Paving the road to prosperity: Productive government spending, takeoff and robust endogenous growth

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Abstract

This study contributes to the debate on how government spending shapes the growth process. We take the analysis in three new directions. First, we consider government spending, both as a flow and as a cumulated stock, in a scale-invariant Schumpeterian model of endogenous innovation. Second, we allow public spending to be the catalyst that precipitates an industrial takeoff. Third, we postulate a production structure that generates robust endogenous growth by violating the conventional condition for endogenous growth, namely, that the economy's reduced-form production function must be linear in the accumulated factor. With non-distortionary taxation, increasing productive government spending causes an earlier industrial takeoff and faster economic growth. With distortionary labor-income tax under elastic labor supply, instead, increasing productive government spending has a U-shaped effect on the timing of the industrial takeoff and an inverted-U effect on economic growth. Using cross-country panel data, we document a hump-shaped relationship between productive government spending and economic growth. Calibrating the model to the US, we find that raising productive government spending from its historical value to its growth-maximizing value causes an earlier industrial takeoff by over two decades and an increase in the long-run level of output by about 40%. We also explore the robustness of our results under a consumption tax and a corporate income tax.

JEL classification: E60, O30, O40
Keywords: Government spending, taxation, industrialization, innovation, economic growth

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1 Introduction

The goal of this study is to contribute to the debate on the role of government spending in shaping the growth process. We follow previous studies to focus on the government's provision of productive public services to the private sector. We then take the analysis in three new directions. First, we investigate the role of government spending in a scale-invariant Schumpeterian model of endogenous innovation. Second, we allow public spending to be the catalyst that precipitates the takeoff of the economy. This is a question never raised before in the theoretical work on the growth effects of public spending. Third, we postulate a production structure with government spending that generates robust endogenous growth by violating the conventional condition for endogenous growth identified by Barro (1990), namely, that the economy's reduced-form production function must be linear in the accumulated factors. Consequently, our results expand significantly the conditions under which general-equilibrium growth models produce constant exponential growth in steady state, in particular innovation-driven growth that starts at a specific date, accelerates throughout the secular transition, and in steady state is scale invariant and subject to policy action.

Given our emphasis on the provision of productive public services as the catalyst of the takeoff, a classic example of productive government spending driving our model is that it represents the allocation of resources to building and maintaining a public infrastructure that produces a flow of productive services that enhances the productivity of the private sector.[1] The evidence on the importance of infrastructure is extensive and motivates a similarly extensive literature that discusses the desirability of governments taking an active role in building and maintaining infrastructure. This literature emphasizes that history provides several examples of infrastructure building that arguably triggered dramatic growth accelerations. Examples of empirical studies in this vein are Fernald (1999) and Agrawal *et al.* (2017). They both consider road-building investment in the 20th century and provide evidence that such public infrastructure has significant positive effects on productivity and innovation in the US economy. Similarly, Donaldson and Hornbeck (2016) consider the expansion of the railroad network in the 19th century and find significant effects on the agricultural sector of the US economy. As an example of the policy arguments that build on this literature, consider one of the United Nations Sustainable Development Goals: "to build resilient infrastructure, promote sustainable industrialization and foster innovation", the rationale being that "sustained growth must include industrialization that [. . .] is supported by innovation and resilient infrastructure".[2] This quote describes quite well the main results produced by our theoretical model, which we now outline.

As mentioned, to study the effects of public spending on industrialization and the emergence of innovation, we develop a scale-invariant Schumpeterian growth model that features an endogenous takeoff and a productive structure that is not linear in the accumulated factors, firm knowledge alone in the model's baseline specification and firm knowledge and public capital jointly in the extended specification. The provision of productive government services makes private activity more productive and increases the level of output. As in

[1] Another example of productive government spending is military spending. A recent empirical study by Antolin-Diaz and Surico (2024) shows that military spending in the US indeed has positive effects on R&D and innovation.

[2] https://www.un.org/sustainabledevelopment/infrastructure-industrialization/

Barro (1990), the magnitude of this effect depends on the elasticity of output with respect to government spending in the production function for a final homogenous good. Differently from Barro (1990), our Schumpeterian model generates endogenous growth even when this elasticity is zero due to the property of robust endogenous growth first reported in Peretto (2018) in a model that does not consider productive government spending. Extending that framework, we make this elasticity positive to enable productive government spending to have a positive effect on firm size and economic growth and to explore how varying this elasticity generates different cases for the dynamics of the economy. We then investigate how public spending and its financing affect the timing of the takeoff and the overall shape of the secular growth process.

Given that government needs to collect tax revenue to finance its spending, the modelling of taxation plays an important role in our growth-theoretic analysis. We first consider a simple case in which government spending is financed by a non-distortionary labor income tax under inelastic labor supply. In this case, higher productive government spending yields an earlier industrial takeoff and a higher transitional growth rate by increasing firm size in the short run. It does not, however, affect economic growth in the long run (steady state) due to scale invariance, i.e., the sterilization of the strong scale effect in our Schumpeterian growth model with endogenous market structure.[3]

We then consider the more realistic case in which government spending is financed by a distortionary labor income tax that reduces employment under elastic labor supply. In this case, although productive government spending continues to have no effect on the steady-state growth rate due to the model's scale invariance, it has a U-shaped effect on the timing of industrial takeoff and a hump-shaped effect on the transitional growth rate. To link this theoretical result to the data, we revisit the cross-country evidence and document with the most recent panel data a hump-shaped relationship between productive government spending and economic growth. We also calibrate our model to the US as a representative advanced economy and find that raising productive government spending from its historical value to its growth-maximizing value causes an earlier industrial takeoff by over two decades and an increase in the long-run level of output by roughly 40%.

Given the importance of different tax instruments in different time periods, we explore the robustness of our results under other tax instruments: a consumption tax and a corporate income tax. We find that when a consumption tax finances government spending, increasing the tax rate leads to an earlier industrial takeoff and a higher transitional growth rate but does not affect steady-state growth. These results are in line with the effects of the non-distortionary labor income tax because the consumption tax does not affect the equilibrium level of employment, so that in the short run only the positive effect of productive government spending is at work.

When a corporate income tax finances government spending, increasing the tax rate has a U-shaped effect on the timing of industrial takeoff. This result is also in line with the effects of the distortionary labor income tax, except that in this case the negative effect arises from the reduction of firm entry (rather than employment) caused by the corporate income tax. The more important difference is that the effect of productive government spending on the steady-state growth rate in the industrial era becomes positive because the corporate income

[3]See Laincz and Peretto (2006) for a discussion of the scale effect in the Schumpeterian growth model.

tax increases firm size in the long run by discouraging firm entry.

In the final part of the paper, we extend the model to allow for the accumulation of public capital. Specifically, we make the provision of productive public services to the private sector a stock variable instead of a flow variable. This extension brings the model closer to the empirical evidence on infrastructure mentioned above. We find again that raising the government spending ratio hastens the industrial takeoff with inelastic labor supply and has an ambiguous effect on the timing of the industrial takeoff with elastic labor supply, while it has no affect on the steady-state growth rate regardless of whether labor supply is elastic or perfectly inelastic. However, the inclusion of public capital changes fundamentally the transitional dynamics of the model and produces new insights. In particular, the accumulation of public capital alone fuels the growth of the economy in the pre-industrial era, since the private sector does not invest in innovation. However, this process is subject to decreasing returns to scale (DRS) to public capital and slows down as long as private sector technology remains constant. In other words, public capital accumulation is not an engine of long-run growth. Yet, it is the catalyst of industrialization because it expands the size of the market and thus creates the conditions for private sector innovation. When industrialization occurs, the growth rate of public capital stops falling and starts growing due to the fact that technological innovation by the private sector raises output, and thereby tax revenue, so fast that it compensates for the falling average product of public capital. This interdependence between the accumulation of public and private productive assets shapes the growth path of the economy in the industrial era. Scale invariance, however, still holds and in steady state the growth rate does not depend on the public spending ratio. Therefore, this extended model as well produces effects that agree with the evidence of a hump-shaped relationship between productive government spending and economic growth that we discussed above.

Our study relates to the literature that examines how government spending affects economic growth. Barro (1990) introduces government spending as a way to obtain an AK endogenous growth model that features constant returns to scale to physical capital as an equilibrium outcome. This growth-theoretic framework is an excellent point of departure to illustrate the novelty of our approach because it is analytically transparent and delivers a convincing fundamental insight about the positive contribution that government expenditure can make to economic growth. Specifically, the model produces a hump-shaped relation between economic growth and government spending as a fraction of GDP, a relation that reflects the competing effects of productive public services and distortionary taxation. Since its development more than three decades ago, this insight has stood the test of time and has received strong empirical support. Subsequent studies, such as Barro and Sala-i-Martin (1992), Futagami *et al.* (1993), Glomm and Ravikumar (1994), Futagami and Mino (1995), Turnovsky (1996, 2000), Futagami *et al.* (2008), Chatterjee and Turnovsky (2012) and Maebayashi *et al.* (2017), explore different ways to model productive government spending in capital-based growth models. Our study complements these contributions by introducing productive government spending to a scale-invariant Schumpeterian innovation-driven growth model with endogenous industrial takeoff.

The empirical literature on the growth effects of public spending is quite large, see for example, Kormendi and Meguire (1985), Ahmed (1986), Aschauer (1989), Levine and Renelt (1992), Evans and Karras (1994), Andres *et al.* (1996), Devarajan *et al.* (1996), Kneller *et al.* (1999) and Folster and Henrekson (2001). Several of these studies identify either a positive, a negative or even an insignificant effect of government spending on economic growth.[4] For our purposes, Kneller *et al.* (1999) stand out because they test explicitly the Barro (1990) model and pay close attention to the financing of public spending. That is, they incorporate the government's budget constraint as an explicit element of the regression analysis. Their main conclusion validates the core prediction of Barro (1990): productive public spending financed with non-distortionary taxes boosts growth, unproductive public spending financed with distortionary taxes lowers growth. Since in the real world, most governments engage in a mixture of productive and non-productive spending and of distortionary and non-distortionary taxes, it is not surprising that many studies that do not check explicitly for the government's budget constraint do not reach clear conclusions on the sign of the growth effect of public spending. Kneller *et al.* (1999), in contrast, account for the government's budget constraint and as a consequence identify cleanly the tradeoff driving the seminal insight in Barro (1990).

Our study also relates to the literature on innovation and economic growth. The seminal study in this literature is Romer (1990), who develops the first R&D-based growth model driven by the development of new products (horizontal innovation). The roughly contemporaneous study by Aghion and Howitt (1992) develops the quality-ladder Schumpeterian growth model in which innovation is driven by the improvement of the quality of products (vertical innovation); see also Grossman and Helpman (1991) and Segerstrom *et al.* (1990). Subsequent studies in this literature combine the two dimensions of innovation to develop the Schumpeterian growth model with endogenous market structure that removes the scale effect; see Smulders and van de Klundert (1995) and Peretto (1998, 1999) for the variant with creative accumulation and Howitt (1999) for the variant with creative destruction.[5] Peretto (2015) builds on these contributions and develops a model that features the endogenous activation of the two dimensions of innovation when the economy crosses dimension-specific thresholds or market size, a property that produces an endogenous takeoff. This study expands the scope of this literature by exploring the effects of productive government spending in the scale-invariant Schumpeterian growth model with endogenous market structure and endogenous takeoff.

