singleton-outcome setting.

As an immediate consequence, we recover the main result of Kumar et al. (2021), which states that in environments with singleton outcomes and *undirected* graphs, Property DL characterizes local-global equivalence (LGE) environments.[4]

Consider the setting where the set of outcomes is the collection of all non-empty subsets of A, i.e., $\tilde{A} = 2^A \setminus \{\emptyset\}$, and preferences over outcomes are given by the lexicographic extension of preferences over A. We define an environment as *adjacent* if the edge set of the graph G satisfies the condition that $(P, P') \in E(G)$ if and only if P' can be obtained from P by swapping two consecutively ranked alternatives—i.e., P and P' differ only by a single adjacent transposition.[5]

[4]Kumar et al. (2021) refer to this as Property L, since they restrict attention to undirected graphs.

[5]Note that in an adjacent environment, if $(P, P') \in E(G)$, then $(P', P) \in E(G)$ as well. Hence, the graph G may naturally be viewed as undirected.

In Proposition 4.4, we show that the adjacent environment satisfies DLGE when the domain is *unrestricted*, that is, when every strict preference ordering over A is admissible. However, we further demonstrate that DLGE fails in the adjacent environment when the domain is restricted to *single-peaked* preferences and the upper bound κ on the number of alternatives in an outcome satisfies $\kappa \geq 2$.[6] The special case where $\kappa = 1$—that is, outcomes are singleton sets—has been extensively studied in the literature. It has been shown (see Carroll (2012) and Sato (2013)) that, under this restriction, the adjacent environment does satisfy DLGE.

We next turn to the analysis of *k-push-down* and *k-push-up* environments. A preference $P' \in \mathcal{D}$ is said to be obtained from another preference $P \in \mathcal{D}$ via a *push-down* (respectively, *push-up*) operation if exactly one alternative is moved to the last (respectively, first) position in P', while the relative ordering of all other alternatives remains unchanged between P and P'. An environment is called a *k-push-down* (respectively, *k-push-up*) environment if $(P, P') \in E(G)$ if and only if P' can be obtained from P through a sequence of at most k push-down (respectively, push-up) operations.[7]

We show that a k-push-down (respectively, k-push-up) environment satisfies DLGE if and only if $k \geq m - 1$, where m is the number of alternatives. Furthermore, we establish a more general result: a *k_1-push-up* $\cup$ *k_2-push-down* environment satisfies DLGE if and only if $k_1 + k_2 \geq m - 1$. As the name suggests, such an environment includes an edge between two preferences if one can be obtained from the other through either a sequence of at most k_1 push-up operations or a sequence of at most k_2 push-down operations.

[6]The single-peaked domain consists of all preferences that are single-peaked with respect to a fixed linear order over the alternatives. See Moulin (1980) for a formal definition.

[7]That is, there exists a sequence $(P = P^1, P^2, \ldots, P^l = P')$ of preferences with $l \leq k$, such that for each $i \in \{2, \ldots, l\}$, P^i is obtained from P^{i-1} via a push-down (respectively, push-up) operation.

In the special case where outcomes are singleton sets (i.e., $\tilde{A} = A$) and the domain is unrestricted, we further show that:

- A k-push-down environment satisfies DLGE if and only if $k \geq m - 1$;
- A k-push-up environment satisfies DLGE if and only if $k \geq \min\{2, m - 1\}$;
- A k_1-push-up $\cup$ k_2-push-down environment satisfies DLGE if and only if $k_1, k_2 \geq 1$.

1.1 Related Literature

Kumar et al. (2021) investigate environments in which each outcome is a singleton set. Focusing on settings with *undirected* graphs, they provide a complete characterization of environments satisfying local-global equivalence (LGE). Altuntaş et al. (2023) extend the analysis to more general settings where the set of outcomes consists of all non-empty subsets of A, preferences are extended lexicographically, and the domain is unrestricted. They show that the adjacent environment satisfies DLGE. Additionally, they examine *k-push-down* and *k-push-up* environments, proving that a k-push-down (respectively, k-push-up) environment satisfies DLGE if and only if $k \geq m - 1$. Their results are derived through separate and specific arguments for each case.

By contrast, in this paper we provide a unified and more general characterization (Theorem 3.1), from which these results follow as natural corollaries. It is also worth emphasizing that our proof technique is conceptually and technically distinct from that employed by Altuntaş et al. (2023).

Recently, Cho and Park (2023) study environments in which outcomes are singleton sets and the preference domain satisfies $|\mathcal{D}| \geq 3$ (implying $m \geq 3$). The main result of their paper establishes that *Property DL* characterizes all DLGE environments in this setting. We obtain the same characterization as a consequence of our more general framework—specifically, as Corollary 4.1 derived from Theorem 3.1. In addition, Cho and Park (2023) show that a k-push-down environment satisfies DLGE if and only if $k \geq m - 1$, and that a k-push-up environment satisfies DLGE if and only if $k \geq 2$. These results correspond to Proposition 4.1 and Proposition 4.2, respectively, in the present paper. The contributions of the two papers are independent. In particular, the techniques employed in proving these results differ substantially in both approach and construction.

Carroll (2012) establishes that the adjacent environment satisfies DLGE for a variety of domains, including the unrestricted domain, the single-peaked domain, and successive single-crossing domains. Sato (2013) subsequently generalizes this result by providing a sufficient condition for DLGE in adjacent environments. All of these results are encompassed as special cases of Corollary 4.1 in this paper, which follows from our more general characterization of DLGE environments.

2. THE MODEL

Let A denote a finite set of alternatives with $|A| = m \geq 2$. We assume without loss of generality that there is a single voter, which is a standard practice for the type of analysis we do in this paper.

A type of the agent is identified by a *strict* preference over A. The set of all possible strict preferences is denoted by $\mathcal{P}$. A *domain* $\mathcal{D}$ is a subset of the set of all preferences $\mathcal{P}$.

Let $\tilde{A}$ be the set of outcomes consisting of bundles (of alternatives) whose size is less than or equal to κ, for some $\kappa \in \{1, \ldots, m\}$, that is, there exists $\kappa \in \{1, \ldots, m\}$ such that $\tilde{A} = \{S \subseteq A \text{ such that } |S| \leq \kappa\}$. Each preference P in $\mathcal{D}$ induces a unique preference $\eta(P)$ on $\tilde{A}$. Whenever it is clear from the context, for simplicity, we often denote $\eta(P)$ by $\tilde{P}$. Let $\widetilde{\mathcal{D}} = \{\eta(P) : P \in \mathcal{D}\}$ be the set of all induced preferences over $\tilde{A}$.

An *environment* is denoted by $\langle A, \mathcal{D}, G, \tilde{A}, \eta \rangle$, where $G = \langle \mathcal{D}, \mathcal{E}(G) \rangle$ is a *directed* graph on $\mathcal{D}$ with the set of directed edges $\mathcal{E}(G) \subseteq \mathcal{D} \times \mathcal{D}$ and $\eta : \mathcal{D} \to \widetilde{\mathcal{D}}$ is the preference extension function. A Social Choice Function (SCF) on $\mathcal{D}$ is a map $f : \mathcal{D} \to \tilde{A}$.

Definition 2.1. Consider an environment $\langle A, \mathcal{D}, G, \tilde{A}, \eta \rangle$. An SCF $f : \mathcal{D} \to \tilde{A}$ is locally manipulable at P if there exists $P' \in \mathcal{D}$ with $(P, P') \in \mathcal{E}$ such that $f(P')\tilde{P}f(P)$.[8] The SCF f is locally strategy-proof if it is not locally manipulable at any $P \in \mathcal{D}$.

Definition 2.2. An SCF $f : \mathcal{D} \to \tilde{A}$ is manipulable at P if there exists $P' \in \mathcal{D}$ such that $f(P')\tilde{P}f(P)$. The SCF f is strategy-proof if it is not manipulable at any $P \in \mathcal{D}$.

A strategy-proof SCF is clearly locally strategy-proof. We investigate the structure of environments where the converse also holds.

Definition 2.3. The environment $\langle A, \mathcal{D}, G, \tilde{A}, \eta \rangle$ satisfies directed-local-global equivalence (DLGE) if every locally strategy-proof SCF $f : \mathcal{D} \to \tilde{A}$ is strategy-proof.

REMARK 2.1. Note that if an environment $\langle A, \mathcal{D}, G, \tilde{A}, \eta \rangle$ satisfies DLGE, then the environment $\langle A, \mathcal{D}, G', \tilde{A}, \eta \rangle$ also satisfies DLGE where $\mathcal{E}(G) \subseteq \mathcal{E}(G')$.

3. THE MAIN RESULT

Let $1 \leq \kappa \leq m$. We consider outcomes that are bundles/subsets (of size $\leq \kappa$) of alternatives in A, i.e., $\tilde{A} = \{S \subseteq A : S \neq \emptyset \text{ and } |S| \leq \kappa\}$, and the preference extension η is lexicographic. We denote the top-ranked alternative in a subset

[8] Recall that $\tilde{P}$ is the shorthand notation for $\eta(P)$.

of alternatives S according to a preference P by $\tau_P(S)$. Formally, for any $P \in \mathcal{P}$ and $S \subseteq A$, define $\tau_P(S) = x$ if $x \in S$ and xPy for every $y \in S \setminus \{x\}$. We now formally define the lexicographic extension. For subsets $S, S' \in \tilde{A}$ and preference $P \in \mathcal{P}$, we have $S\tilde{P}S'$ if and only if one of the following two holds:
(i) $S' \subsetneq S$,
(ii) $\tau_P(S \setminus S')P\tau_P(S' \setminus S)$.

In the following, we present a necessary and sufficient condition on $\langle A, \mathcal{D}, G\rangle$ so that the environment $\langle A, \mathcal{D}, G, \tilde{A}, \eta\rangle$ satisfies DLGE.

Notice that for any $P \in \mathcal{P}$, $\tilde{P}$ is a unique extension and is a strict preference over the bundles in $\tilde{A}$.[9] So, it is justified to assume that agents submit only their preferences over A. Hence, the notion of localness is decided on the basis of these preferences only. In this framework, we ask a natural question: what are *all* $\langle A, \mathcal{D}, G\rangle$ such that the environment $\langle A, \mathcal{D}, G, \tilde{A}, \eta\rangle$ satisfies DLGE. We provide this answer in our next theorem where we characterize $\langle A, \mathcal{D}, G\rangle$ such that the environment $\langle A, \mathcal{D}, G, \tilde{A}, \eta\rangle$ satisfies DLGE. For this purpose, we introduce the notion of Strong-Directed Lower Contour Set no-restoration property (Property SDL).

For a preference P, and two disjoint bundles S and S', we write SPS' if there is *some* alternative in S which is preferred to *all* the alternatives in S' according to P, that is, SPS' if there exists $s \in S$ such that sPs' for all $s' \in S'$.[10] Also, for any bundle $S \subseteq A$ and a preference $P \in \mathcal{P}$, define $L(S, P) = \{x \in A \setminus S : SPx\}$.

Let $G = \langle \mathcal{D}, \mathcal{E}\rangle$. For $P, P' \in \mathcal{D}$, a *path* from P to P' is defined as $\pi = (P^1, \ldots, P^t)$, consisting of a sequence of *distinct* vertices in $\mathcal{D}$ satisfying the property that $P^1 = P, P^t = P'$ and $(P^k, P^{k+1}) \in \mathcal{E}$ for all $k = 1, \ldots t-1$.[11]

[9] Recall that by $\tilde{P}$ we denote $\eta(P)$.

[10] We denote a bundle containing only one alternative by writing a instead of writing $\{a\}$.

[11] In other words, repetitions of vertices in a path are ruled out and also each pair of consecutive vertices form a directed edge.

Let $\Pi(P, P')$ denote the set of all paths from P to P' in G. For any path $\pi = (P^1, \ldots, P^s, P^{s+1}, \ldots, P^t)$, we let $\pi|_{[P^s, P^t]}$ denote the sub-path $(P^s, P^{s+1}, \ldots, P^t)$. We say G is *connected* if for every pair of vertices P, P' in G, there exists a path from P to P' i.e. $\Pi(P, P') \neq \emptyset$ for all $(P, P') \in \mathcal{D} \times \mathcal{D}$.[12]

Definition 3.1. Let G be a graph on $\mathcal{D}$. For any $P, P' \in \mathcal{D}$, $S \in \tilde{A}$ and $b \in L(S, P)$, a path $\pi = (P = P^1, \ldots, P^k = P')$ from P to P' satisfies $\{S, b\}$-restoration if there exist integers q and r with $1 < q < r \leq k$ and $s, s' \in S$ (not necessarily distinct) such that $\{s, s'\}P^1 b$, $bP^q\{s, s'\}$, and $\{s, s'\}P^r b$.

REMARK 3.1. It is worth mentioning that there exist multiple ways to define the notion of $\{S, b\}$-restoration for a set of alternatives S and an alternative $b \in A \setminus S$ with SPb. However, it is important to emphasize that the definition of $\{S, b\}$-restoration provided in Definition 3.1 is the *only* formulation that yields the desired characterization result presented in Theorem 3.1. This highlights that our result is not a routine extension of the main theorem in Kumar et al. (2021), but rather relies on a carefully constructed framework tailored to our setting.

Definition 3.2. We say that a graph G on $\mathcal{D}$ satisfies Strong-Directed Lower Contour Set no-restoration property (Property SDL) if for all $P, P' \in \mathcal{D}$ and for all $S \in \tilde{A}$, there exists a path π from P to P' such that for all $b \in L(S, P)$, the path π does *not* satisfy $\{S, b\}$-restoration.

It should be noted that the choice of the path π depends on the preferences P, P' and the set S, but does not depend on the alternative b. In particular, the same path π will have no $\{S, b\}$-restoration for all $b \in L(S, P)$.

REMARK 3.2. It is worth noting that Property SDL is a significant strengthening of Property DL. This is because a path having no-restoration (with respect

[12]Since we are considering directed edges, a path from P to P' might not necessarily be a path from P' to P.

to every pair of alternatives) can have $\{S, b\}$-restoration for some bundle S. For example, let $A = \{a, b, c\}$ and consider the path $\pi = (P^1, P^2, P^3)$ where $P^1 = abc, P^2 = bac$ and $P^3 = cba$.[13] Clearly, the path π has no restoration with respect to any pair of alternatives. However, π has an $\{S, b\}$-restoration for $S = \{a, c\}$.

Two preferences P and P' are completely opposite if, for all $a, b \in A$, we have aPb if and only if $bP'a$.

REMARK 3.3. Consider a domain $\mathcal{D} = \{P, P'\}$ where P and P' are completely opposite preferences and suppose that the outcomes are singleton (η is the identity function). Then, the necessary and sufficient condition for $\langle A, \mathcal{D}, G, \tilde{A}, \eta \rangle$ to satisfy DLGE is $\mathcal{E}(G) \neq \emptyset$. In other words, presence of at least one directed edge is both necessary and sufficient for the environment to be DLGE. It follows from the fact that in this situation, an SCF is manipulable from P to P' if and only if it is manipulable from P' to P. Therefore, in this section, whenever we deal with singleton outcomes, we assume that $\mathcal{D} \neq \{P, P'\}$, where P and P' are completely opposite preferences.

Theorem 3.1. *An environment $\langle A, \mathcal{D}, G, \tilde{A}, \eta \rangle$ satisfies DLGE if and only if G satisfies Property SDL.*

The proof of this theorem is relegated to Appendix A.1. The proof of the necessity part is a highly non-trivial generalization and that of the sufficiency part is completely new as compared to the corresponding parts of the proof of Theorem 1 in Kumar et al. (2021). The logic in the proof (for both necessity and sufficiency) of Theorem 1 in Kumar et al. (2021) crucially requires undirected graphs and in particular, it fails for directed graphs. Apart from generalizing the proofs for directed graphs, we allow for bundles as outcomes as well. It is worth

[13] By the preference $P^1 = abc$, we mean that a is strictly preferred to b according to the preference P^1 and b is strictly preferred to c according to the preference P^1.

emphasizing that the proof of Theorem 3.1 is both conceptually and technically different from the proof of Theorem 1 in Kumar et al. (2021).

4. Applications

4.1 The case of singleton outcomes

In this subsection, we consider the situation where $\tilde{A} = A$ (and hence $\eta(P) = P$ for all $P \in \mathcal{D}$), and consequently, we denote an environment by $\langle A, \mathcal{D}, G\rangle$. We provide a necessary and sufficient condition on the directed graph G so that the environment $\langle A, \mathcal{D}, G\rangle$ satisfies directed-local-global equivalence (DLGE). We define the reduced version of Property SDL when outcomes are singletons.

Definition 4.1. Let G be a graph on $\mathcal{D}$ and let $a, b \in A$. A path $\pi = (P^1, P^2, \ldots, P^t)$ in G satisfies no $\{a, b\}$-restoration if the relative ranking of a and b is reversed [14] at most once along π i.e. there do not exist integers q, r and s with $1 \leq q < r < s \leq t$ such that either (i) $aP^q b, bP^r a$ and $aP^s b$ or (ii) $bP^q a, aP^r b$ and $bP^s a$.[15]

For any $P \in \mathcal{D}$ and $a \in A$, the lower contour set of a at P is the set of alternatives strictly worse than a according to P, i.e. $L(a, P) = \{b \in A : aPb\}$.

Definition 4.2. The environment $\langle A, \mathcal{D}, G\rangle$ satisfies the Directed-Lower Contour Set no-restoration property (Property DL) if, for all $P, P' \in \mathcal{D}$ and $a \in A$, there exists a path $\pi \in \Pi(P, P')$ such that for all $b \in L(a, P)$, the path π in G satisfies no $\{a, b\}$-restoration.

We obtain the following result as a corollary of Theorem 3.1. The same result is proved independently in Cho and Park (2023) (see Theorem 1) when $|\mathcal{D}| \geq 3$.

[14] A pair of alternatives a, b are reversed in the pair of preferences P and P' if they are ranked differently in P and P'.

[15] It is worth emphasizing that in our definition of "$\{a, b\}$-restoration", we are *not* referring to an ordered pair (a, b). Thus $\{a, b\}$-restoration and $\{b, a\}$-restoration are the same in our definition. We use expressions such as "the path has no $\{a, b\}$-restoration" and "the path has no restoration for the pair $\{a, b\}$" interchangeably.

Corollary 4.1. *An environment satisfies DLGE if and only if it satisfies Property DL.*

4.1.1 k-PUSH-DOWN STRATEGY-PROOFNESS

We introduce the k-push-down environment for the unrestricted domain. We first define the notion of 'push-down'. We say a preference P' is obtained from P by a push-down (and thereby forms a directed edge in the push-down environment $\mathcal{E}^{down}$), if an alternative is moved to the last rank from P to P' while keeping the relative ranking of all other alternatives unchanged. Formally, for $P, P' \in \mathcal{P}$, we say that $(P, P') \in \mathcal{E}^{down}$ if there exists $x \in A$ such that

(i) $aP'x$ for all $a \in A \setminus \{x\}$, and

(ii) for all $a, b \in A \setminus \{x\}$, aPb if and only if $aP'b$.

A k-push-down environment $\mathcal{E}^{k\text{-}down}$ is defined as the one where (P, P') is a (directed) edge if and only if P' can be obtained from P by applying the push-down action at most k times. Formally, for $k \geq 1$, $(P, P') \in \mathcal{E}^{k\text{-}down}$ if and only if there exists $l \in \{1, \ldots, k\}$ and a sequence of preferences $(P^1, \ldots, P^{l+1})$ in $\mathcal{P}$ such that

(i) $P^1 = P$,

(ii) $P^{l+1} = P'$, and

(iii) $(P^j, P^{j+1}) \in \mathcal{E}^{down}$ for each $j \in \{1, \ldots, l\}$.

It is worth noting that the graph $G = \langle \mathcal{P}, \mathcal{E}^{k\text{-}down} \rangle$ can be obtained from the graph $G = \langle \mathcal{P}, \mathcal{E}^{down} \rangle$ by adding edges from any preference P to another preference P' if there exists a path from P to P' of length (number of edges) at most k.

The following proposition provides a necessary and sufficient condition for DLGE for k-push-down strategy-proofness. Cho and Park (2023) (Proposition 4) prove this result independently.

Proposition 4.1. *The environment $\langle A, \mathcal{P}, G \rangle$, where $G = \langle \mathcal{P}, \mathcal{E}^{k\text{-}down} \rangle$, satisfies DLGE if and only if $k \geq m-1$.*

The proof of this proposition is relegated to Appendix A.2.

4.1.2 k-PUSH-UP STRATEGY-PROOFNESS

As the name sounds, k-push-up environment is symmetrically opposite of the k-push-down environment. As for the k-push-down environment, we define this environment for the unrestricted domain. Let us first define the 'push-up' environment $\mathcal{E}^{up}$. For two preferences P and P', we say that $(P, P') \in \mathcal{E}^{up}$ if P' can be obtained from P by pushing up one alternative to the first rank while keeping the relative ranking of all other alternatives unchanged. Formally, for $P, P' \in \mathcal{P}$, we say that $(P, P') \in \mathcal{E}^{up}$ if there exists $x \in A$ such that

(i) $xP'a$ for all $a \in A \setminus \{x\}$, and

(ii) for all $a, b \in A \setminus \{x\}$, aPb if and only if $aP'b$.

The $\mathcal{E}^{k\text{-}up}$ environment is obtained by putting a directed edge from P to P' if and only if P' can be obtained from P by applying the push-up action at most k times. Formally, $(P, P') \in \mathcal{E}^{k\text{-}up}$ if there exist $l \in \{1, \ldots, k\}$ and a sequence of preferences $(P^1, \ldots, P^{l+1})$ in $\mathcal{P}$ such that

(i) $P^1 = P$,

(ii) $P^{l+1} = P'$, and

(iii) $(P^j, P^{j+1}) \in \mathcal{E}^{up}$ for each $j \in \{1, \ldots, l\}$.

As we have remarked for the k-push-down case, the graph $G = \langle \mathcal{P}, \mathcal{E}^{k\text{-}up} \rangle$ can be obtained from the graph $G = \langle \mathcal{P}, \mathcal{E}^{up} \rangle$ by adding edges from any preferences P to another preference P' if there exists a path from P to P' of length at most k.

Proposition 4.2. *The environment $\langle A, \mathcal{P}, G \rangle$, where $G = \langle \mathcal{P}, \mathcal{E}^{k\text{-}up} \rangle$, satisfies DLGE if and only if $k \geq \min\{2, m-1\}$.*

The proof of this proposition is relegated to Appendix A.3.

4.1.3 Push-down or push-up strategy-proofness

Similar to k-push-down (or push-up) environments, we consider the unrestricted domain for the 'Push-down or push-up' environment. As the name sounds, there is a directed edge from P to P' in this environment if and only if P' can be obtained from P either by pushing up one alternative to the first rank or by pushing down one alternative to the last rank, while keeping the relative ranking of all other alternatives unchanged, that is, when $(P, P') \in \mathcal{E}^{up} \cup \mathcal{E}^{down}$.