Therefore, our study also relates to the branch of the literature that examines the effects of fiscal policy on innovation and economic growth. For example, Lin and Russo (1999), Zeng and Zhang (2002), Peretto (2003, 2007a, 2007b, 2011), Haruyama and Itaya (2006), Chen *et al.* (2017), Jaimovich and Rebelo (2017), Akcigit *et al.* (2022) and Arawatari *et al.* (2023) explore the effects of various fiscal policy instruments in different variants of the innovation-driven growth model. Among these studies, Peretto (2007b) also studies the effects of distortionary taxes that finance productive public spending in the scale-invariant Schumpeterian growth model with endogenous market structure; however, he considers only local dynamics around the steady state and follows the literature's conventional wisdom in postulating a productive structure that is linear in firm knowledge. Introducing agents with heterogeneous R&D abilities to the model in Romer (1990), Arawatari *et al.* (2023) consider the effect of productive government spending on innovation and show that it becomes non-

[4] For a recent exhaustive review of this literature, see Arawatari *et al.* (2023).

[5] Garcia-Macia *et al.* (2019) provide evidence that innovation is mostly driven by the creative accumulation of incumbent firms.

linear due to the presence of heterogeneous agents. Our study contributes to this literature by exploring the effects of productive government spending and various tax instruments on the endogenous transition of an economy from pre-industrial stagnation to innovation-driven growth.

Finally, our study also relates to the literature on endogenous takeoff and economic growth. Galor and Weil (2000) develop the seminal model of Unified Growth Theory (UGT) to capture the transition from a pre-industrial economy to a modern economy with technological progress. Subsequent studies include Galor and Moav (2002), Galor and Mountford (2008), Galor *et al.* (2009) and Ashraf and Galor (2011) and provide supportive evidence for UGT.[6] In a related branch of the literature, Peretto (2015) develops a Schumpeterian growth model with endogenous takeoff to capture the transition from a pre-industrial economy to a modern economy with innovation-driven growth.[7] Our study contributes to this branch of the literature by being the first study to explore the effects of productive government spending on the industrialization of an economy in a Schumpeterian growth model with endogenous takeoff.

The rest of this study is organized as follows. Section 2 provides some stylized facts. Section 3 presents the Schumpeterian growth model. Section 4 explores the effects of productive government spending. Section 5 considers two extensions with different tax instruments. Section 6 extends the approach to public capital. The final section concludes.

2 Stylized facts

In this section, we revisit the stylized facts from cross-country panel data that motivate our study. There exists an established empirical literature that examines the relationship between government spending and economic growth. While many of the early studies found either a positive, a negative or even an insignificant effect of government spending on economic growth, the more recent studies that account for the government's budget constraint find a hump-shaped relationship between government spending and economic growth; see Coayla (2021) for a recent review.[8] As stated, this result validates the Barro (1990) insight. It is useful to review the existing evidence with recent data and provide fresh support for the hump-shaped relation between public spending and economic growth. This relation is the key empirical fact that informs and disciplines our theoretical exercise.[9]

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[6] Galor (2005, 2011) provides a comprehensive review of UGT.

[7] See also Iacopetta and Peretto (2021), Chu, Fan and Wang (2020), Chu, Kou and Wang (2020), Chu, Furukawa and Wang (2022), Chu, Peretto and Wang (2022), Chu and Peretto (2023) and Chu, Peretto and Xu (2023), who explore other mechanisms, such as corporate governance, status-seeking culture, intellectual property rights, rent-seeking government, agricultural revolution, income inequality and international trade, of endogenous takeoff in the Schumpeterian growth model.

[8] Most of these studies are based on a single country or a small number of countries. A notable exception is Asimakopoulos and Karavias (2016), who also consider cross-country panel data.

[9] In this study, we are interested in additional aspects of the growth path that the literature has not examined before, most prominently the timing of the transition from stagnation to growth and the potential role of public spending in explaining large difference in income across countries as the result of the secular cumulation of differences in growth rates. Ideally, we would examine these other aspects with data on the timing of the takoff in various countries. Such data, unfortunately, is not available.

Our theoretical model exhibits a hump-shaped effect of productive government spending on the contemporaneous growth rate under distortionary taxation as long as the economy hasn't reached yet the balanced growth path. Since we are not necessarily interested in the specific tax instrument that induces the non-monotonic relationship, we use the following empirical specification to document the hump-shaped relation between public spending and growth:

$$g_{it} = \vartheta_1 \gamma_{it} + \vartheta_2 \gamma_{it}^2 + \vartheta_3 y_{it-1} + \varphi_i + \varrho_t + \epsilon_{it},$$

where g_{it} denotes the average annual growth rate (of real GDP, real GDP per capita or real GDP per worker) in country i in period t, γ_{it} denotes the average value of productive government spending, defined as government spending on education, health, defence, and economic affairs as in Devarajan *et al.* (1996), as a share of GDP in country i at period t, γ_{it}^2 denotes the quadratic term of γ_{it} in country i at period t, and y_{it-1} is the log value of per capita GDP in country i at the beginning of period t to capture the country's initial income level. φ_i is the country fixed effect, ϱ_t is the period fixed effect, and ϵ_{it} is the error term. The data is from 1975 to 2015, and we consider five years as a period to remove cyclical fluctuations. So, we have 8 periods. After merging data from the IMF Government Finance Statistics and the Penn World Table, we have a sample of 189 observations covering 59 countries. We report the summary statistics of the variables in Appendix B.

Table 1 reports our baseline regression results. The dependent variable in column (1) is the average annual growth rate of real GDP, capturing output growth. The dependent variable in column (2) is the average annual growth rate of real GDP per capita, capturing income growth. The dependent variable in column (3) is the average annual growth rate of real GDP per worker, capturing labor productivity growth. In all columns, the coefficients on productive government spending are significantly positive, whereas the coefficients on the quadratic term are significantly negative. Therefore, the empirical relationship between productive government spending and economic growth follows a hump-shaped pattern.

Table 1: Effects of productive government spending on economic growth

	(1)	(2)	(3)
	GDP growth	per capita GDP growth	per worker GDP growth
γ_{it}	1.748***	1.547***	1.284***
	(0.361)	(0.347)	(0.269)
γ_{it}^2	-5.599***	-5.035***	-3.773***
	(1.189)	(1.200)	(0.798)
y_{it-1}	-0.082***	-0.085***	-0.071***
	(0.015)	(0.015)	(0.007)
Country fixed effects	Yes	Yes	Yes
Period fixed effects	Yes	Yes	Yes
Observations	189	189	189
R-squared	0.706	0.687	0.717

Note: *** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$. Cluster-robust standard errors are in parentheses. The dependent variable in column (1) is the average annual growth rate of real GDP. The dependent variable in column (2) is the average annual growth rate of real GDP per capita. The dependent variable in column (3) is the average annual growth rate of real GDP per worker.

In Table 1, we do not control for other explanatory variables. To mitigate omitted variable bias, we introduce a vector of control variables $\varkappa_{it}$ to our regression model as follows:

$$g_{it} = \vartheta_1 \gamma_{it} + \vartheta_2 \gamma_{it}^2 + \vartheta_3 y_{it-1} + \Gamma \varkappa_{it} + \varphi_i + \varrho_t + \epsilon_{it}.$$

Specifically, we control for the log value of population size, the log value of capital stock, and the degree of trade openness (measured by the average ratio of export plus import to GDP).[10] Population size captures the scale effect, whereas the capital stock captures the effect of capital accumulation. Trade openness captures the effect of international trade. Table 2 reports the regression results. As before, the dependent variables in the three columns are the average annual growth rates of real GDP, real GDP per capita, and real GDP per worker, respectively. In all columns, the coefficients on productive government spending are significantly positive, whereas the coefficients on the quadratic term are significantly negative. Therefore, the hump-shaped relationship between productive government spending and economic growth remains robust to controlling for additional explanatory variables $\varkappa_{it}$.

Table 2: Effects of productive government spending on economic growth (with controls)

	GDP growth (1)	per capita GDP growth (2)	per worker GDP growth (3)
γ_{it}	1.435***	1.263***	1.159***
	(0.483)	(0.456)	(0.303)
γ_{it}^2	-4.761***	-4.293***	-3.464***
	(1.564)	(1.549)	(0.945)
y_{it-1}	-0.106***	-0.113***	-0.090***
	(0.016)	(0.016)	(0.015)
Control variables	Yes	Yes	Yes
Country fixed effects	Yes	Yes	Yes
Period fixed effects	Yes	Yes	Yes
Observations	189	189	189
R-squared	0.754	0.744	0.746

Note: *** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$. Cluster-robust standard errors are in parentheses. The dependent variable in column (1) is the average annual growth rate of real GDP. The dependent variable in column (2) is the average annual growth rate of real GDP per capita. The dependent variable in column (3) is the average annual growth rate of real GDP per worker. The additional control variables are the log value of population size, the log value of capital stock, and the degree of trade openness.

To test whether the effect of productive government spending on the growth rate vanishes as the economy approaches the balanced growth path, we divide the full sample into two subsamples using the top 25th percentile of log GDP per capita as the income threshold, thereby defining countries that have reached the top quartile of income as economies on the balanced growth path. Table B2 in Appendix B presents the baseline subsample regressions, which show that for countries that haven't reached the balanced growth path, the coefficients on productive government spending are significantly positive, while the coefficients

[10] We consider the average value within each period.

on the quadratic term are significantly negative, indicating a humps-shaped relationship on the transition path. In contrast, for the high-income group, the estimated coefficients are smaller in magnitude and statistically insignificant, consistent with a zero effect of productive government spending on the balanced growth path. These results remain robust when we incorporate additional control variables in our regressions, as shown in Table B3.

3 A Schumpeterian model with productive government spending

We introduce government spending modeled as in Barro (1990) in the Schumpeterian growth model with endogenous takeoff developed in Peretto (2015). In the benchmark case, we consider labor income taxation as it is distortionary under elastic labor supply. The model features both entry of new products (horizontal innovation) and improvement of the quality of existing products (vertical innovation). We characterize the entire transition path, from pre-industrial stagnation to endogenous innovation-driven growth. The model also has the property of robust endogenous growth first discussed in Peretto (2018) because the elasticity of output with respect to the quality of products in the aggregate production function does not need to be one to generate endogenous growth.

3.1 Household

There is a representative household with $L_t = L_0 e^{\lambda t}$ identical members (population), where $L_0 = 1$ and the parameter $\lambda \in (0, \infty)$ is the growth rate of the mass household members. The household has lifetime utility function

$$U = \int_0^\infty e^{-(\rho-\lambda)t} \left[\ln c_t + \eta \ln (1 - l_t)\right] dt, \tag{1}$$

where the parameter $\rho > \lambda$ is the subjective discount rate. The variable c_t denotes consumption per capita of a final good (our numeraire good). Accordingly, aggregate consumption is $C_t \equiv c_t L_t$. Finally, the variable $l_t \in (0, 1]$ is the fraction of time that each member of the household allocates to work and the parameter $\eta \geq 0$ determines the importance of leisure relative to consumption.