Proposition 4.3. *The environment $\langle A, \mathcal{P}, G \rangle$, where $G = \langle \mathcal{P}, \mathcal{E}^{up} \cup \mathcal{E}^{down} \rangle$, satisfies DLGE.*

The proof of this proposition is relegated to Appendix A.4.

It may be noted that unlike the k-push-down (or, push-up) environments, the $\mathcal{E}^{up} \cup \mathcal{E}^{down}$ environment is defined without any reference to a parameter k. We fill this apparent gap in the next remark. Let us define the $\mathcal{E}^{k_1\text{-}up} \cup \mathcal{E}^{k_2\text{-}down}$ environment where (P, P') is an edge if and only P' can be obtained from P either by applying the push-up action at most k_1 times or by push-down action at most k_2 times.

Remark 4.1. Take $(P, P') \in \mathcal{E}^{up} \cup \mathcal{E}^{down}$. Then, $(P, P') \in \mathcal{E}^{up}$ or $(P, P') \in \mathcal{E}^{down}$. If $(P, P') \in \mathcal{E}^{up}$ (or, $(P, P') \in \mathcal{E}^{down}$), then by definition $(P, P') \in \mathcal{E}^{k_1\text{-}up}$ for any

$k_1 \geq 1$ (respectively, $(P, P') \in \mathcal{E}^{k_2\text{-}down}$ for any $k_2 \geq 1$). Hence, $(P, P') \in \mathcal{E}^{k_1\text{-}up} \cup \mathcal{E}^{k_2\text{-}down}$ for every $k_1, k_2 \geq 1$. By Remark 2.1 and Proposition 4.3, it follows that the environment $\langle A, \mathcal{P}, G\rangle$, where $G = \langle \mathcal{P}, \mathcal{E}^{k_1\text{-}up} \cup \mathcal{E}^{k_2\text{-}down}\rangle$, satisfies DLGE for every $k_1, k_2 \geq 1$.

4.2 The case of arbitrary bundle outcomes and lexicographic extension

Throughout this subsection, we consider the situation where every non-empty subset of alternatives can be an outcome, that is, $\tilde{A} = \{S \subseteq A : S \neq \emptyset\}$ (or in other words, $\kappa = m$) and the preference extension η is lexicographic. We provide four applications of Theorem 3.1. The first three results can be found in Altuntaş et al. (2023) as well; however our proofs are different.

We begin with making an observation.

Observation 4.1. *Consider an environment $\langle A, \mathcal{D}, G, \tilde{A}, \eta\rangle$ that satisfies DLGE. Then, the tuple $\langle A, \mathcal{D}, G\rangle$, viewed as an environment as defined in Section 4.1, also satisfies DLGE. This is because an SCF $f : \mathcal{D} \to A$ can also be viewed as an SCF $\tilde{f} : \mathcal{D} \to \tilde{A}$ where $\tilde{f}(P) = f(P)$ for every $P \in \mathcal{D}$, and hence, f is (locally) strategy-proof if and only if $\tilde{f}$ is (locally) strategy-proof.*

4.2.1 Adjacent strategy-proofness

For any $P \in \mathcal{P}$ and $a \in A$, let us define the rank of a at P, denoted by $r(a, P)$, as $|\{x \in A : xPa\}| + 1$. In other words, $r(a, P) = k$ if and only if $|\{x \in A : xPa\}| = k - 1$.

Let $G = \langle \mathcal{P}, \mathcal{E}^{adj}\rangle$ where $(P, P') \in \mathcal{E}^{adj}$ if there exist distinct $x, y \in A$ such that

(i) $r(x, P) + 1 = r(x, P')$,

(ii) $r(y, P) - 1 = r(y, P')$, and

(iii) $r(z,P) = r(z,P')$ for each $z \in A \setminus \{x,y\}$.

Proposition 4.4. *(Altuntaş et al. (2023)) The environment $\langle A, \mathcal{P}, G, \tilde{A}, \eta \rangle$, where $G = \langle \mathcal{P}, \mathcal{E}^{adj} \rangle$, satisfies DLGE.*

The proof of this proposition is relegated to Appendix A.5.

In what follows, we prove a negative result and show that adjacent environments no longer satisfy DLGE if we consider single-peaked domains instead of the unrestricted domain. Let $m \geq 3$ and assume that $A = \{a_1, \ldots, a_m\}$. Fix any linear order $\prec$ over A given by $a_1 \prec a_2 \prec a_3 \prec \cdots \prec a_m$. Let $\mathcal{S}$ be the set of all single-peaked preferences, called the single-peaked domain, with respect to the linear order $\prec$.[16]

Proposition 4.5. *The environment $\langle A, \mathcal{S}, G, \tilde{A}, \eta \rangle$, where $G = \langle \mathcal{S}, \mathcal{E}^{adj} \rangle$, does not satisfy DLGE.*

Proof. Let $\mathcal{S}$ be the set of all single-peaked preferences with respect to a linear order $a_1 \prec a_2 \prec a_3 \prec \ldots \prec a_m$. Consider the preference $P \in \mathcal{S}$ such that $r(a_1, P) = 1$ and any other preference $P' \in \mathcal{S}$ such that $r(a_3, P') = 1$. Notice that every path from P to P' must have $\{Z, a_2\}$-restoration where $Z = \{a_1, a_3\}$.[17] Also, note that $a_2 \in L(Z, P)$. Therefore, G violates Property *SDL*. Hence, by Theorem 3.1 it follows that $\langle A, \mathcal{S}, G, \tilde{A}, \eta \rangle$ does *not* satisfy DLGE.

■

4.2.2 k-PUSH-DOWN STRATEGY-PROOFNESS

Proposition 4.6. *(Altuntaş et al. (2023)) The environment $\langle A, \mathcal{P}, G, \tilde{A}, \eta \rangle$, where $G = \langle \mathcal{P}, \mathcal{E}^{k\text{-}down} \rangle$, satisfies DLGE if and only if $k \geq m - 1$.*

[16] See Moulin (1980) for a formal definition of single-peaked preferences.

[17] The fact that there exists a path from any preference $P \in \mathcal{S}$ to any other preference $P' \in \mathcal{S}$ is proven in Carroll (2012) and Sato (2013).

Proof. **If part:** As observed in Altuntaş et al. (2023), if $k \geq m-1$, then $(P, P') \in \mathcal{E}^{k\text{-}down}$ for every $P, P' \in \mathcal{P}$. Therefore, $\langle A, \mathcal{P}, G, \tilde{A}, \eta \rangle$ satisfies DLGE trivially.

Only if part: Suppose $k \leq m-2$. By Proposition 4.1, the environment $\langle A, \mathcal{P}, G \rangle$ does not satisfy DLGE. Therefore by Observation 4.1, the environment $\langle A, \mathcal{P}, G, \tilde{A}, \eta \rangle$ does not satisfy DLGE.

■

4.2.3 k-PUSH-UP STRATEGY-PROOFNESS

Proposition 4.7. *(Altuntaş et al. (2023)) The environment $\langle A, \mathcal{P}, G, \tilde{A}, \eta \rangle$, where $G = \langle \mathcal{P}, \mathcal{E}^{k\text{-}up} \rangle$, satisfies DLGE if and only if $k \geq m-1$.*

The proof of this proposition is relegated to Appendix A.6.

4.2.4 k_1-PUSH-UP OR k_2-PUSH-DOWN STRATEGY PROOFNESS

Let $G = \langle \mathcal{P}, \mathcal{E}^{k_1\text{-}up} \cup \mathcal{E}^{k_2\text{-}down} \rangle$ as defined in Subsubsection 4.1.3. In this subsubsection, we characterize values of k_1 and k_2 such that the environment $\langle A, \mathcal{P}, G, \tilde{A}, \eta \rangle$ satisfies DLGE.

Proposition 4.8. *The environment $\langle A, \mathcal{P}, G, \tilde{A}, \eta \rangle$, where $G = \langle \mathcal{P}, \mathcal{E}^{k_1\text{-}up} \cup \mathcal{E}^{k_2\text{-}down} \rangle$, satisfies DLGE if and only if $k_1 + k_2 \geq m-1$.*

The proof of this proposition is relegated to Appendix A.7.

Recall the environment (as defined in Subsubsection 4.1.3) $\langle A, \mathcal{P}, G \rangle$ where $G = \langle \mathcal{P}, \mathcal{E}^{up} \cup \mathcal{E}^{down} \rangle$. We have shown in Proposition 4.3 that $\langle A, \mathcal{P}, G \rangle$ satisfies DLGE. However, we now state a corollary of Proposition 4.8 which shows that for $m \geq 4$, the environment $\langle A, \mathcal{P}, G, \tilde{A}, \eta \rangle$, where $G = \langle \mathcal{P}, \mathcal{E}^{up} \cup \mathcal{E}^{down} \rangle$ does *not* satisfy DLGE. This clarifies that η plays a crucial role for an environment to satisfy DLGE.

Corollary 4.2. *Let $G = \langle \mathcal{P}, \mathcal{E}^{up} \cup \mathcal{E}^{down} \rangle$. For $m \geq 4$, the environment $\langle A, \mathcal{P}, G, \tilde{A}, \eta \rangle$ does not satisfy DLGE. For $m = 3$ and $m = 2$, the environment $\langle A, \mathcal{P}, G, \tilde{A}, \eta \rangle$ satisfies DLGE.*

A. Appendix

A.1 Proof of Theorem 3.1

Proof. **Sufficiency**: Suppose $G = \langle \mathcal{D}, \mathcal{E} \rangle$ satisfies Property SDL but $\langle A, \mathcal{D}, G, \tilde{A}, \eta \rangle$ fails DLGE. There exists a locally strategy-proof SCF $f : \mathcal{D} \to \tilde{A}$ that is not strategy-proof. Therefore, there exists $P^0, P^1 \in \mathcal{D}$ such that $f(P^1)\tilde{P}^0 f(P^0)$. Without loss of generality assume that $P^1 \in \mathcal{D}$ is such that $f(P^1)$ is the $\tilde{P}^0$-maximal outcome in the set of outcomes that is an image under f at some preference, i.e., $f(P^1) = \max_{\tilde{P}^0}\{S \in \tilde{A} : f(P) = S \text{ for some } P \in \mathcal{D}\}$.[18] Let $f(P^1) = S^1$.

Since G satisfies Property SDL, we conclude that there exists a path $\pi \equiv (\hat{P}^1 = P^0, \hat{P}^2, \ldots, \hat{P}^t = P^1)$ in G such that for all $z \in L(S^1, P^0)$, the path π has no $\{S^1, z\}$-restoration.

Consider the path $\eta(\pi) \equiv (\eta(\hat{P}^1), \eta(\hat{P}^2), \ldots, \eta(\hat{P}^t))$. Notice that $\eta(\pi)$ is a path (in $\widetilde{G}$) from $\eta(\hat{P}^1)$ to $\eta(\hat{P}^t)$.

Searching the path π backwards from $\hat{P}^t$ to $\hat{P}^1$, let $\hat{P}^s$ be the first vertex such that $f(\hat{P}^s) = S^2 \neq S^1$ i.e. $f(\hat{P}^k) = S^1$ for all $s < k \leq t$. Note that $\hat{P}^s$ always exists since $f(\hat{P}^t) \neq f(\hat{P}^1)$ and $\hat{P}^s \neq \hat{P}^t$.

Claim A.1. *$S^2 \eta(\hat{P}^s) S^1$ and $S^2 \eta(P^1) S^1$.*

Proof: Consider $\hat{P}^s$ and $\hat{P}^{s+1}$. Since $(\hat{P}^s, \hat{P}^{s+1}) \in \mathcal{E}$ and $f(\hat{P}^s) = S^2 \neq S^1 = f(\hat{P}^{s+1})$, local strategy-proofness implies $S^2 \eta(\hat{P}^s) S^1$. Finally, we show that

[18] Recall that $\tilde{P}^0 = \eta(P^0)$. We use the notation $\tilde{P}$ and $\eta(P)$ interchangeably for $P \in \mathcal{D}$. Also, we use the notation $\tilde{P}^1$ (instead of $\widetilde{P^1}$) to denote $\eta(P^1)$.

$S^2\eta(P^1)S^1$. Suppose, contrary to the claim, we have $S^1\eta(P^1)S^2$. Due to the without loss of generality assumption on P^1 and the fact that $f(\hat{P}^s) = S^2$, it must be the case that $S^1\eta(P^0)S^2$. Therefore, we have $S^1\eta(P^0)S^2$, $S^2\eta(\hat{P}^s)S^1$ and $S^1\eta(P^1)S^2$. Moreover, since η is a lexicographic preference extension it follows that neither $S^1 \subseteq S^2$ nor $S^2 \subseteq S^1$ holds. Let $\tau_{\hat{P}^s}(S^2 \setminus S^1) = z$, $\tau_{P^0}(S^1 \setminus S^2) = x$ and $\tau_{P^1}(S^1 \setminus S^2) = y$.[19]. Since $S^1\eta(P^0)S^2$, it follows that xP^0z because η is a lexicographic preference extension. Hence, $z \in L(S^1, P^0)$. This, together with the facts that η is a lexicographic preference extension, $S^1\eta(P^0)S^2$, $S^2\eta(\hat{P}^s)S^1$ and $S^1\eta(P^1)S^2$, it must be the case that $\{x,y\}P^0z$, $z\hat{P}^s\{x,y\}$, and $\{x,y\}P^1z$. This implies that the path π satisfies $\{S^1,z\}$-restoration where $z \in L(S^1, P^0)$. This leads to a contradiction to our assumption that π contains no $\{S^1,z\}$-restoration for any $z \in L(S^1, P^0)$. Therefore, $S^2\eta(P^1)S^1$. This completes the proof of the claim. $\square$

For notational consistency, let us rename the preference $\hat{P}^s$ as P^2. We return to the proof of sufficiency. Consider P^1 and P^2. Note that $f(P^1) = S^1$ and $f(P^2) = S^2$. Since $S^2\eta(P^2)S^1$ and $f(P^2) = S^2$, it follows that $\{S \in \tilde{A} : f(P) = S \text{ for some } P \in \mathcal{D} \text{ such that } f(P)\tilde{P}S^1\} \neq \emptyset$. Without loss of generality assume that $P^2 \in \mathcal{D}$ is such that S^2 is the $\tilde{P}^1$-maximal outcome in the set of outcomes that is an image under f at some preference P such that S^1 lies below $f(P)$ at $\tilde{P}$, i.e., $S^2 = \max_{\tilde{P}^1}\{S \in \tilde{A} : f(P) = S \text{ for some } P \in \mathcal{D} \text{ such that } f(P)\tilde{P}S^1\}$.

It follows from the Property SDL that there exists a path $\bar{\pi} = (\bar{P}^1 = P^1, \ldots, \bar{P}^l = P^2)$ in G having no $\{S^2,z\}$-restoration for all $z \in L(S^2, P^1)$. Once again, searching the path $\bar{\pi}$ backwards from $\bar{P}^l$ to $\bar{P}^1$, we can identify the first vertex $\bar{P}^{\bar{s}}$ such that $f(\bar{P}^{\bar{s}}) = S^3 \neq S^2$.

Claim A.2. *$S^3\eta(\bar{P}^{\bar{s}})S^2$, $S^3\eta(P^2)S^2$ and $S^2\eta(\bar{P}^{\bar{s}})S^1$.*

Proof: Using similar arguments in Claim A.1, it follows that $S^3\eta(\bar{P}^{\bar{s}})S^2$ and

[19] x and y might not be distinct.

$S^3\eta(P^2)S^2$.

Finally, we show that $S^2\eta(\bar{P}^{\bar{s}})S^1$. Suppose, contrary to the claim, we have $S^1\eta(\bar{P}^{\bar{s}})S^2$. By Claim A.1, we have $S^2\eta(P^1)S^1$ and $S^2\eta(P^2)S^1$. Therefore, we have $S^2\eta(P^1)S^1$, $S^1\eta(\bar{P}^{\bar{s}})S^2$ and $S^2\eta(P^2)S^1$. Moreover, since η is a lexicographic preference extension it follows that neither $S^1 \subseteq S^2$ nor $S^2 \subseteq S^1$ holds. Let $\tau_{\bar{P}^{\bar{s}}}(S^1 \setminus S^2) = z$, $\tau_{P^1}(S^2 \setminus S^1) = x$ and $\tau_{P^2}(S^2 \setminus S^1) = y$.[20]. Since $S^2\eta(P^1)S^1$, it follows that xP^1z because η is a lexicographic preference extension. Hence, $z \in L(S^2, P^1)$. This, together with the facts that η is a lexicographic preference extension, $S^2\eta(P^1)S^1$, $S^1\eta(\bar{P}^{\bar{s}})S^2$ and $S^2\eta(P^2)S^1$, it must be the case that $\{x, y\}P^1z$, $z\bar{P}^{\bar{s}}\{x,y\}$, and $\{x,y\}P^2z$. This implies that the path $\bar{\pi}$ satisfies $\{S^2,z\}$-restoration where $z \in L(S^2, P^1)$. This leads to a contradiction to our assumption that $\bar{\pi}$ contains no $\{S^2,z\}$-restoration for any $z \in L(S^2, P^1)$. Therefore, $S^2\eta(\bar{P}^{\bar{s}})S^1$. This completes the proof of the claim. □

Therefore, $S^3\eta(\bar{P}^{\bar{s}})S^2\eta(\bar{P}^{\bar{s}})S^1$ and $S^3\eta(P^2)S^2\eta(P^2)S^1$. Hence, S^3, S^2 and S^1 are all distinct outcomes.

For notational consistency, let us rename the preference $\bar{P}^{\bar{s}}$ as P^3. As previously argued, we can assume without loss of generality that $S^3 = \max_{\tilde{P}^2}\{S \in \tilde{A} : f(P) = S \text{ for some } P \in \mathcal{D} \text{ such that } f(P)\tilde{P}S^2\tilde{P}S^1\}$. Notice that $\{S \in \tilde{A} : f(P) = S \text{ for some } P \in \mathcal{D} \text{ such that } f(P)\tilde{P}S^2\tilde{P}S^1\} \neq \emptyset$ because $S^3\tilde{P}^3S^2\tilde{P}^3S^1$ and $f(P^3) = S^3$. Therefore, we have three distinct outcomes $S^1, S^2, S^3 \in \tilde{A}$ and a pair of distinct preferences $P^2, P^3 \in \mathcal{D}$ such that:

(i) $f(P^2) = S^2$,

(ii) $f(P^3) = S^3$,

(iii) $S^3\tilde{P}^2S^2\tilde{P}^2S^1$,

(iv) $S^3\tilde{P}^3S^2\tilde{P}^3S^1$, and

[20] As we note in the proof of Claim A.1, x and y might not be distinct.

(v) $S^3 = \max\limits_{\tilde{P}^2}\{S \in \tilde{A} : f(P) = S \text{ for some } P \in \mathcal{D} \text{ such that } f(P)\tilde{P}S^2\tilde{P}S^1\}$.

Repeating these arguments and making repeated use of arguments in Claim A.2, along with the fact that $\tilde{A}$ is finite (say $|\tilde{A}| = n$), we conclude that there exist distinct outcomes $S^1, \ldots, S^n \in \tilde{A}$ and a pair of distinct preferences $P^{n-1}, P^n \in \mathcal{D}$ such that:

(i) $f(P^{n-1}) = S^{n-1}$,

(ii) $f(P^n) = S^n$,

(iii) $S^n\tilde{P}^{n-1}S^{n-1}\tilde{P}^{n-1}S^j$ for every $j \in \{1, \ldots, n-2\}$, and

(iv) $S^n\tilde{P}^nS^{n-1}\tilde{P}^nS^j$ for every $j \in \{1, \ldots, n-2\}$.

Since $|\tilde{A}| = n, S^n$ is the first-ranked outcome in $\tilde{P^n}$ because $S^n\tilde{P}^nS^{n-1}\tilde{P}^nS^j$ for every $j \in \{1, \ldots, n-2\}$.

Note that $S^n = \max\limits_{\tilde{P}^{n-1}}\{S \in \tilde{A} : f(P) = S \text{ for some } P \in \mathcal{D} \text{ such that } f(P)\tilde{P}S^{n-1}\tilde{P}S^j \text{for every} j \in \{1, \ldots, n-2\}\}$ because $S^n\tilde{P}^{n-1}S^{n-1}\tilde{P}^{n-1}S^j$ for every $j \in \{1, \ldots, n-2\}$ and $f(P^n) = S^n$. The earlier arguments in Claim A.2 can be applied to the pair of distinct preferences P^{n-1}, P^n to infer the existence of an outcome S^{n+1} with the property $S^{n+1}\tilde{P}^nS^n$. However this is impossible since S^n is first-ranked in P^n. We have reached a contradiction. Hence, $\langle A, \mathcal{D}, G, \tilde{A}, \eta\rangle$ must satisfy DLGE.

Necessity: Let $\langle A, \mathcal{D}, G, \tilde{A}, \eta\rangle$ be an environment satisfying DLGE. We show that G satisfies Property SDL. First, we begin with a claim.

Claim A.3. *G is connected.*

Proof: We need to show $\Pi(P, P') \neq \emptyset$ for all $(P, P') \in \mathcal{D} \times \mathcal{D}$. We consider two cases:

Case (i): Let $P, P' \in \mathcal{D}$ such that there exist $S, S' \in \tilde{A}$ with $S\tilde{P}S'$ and $S\tilde{P}'S'$. We show that $\Pi(P, P') \neq \emptyset$.

Assume for contradiction that $\Pi(P, P') = \emptyset$. Define an SCF $f : \mathcal{D} \to \tilde{A}$ as follows:

(i) $f(P) = S'$,

(ii) $f(\hat{P}) = S'$ if $\Pi(P, \hat{P}) \neq \emptyset$, and

(iii) $f(\hat{P}) = \max\limits_{\eta(\hat{P})}(\{S, S'\})$ otherwise, where $\max\limits_{\eta(\hat{P})}(\{S, S'\}) = S$ if $S\eta(\hat{P})S'$ and $\max\limits_{\eta(\hat{P})}(\{S, S'\}) = S'$ if $S'\eta(\hat{P})S$.

Since $\Pi(P, P') = \emptyset$ and $S\eta(P')S'$, by definition $f(P') = S$. Also, since $S\eta(P)S'$, $S = f(P')\eta(P)f(P) = S'$ which establishes that f is not strategy-proof. Now we show that f is locally strategy-proof which will lead to a contradiction to the assumption that $\langle A, \mathcal{D}, G, \tilde{A}, \eta \rangle$ satisfies DLGE, thereby establishing that $\Pi(P, P') \neq \emptyset$.