The household maximizes utility subject to the asset-accumulation equation

$$\dot{a}_t = (r_t - \lambda)a_t + (1 - \tau_t)w_t l_t - c_t, \tag{2}$$

where a_t is the real value of assets held by each member of the household and the aggregate value of assets is $A_t \equiv a_t L_t$. r_t is the real interest rate and w_t is the real wage rate. The government levies a tax rate τ_t on wage income. Dynamic optimization yields the consumption growth rate

$$\frac{\dot{c}_t}{c_t} = r_t - \rho \tag{3}$$

and the labor supply of each individual

$$l_t = 1 - \frac{\eta c_t}{(1 - \tau_t)w_t}. \tag{4}$$

3.2 Final good

The final good is produced by competitive firms with the technology

$$Y_t = \int_0^{N_t} X_t^\theta(i) \left[Z_t^\alpha(i) \left(\frac{G_t}{L_t} \right)^\kappa \frac{L_{y,t}}{N_t^{1-\sigma}} \right]^{1-\theta} di, \tag{5}$$

where $\{\theta, \sigma\} \in (0, 1)$, $\alpha > 0$ and $\kappa \geq 0$. We interpret this Cobb-Douglas production structure as featuring constant returns to scale (CRS) with respect to two rival inputs: intermediate goods and augmented labor. It then follows directly that $\theta \in (0, 1)$. For the purposes of this study, the core property of this structure is that it features two augmentation terms: the quality $Z_t(i)$ of intermediate goods and the provision of government services G_t. Moreover, it features two forms of congestion due to the rival nature of the physical inputs intermediate goods and labor $L_{y,t}$.

Specifically, $X_t(i)$ is the quantity of non-durable intermediate good $i \in [0, N_t]$, where N_t is the variety of intermediate goods available at time t. We allow for partial congestion of labor across intermediate goods (one intermediate good cannot be used by all the workers) and partial congestion of intermediate goods across units of labor (one worker cannot use all of the intermediate goods) and measure this effect with the parameter $\sigma \in (0, 1)$, which removes the scale effect from the model.

Next, the contribution of intermediate good i to the productivity of labor, the model's first augmentation term, depends on its own quality $Z_t(i)$ with parameter $\alpha > 0$. This representation defines quality as the contribution of an intermediate good to increasing the flow of labor services obtained by the good's user (the final producer) from each unit of labor supplied by the household. The second augmentation term is similar: it is the contribution of government services G_t to increasing the flow of labor services obtained by the final producer from each unit of labor supplied by the household. We capture the magnitude of this channel with the parameter $\kappa \geq 0$. We also make public services subject to congestion on a per capita basis as in, among many others, Peretto (2007b).[11]

This production structure delivers endogenous growth even with $\kappa = 0$, a restriction that reduces the model to a variant of the one in Peretto (2015), further elaborated in Peretto (2018), which does not consider productive government spending. In this study, we introduce G with $\kappa > 0$ to investigate the interaction between productive government spending and endogenous innovation-driven growth. Crucially, we do not impose a priori restrictions designed to produce steady-state constant endogenous growth on the parameters $\{\sigma, \alpha, \kappa\}$, but we will derive such restrictions later on from the equilibrium of the model under the criterion that the model delivers a transition from initial stagnation to the takeoff and then to the steady state with endogenous growth driven by the accumulation of knowledge by firms. The analysis will reveal that the model requires an upper bound on σ, reflecting market share dilution in equilibrium, and allows for $\alpha + \kappa$ being less than, equal to or greater than one because our model generates endogenous growth regardless of whether the

[11] We could specify this form of congestion as partial by raising L_t to some additional parameter ranging from zero to one. The results would not change in any interesting way. Note also that our Cobb-Douglas structure nests congestion of public services across intermediate good in the parameter σ already characterized.

aggregate production function in equilibrium is concave, linear or convex in the average level of quality Z_t, which is the model's key accumulated factor.

To close this subsection, we now characterize the behavior of the representative final producer. Profit maximization yields the conditional demand for labor,

$$L_{y,t} = (1-\theta)\frac{Y_t}{w_t}, \tag{6}$$

and the conditional demand for each intermediate good,

$$X_t(i) = \left[\frac{\theta}{P_t(i)}\right]^{1/(1-\theta)} Z_t^{\alpha}(i)\left(\frac{G_t}{L_t}\right)^{\kappa}\frac{L_{y,t}}{N_t^{1-\sigma}}, \tag{7}$$

where $P_t(i)$ is the price of good i. Accordingly, the final producer pays $(1-\theta)Y_t = w_t L_{y,t}$ for labor and $\theta Y_t = \int_0^{N_t} P_t(i) X_t(i)\,di$ for intermediate goods.

3.3 Intermediate goods and in-house R&D

A monopolistic firm produces differentiated intermediate good i with a linear technology that uses $X_t(i)$ units of final good to produce $X_t(i)$ units of intermediate good i at quality $Z_t(i)$. This implies that the marginal cost of production is one. The firm also pays $\phi Z_t^{\alpha}(i) Z_t^{1-\alpha}$ units of final good as a fixed operating cost, where $Z_t \equiv \int_0^{N_t} Z_t(j)\,dj/N_t$ is the average quality of all intermediate goods.[12] Finally, to improve the quality of its product, the monopolistic firm devotes $I_t(i)$ units of final good to in-house R&D with the innovation technology

$$\dot{Z}_t(i) = I_t(i). \tag{8}$$

With this structure, the monopolist's before-R&D profit flow is

$$\Pi_t(i) = [P_t(i) - 1]\,X_t(i) - \phi Z_t^{\alpha}(i) Z_t^{1-\alpha} \tag{9}$$

and the value of the monopolistic firm is

$$V_t(i) = \int_t^{\infty} \exp\left(-\int_t^{s} r_u du\right)[\Pi_s(i) - I_s(i)]\,ds. \tag{10}$$

The monopolistic firm maximizes (10) subject to (7) and (8).

It is important to note here that this firm-level problem is well defined if and only if it features concavity with respect to the firm-specific state variable $Z_t(i)$. This is the case when $0 < \alpha < 1$. This is the first restriction implied by the model's structure for our core parameters. It is worth stressing that it is a restriction that has nothing to do with the ability of the model to generate endogenous growth but it stems from the model's deeper micro structure, specifically, the requirement that the investment problem of the typical intermediate firm be well-defined.

[12] Our results are robust to a more general specification for the fixed operating cost: $\phi Z_t^{\chi}(i) Z_t^{1-\chi}$, where $\chi \in (0,1)$. Given that it is a fixed operating cost, it is independent of the quantity $X_t(i)$ of production.

Dynamic optimization of the monopolistic firm yields the unconstrained profit-maximizing markup ratio $1/\theta$ (see Appendix A). However, we follow Chu *et al.* (2020) to allow for diffusion of knowledge from monopolistic firms to competitive fringe firms, which can produce $X_t(i)$ with the same quality $Z_t(i)$ but at the higher marginal cost $\mu > 1$. To price these fringe firms out of the market, the monopolistic firm sets

$$P_t(i) = \min\{\mu, 1/\theta\} = \mu, \tag{11}$$

where we assume $\mu < 1/\theta$. The firm's optimization problem also yields the rate of return to in-house R&D

$$r_t^q(i) = \alpha \left\{ (\mu - 1) \frac{X_t(i)}{Z_t(i)} - \phi \left[\frac{Z_t}{Z_t(i)} \right]^{1-\alpha} \right\},$$

which is increasing in quality-adjusted firm size $X_t(i)/Z_t(i)$.

3.4 Entrants

A new firm pays βX_t units of final good (where $\beta > 0$ is an entry-cost parameter) to develop a new differentiated good with average quality Z_t and start serving the market.[13] Once in the market, the new firm behaves like the incumbent firm characterized above. Therefore, at any point in time the value of all firms — incumbents and entrants — is governed by the asset-pricing equation

$$r_t = \frac{\Pi_t - I_t}{V_t} + \frac{\dot{V}_t}{V_t}. \tag{12}$$

When entry is positive, the free-entry condition

$$V_t = \beta X_t \tag{13}$$

holds. Substituting (7), (8), (9), (11) and (13) into (12) yields the return to entry as

$$r_t^e = \frac{1}{\beta} \left(\mu - 1 - \frac{\phi + z_t}{X_t/Z_t} \right) + \frac{\dot{X}_t}{X_t},$$

where $z_t \equiv \dot{Z}_t/Z_t$ is the growth rate of average quality. The return to entry r_t^e is also increasing in quality-adjusted firm size X_t/Z_t.

3.5 Government

The government balances its fiscal budget at each point in time and finances its spending with the revenues from a flat rate tax on labor income. Therefore, the government's budget constraint is

$$G_t = \tau_t w_t L_{y,t}. \tag{14}$$

[13]This characterization of entry preserves the symmetry of the intermediate goods market equilibrium at all times. Generalizing the entry cost βX_t to make it dependent on quality Z_t would complicate the dynamics of the model.

Following Barro (1990), we focus on spending as the key policy variable and thus assume that the government sets

$$G_t = \gamma Y_t, \tag{15}$$

where $\gamma \in (0, 1-\theta)$ is the fiscal policy instrument. Consequently, in our analysis we take γ as exogenous and τ_t as the endogenous tax rate that balances the fiscal budget (14).

3.6 Equilibrium

The equilibrium of this economy is a time path of allocations $\{A_t, C_t, l_t, Y_t, X_t, I_t, G_t\}$, prices $\{r_t, w_t, P_t(i), V_t\}$ and labor income tax rate τ_t such that:

- the household chooses consumption c_t and labor supply l_t to maximize utility taking $\{r_t, w_t, \tau_t\}$ as given;
- competitive firms produce Y_t to maximize profits taking $\{w_t, P_t(i)\}$ as given;
- monopolistic intermediate-good firms choose $\{P_t(i), I_t\}$ to maximize V_t taking r_t as given;
- entrants make entry decisions taking the maximized value V_t as given;
- the aggregate value of monopolistic firms equals the household's wealth, $a_t L_t = N_t V_t$;
- the government balances the fiscal budget, $G_t = \tau_t w_t L_{y,t}$;
- the labor market clears, $L_{y,t} = l_t L_t$;
- the market for the final good clears, $Y_t = C_t + G_t + N_t (X_t + \phi Z_t + I_t) + \dot{N}_t \beta X_t$.

3.7 Aggregation

Under the conditions discussed in Peretto (2015), the equilibrium of this model is symmetric: intermediate firms charge the same price, produce the same quantity and grow at the same rate. In such equilibrium, (7), (11) and (15) yield the reduced-form aggregate production function

$$Y_t = \left[\left(\frac{\theta}{\mu} \right)^{\theta/(1-\theta)} \gamma^{\kappa} l_t N_t^{\sigma} Z_t^{\alpha} \right]^{1/(1-\kappa)} L_t, \tag{16}$$

where the elasticity of output with respect to product variety, N_t, is $\sigma/(1-\kappa)$ and the elasticity of output with respect to (average) product quality, Z_t, is $\alpha/(1-\kappa)$. For this to be a sensible representation of production, we must impose $\kappa < 1$ so that these two elasticities are finite and positive.