Pick any two preferences $\hat{P}, \tilde{P} \in \mathcal{D}$ and without loss of generality assume that $(\hat{P}, \tilde{P}) \in \mathcal{E}$. We need to show that either $f(\hat{P}) = f(\tilde{P})$ or $f(\hat{P})\eta(\hat{P})f(\tilde{P})$. If $\hat{P} = P$, then by definition $f(\hat{P}) = f(\tilde{P}) = S'$. If $\hat{P} \neq P$ is such that $\Pi(P, \hat{P}) \neq \emptyset$. Let $\pi = (P^1, \ldots, P^t) \in \Pi(P, \hat{P})$. Then by definition $f(\hat{P}) = S'$. Since $(\hat{P}, \tilde{P}) \in \mathcal{E}$, construct a new path $\bar{\pi} = (P^1, \ldots, P^t, \tilde{P}) \in \Pi(P, \tilde{P})$. Consequently, by the definition of f, we conclude that $f(\tilde{P}) = S' = f(\hat{P})$. Finally, if $\hat{P} \neq P$ is such that $\Pi(P, \hat{P}) = \emptyset$. Then by the definition of f, we have $f(\hat{P}) = \max\limits_{\eta(\hat{P})}(\{S, S'\})$. This, together with the fact that $f(\tilde{P}) \in \{S, S'\}$, implies either $f(\hat{P}) = f(\tilde{P})$ or $f(\hat{P})\eta(\hat{P})f(\tilde{P})$. Hence, f is locally strategy-proof.

Case (ii): Let $P, P' \in \mathcal{D}$ such that Case (i) does not hold. Then, it must be the case that the outcomes are singletons (that is, $k = 1$). Also, P and P' must be completely opposite preferences. Therefore, by Remark 3.3 it follows that $\mathcal{D} \neq \{P, P'\}$. Hence, there exists a preference $\bar{P} \in \mathcal{D} \setminus \{P, P'\}$. Notice that neither P and $\bar{P}$ nor $\bar{P}$ and P' are completely opposite preferences. This, together with the fact that outcomes are singletons (that is, $k = 1$), by Case (i), $\Pi(P, \bar{P}) \neq \emptyset$ and $\Pi(\bar{P}, P') \neq \emptyset$. This implies that $\Pi(P, P') \neq \emptyset$.

Therefore, we establish that $\Pi(P, P') \neq \emptyset$ for all $(P, P') \in \mathcal{D} \times \mathcal{D}$. Hence G is connected. $\square$

We define a class of SCFs that we will employ repeatedly in the proof.

Definition A.1. Fix an environment $\langle A, \mathcal{D}, G, \tilde{A}, \eta \rangle$. Let $S \in \tilde{A}$, a pair of preferences $\hat{P}, P^0 \in \mathcal{D}$ such that $L(S, \hat{P}) \cap L(S, P^0) \neq \emptyset$ and let B be a non-empty set with $B \subseteq L(S, \hat{P}) \cap L(S, P^0)$. An SCF $f : \mathcal{D} \to \tilde{A}$ is monotonic with respect to the ordered tuple $(S, B, \hat{P}, P^0)$ if

(i) $f(P) = S$ if there is a path $\pi \in \Pi(P, \hat{P})$ such that for all (not necessarily distinct) $s, s' \in S$ and for all $b \in B$ with $\{s, s'\}\hat{P}b$ and $\{s, s'\}P^0 b$, $\{s, s'\}\bar{P}b$ for all $\bar{P} \in \pi$, and

(ii) $f(P) = \max\limits_{\eta(P)}(W)$ otherwise, where $W = \{\{b\} \cup S \setminus \{s, s'\} \mid$ (not necessarily distinct) $s, s' \in S, b \in B$ with $\{s, s'\}\hat{P}b$ and $\{s, s'\}P^0 b\}$.[21]

Note that $f(\hat{P}) = S$. The next lemma shows that SCF f of Definition A.1 is locally strategy-proof.

Lemma A.1. *Suppose $f : \mathcal{D} \to \tilde{A}$ is monotonic with respect to $(S, B, \hat{P}, P^0)$. Then f is locally strategy-proof.*[22]

Proof: Pick an arbitrary pair $P, P' \in \mathcal{D}$ and without loss of generality assume that $(P, P') \in \mathcal{E}$. We show either $f(P) = f(P')$, or $f(P)\eta(P)f(P')$ establishing local strategy-proofness.

Let $\mathcal{D}_S = \{\bar{P} \in \mathcal{D} : f(\bar{P}) = S\}$ denote the set of preferences which are associated to S at SCF f. There are four cases to consider.

[21]Note that if $|S| = l$, then for each $w \in W$, $l - 1 \leq |w| \leq l$. Hence, $W \subseteq \tilde{A}$.

[22]In the Definition A.1, if we use $\Pi(\hat{P}, P)$ instead of $\Pi(P, \hat{P})$, then the monotonic SCF is not necessarily locally strategy-proof. More specifically, the arguments in Case 4 of the proof does not always hold.

Case 1: $P, P' \in \mathcal{D}_S$. Then $f(P) = f(P') = S$.

Case 2: $P, P' \notin \mathcal{D}_S$. Then $f(P) = \max_{\eta(P)}(W)$ and $f(P') = \max_{\eta(P')}(W)$. Hence, either $f(P) = f(P')$ or $f(P)\eta(P)f(P')$ and $f(P')\eta(P')f(P)$ must hold.

Case 3: $P \in \mathcal{D}_S$ and $P' \notin \mathcal{D}_S$. Thus, $f(P) = S \neq S' = \max_{\eta(P')}(W) = f(P')$. Since $P \in \mathcal{D}_S$, there exists a path $\pi = (P^1, \ldots, P^t) \in \Pi(P, \hat{P})$ such that for all (not necessarily distinct) $s, s' \in S$ and for all $b \in B$ with $\{s, s'\}\hat{P}b$ and $\{s, s'\}P^0 b$, $\{s, s'\}P^l b$ for all $1 \leq l \leq t$ (recall Definition A.1). Since η is a lexicographic preference extension and $P^1 = P$, we have $S\eta(P)S'$. Therefore, $f(P)\eta(P)f(P')$.

Case 4: $P \notin \mathcal{D}_S$ and $P' \in \mathcal{D}_S$. Thus, $f(P') = S \neq \bar{S} = \max_{\eta(P)}(W) = f(P)$. Since $P' \in \mathcal{D}_S$, there exists a path $\pi = (P^1, \ldots, P^t) \in \Pi(P', \hat{P})$ such that for all (not necessarily distinct) $s, s' \in S$ and for all $b \in B$ with $\{s, s'\}\hat{P}b$ and $\{s, s'\}P^0 b$, $\{s, s'\}P^l b$ for all $1 \leq l \leq t$. Next, suppose $S\eta(P)\bar{S}$. Since $\bar{S} = \max_{\eta(P)}(W)$, it follows that for all (not necessarily distinct) $s, s' \in S$ and for all $b \in B$ with $\{s, s'\}\hat{P}b$ and $\{s, s'\}P^0 b$, $\{s, s'\}Pb$. Observe that P must be distinct from the vertices in the path π; otherwise we would contradict the hypothesis that $P \notin \mathcal{D}_S$. Since $(P, P') \in \mathcal{E}$, we now have a new path $\bar{\pi} = (P, P^1, \ldots, P^t) \in \Pi(P, \hat{P})$ such that for all (not necessarily distinct) $s, s' \in S$ and for all $b \in B$ with $\{s, s'\}\hat{P}b$ and $\{s, s'\}P^0 b$, $\{s, s'\}\bar{P}b$ for all $\bar{P} \in \bar{\pi}$. Consequently, Definition A.1 implies $f(P) = S$. This contradicts our initial assumption that $f(P) = \bar{S}$. Therefore, $\bar{S}\eta(P)S$.

This completes the proof of the lemma. $\square$

Now, we show that G satisfies Property SDL. Assume for contradiction that G violates Property SDL i.e. there exist $P^0, P^1 \in \mathcal{D}$ and $S \in \tilde{A}$ such that every path of $\Pi(P^0, P^1)$ has an $\{S, x\}$-restoration for some $x \in L(S, P^0)$. In view of Claim A.3, this statement cannot hold vacuously.

Let Γ be the set of alternatives in $L(S, P^0)$ that appear in some restoration with

S on some path of $\Pi(P^0, P^1)$:

$$\Gamma = \left\{ x \in L(S, P^0) : \text{there exists } \pi \in \Pi(P^0, P^1) \text{ with } \{S, x\}\text{-restoration} \right\}.$$

Then, the hypothesis for the contradiction can be restated as follows: each path of $\Pi(P^0, P^1)$ has an $\{S, x\}$-restoration for some $x \in \Gamma$.

For a specific path $\pi \in \Pi(P^0, P^1)$, let Γ_1^π denote the set of alternatives in $L(S, P^0)$ that appear in some restoration with S on the path π, i.e.

$$\Gamma_1^\pi = \{x \in L(S, P^0) : \pi \text{ has } \{S, x\}\text{-restoration}\}.$$

Let $\Gamma^1 \subseteq [\Gamma \cap L(S, P^1)]$ be the set of alternatives such that for *every* path $\pi = (\hat{P}^1, \ldots, \hat{P}^t) \in \Pi(P^0, P^1)$, there exists (not necessarily distinct) $s, s' \in S$, $x \in \Gamma^1$ and $1 < r < t$ with $\{s, s'\} P^0 x, x \hat{P}^r \{s, s'\}$ and $\{s, s'\} P^1 x$. Notice that here π has an $\{S, x\}$-restoration with an additional constraint that $\{s, s'\} P^1 x$ must also hold. Therefore, either $\Gamma^1 \neq \emptyset$ or $\Gamma^1 = \emptyset$ must hold. We show that each of the two possible cases $\Gamma^1 \neq \emptyset$ and $\Gamma^1 = \emptyset$ leads to a contradiction.

Case A: $\Gamma^1 \neq \emptyset$.

Let $f : \mathcal{D} \to \tilde{A}$ be the SCF which is monotonic with respect to (S, Γ^1, P^1, P^0). Note that f is well-defined since $\emptyset \neq \Gamma^1 \subseteq L(S, P^1) \cap L(S, P^0)$. According to Lemma A.1, f is locally strategy-proof. We show that f is not strategy-proof.

According to Definition A.1, $f(P^1) = S$. Pick an arbitrary path $\pi = (\hat{P}^1, \ldots, \hat{P}^t) \in \Pi(P^0, P^1)$. By the hypothesis for contradiction, there exists (not necessarily distinct) $s, s' \in S, x \in \Gamma^1$ and $1 < r < t$ with $\{s, s'\} P^0 x, x \hat{P}^r \{s, s'\}$ and $\{s, s'\} P^1 x$. Since π was chosen arbitrarily, Definition A.1 implies that $f(P^0) = \max_{\eta(P^0)}(W)$, where $W = \{\{b\} \cup S \setminus \{s, s'\} \mid$ (not necessarily distinct) $s, s' \in S, b \in B$ with $\{s, s'\} P^1 b$ and $\{s, s'\} P^0 b\}$. Notice that for each $w \in W$,

$S\eta(P^0)w$. Therefore, $S\eta(P^0)f(P^0)$. Hence, f is not strategy-proof and we have a contradiction to the assumption that $\langle A, \mathcal{D}, G, \tilde{A}, \eta \rangle$ satisfies DLGE.

This argument establishes that Case A cannot occur.

Case B: $\Gamma^1 = \emptyset$. Therefore, there exists a path $\pi^1 = (\hat{P}^1, \ldots, \hat{P}^t) \in \Pi(P^0, P^1)$ such that for all $x \in \Gamma_1^{\pi^1}$ and all (not necessarily distinct) $s, s' \in S$ with $\{s, s'\}P^0 x$, $x\hat{P}^q\{s, s'\}, \{s, s'\}\hat{P}^r x$ for some integers $1 \leq q < r \leq t$, $xP^1\{s, s'\}$.[23] Define $V = \{v \in \{1, \ldots, t-1\}$: there exists $x \in \Gamma_1^{\pi^1}$, (not necessarily distinct) $s, s' \in S$, integers $1 \leq q < v$ with $\{s, s'\}\hat{P}^1 x, x\hat{P}^q\{s, s'\}$ and $\{s, s'\}\hat{P}^v x\}$. Note that $V \neq \emptyset$ because the path π^1 has $\{S, x\}$-restoration for some $x \in \Gamma_1^{\pi^1}$. Let $P^2 = \hat{P}^{\max V}$. Note that $P^2 \neq P^0$. By the definition of P^2, it follows that for all $x \in \Gamma_1^{\pi^1}$, all (not necessarily distinct) $s, s' \in S$ with $\{s, s'\}\hat{P}^1 y$, $y\hat{P}^q\{s, s'\}$ and $\{s, s'\}\hat{P}^r y$ for some $1 \leq q < r \leq \max V$, $y\bar{P}\{s, s'\}$ for all $\bar{P} \in \pi^1|_{[P^2, P^1]} \setminus \{P^2\}$.

Let Z be the (non-empty) subset of alternatives in $\Gamma_1^{\pi^1}$ such that

(i) for all $z \in Z$, there exists (not necessarily distinct) $s, s' \in S$, integer $1 \leq q < \max V$ with $\{s, s'\}\hat{P}^1 z$, $z\hat{P}^q\{s, s'\}$ and $\{s, s'\}P^2 z$,

(ii) for all $y \in \Gamma_1^{\pi^1} \setminus Z$ (if $Z \neq \Gamma_1^{\pi^1}$), all (not necessarily distinct) $s, s' \in S$ with $\{s, s'\}\hat{P}^1 y$, $y\hat{P}^q\{s, s'\}$ and $\{s, s'\}\hat{P}^r y$ for some $1 \leq q < r < \max V$, $yP^2\{s, s'\}$.

For any two preferences $P, P' \in \mathcal{D}$, define
$L(S, P, P') = \{x \in L(S, P) \cap L(S, P')$: there exists (not necessarily distinct) $s, s' \in S$ with $\{s, s'\}Px$ and $\{s, s'\}P'x\}$.

Claim A.4. *$\Gamma \cap L(S, P^0, P^1)$ is a strict subset of $\Gamma \cap L(S, P^0, P^2)$.*

Proof: It follows from the definition of Z that if $\Gamma \cap L(S, P^0, P^1) \subseteq \Gamma \cap L(S, P^0, P^2)$, then $\Gamma \cap L(S, P^0, P^1)$ must be a strict subset of $\Gamma \cap L(S, P^0, P^2)$. Suppose it is

[23] Note that the path π^1 has $\{S, x\}$-restoration only for all $x \in \Gamma_1^{\pi^1}$

not the case that $\Gamma \cap L(S, P^0, P^1) \subseteq \Gamma \cap L(S, P^0, P^2)$ i.e. there exists $x \in \Gamma \cap L(S, P^0, P^1)$ and (not necessarily distinct) $s, s' \in S$ such that $\{s, s'\}P^0x$, $\{s, s'\}P^1x$ and $xP^2\{s, s'\}$. This implies that the path π^1 has $\{S, x\}$-restoration such that $\{s, s'\}P^0x, xP^2\{s, s'\}$ and $\{s, s'\}P^1x$. Hence, $x \in \Gamma^1$. This contradicts the hypothesis $\Gamma^1 = \emptyset$. $\square$

Claim A.5. *For every $\hat{\pi} \in \Pi(P^0, P^2)$, there exists $x \in \Gamma$ such that $\hat{\pi}$ has $\{S, x\}$-restoration.*

Proof: Suppose there exists $\hat{\pi} \in \Pi(P^0, P^2)$ and $\hat{\pi}$ has no $\{S, x\}$-restoration for any $x \in \Gamma$. Clearly P^2 is a vertex common to both $\hat{\pi}$ and $\pi^1|_{[P^2,P^1]}$. Starting from P^1, proceed along the pseudo path which is the reverse of $\pi^1|_{[P^2,P^1]}$.[24] Let P^* be the first vertex in this reverse pseudo path which also belongs to $\hat{\pi}$. From our earlier remark, such a vertex must exist (it could be P^2). Now combine the sequences of vertices $\hat{\pi}|_{[P^0,P^*]}$ and $\pi^1|_{[P^*,P^1]}$ to form the vertex sequence $\bar{\pi}$. By construction, $\bar{\pi}$ contains no repetition of vertices so that it is a path and $\bar{\pi} \in \Pi(P^0, P^1)$.

For convenience, let $\bar{\pi} = (\bar{P}^1, \ldots, \bar{P}^k, \ldots, \bar{P}^t)$ where $\bar{P}^k = P^*$, $\hat{\pi}|_{[P^0,P^*]} = (\bar{P}^1, \ldots, \bar{P}^k)$ and $\pi^1|_{[P^*,P^1]} = (\bar{P}^k, \ldots, \bar{P}^t)$. Since $\bar{\pi} \in \Pi(P^0, P^1)$, the hypothesis for the contradiction of the necessity part of Theorem 3.1 implies $\Gamma_1^{\bar{\pi}} \neq \emptyset$. Therefore, there exists $b \in \Gamma$ such that $\bar{\pi}$ has $\{S, b\}$-restoration. Since $\hat{\pi}$ has no $\{S, b\}$-restoration and by the definition of Z and P^2, it follows that there exists (not necessarily distinct) $s, s' \in S$ such that $\{s, s'\}\bar{P}^1b, b\bar{P}^k\{s, s'\}$ and $\{s, s'\}\bar{P}^tb$. Now refer back to the path π^1. Since $\{s, s'\}P^0b, bP^*\{s, s'\}$ and $\{s, s'\}P^1b$, the path π^1 has $\{S, b\}$-restoration and hence, $b \in \Gamma^1$. This contradicts the hypothesis $\Gamma^1 = \emptyset$. $\square$

[24]For a path $\pi = (P^1, \ldots, P^t)$, we say that $(P^t, \ldots, P^1)$ is the pseudo path which is the reverse of π. We use the word "pseudo" here because $(P^t, \ldots, P^1)$ might not be a path as we are in the directed graph setting.

We can now replace P^1 by P^2 in our earlier arguments and define Γ^2 in the same way as we defined Γ^1. Once again, there are two possibilities, $\Gamma^2 \neq \emptyset$ and $\Gamma^2 = \emptyset$. The former case leads to an immediate contradiction using the arguments in Case A. In the latter case, we can apply Claims A.4 and A.5 to infer the existence of P^3 such that (i) $\Gamma \cap L(S, P^0, P^2)$ is a strict subset of $\Gamma \cap L(S, P^0, P^3)$, and (ii) every path $\pi \in \Pi(P^0, P^3)$ has $\{S, x\}$-restoration for some $x \in \Gamma$. Repeating the argument, it follows that the only way to avoid a contradiction via Case A is to find an infinite sequence of vertices $P^1, P^2, \ldots P^n, \ldots$ such that

$$[\Gamma \cap L(S, P^0, P^1)] \subset [\Gamma \cap L(S, P^0, P^2)] \subset \cdots \subset [\Gamma \cap L(S, P^0, P^n)] \cdots .$$ [25]

However this is impossible in view of the finiteness of G. Thus Case B cannot occur either and the proof is complete. ■

A.2 Proof of Proposition 4.1

Proof. **If part:** As observed in Altuntaş et al. (2023), if $k \geq m - 1$, then $(P, P') \in \mathcal{E}^{k\text{-}down}$ for every $P, P' \in \mathcal{P}$. Therefore, $\langle A, \mathcal{P}, G \rangle$ satisfies DLGE trivially.

Only if part: Suppose $k \leq m - 2$. Since $k \geq 1$, it follows that $m \geq 3$. We show that $\langle A, \mathcal{P}, G \rangle$ does not satisfy Property DL, thereby using Corollary 4.1 we conclude that $\langle A, \mathcal{P}, G \rangle$ does not satisfy DLGE.

Let $P, P' \in \mathcal{P}$ and $x, y \in A$ be such that $xPyPz$ and $yP'xP'z$ for every $z \in A \setminus \{x, y\}$. We show that every path from P to P' has an $\{x, z\}$-restoration for some $z \in L(x, P)$ which would establish that $\langle A, \mathcal{P}, G \rangle$ does not satisfy Property DL. Pick any path $\pi = (P^1 = P, \ldots, P^t = P')$ from P to P'. Notice that $t \geq 3$ because of the facts that $k \leq m - 2$ and π is a path from P to P'. Since xPy,$yP'x$ and $k \leq m - 2$, there must exist $r \in \{2, \ldots, t-1\}$ and $z \in A \setminus \{x, y\}$ such

[25] Each of the subset relations is strict.

that zP^rx. Therefore, xP^1z, zP^rx and xP^tz which implies that π has an $\{x,z\}$-restoration. Since π is an arbitrary path from P to P', it follows that $\langle A, \mathcal{P}, G\rangle$ does not satisfy Property DL. This completes the proof of the only if part of the proposition. ∎

A.3 Proof of Proposition 4.2

Proof. First we consider the case where $m = 2$. Hence, $\min\{2, m-1\} = 1$. Since $m = 2$, it must be the case that $\mathcal{P} = \{P, P'\}$ where P and P' are opposite preferences and both $(P, P'), (P', P) \in \mathcal{E}^{up}$. Therefore, by Remark 3.3, it follows that $\langle A, \mathcal{P}, G\rangle$ satisfies DLGE if and only if $k \geq 1$.

Next, we consider the case where $m \geq 3$. Therefore, $\min\{2, m-1\} = 2$. We show that $\langle A, \mathcal{P}, G\rangle$ satisfies DLGE if and only if $k \geq 2$.

Only if part: Suppose $k = 1$, then we show that $\langle A, \mathcal{P}, G\rangle$ does not satisfy Property DL, thereby using Corollary 4.1 we conclude that $\langle A, \mathcal{P}, G\rangle$ does not satisfy DLGE.

Let $P, P' \in \mathcal{P}$ and $x, y, z \in A$ be such that $xPaPyPz$ and $xP'aP'zP'y$ for every $a \in A \setminus \{x, y, z\}$. We show that every path from P to P' has an $\{x,z\}$-restoration which would establish that G does not satisfy Property DL. Pick any path $\pi = (P^1 = P, \ldots, P^t = P')$ from P to P'. Notice that $t \geq 3$ because of the facts that $k = 1$ and π is a path from P to P'. Since yPz, $zP'y$ and $k = 1$, there must exist $r \in \{2, \ldots, t-1\}$ such that zP^rx. Therefore, xP^1z, zP^rx and xP^tz which implies that π has an $\{x,z\}$-restoration. Since π is an arbitrary path from P to P', it follows that $\langle A, \mathcal{P}, G\rangle$ does not satisfy Property DL. This completes the proof of the only if part of the proposition.