As stated in the introduction, our model generates endogenous growth even when $\kappa = 0$, in which case the two elasticities are $\{\sigma, \alpha\} < 1$. An important property of this class of models is that product variety expansion is not an engine of endogenous growth because of the fixed operating cost borne by firms. Therefore, whether the model produces endogenous growth or not depends only on the elasticity α. Given $\alpha < 1$, the conventional wisdom in the literature says that this model cannot generate endogenous growth. Peretto (2018) has challenged such conventional wisdom in a simpler variant of this model. In this study, we generalize the challenge by nesting the basic model in a structure that (a) allows for the additional labor augmentation channel via government spending in the spirit of Barro (1990) and (b) removes the knowledge spillover term that early versions of the theory posited.[14]

For future use, we close this discussion with the derivation the growth rate of final output per capita, $y_t = Y_t/L_t$. According to the production function (16), the growth rate is

$$g_t \equiv \frac{\dot{y}_t}{y_t} = \frac{1}{1-\kappa}\left(\sigma n_t + \alpha z_t + \frac{\dot{l}_t}{l_t}\right) \tag{17}$$

and consists of three components: the variety growth rate $n_t \equiv \dot{N}_t/N_t$; the quality growth rate z_t; and the growth rate of individual labor supply $\dot{l}_t/l_t$.

4 Productive government spending and takeoff

In this section, we first solve for the entire path of the economy from stagnation to steady-state growth. The economy experiences four stages of economic growth governed by the evolution of firm size. It begins in a pre-industrial era in which the growth rate of final output per capita is zero. It then enters the industrial era, which consists of two phases. In the first phase, the entry of new firms that bring to market new products drives the growth rate of output per capita. In the second phase, the improvement of the quality of existing products by existing firms adds its contribution to economic growth and produces an acceleration of the growth rate.[15] The economy finally converges to a balanced growth path that features constant growth in output per capita fueled by both vertical and horizontal innovation.

Next, we show that productive government spending shapes this process of phase transitions and convergence, by determining the timing of the first phase transition (i.e., the endogenous takeoff of the economy) and the timing of the second phase transition (i.e., the activation of vertical innovation), which further accelerates economic growth. Importantly, we find that due to the model's scale invariance, productive government spending does not affect the steady-state growth rate.

[14] Specifically, those models specify the augmentation term for quality as $Z_t^{\alpha}(i)\, Z_t^{1-\alpha}$, which in symmetric equilibrium becomes Z_t. This specification accepts the conventional wisdom that endogenous growth requires production Y_t to be linear in Z_t. Here, we reject such conventional wisdom and remove the second term from this expression, obtaining nevertheless endogenous growth even under $\kappa = 0$.

[15] We consider the realistic case in which the activation of variety innovation happens before the activation of quality innovation. See Peretto (2015) for a comprehensive discussion of this property of the baseline growth model.

4.1 Dynamics

Our characterization of government spending in (15) and the reduced-form production function (16) yield that per capita public spending is

$$\frac{G_t}{L_t} = \left[\left(\frac{\theta}{\mu} \right)^{\theta/(1-\theta)} \gamma l_t N_t^{\sigma} Z_t^{\alpha} \right]^{1/(1-\kappa)} . \tag{18}$$

Using (7), (11) and (18), we express quality-adjusted firm size as

$$\frac{X_t(i)}{Z_t(i)} = \frac{X_t}{Z_t} = \left[\left(\frac{\theta}{\mu} \right)^{\frac{1}{1-\theta}-\kappa} \gamma^{\kappa} l_t \right]^{\frac{1}{1-\kappa}} \frac{Z_t^{\frac{\alpha}{1-\kappa}-1} L_t}{N_t^{1-\frac{\sigma}{1-\kappa}}}.$$

We then define the following composite variable:

$$x_t \equiv \frac{1}{(\gamma^{\kappa} l_t)^{\frac{1}{1-\kappa}}} \frac{X_t}{Z_t} = \left(\frac{\theta}{\mu} \right)^{\frac{1-\kappa(1-\theta)}{(1-\kappa)(1-\theta)}} \frac{Z_t^{\frac{\alpha}{1-\kappa}-1} L_t}{N_t^{1-\frac{\sigma}{1-\kappa}}}, \tag{19}$$

which is the state-variable component of quality-adjusted firm size $X_t/Z_t = (\gamma^{\kappa} l_t)^{\frac{1}{1-\kappa}} x_t$. This state variable x_t compresses the three state variables L_t (population), Z_t (average quality) and N_t (mass of products/firms) into the ratio $Z_t^{\alpha/(1-\kappa)-1} L_t / N_t^{1-\sigma/(1-\kappa)}$ and, therefore, makes the analysis of the model's dynamics simple. Moreover, this expression shows that for the model to exhibit the sensible property that in equilibrium firm size is decreasing in the mass of firms, we must assume $\sigma/(1-\kappa) < 1$. This is the second restrictions on the parameters that we impose not to obtain endogenous growth but to ensure that the model's microstructure produces realistic properties in equilibrium. Our model generates a stationary long-run level of employment per firm L_t/N_t in the special case $\kappa = 1 - \alpha$ and $\sigma = 0$. Importantly, it generates a stationary quality-adjusted firm size $X_t/Z_t = (\gamma^{\kappa} l_t)^{\frac{1}{1-\kappa}} x_t$ even without these parameter restrictions.

To see in more detail the previous point, we use equation (19) to write the rate of return to quality improvement as

$$r_t^q = \alpha \left[(\mu - 1)(\gamma^{\kappa} l_t)^{\frac{1}{1-\kappa}} x_t - \phi \right] \tag{20}$$

and the rate of return to entry as

$$r_t^e = \frac{1}{\beta} \left[\mu - 1 - \frac{\phi + z_t}{(\gamma^{\kappa} l_t)^{\frac{1}{1-\kappa}} x_t} \right] + \frac{\dot{x}_t}{x_t} + z_t + \frac{1}{1-\kappa} \frac{\dot{l}_t}{l_t}. \tag{21}$$

Both rates of return are increasing in quality-adjusted firm size $(\gamma^{\kappa} l_t)^{1/(1-\kappa)} x_t$ and are thus decreasing in the mass of firms for $\sigma/(1-\kappa) < 1$. This property captures the main force driving this class of models: as the mass of firms rises, each firm captures a smaller share of the market and experiences falling profitability and thereby a weaker incentive to innovate.

The government budget constraint, $G_t = \tau_t w_t L_{y,t}$, yields the labor income tax rate

$$\tau_t = \tau = \frac{\gamma}{1-\theta}, \tag{22}$$

which is increasing in the government spending ratio γ. The combination of labor supply (4) and labor demand (6) yields the equilibrium fraction of time allocated to work

$$l_t = \left(1 + \frac{\eta}{1-\theta-\gamma}\frac{c_t}{y_t}\right)^{-1}. \tag{23}$$

This equation says that given the consumption-output ratio c_t/y_t, the fraction of time allocated to work l_t is decreasing in the government spending ratio γ via the higher tax rate τ.

Finally, equation (19) yields the equilibrium law of motion of the state variable x_t,

$$\frac{\dot{x}_t}{x_t} = \lambda + \left(\frac{\alpha}{1-\kappa} - 1\right) z_t - \left(1 - \frac{\sigma}{1-\kappa}\right) n_t. \tag{24}$$

In this expression, the entry rate n_t and the quality growth rate z_t are either zero or increasing functions of quality-adjusted firm size $(\gamma^\kappa l_t)^{1/(1-\kappa)} x_t$ (as we show below). If these two functions have the required properties, the composite variable x_t converges to its constant steady-state value. Thus, the core of the analysis in this section is the characterization of these two functions as equilibrium objects.

4.2 The pre-industrial era

We follow the configuration of the pre-industrial intermediate goods sector in Chu *et al.* (2022). In the pre-industrial era, initial demand for each intermediate good is insufficient for a would-be monopolist operating the increasing-returns technology characterized in Section 3 to earn positive profit (see Appendix A for details). As a result, competitive firms produce the existing N_0 intermediate goods. They make zero profit at the limit price $P_t(i) = \mu$ and consequently have zero stock-market value.[16] Anticipating such zero value, entrepreneurs do not pay the sunk entry cost, which implies that there is no variety innovation (no entry of products). Therefore, all technologies in the pre-industrial era exhibit constant returns to scale, and x_t grows solely due to exogenous population growth (i.e., $\dot{x}_t/x_t = \lambda$). In this pre-industrial era, the initial mass of intermediate goods N_0 is exogenous and predetermined, whereas the market structure in each product line (i.e., the number of firms and the size of each firm) is indeterminate.

The demand for intermediate goods eventually becomes sufficiently high for a would-be monopolist operating the increasing-returns technology to earn positive profit. However, although the increasing returns technology becomes viable, agents do not deploy it yet because this technology requires the sunk entry cost. In other words, only innovation allows a new firm to monopolize an existing market. Therefore, the pre-industrial era ends only when the present value of monopolistic firms becomes sufficiently high for the free-entry condition (13) to hold.

As a result of the pre-industrial market structure outlined above, in the pre-industrial era the household's financial wealth is zero and the household's consumption is $c_t = (1-\tau_t)w_t l_t =$

[16] Extending the baseline model to allow for positive monopolistic profits in the pre-industrial era complicates the pre-industrial dynamics but does not change the main results of the paper. Derivations are available upon request.

$(1-\theta-\gamma)y_t$, which yields

$$\frac{c}{y} = \left(\frac{c}{y}\right)_0 \equiv 1-\theta-\gamma. \tag{25}$$

Substituting this result into (23) yields

$$l = l_0 \equiv \frac{1}{1+\eta}. \tag{26}$$

This says that the equilibrium fraction of time allocated to work in the pre-industrial era is stationary and independent of the government spending ratio γ. The associated growth rate of output per capita is

$$g_t = \frac{1}{1-\kappa}\left(\sigma n_t + \alpha z_t + \frac{\dot{l}_t}{l_t}\right) = 0 \tag{27}$$

because $n_t = z_t = \dot{l}_t/l_t = 0$.

4.3 The industrial era: phase 1

Horizontal innovation (but not yet vertical innovation) starts when firm size grows sufficiently large. This event marks the beginning of the industrial era. In the first phase of this era, we have a positive variety growth rate $n_t > 0$ and a zero quality growth rate $z_t = 0$. To see this, note first that when the free-entry condition holds, the consumption-output ratio c_t/y_t and the fraction of time allocated to work l_t jump to the following steady-state values (derived in Appendix A):

$$\left(\frac{c}{y}\right)^* = 1-\theta-\gamma+\frac{\beta\theta}{\mu}(\rho-\lambda); \tag{28}$$

$$l^* = \frac{1}{1+\eta\left[1+\frac{\beta\theta(\rho-\lambda)}{\mu(1-\theta-\gamma)}\right]}. \tag{29}$$

The second equation shows that the equilibrium fraction of time allocated to work in the industrial era is decreasing in the government spending ratio γ.

In the first phase of the industrial era, the growth rate of output per capita is $g_t = \sigma n_t/(1-\kappa)$ because $z_t = 0$. Using the fact that in equilibrium $r_t^e = r_t = \rho + g_t = \rho + \sigma n_t/(1-\kappa)$, we derive the growth rate of product variety n_t as

$$n_t = \frac{1}{\beta}\left[\mu - 1 - \frac{\phi}{(\gamma^\kappa l^*)^{\frac{1}{1-\kappa}} x_t}\right] + \lambda - \rho, \tag{30}$$

which is increasing in firm size $(\gamma^\kappa l^*)^{1/(1-\kappa)} x_t$ and is positive if and only if

$$x_t > \frac{\phi}{(\gamma^\kappa l^*)^{\frac{1}{1-\kappa}}\left[\mu - 1 - \beta(\rho-\lambda)\right]} \equiv x_N, \tag{31}$$

where x_N is decreasing in $\gamma^\kappa l^*$. Given this activation threshold, we use the fact that in the pre-industrial era $\dot{x}_t/x_t = \lambda$ to compute the time of the industrial takeoff,

$$T_N = \frac{1}{\lambda}\log\left(\frac{x_N}{x_0}\right), \tag{32}$$

where x_0 is the initial value of the composite state variable x_t. This result says that given $x_t = x_0$ at time $t = 0$, it takes T_N years for the economy to reach the threshold x_N and thus experience the industrial takeoff.

With these expressions in hand, we can investigate the effects of government spending. If labor supply is perfectly inelastic (i.e., $\eta = 0$), the term $\gamma^\kappa l^*$ is monotonically increasing in the government spending ratio γ because $l^* = 1$. In this case, raising productive government spending leads to an earlier takeoff by decreasing x_N and a higher growth rate by increasing n_t. If labor supply is elastic (i.e., $\eta > 0$), the term $\gamma^\kappa l^*$ becomes a hump-shaped function of the government spending ratio γ because l^* is decreasing in γ. In this case, raising productive government spending γ has a U-shaped effect on the date of the industrial takeoff, T_N, and a hump-shaped effect on the transitional growth rate g_t.

To fully characterize this phase, we note that equations (24) and (30) yield that the dynamics of the economy are governed by the linear differential equation

$$\dot{x}_t = \frac{\left(1 - \frac{\sigma}{1-\kappa}\right)\frac{\phi}{\beta}}{(\gamma^\kappa l^*)^{\frac{1}{1-\kappa}}} - \left[\left(1 - \frac{\sigma}{1-\kappa}\right)\left(\frac{\mu - 1}{\beta} - \rho\right) - \frac{\sigma\lambda}{1-\kappa}\right] x_t, \tag{33}$$

where we argued above that $\sigma/\left(1-\kappa\right) < 1$.

4.4 The industrial era: phase 2

When firm size is sufficiently large, horizontal and vertical innovation occur simultaneously. This is the second phase of the industrial era. Given active horizontal innovation, the consumption-output ratio and the fraction of time allocated to work l_t remain at the steady-state values (28)-(29). Therefore, we can use the relation $r_t^q = r_t = \rho + g_t$ to write the growth rate g_t as

$$g_t = \alpha\left[(\mu - 1)(\gamma^\kappa l^*)^{\frac{1}{1-\kappa}} x_t - \phi\right] - \rho, \tag{34}$$

which is linearly increasing in firm size $(\gamma^\kappa l^*)^{1/(1-\kappa)} x_t$. Therefore, raising productive government spending γ has a hump-shaped (a positive) effect on the transitional growth rate g_t if labor supply is elastic (perfectly inelastic).

Next, we use the fact that $r_t^e = r_t = \rho + g_t = \rho + \sigma n_t/(1-\kappa) + \alpha z_t/(1-\kappa)$ to write the entry process driving the dynamics of x_t as

$$n_t = \frac{1}{\beta}\left[\mu - 1 - \frac{\phi + z_t}{(\gamma^\kappa l^*)^{\frac{1}{1-\kappa}} x_t}\right] + \lambda - \rho. \tag{35}$$

Manipulating (34), (35) and $g_t = \sigma n_t/(1-\kappa) + \alpha z_t/(1-\kappa)$ yields the growth rate of quality z_t as a function of the state variable x_t, namely, $z_t = z\left(x_t\right)$, where

$$z\left(x_t\right) \equiv \frac{\left[(\mu - 1)(\gamma^\kappa l^*)^{\frac{1}{1-\kappa}} x_t - \phi\right]\left[\alpha(1-\kappa) - \frac{\sigma}{\beta(\gamma^\kappa l^*)^{\frac{1}{1-\kappa}} x_t}\right] - \rho(1-\kappa) + \sigma\left(\rho - \lambda\right)}{\alpha - \frac{\sigma}{\beta(\gamma^\kappa l^*)^{1/(1-\kappa)} x_t}}. \tag{36}$$

This expression says that quality growth is positive if and only if $x_t > x_Z$, where

$$x_Z \equiv \arg\underset{x}{\text{solve}}\left\{\frac{(\mu - 1)(\gamma^\kappa l^*)^{\frac{1}{1-\kappa}} x - \phi}{\rho(1-\kappa) - \sigma\left(\rho - \lambda\right)}\left[\alpha(1-\kappa) - \frac{\sigma}{\beta(\gamma^\kappa l^*)^{\frac{1}{1-\kappa}} x}\right] = 1\right\} \tag{37}$$

and, as argued earlier, we work with a configuration of parameters that yields $x_Z > x_N$. Substituting (36) in (35) and rearranging terms, we write $n_t = n(x_t)$, where

$$n(x_t) \equiv \frac{[\kappa(\mu-1)-\beta(\rho-\lambda)](\gamma^{\kappa}l^{*})^{\frac{1}{1-\kappa}}x_t + (1-\kappa)\rho/\alpha - \kappa\phi}{\beta(\gamma^{\kappa}l^{*})^{\frac{1}{1-\kappa}}x_t - \sigma/\alpha}, \tag{38}$$

which expresses the rate of entry as a function of the state variable x_t. Finally, using (36) and (38), we write the equilibrium law of motion

$$\frac{\dot{x}_t}{x_t} = \lambda + \left(\frac{\alpha}{1-\kappa}-1\right)z(x_t) - \left(1-\frac{\sigma}{1-\kappa}\right)n(x_t). \tag{39}$$

This equation is non-linear but relatively straightforward to study.

Summing up the results of the analysis of each phase, we have reduced a seemingly complex model to a representation of the equilibrium dynamics that consists of a piece-wise differential equation in the composite state variable x_t. The equation has the properties that the first two pieces in the pre-industrial era and the first phase of the industrial era are linear while the last piece in the second phase of the industrial era is non-linear but not particularly challenging. Armed with this representation, we next discuss the conditions under which the model converges to a steady state with endogenous growth and how the process depends on government spending.

4.5 Convergence to the balanced growth path

We showed above that the state variable x_t grows exponentially in the pre-industrial era due to the exogenous growth of the population. Therefore, the economy experiences the takeoff in finite time as long as the threshold x_N is finite. This property gives us the third restriction that we impose to characterize the global dynamics of the model, namely, $\mu - 1 > \beta(\rho - \lambda)$. This restriction simply says that the fundamentals are such that the gross profit rate earned by monopolistic firms can cover the flow cost due to the decision to undertake entry. Specifically, the right hand side of the inequality is the initial sunk cost of entry, β, multiplied by the interest rate that the firm must pay at each point in time to finance that initial expenditure (think of the entrant firm taking a loan to finance β or, equivalently, issuing equity that must promise the market rate of return).

In phase 1 of the industrial era, the economy obeys the linear differential equation (33), which says that the economy crosses the threshold x_Z in finite time if (33) has the property

$$\dot{x}_t(x_Z) = \frac{\left(1-\frac{\sigma}{1-\kappa}\right)\frac{\phi}{\beta}}{(\gamma^{\kappa}l^{*})^{\frac{1}{1-\kappa}}} - \left[\left(1-\frac{\sigma}{1-\kappa}\right)\left(\frac{\mu-1}{\beta}-\rho\right) - \frac{\sigma\lambda}{1-\kappa}\right]x_Z > 0.$$

This is simply a restriction on the parameters that ensures that the process of entry does not saturate the market so much that incumbent firms cannot cross the threshold of profitability that activates in-house quality innovation. It is the analog of the condition discussed above that guarantees that the first phase transition occurs. In other words, this condition guarantees that the second phase transition occurs. On reflection, we can express these conditions in more compact terms. We rewrite the inequality above as

$$\rho\beta + \left[\frac{\left(1 - \frac{\sigma}{1-\kappa}\right)\frac{\phi}{\beta}}{(\gamma^{\kappa} l^{*})^{\frac{1}{1-\kappa}} x_Z} + \frac{\sigma\lambda}{1-\kappa}\right] \frac{\beta}{1 - \frac{\sigma}{1-\kappa}} > \mu - 1.$$

Combining this inequality with the inequality derived for the first phase transition, we obtain

$$\rho\beta + \left[\frac{\left(1 - \frac{\sigma}{1-\kappa}\right)\frac{\phi}{\beta}}{(\gamma^{\kappa} l^{*})^{\frac{1}{1-\kappa}} x_Z} + \frac{\sigma\lambda}{1-\kappa}\right] \frac{\beta}{1 - \frac{\sigma}{1-\kappa}} > \mu - 1 > \beta(\rho - \lambda) \tag{40}$$

as the *sufficient* condition for the dynamics of the model to yield the full sequence: pre-industrial era → industrial era phase 1 → industrial era phase 2.

Perhaps intuitively, the conditions just discussed are subsumed in the conditions that we obtain by looking directly at the model's global dynamics, paying special attention to what happens in phase 2 of the industrial era. Figure 1 plots the phase diagrams for the three cases ($\alpha + \kappa = 1$, $\alpha + \kappa > 1$ and $\alpha + \kappa < 1$) and shows that in each case x_t can converge to the steady-state value x^* that features both quality improvement ($z^* > 0$) and variety expansion ($n^* > 0$) in the long run. Note that under the sufficient condition (40) in all three panels the piece of the differential equation in the interval $[x_N, x_Z]$ is above the horizontal axis.[17] The conditions for the model to produce endogenous growth conventionally defined, therefore, are the conditions under which in each case the steady state x^* exists and is the global attractor of the economy's equilibrium dynamics. We now characterize each case.