If part: Let $k \geq 2$. It is enough to consider the case $k = 2$ and show that $\langle A, \mathcal{P}, G\rangle$ satisfies Property DL, then using Corollary 4.1 we conclude that $\langle A, \mathcal{D}, G\rangle$

satisfies DLGE.[26]

We use the following notations in the proof. For any $\hat{P} \in \mathcal{P}$ and $a \in A$, let $\hat{P}^a \in \mathcal{P}$ be such that $a\hat{P}^a z$ for all $z \in A \setminus \{a\}$ and $x\hat{P}y$ if and only if $x\hat{P}^a y$ for every $x, y \in A \setminus \{a\}$. Thus, $\hat{P}^a$ is obtained from $\hat{P}$ by moving a to the top and keeping the relative ordering of all other alternatives unchanged. Notice that if $\hat{P} \neq \hat{P}^a$, then $(\hat{P}, \hat{P}^a) \in \mathcal{E}^{2\text{-}up}$. Also, for any $\hat{P} \in \mathcal{P}$ and $a, b \in A$, let $\hat{P}^a_b \in \mathcal{P}$ be such that $a\hat{P}^a_b b\hat{P}^a_b z$ for all $z \in A \setminus \{a, b\}$ and $x\hat{P}y$ if and only if $x\hat{P}^a_b y$ for every $x, y \in A \setminus \{a, b\}$. Notice that if $\hat{P} \neq \hat{P}^a_b$, then $(\hat{P}, \hat{P}^a_b) \in \mathcal{E}^{2\text{-}up}$.

Now, we show that $\langle A, \mathcal{P}, G \rangle$ satisfies Property DL. Pick any $P, P' \in \mathcal{P}$ and any $a \in A$. Let $B = \{x \in A \mid aP'x\} \equiv \{b_1, \ldots, b_l\}$ and $C = \{x \in A \mid xP'a\} \equiv \{c_1, \ldots, c_{m-l-1}\}$ be such that $c_1 P' \ldots P' c_{m-l-1} P' a P' b_1 \ldots P' b_l$. We now construct a path π from P to P' such that for all $b \in L(a, P)$, the path π satisfies no $\{a, b\}$- restoration. We proceed in two steps described below where we first arrange alternatives in B (according to P') and then we arrange alternatives in C (according to P'):

Step 1: Suppose $B \neq \emptyset$. Starting from P, first we arrange b_i for every $i \in \{1, \ldots, l\}$ according to the preference P'. Define the path $\pi_B = (P^0 = P, P^1, \ldots, P^l)$ where $P^i = {P^{i-1}}^a_{b_{l-i+1}}$ for each $i \in \{1, \ldots, l\}$.[27] Thereafter, proceed to Step 2.

If $B = \emptyset$, then define the path $\pi_B = \emptyset$. Thereafter, go to Step 2 and start from $P^l = P$ as $C \neq \emptyset$[28].

Step 2: Suppose $C \neq \emptyset$. Starting from P^l, we now arrange c_i for every $i \in \{1, \ldots, m-l-1\}$ according to the preference P'. Define the path $\pi_C = (\bar{P}^0 = P^l,$

[26] It is enough to consider the case $k = 2$ because of the Remark 2.1 and the fact that $\mathcal{E}^{2\text{-}up} \subseteq \mathcal{E}^{k\text{-}up}$ for any $k \geq 3$.

[27] Note that P^0 might be the same as P^1. In this case, consider the path $\pi_B = (P^0 = P^1 = P, P^2, \ldots, P^l)$.

[28] $C \neq \emptyset$ because both $C = \emptyset$ and $B = \emptyset$ cannot hold.

$\bar{P}^1 \ldots, \bar{P}^{m-l-1})$ where $\bar{P}^i = \bar{P}^{i-1^{c_{m-l-i}}}$ for each $i \in \{1, \ldots, m-l-1\}$.[29] Note that $\bar{P}^{m-l-1} = P'$.

If $C = \emptyset$, then it must be the case that $l = m - 1$ and $P^l = P'$. Define the path $\pi_C = \emptyset$.

Now, we combine the two paths obtained in Step 1 and Step 2. Define the path $\pi = (\pi_B, \pi_C) \equiv (P^0 = P, P^1, \ldots, P^l = \bar{P}^0, \bar{P}^1 \ldots, \bar{P}^{m-l-1} = P')$. Note that π is a path from P to P' such that for all $b \in L(a, P)$, the path π satisfies no $\{a, b\}$-restoration. This is because for any $b \in L(a, P)$, $a\hat{P}b$ for every $\hat{P}$ in the path π_B and the fact that the relative ranking between a and b changes at most once along the path π_C. Since P, P', a was chosen arbitrarily, it follows that $\langle A, \mathcal{P}, G \rangle$ satisfies Property DL.

■

A.4 Proof of Proposition 4.3

Proof. First, we define some notations that we will use in the proof. For any $\hat{P} \in \mathcal{P}$ and $a \in A$, let $\hat{P}_a \in \mathcal{P}$ be such that $z\hat{P}_a a$ for all $z \in A \setminus \{a\}$ and $x\hat{P}_a y$ if and only if $x\hat{P}y$ for every $x, y \in A \setminus \{a\}$. Notice that if $\hat{P} \neq \hat{P}_a$, then $(\hat{P}, \hat{P}_a) \in \mathcal{E}^{down}$. Similarly, for any $\hat{P} \in \mathcal{P}$ and $a \in A$, we have $(\hat{P}, \hat{P}^a) \in \mathcal{E}^{up}$ if $\hat{P} \neq \hat{P}^a$.[30]

Now we proceed to the proof. We show that $\langle A, \mathcal{P}, G \rangle$ satisfies Property DL, then using Corollary 4.1 we conclude that $\langle A, \mathcal{P}, G \rangle$ satisfies DLGE. Pick any $P, P' \in \mathcal{P}$ and any $a \in A$. Let $B = \{x \in A \mid aP'x\} = \{b_1, \ldots, b_l\}$ and $C = \{x \in A \mid xP'a\} = \{c_1, \ldots, c_{m-l-1}\}$ be such that $c_1 P' \ldots P' c_{m-l-1} P' a P' b_1 \ldots P' b_l$. We now construct a path π from P to P' such that for all $b \in L(a, P)$, the path π satisfies no $\{a, b\}$- restoration. We proceed in two steps:

[29] Note that $\bar{P}^0$ might be the same as $\bar{P}^1$. In this case, consider the path $\pi_C = (\bar{P}^0 = \bar{P}^1 = P^l$, $\bar{P}^2 \ldots, \bar{P}^{m-l-1})$.

[30] Recall the definition of $\hat{P}^a$, defined in the proof (if part) of Proposition 4.2.

Step 1: Suppose $B \neq \emptyset$. Starting from P, first we arrange b_i for every $i \in \{1, \ldots, l\}$ according to the preference P'. Define the path $\pi_B = (P^0 = P, P^1, \ldots, P^l)$ where $P^i = {P^{i-1}}_{b_i}$ for each $i \in \{1, \ldots, l\}$.[31] Thereafter, proceed to Step 2.

If $B = \emptyset$, then define the path $\pi_B = \emptyset$. Thereafter, go to Step 2 and start from $P^l = P$ as $C \neq \emptyset$[32].

Step 2: Suppose $C \neq \emptyset$. Starting from P^l, we now arrange c_i for every $i \in \{1, \ldots, m-l-1\}$ according to the preference P'. Define the path $\pi_C = (\bar{P}^0 = P^l, \bar{P}^1 \ldots, \bar{P}^{m-l-1})$ where $\bar{P}^i = {\bar{P}^{i-1}}^{c_{m-l-i}}$ for each $i \in \{1, \ldots, m-l-1\}$.[33] Note that $\bar{P}^{m-l-1} = P'$.

If $C = \emptyset$, then it must be the case that $l = m-1$ and $P^l = P'$. Define the path $\pi_C = \emptyset$.

Now, we combine the two paths obtained in Step 1 and Step 2. Define the path $\pi = (\pi_B, \pi_C) \equiv (P^0 = P, P^1, \ldots, P^l = \bar{P}^0, \bar{P}^1 \ldots, \bar{P}^{m-l-1} = P')$. Note that π is a path from P to P' such that for all $b \in L(a, P)$, the path π satisfies no $\{a, b\}$-restoration. This is because for any $b \in L(a, P)$, $a\hat{P}b$ for every $\hat{P}$ in the path π_B and the fact that the relative ranking between a and b changes at most once along the path π_C. Since P, P', a was chosen arbitrarily, it follows that $\langle A, \mathcal{P}, G \rangle$ satisfies Property DL. ■

A.5 Proof of Proposition 4.4

Proof. We show that G satisfies Property SDL, then using Theorem 3.1 we conclude that $\langle A, \mathcal{P}, G, \tilde{A}, \eta \rangle$ satisfies DLGE. Given any $P \in \mathcal{P}$ and $a \in A$, let

[31] Note that P^0 might be the same as P^1. In this case, consider the path $\pi_B = (P^0 = P^1 = P, P^2, \ldots, P^l)$.

[32] $C \neq \emptyset$ because both $C = \emptyset$ and $B = \emptyset$ cannot hold.

[33] Note that $\bar{P}^0$ might be the same as $\bar{P}^1$. In this case, consider the path $\pi_C = (\bar{P}^0 = \bar{P}^1 = P^l, \bar{P}^2 \ldots, \bar{P}^{m-l-1})$.

$P^{(a,l)} \in \mathcal{P}$ be such that $r(a, P^{(a,l)}) = l$ and xPy if and only if $xP^{(a,l)}y$ for every $x, y \in A \setminus \{a\}$ where $l \in \{1, \ldots, m\}$.

Pick any $P, P' \in \mathcal{P}$ and without loss of generality assume that $A = \{a_1, \ldots, a_m\}$ with $a_1 P' a_2 P' \ldots P' a_m$. Now we construct a specific path from P to P'. We proceed in the following steps:

Step 1: Suppose $r(a_1, P) = k > 1$, then define the path $\pi_1 = (\hat{P}_{1_1}, \ldots, \hat{P}_{1_k})$ where $\hat{P}_{1_1} = P$ and $\hat{P}_{1_l} = \hat{P}_{1_{l-1}}^{(a_1, k-(l-1))}$ for each $l \in \{2, \ldots, k\}$. Verbally speaking, the path π_1 is constructed by only improving the rank of a_1 by 1 at each consecutive vertices, starting from $\hat{P}_{1_1}$ and ending at the preference ($\hat{P}_{1_k}$) where a_1 is the top ranked alternative.

If $r(a_1, P) = 1$, then define the path $\pi_1 = \emptyset$. Thereafter, go to the next step and start with $\hat{P}_{1_k} = P$.

Step 2: Suppose $r(a_2, \hat{P}_{1_k}) = r > 2$, then define the path $\pi_2 = (\hat{P}_{2_1}, \ldots, \hat{P}_{2_{r-1}})$ where $\hat{P}_{2_1} = \hat{P}_{1_k}$ and $\hat{P}_{2_l} = \hat{P}_{2_{l-1}}^{(a_2, r-(l-1))}$ for each $l \in \{2, \ldots, r-1\}$. Verbally speaking, the path π_2 is constructed by only improving the rank of a_2 by 1 at each consecutive vertices, starting from $\hat{P}_{1_k}$ and ending at the preference ($\hat{P}_{2_{r-1}}$) where a_2 is the second ranked alternative.

If $r(a_2, \hat{P}_{1_k}) = 2$, then define the path $\pi_2 = \emptyset$. Thereafter, go to the next step and start with $\hat{P}_{2_{r-1}} = \hat{P}_{1_k}$.

Step 3: Suppose $r(a_3, \hat{P}_{2_{r-1}}) = q > 3$, then define the path $\pi_3 = (\hat{P}_{3_1}, \ldots, \hat{P}_{3_{q-2}})$ where $\hat{P}_{3_1} = \hat{P}_{2_{r-1}}$ and $\hat{P}_{3_l} = \hat{P}_{3_{l-1}}^{(a_3, q-(l-1))}$ for each $l \in \{2, \ldots, q-2\}$. Verbally speaking, the path π_3 is constructed by only improving the rank of a_3 by 1 at each consecutive vertices, starting from $\hat{P}_{2_{r-1}}$ and ending at the preference ($\hat{P}_{3_{q-2}}$) where a_3 is the third ranked alternative.

If $r(a_3, \hat{P}_{2_{r-1}}) = 3$, then define the path $\pi_3 = \emptyset$. Thereafter, go to the next step and start with $\hat{P}_{3_{q-2}} = \hat{P}_{2_{r-1}}$.

Similarly, we proceed till $m-1$ steps and get hold of the paths $\pi_1, \ldots, \pi_{m-1}$.

Notice that at the end of Step $m-1$, the final preference of the path π_{m-1} is the preference P'.

Now define the path $\pi = (\pi_1, \dots, \pi_{m-1})$. Let $\pi = (P^1, \dots, P^t)$. Observe that π is a path from P to P'. Also, for any non-empty subset $S \subseteq A$, the path π does not satisfy $\{S, b\}$-restoration for all $b \in L(S, P)$. This is because for any non-empty subset $S \subseteq A$ and any $b \in L(S, P)$, if $\{s, s'\}P^i b$ and $bP^j\{s, s'\}$ for some $i, j \in \{1, \dots, t\}$ with $i < j$ and some $s, s' \in S$, then by the construction of the path π, it follows that $bP^q\{s, s'\}$ for every $q \in \{j+1, \dots, t\}$. Therefore, the path π does not satisfy $\{S, b\}$-restoration.

Since $P, P' \in \mathcal{P}$ was chosen arbitrarily, it follows that G satisfies Property SDL. Using Theorem 3.1 we conclude that $\langle A, \mathcal{P}, G, \tilde{A}, \eta \rangle$ satisfies DLGE.

■

A.6 Proof of Proposition 4.7

Proof. **If part:** As observed in Altuntaş et al. (2023), if $k \geq m-1$, then $(P, P') \in \mathcal{E}^{k\text{-}up}$ for every $P, P' \in \mathcal{P}$. Therefore, $\langle A, \mathcal{P}, G, \tilde{A}, \eta \rangle$ satisfies DLGE trivially.

Only if part: Suppose $k \leq m-2$. We show that G does not satisfy Property SDL, thereby using Theorem 3.1 we conclude that $\langle A, \mathcal{P}, G, \tilde{A}, \eta \rangle$ does not satisfy DLGE.

Since $1 \leq k \leq m-2$, it must be the case that $m \geq 3$. Let $P, P' \in \mathcal{P}$ and $x, y \in A$ be such that $zPxPy$ and $zP'yP'x$ for every $z \in A \setminus \{x, y\}$. Let $S = A \setminus \{x, y\}$. Note that $S \neq \emptyset$ and $y \in L(S, P)$. We show that every path from P to P' has an $\{S, y\}$-restoration, thereby establishing that G does not satisfy Property SDL. Pick any path $\pi = (P^1 = P, \dots, P^t = P')$ from P to P'. Notice that $t \geq 3$ because of the facts that $k \leq m-2$ and π is a path from P to P'. Since xPy,$yP'x$ and $k \leq m-2$, there must exist $r \in \{2, \dots, t-1\}$ and $z \in S$ such that yP^rz. Therefore,

zP^1y, yP^rz and zP^ty which implies that π has an $\{S, y\}$-restoration. Since π is an arbitrary path from P to P', it follows that G does not satisfy Property SDL. This completes the proof of the only if part of the proposition. ∎

A.7 Proof of Proposition 4.8

Proof. **If part:** First, we define some notations that we will be using in the proof. For any $P, \hat{P} \in \mathcal{P}$ and $B \subseteq A$ with $1 \leq |B| \leq m-1$, let $\hat{P}^{B_P} \in \mathcal{P}$ be such that

(i) $a\hat{P}^{B_P}b$ if and only if aPb for every $a, b \in B$,

(ii) $x\hat{P}^{B_P}y$ if and only if $x\hat{P}y$ for every $x, y \in A \setminus B$, and

(iii) $b\hat{P}^{B_P}x$ for each $b \in B$ and each $x \in A \setminus B$.

Note that if $|B| = k_1$ where $1 \leq k_1 \leq m-1$, then $(\hat{P}, \hat{P}^{B_P}) \in \mathcal{E}^{k_1\text{-}up}$.

Similarly, For any $P, \hat{P} \in \mathcal{P}$ and $C \subseteq A$ with $1 \leq |C| \leq m-1$, let $\hat{P}_{C_P} \in \mathcal{P}$ be such that

(i) $a\hat{P}_{C_P}b$ if and only if aPb for every $a, b \in C$,

(ii) $x\hat{P}_{C_P}y$ if and only if $x\hat{P}y$ for every $x, y \in A \setminus C$, and

(iii) $x\hat{P}_{C_P}c$ for every $c \in C$ and $x \in A \setminus C$.

Note that if $|C| = k_2$ where $1 \leq k_2 \leq m-1$, then $(\hat{P}, \hat{P}_{C_P}) \in \mathcal{E}^{k_2\text{-}down}$.

Now we prove the if part of the proposition. Let $k_1 + k_2 \geq m-1$. It is enough to consider k_1 and k_2 such that $k_1 + k_2 = m-1$ and show that G satisfies Property SDL, then using Theorem 3.1 we conclude that $\langle A, \mathcal{P}, G, \tilde{A}, \eta \rangle$ satisfies DLGE.[34]

[34] It is enough to consider k_1 and k_2 such that $k_1 + k_2 = m-1$ because of the Remark 2.1 and the fact that for any $k_1^* + k_2^* > m-1$, there exists k_1, k_2 with $k_1 + k_2 = m-1$ and $\mathcal{E}^{k_1\text{-}up} \cup \mathcal{E}^{k_2\text{-}down} \subseteq \mathcal{E}^{k_1^*\text{-}up} \cup \mathcal{E}^{k_2^*\text{-}down}$.

Pick any $P, P' \in \mathcal{P}$ and without loss of generality assume that $A = \{a_1, \ldots, a_m\}$ with $a_1 P' a_2 P' \ldots P' a_m$. We construct a specific path from P to P' as described below:

Let $\pi = (P^1 = P, P^2, P^3)$ where $P^2 = P^{B_{P'}}$ and $P^3 = P^2_{C_{P'}}$ such that $B = \{a_1, \ldots, a_{k_1}\}$ and $C = \{a_{k_1+2}, \ldots, a_m\}$. Notice that by contruction, $P^3 = P'$ and π is a path in G. For any $S \subseteq A$, the path π does not satisfy $\{S, b\}$-restoration for all $b \in L(S, P)$. This is because for any $b \in L(S, P)$, if $\{s, s'\} P^1 b$ and $b P^2 \{s, s'\}$ for some $s, s' \in S$, then by the construction of the path π, it follows that $b P^3 \{s, s'\}$. Hence, the path π does not satisfy $\{S, b\}$-restoration for all $b \in L(S, P)$.

Since $P, P' \in \mathcal{P}$ was chosen arbitrarily, it follows that G satisfies Property SDL. This completes the proof of if part of the proposition.

Only if part: Suppose $k_1 + k_2 \leq m - 2$. We show that G does not satisfy Property SDL, thereby using Theorem 3.1 we conclude that $\langle A, \mathcal{P}, G, \tilde{A}, \eta \rangle$ does not satisfy DLGE.

Since $k_1 \geq 1, k_2 \geq 1$ and $k_1 + k_2 \leq m - 2.$, it must be the case that $m \geq 4$. Let $P, P' \in \mathcal{P}$ and $x, y \in A$ be such that

(i) $r(x, P) = k_1 + 1$,

(ii) $r(y, P) = k_1 + 2$,

(iii) $r(x, P') = k_1 + 2$,

(iv) $r(y, P') = k_1 + 1$, and

(v) $r(z, P) = r(z, P')$ for each $z \in A \setminus \{x, y\}$.

Let $S = \{a \in A : r(a, P) \leq k_1 + 1\}$. Notice that by the construction of the set S, it follows that for every $z \in L(S, P)$, it must be the case that sPz for each $s \in S$. Also, notice that by the choice of P', it follows that for every $z \in L(S, P) \setminus \{y\}$, it must be the case that $sP'z$ for each $s \in S$. We show that every path from P to P'

has an $\{S, z\}$-restoration for some $z \in L(S, P)$, thereby establishing that G does not satisfy Property SDL. Pick any path $\pi = (P^1 = P, \ldots, P^t = P')$ from P to P'. Let $\alpha = \min\{i \in \{1, \ldots, t\} : r(y, P^i) \neq k_1 + 2\}$. Since π is a path from P to P' and $r(y, P') \neq k_1 + 2$, the set $\{i \in \{1, \ldots, t\} : r(y, P^i) \neq k_1 + 2\}$ is non-empty and α always exists. Notice that $\alpha \geq 2$ because $r(y, P^1 = P) = k_1 + 2$. We distinguish two cases:

Case 1: Suppose $r(y, P^\alpha) > k_1 + 2$. Then it must be the case that $(P^{\alpha-1}, P^\alpha) \in \mathcal{E}^{k_1\text{-}up}$ and there exists $z \in L(S, P) \setminus \{y\}$ such that $r(z, P^\alpha) \leq k_1$. This implies that there exists $s \in S$ such that $zP^\alpha s$. Also, $sP^1 z$ and $sP^t z$. Therefore, the path π satisfies (S, z)-restoration where $z \in L(S, P)$. This completes the proof for the first case.

Case 2: Suppose $r(y, P^\alpha) < k_1 + 2$. Then it must be the case that $(P^{\alpha-1}, P^\alpha) \in \mathcal{E}^{k_2\text{-}down}$ and there exists $s \in S$ such that $r(s, P^\alpha) \geq m - k_2 + 1$. This implies that there exists $z \in L(S, P) \setminus \{y\}$ such that $zP^\alpha s$. Also, $sP^1 z$ and $sP^t z$. Therefore, the path π satisfies (S, z)-restoration where $z \in L(S, P)$. This completes the proof for the second case.

Since π is an arbitrary path from P to P', it follows that G does not satisfy Property SDL. This completes the proof of the only if part of the proposition. ■

REFERENCES

ALTUNTAŞ, A., W. PHAN, AND Y. TAMURA (2023): "Some characterizations of generalized top trading cycles," *Games and Economic Behavior*.

CARROLL, G. (2012): "When are local incentive constraints sufficient?" *Econometrica*, 80, 661–686.

CHO, W. J. AND C. PARK (2023): "The Local-global Equivalence on General Networks," https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4551159, unpublished manuscript.

KUMAR, U., S. ROY, A. SEN, S. YADAV, AND H. ZENG (2021): "Local-global equivalence in voting models: A characterization and applications," *Theoretical Economics*, 16, 1195–1220.

MISHRA, D., A. PRAMANIK, AND S. ROY (2016): "Local incentive compatibility with transfers," *Games and Economic Behavior*, 100, 149–165.

MOULIN, H. (1980): "On strategy-proofness and single peakedness," *Public Choice*, 35, 437–455.

SATO, S. (2013): "A sufficient condition for the equivalence of strategy-proofness and nonmanipulability by preferences adjacent to the sincere one," *Journal of Economic Theory*, 148, 259–278.