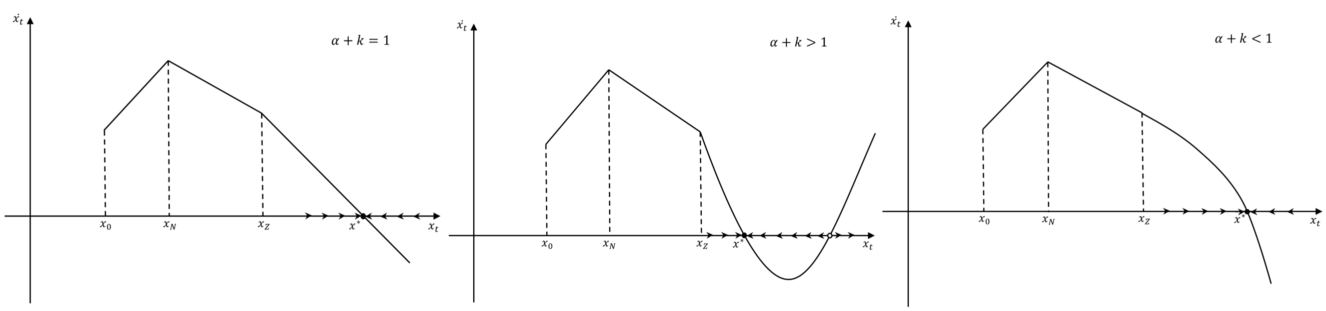

Figure 1a: $\alpha + \kappa = 1$ Figure 1b: $\alpha + \kappa > 1$ Figure 1c: $\alpha + \kappa < 1$

Case 1. Consider $\alpha + \kappa = 1$. In Appendix A, we show that the steady-state value of the state variable x_t is

$$x^* = \frac{1}{(\gamma^{\kappa} l^{*})^{\frac{1}{1-\kappa}}} \frac{(\kappa\phi - \rho)\left(1 - \frac{\sigma}{1-\kappa}\right) - \frac{\sigma\lambda}{\alpha}}{[\kappa(\mu - 1) - \beta(\rho - \lambda)]\left(1 - \frac{\sigma}{1-\kappa}\right) - \beta\lambda}. \tag{41}$$

The associated growth rate of income per capita is

$$g^* = \alpha\left[(\mu - 1)\frac{(\kappa\phi - \rho)\left(1 - \frac{\sigma}{1-\kappa}\right) - \frac{\sigma\lambda}{\alpha}}{[\kappa(\mu - 1) - \beta(\rho - \lambda)]\left(1 - \frac{\sigma}{1-\kappa}\right) - \beta\lambda} - \phi\right] - \rho, \tag{42}$$

[17] In an online appendix (see Appendix C), we show the phase diagrams for other possibilities.

which is fueled by the variety growth rate $n^* = (1-\kappa)\lambda/(1-\kappa-\sigma) > 0$ and the quality growth rate $z^* = g^* - (\sigma/\alpha)n^*$. Note that the steady-state firm size, $(\gamma^\kappa l^*)^{1/(1-\kappa)}x^*$, the steady-state growth rate, g^*, and the steady-state rate of quality innovation, z^*, are all independent of the government spending ratio γ due to the model's scale invariance.

The conditions for endogenous growth are (i) that the values x^*, g^* and z^* exist and are positive and (ii) that the model's dynamics allow the state variable x_t to converge to the steady state x^*. Inspecting the phase diagrams shows that under condition (40) if the steady state x^* exists, then it is locally stable and, therefore, the global attractor of the full dynamical system. We provide a more formal characterization of this property in the proof of Proposition 1 below. The condition for $x^* > 0$ is

$$\left(1 - \frac{\sigma}{1-\kappa}\right) > \max\left\{\frac{\sigma\lambda}{\alpha\left(\kappa\phi - \rho\right)}, \frac{\beta\lambda}{\kappa(\mu-1) - \beta(\rho-\lambda)}\right\}, \tag{43}$$

which says that both the numerator and the denominator of (41) are positive. The conditions for $g^* > 0$ and $z^* > 0$ add, respectively, the inequalities:

$$\alpha\left[(\mu-1)\frac{\left(\kappa\phi-\rho\right)\left(1-\frac{\sigma}{1-\kappa}\right) - \frac{\sigma\lambda}{\alpha}}{\left[\kappa(\mu-1) - \beta(\rho-\lambda)\right]\left(1-\frac{\sigma}{1-\kappa}\right) - \beta\lambda} - \phi\right] > \rho;$$

$$\alpha\left[(\mu-1)\frac{\left(\kappa\phi-\rho\right)\left(1-\frac{\sigma}{1-\kappa}\right) - \frac{\sigma\lambda}{\alpha}}{\left[\kappa(\mu-1) - \beta(\rho-\lambda)\right]\left(1-\frac{\sigma}{1-\kappa}\right) - \beta\lambda} - \phi\right] > \rho + \frac{\sigma\lambda}{1-\kappa-\sigma}. \tag{44}$$

Note that the latter implies the former, which we can thus ignore. Summarizing, the conditions that ensure that the economy converges to x^* under the model's equilibrium dynamics, where x^* exhibits endogenous growth, are (40), (43) and (44). These are *inequality* restrictions. Nowhere the model requires an equality restriction representing a knife-edge condition on the parameters.

Case 2 and Case 3. To characterize the next two cases, we derive in Appendix A the steady-state firm size and the steady-state growth rate given, respectively, by:

$$(\gamma^\kappa l^*)^{1/(1-\kappa)}x^* = \frac{a_2 - \sqrt{a_2^2 - 4a_1a_3}}{2a_1}; \tag{45}$$

$$g^* = \alpha\left[(\mu-1)\frac{a_2 - \sqrt{a_2^2 - 4a_1a_3}}{2a_1} - \phi\right] - \rho. \tag{46}$$

To work with compact notation, we defined the coefficients:

$$a_1 \equiv \alpha\beta(\mu-1)(1-\kappa)(\alpha+\kappa-1);$$

$$\begin{aligned} a_2 \equiv\ & (\alpha+\kappa-1)\{[\mu-1-\beta(\rho-\lambda)]\sigma + \beta(1-\kappa)(\rho+\phi\alpha)\} \\ & + \alpha(1-\kappa-\sigma)[(\mu-1)\kappa - \beta(\rho-\lambda)] - \alpha\beta\lambda(1-\kappa); \end{aligned}$$

$$a_3 \equiv (\alpha+\kappa-1)\phi\sigma + (1-\kappa-\sigma)[\alpha\phi\kappa - (1-\kappa)\rho] - \sigma\lambda(1-\kappa).$$

Note that once again both the steady-state firm size, $(\gamma^{\kappa} l^*)^{1/(1-\kappa)} x^*$, and the steady-state growth rate, g^*, are independent of the government spending ratio γ due to the model's scale invariance. We now examine each case using these expressions. Before doing so, we note the new property that arises in these cases, namely, the equation

$$\lambda + \left(\frac{\alpha}{1-\kappa} - 1\right) z^* = \left(1 - \frac{\sigma}{1-\kappa}\right) n^*,$$

which says that there exists a relation between variety growth and quality growth dictated by the sign of the coefficient $\alpha + \kappa - 1$. Recall that $1 > \sigma/(1-\kappa)$. This new property yields that since $g^* = (\sigma n^* + \alpha z^*)/(1-\kappa)$, the breakdown of economic growth in its two components — variety and quality growth — is:

$$z^* = \frac{(1-\kappa) g^* - \frac{\sigma\lambda(1-\kappa)}{1-\kappa-\sigma}}{\frac{\sigma(\alpha+\kappa-1)}{1-\kappa-\sigma} + \alpha};$$

$$n^* = \frac{\lambda(1-\kappa) + (\alpha+\kappa-1) z^*}{1-\kappa-\sigma}.$$

The new result here is that for $\alpha + \kappa \neq 1$ the rate of entry is not pinned down by the rate of growth of the population but is jointly determined with the rate of quality growth. In Appendix A, we provide a more detailed discussion of the two cases: $\alpha + \kappa > 1$ and $\alpha + \kappa < 1$.

4.6 Summary of results

We can summarize our result on the main global dynamics in Proposition 1.

Proposition 1 *Assume that $x_0 < x_N < x_Z$. Then, the economy begins in the pre-industrial era with no innovation of any kind. It then experiences an industrial takeoff and enters the first phase of the industrial era where horizontal innovation alone fuels industrial growth. After that, the economy enters the second phase of the industrial era with both vertical and horizontal innovation and converges to the balanced growth path.*

Proof. See Appendix A. ■

In Proposition 2, we summarize the effects of productive government spending, which depend on whether labor supply is elastic or not. If labor supply is perfectly inelastic, then the labor income tax has no distortionary effect. In this case, raising productive government spending causes an earlier industrial takeoff and increases the transitional growth rate by enlarging firm size in the short run; however, it has no effect on long-run economic growth due to the absence of the scale effect. If labor supply is elastic, then the labor income tax has a distortionary effect on employment. In this case, raising productive government spending has a U-shaped effect on the timing of industrial takeoff and an inverted-U effect on the transitional growth rate.

Proposition 2 *The effects of productive government spending are as follows. If labor supply is perfectly inelastic (i.e., $\eta = 0$), then raising productive government spending γ leads to an earlier industrial takeoff and a higher transitional growth rate g_t during the industrial era. If labor supply is elastic (i.e., $\eta > 0$), then there exists a threshold value $\tilde{\gamma} \in (0, 1-\theta)$ such that raising productive government spending γ leads to an earlier (a delayed) industrial takeoff when $\gamma < \tilde{\gamma}$ ($\gamma > \tilde{\gamma}$). During the industrial era, i.e., $x_t \in (x_N, x^*)$, an increase in γ increases (decreases) the transitional growth rate g_t when $\gamma < \tilde{\gamma}$ ($\gamma > \tilde{\gamma}$). In the long run, the government spending share γ does not affect the steady-state growth rate (regardless of whether labor supply is elastic or perfectly inelastic).*

Proof. See Appendix A. ■

It is useful to note that the threshold level $\tilde{\gamma}$ of government spending that gives rise to the earliest takeoff and the highest transition growth rate is the same. Once again, due to the endogenous market structure removing the scale effect, any changes in government spending have no effect on long-run economic growth.

4.7 Quantitative illustration

In this section, we calibrate the model to the US as a representative advanced economy, due to its data availability, and perform a quantitative analysis to gauge the magnitudes of the effects that the model produces. The model features the following 11 parameters: $\{\kappa, \lambda, \alpha, \mu, \gamma, \theta, \rho, \beta, \phi, \eta, \sigma\}$. In the reduced-form production function (5), the exponent on productive government spending G_t is κ; therefore, κ is the key parameter that determines the strength of the effects of productive government spending on the timing of industrial takeoff and the transitional growth rate. In light of its importance, we consider a wide range of values for $\kappa \in [0.10, 0.55]$.[18] Then, we set $\lambda = 1.6\%$, equal to the average growth rate of employment in 1978-2019 from the Business Dynamics Statistics (BDS). We follow Iacopetta and Peretto (2021) to set the elasticity of profit with respect to own knowledge α to 0.333. We set markup ratio μ to 1.3, which is within the range of aggregate markup ratios estimated in De Loecker *et al.* (2020). To capture productive government spending γ, we consider the sum of government expenditures on education, health, defence and economic affairs as a share of GDP in the US, which is about 0.2 on average in recent decades. The labor income share $1-\theta$ is around 60%, so that $\theta = 0.4$. We set the discount rate ρ to a conventional value of 0.03. Then, we calibrate the remaining parameters $\{\beta, \phi, \eta, \sigma\}$ by matching the following moments. The parameters $\{\beta, \phi\}$ mainly target a long-run growth rate of output per capita of 2% and R&D as a share of output of 2.7%. The relative importance of leisure η matches labor supply as a share of labor endowment of 0.3. The social return to variety σ matches a net firm entry rate of 1% (also from the BDS). We summarize the calibrated parameter values in Table 3, which shows that we have $\alpha + \kappa < 1$ for our entire range of values for κ.

[18] As κ rises above 0.566, σ would become negative, which is empirically implausible because a negative σ implies that variety innovation contributes negatively to economic growth.

Table 3. Calibrated parameter values

κ	λ	α	μ	γ	θ	ρ	β	ϕ	η	σ	$\tilde{\gamma}$
0.100	0.016	0.333	1.300	0.200	0.400	0.030	5.837	0.116	2.195	0.933	0.359
0.250	0.016	0.333	1.300	0.200	0.400	0.030	5.837	0.116	2.195	0.633	0.435
0.400	0.016	0.333	1.300	0.200	0.400	0.030	5.837	0.116	2.195	0.333	0.466
0.550	0.016	0.333	1.300	0.200	0.400	0.030	5.837	0.116	2.195	0.033	0.485

Table 3 also computes the value of productive government spending share $\tilde{\gamma}$ that maximizes $\gamma^{\kappa} l^*$ under different values of κ. It is useful to note from (31), (30) and (34) that a larger $\gamma^{\kappa} l^*$ implies a lower industrial-takeoff threshold x_N and a higher transitional growth rate g_t in the two phases of the industrial era. For the value of κ, we choose 0.1, which is a conservative value within the range of empirical estimates reported in Ramey (2021). In this case, the growth-maximizing productive government spending share is $\tilde{\gamma} = 0.359$. Furthermore, under this set of parameter values, the growth rate of average employment L_t/N_t is $\lambda - n^* = 0.6\%$, which is in line with the annual growth rate of average employment per firm in the BDS of 0.62% from 1978 to 2019. Furthermore, given that in-house R&D share of output $N_t I_t / Y_t$ is a constant in the steady-state, the model-produced growth rate of firms' R&D spending is $g_I^* \equiv \dot{I}_t / I_t = g^* + \lambda - n^* = 2.6\%$, which is roughly in line with the empirical estimate of 2.9%.[19] Figure 2 simulates the time path of the equilibrium growth rate g_t under the growth-maximizing productive government spending share $\tilde{\gamma} = 0.359$ and also two counterfactual growth paths with $\gamma \in \{0.010, 0.500\}$. Given the number of years on the horizontal axis, Figure 2 shows that when the productive government spending share increases from the historical value of 0.010 in the early 19th century[20] to the growth-maximizing value of 0.359, the industrial takeoff of the economy occurs earlier by 23 years. However, further increasing the productive government spending share to 0.500 then delays the takeoff by 4 years. We also compute how the balanced-growth level of output would change when the government raises its spending share γ from its historical value of 0.01 to the growth-maximizing value $\tilde{\gamma}$ of 0.359 for the benchmark case of $\kappa = 0.1$. This exercise shows that although the growth rate would converge to the same steady-state value of 2%, the balanced-growth level of output would increase by 38.6%.

[19] From OECD statistics, the average annual growth rate of nominal R&D expenditures is 6.1% in the US, which translates to a real growth rate of 3.9% and a real growth rate of R&D per firm of 2.9% (recall that the net entry rate of firms is 1%).

[20] Data source: IMF Government Finance Statistics and IMF Public Finances in Modern History Database. As in Section 2, we follow Devarajan *et al.* (1996) to define productive government spending as the sum of government expenditures on education, health, defence, and economic affairs. The IMF Government Finance Statistics provides data on these four items from 1970 to 2021, and their average share of US government expenditures is 0.55. The IMF Public Finances in Modern History Database provides data on total government expenditures as a share of US GDP from 1800 to 2022. We multiply this data by 0.55 to obtain an estimate for productive government spending share of US GDP in 1800, by assuming that the expenditure share of the above four items within the government budget remains the same as modern data.

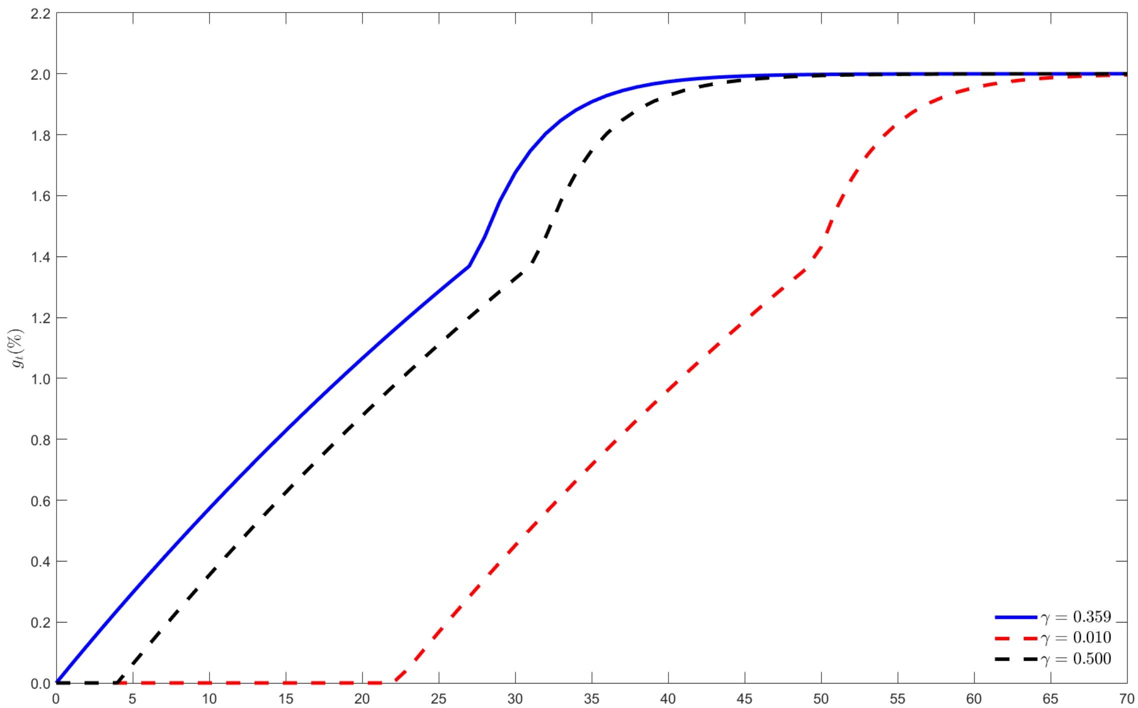

Figure 2: Deviating from the growth-maximizing value $\tilde{\gamma}$.

5 Other taxes

Given the importance of other tax instruments at different points in time, we consider two extensions of the model with different tax instruments in this section. Section 5.1 considers a consumption tax. Section 5.2 considers a corporate income tax.

5.1 Consumption tax

This section replaces the labor income tax with a consumption tax and explores the effects of productive government spending on the dynamics of the economy from pre-industrial stagnation to innovation-driven growth. To conserve space, we do not repeat all the derivations but focus on the equations that are different from the baseline model.

The household's asset-accumulation equation becomes

$$\dot{a}_t = (r_t - \lambda)a_t + w_t l_t - (1 + \tau_{c,t})c_t,$$

where $\tau_{c,t}$ is the consumption tax rate. Individual labor supply becomes

$$l_t = 1 - \eta(1 + \tau_{c,t})\frac{c_t}{w_t}.$$

As for the fiscal budget constraint, it becomes

$$G_t = \tau_{c,t} C_t.$$

The rest of the model remains the same as before. Following the same derivations in the baseline model, we derive equilibrium individual labor supply

$$l_t = \left[1 + \frac{\eta(1+\tau_{c,t})}{1-\theta}\frac{c_t}{y_t}\right]^{-1}.$$

In the pre-industrial era, the consumption-output ratio jumps to

$$\frac{c}{y} = \left(\frac{c}{y}\right)_0 \equiv \frac{1-\theta}{1+\tau_c},$$

where the consumption tax rate is stationary and given by

$$\tau_c = \frac{\gamma}{1-\theta-\gamma},$$

which is increasing in the government spending ratio γ. Therefore, equilibrium individual labor supply in the pre-industrial era jumps to

$$l = l_0 \equiv \frac{1}{1+\eta}.$$

In the industrial era, the consumption-output ratio jumps to

$$\frac{c}{y} = \left(\frac{c}{y}\right)^* \equiv \frac{1}{1+\tau_c^*}\left[1-\theta+\frac{\beta\theta}{\mu}(\rho-\lambda)\right],$$

where the consumption tax rate is stationary and given by

$$\tau_c^* = \frac{\gamma}{1-\theta-\gamma+\beta\theta(\rho-\lambda)/\mu},$$

which is increasing in the government spending ratio γ. Therefore, the equilibrium level of labor in the industrial era jumps to

$$l = l^* \equiv \frac{1}{1+\eta\left[1+\frac{\beta\theta(\rho-\lambda)}{\mu(1-\theta)}\right]},$$

which is independent of the government spending ratio γ because $(1+\tau_c)c/y$ is independent of τ_c. Therefore, raising productive government spending γ continues to have a positive effect on firm size $(\gamma^\kappa l^*)^{1/(1-\kappa)} x_t$ via the term γ^κ, but the negative effect of γ via equilibrium labor l^* in the baseline model disappears because consumption tax does not affect employment l^*.

The dynamics of the economy are captured by (30) to (46) and (A9) to (A11) in Appendix A as in the baseline model. We can now summarize the global dynamics under the consumption tax as follows.

Proposition 3 *When government spending is financed by a consumption tax $\tau_{c,t}$, the effects of productive government spending γ are as follows. In the pre-industrial era, i.e., $x_t \in (x_0, x_N)$, an increase in γ leads to an earlier industrial takeoff. In the industrial era, i.e., $x_t \in (x_N, x^*)$, an increase in γ raises the transitional growth rate g_t but does not affect the steady-state growth rate g^*.*

Proof. From (31), one can show that a larger γ reduces x_N. From (30) and (34), one can show that a larger γ increases g_t in both the first phase and second phase of the industrial era, i.e., $x_t \in (x_N, x^*)$. Then, (46) shows that g^* is independent of γ. ■

5.2 Corporate income tax

This section replaces the consumption tax with a corporate income tax. For simplicity and tractability, we consider the special case with $\kappa = 1 - \alpha$. Before the industrial takeoff, firms make zero corporate income, i.e., $\Pi_t = 0$. Therefore, we keep the labor income tax as in the baseline model to ensure that government spending G_t is positive and financed by labor income tax before the industrial takeoff. Again, we do not repeat all the derivations but focus on the equations that are different from the baseline economy.

Given the corporate income tax, the value of monopolistic firm i becomes

$$V_t(i) = \int_t^{\infty} \exp\left(-\int_t^s r_u du\right)(1 - \tau_{\Pi,s})\left[\Pi_s(i) - I_s(i)\right] ds.$$

The government levies the corporate income tax $\tau_{\Pi,t}$ on the firm's cash flow net of R&D expenditure, which implies that R&D is fully expensible. Then, the asset-pricing equation becomes

$$r_t = \frac{(1 - \tau_{\Pi,t})(\Pi_t - I_t)}{V_t} + \frac{\dot{V}_t}{V_t}.$$

The fiscal budget constraint becomes

$$G_t = \tau w_t L_{y,t} + \tau_{\Pi,t} N_t(\Pi_t - I_t).$$

In the pre-industrial era, it must be the case that $G_t = \tau w_t L_{y,t}$ because $\tau_{\Pi,t} N_t(\Pi_t - I_t) = 0$. In the industrial era, changes in government spending are financed by corporate income tax, and the endogenous corporate income tax is given by

$$\tau_{\Pi,t} = \frac{\gamma - \tau(1 - \theta)}{N_t(\Pi_t - I_t)/Y_t},$$

where we take the government spending ratio γ and the labor-income tax rate τ as exogenous. The rate of return to entry becomes

$$r_t^e = \frac{1 - \tau_{\Pi,t}}{\beta}\left[\mu - 1 - \frac{\phi + z_t}{(\gamma^{1-\alpha} l^*)^{1/\alpha} x_t}\right] + \frac{\dot{x}_t}{x_t} + z_t + \frac{1}{\alpha}\frac{\dot{l}_t}{l_t}.$$

In both the pre-industrial era and the industrial era, the equilibrium individual labor supply is

$$l_t = \left[1 + \frac{\eta}{(1-\theta)(1-\tau)}\frac{c_t}{y_t}\right]^{-1}.$$

In the pre-industrial era, the consumption-output ratio is

$$\frac{c}{y} = \left(\frac{c}{y}\right)_0 \equiv (1-\tau)(1-\theta),$$

and the equilibrium individual labor supply is

$$l = l_0 \equiv \frac{1}{1+\eta}.$$

When the economy enters the industrial era, both the consumption-output ratio c_t/y_t and the equilibrium level of labor l_t jump to their steady-state values:

$$\frac{c}{y} = \left(\frac{c}{y}\right)^* \equiv (1-\theta)(1-\tau) + \frac{\beta\theta}{\mu}(\rho-\lambda),$$

$$l = l^* \equiv \frac{1}{1+\eta\left[1+\frac{\beta\theta(\rho-\lambda)}{\mu(1-\theta)(1-\tau)}\right]},$$

which are independent of $\tau_{\Pi,t}$. In the industrial era, the endogenous corporate income tax can be expressed as (see the derivation in Appendix A)

$$\tau_{\Pi,t} = \frac{\mu[\gamma-(1-\theta)\tau]}{\beta\theta(n_t+\rho-\lambda)+\mu[\gamma-(1-\theta)\tau]},$$

which is decreasing in n_t and increasing in the government spending ratio γ for a given n_t.

In the first phase of the industrial era, the growth rate of output is $g_t = \sigma n_t/\alpha$ and the variety growth rate n_t can be expressed as

$$n_t = \frac{1}{\beta}\left\{\mu - 1 - \frac{\phi}{(\gamma^{1-\alpha}l^*)^{1/\alpha}x_t} - \frac{\mu}{\theta}[\gamma-(1-\theta)\tau]\right\} + \lambda - \rho,$$

which is increasing in firm size $(\gamma^{1-\alpha}l^*)^{1/\alpha}x_t$ as before and is positive if only if

$$x_t > \frac{\phi}{(\gamma^{1-\alpha}l^*)^{1/\alpha}\{\mu-1-\beta(\rho-\lambda)-[\gamma-(1-\theta)\tau]\mu/\theta\}} \equiv x_N.$$

In Appendix A, we show that if the labor income tax rate τ is sufficiently small, then the industrial threshold x_N is a U-shaped function in productive government spending γ, which lowers the industrial threshold x_N (when γ is small) by raising firm size via $\gamma^{1-\alpha}$ but increases the industrial threshold x_N (when γ is large) by discouraging firm entry via corporate income tax $\tau_{\Pi,t}$. Therefore, a small increase in productive government spending γ can reduce the industrial threshold x_N and trigger an immediate industrialization of the economy when x_N falls below x_0.

In the second phase of the industrial era, the growth rate of output becomes $g_t = \sigma n_t/\alpha + z_t$ and converges to the steady state. On the balanced growth path, the steady-state variety growth rate is given by $n^* = \alpha\lambda/(\alpha-\sigma)$, and the steady-state corporate income tax rate is given by

$$\tau_\Pi^* = \frac{\mu[\gamma-(1-\theta)\tau]}{\beta\theta[\rho+\sigma\lambda/(\alpha-\sigma)]+\mu[\gamma-(1-\theta)\tau]},$$

which is increasing in the government spending ratio γ. Finally, the steady-state per capita output growth rate is given by

$$\begin{aligned} g^* &= \frac{\alpha[\beta\phi-(1-\tau_\Pi^*)(\mu-1)]\,[\rho+\sigma\lambda/(\alpha-\sigma)]}{(1-\tau_\Pi^*)(1-\alpha)(\mu-1)-\beta\,[\rho+\sigma\lambda/(\alpha-\sigma)]}-\rho \\ &= \frac{\alpha\{[\beta\phi-(\mu-1)][\rho+\sigma\lambda/(\alpha-\sigma)]+[\gamma-(1-\theta)\tau]\mu\phi/\theta\}}{(1-\alpha)(\mu-1)-\beta[\rho+\sigma\lambda/(\alpha-\sigma)]-[\gamma-(1-\theta)\tau]\mu/\theta}-\rho, \end{aligned}$$

which is increasing in τ_Π^* and, hence, increasing in γ. Intuitively, corporate income tax raises steady-state growth by discouraging firm entry and enlarging firm size in the long run.

We can now summarize the global dynamics under the corporate-income tax in Proposition 4, which focuses on industrial takeoff and long-run growth for simplicity.

Proposition 4 *When government spending is financed by a corporate income tax $\tau_{\Pi,t}$, the effects of productive government spending γ are as follows. In the pre-industrial era, an increase in γ leads to an earlier (a delayed) industrial takeoff if γ is below (above) a threshold. In the industrial era, an increase in γ raises the steady-state growth rate g^*.*

Proof. See Appendix A. ■

6 Public capital

We now modify the baseline model to allow for public capital. Specifically, we make G_t in the production function (5) a stock variable instead of a flow variable. We find again that raising the government spending ratio hastens the industrial takeoff with inelastic labor supply and has an ambiguous effect on the timing of the industrial takeoff with elastic labor supply, while it has no effect on the steady-state growth rate regardless of whether labor supply is elastic or perfectly inelastic. These qualitative results are the same as in the baseline model. However, the inclusion of public capital changes fundamentally the transitional dynamics of the model and produces new insights.

6.1 Equilibrium

Public capital accumulates according to the equation

$$\dot{G}_t = I_{G,t} - \delta_G G_t,$$

where $I_{G,t}$ is public investment financed through taxes and δ_G is the depreciation rate of public capital. The characterization of the final-good sector and the intermediate-good sector are the same as in the baseline model. The government balances its budget at each point in time and finances its spending with the revenue from a labor income tax τ_t. Therefore, the government's budget constraint is $I_{G,t} = \tau_t w_t L_{y,t}$. Accordingly, we have

$$\dot{G}_t = I_{G,t} - \delta_G G_t = \tau_t w_t L_{y,t} - \delta_G G_t = \gamma Y_t - \delta_G G_t, \tag{47}$$

where the policy parameter $\gamma \in (0, 1)$ is now the public capital investment ratio.

The reduced-form aggregate production function is

$$Y_t = \left(\frac{\theta}{\mu}\right)^{\theta/(1-\theta)} \left(\frac{G_t}{L_t}\right)^{\kappa} N_t^{\sigma} Z_t^{\alpha} l_t L_t, \tag{48}$$

where $\{\sigma, \kappa\} \in (0, 1)$. In this specification of the model, it is still the case that the consumption/output ratio, c_t/y_t, jumps to its steady state, given by (25) in the pre-industrial era and (28) in the industrial era. Accordingly, the employment ratio, l_t, jumps to its steady state, given by the value l_0 in equation (26) in the pre-industrial era and by the value l^* in equation (29) in the industrial era. Therefore, log-differentiating (48) with respect to time we obtain the growth rate of output per capita

$$g_t \equiv \frac{\dot{y}_t}{y_t} = \sigma n_t + \alpha z_t + \kappa(g_{G,t} - \lambda), \tag{49}$$

where $g_{G,t} \equiv \dot{G}_t/G_t$ is the growth rate of public capital. Given our characterization of the government's policy, we have the relation

$$g_{G,t} = \gamma \frac{Y_t}{G_t} - \delta_G = \gamma \left(\frac{\theta}{\mu}\right)^{\theta/(1-\theta)} \left(\frac{G_t}{L_t}\right)^{\kappa-1} N_t^{\sigma} Z_t^{\alpha} l_t - \delta_G, \tag{50}$$

in which, differently from the baseline model, the ratio Y_t/G_t is endogenous. In this extension, this ratio is the average product of public capital. To guarantee that the growth rate of public capital is positive at the beginning of time, $g_{G,0} > 0$, we assume that the initial stock of public capital is sufficiently low, $G_0 < (\gamma A_0/\delta_G)^{1/(1-\kappa)}$, where we define $A_0 \equiv (\theta/\mu)^{\theta/(1-\theta)} N_0^{\sigma} Z_0^{\alpha} l_0$.

We now turn to the dynamics of the economy. The composite state variable driving quality-adjusted firm size is

$$x_t \equiv \frac{1}{l_t}\frac{X_t}{Z_t} = \left(\frac{\theta}{\mu}\right)^{\frac{1}{1-\theta}} \frac{\left(\frac{G_t}{L_t}\right)^{\kappa} Z_t^{\alpha-1} L_t}{N_t^{1-\sigma}}. \tag{51}$$

The rates of return to quality innovation and entry are:

$$r_t^q = \alpha\left[(\mu - 1)x_t l_t - \phi\right]; \tag{52}$$

$$r_t^e = \frac{1}{\beta}\left(\mu - 1 - \frac{\phi + z_t}{x_t l_t}\right) + \frac{\dot{x}_t}{x_t} + z_t. \tag{53}$$

In the pre-industrial era, firm size is not sufficiently large there is no innovation of any kind. Therefore, the growth rate of output per capita is $g_t = \kappa(g_{G,t} - \lambda)$. Note what this expression says: in the pre-industrial era the engine of output per capita growth is public capital accumulation. This property expands vastly the paper's perspective on public spending.

6.2 Pre-industrial-era dynamics

We now work out the details of the mechanism. We denote public capital per capita as $k_t \equiv G_t/L_t$ and write the differential equation governing its dynamics in the pre-industrial era as

$$\dot{k}_t = \gamma A_0 k_t^{\kappa} - (\delta_G + \lambda) k_t.$$

This is a Bernoulli differential equation that can be transformed into a linear differential equation with the change of variable $b_t = k_t^{1-\kappa}$ to obtain $\dot{b}_t = (1-\kappa)\gamma A_0 - (1-\kappa)(\delta_G + \lambda) b_t$. Therefore, we obtain the solution

$$\frac{G_t}{L_t} = k_t = \left[\frac{\gamma A_0}{\delta_G + \lambda} \left(1 - e^{-(1-\kappa)(\delta_G+\lambda)t}\right) + G_0^{1-\kappa} e^{-(1-\kappa)(\delta_G+\lambda)t} \right]^{\frac{1}{1-\kappa}}.$$

This equation shows that an increase in γ shifts up the entire time path of public capital per capita. Furthermore, in the pre-industrial era, firm size is

$$x_t l_0 = \left(\frac{\theta}{\mu}\right)^{\frac{1}{1-\theta}} \left[\frac{\gamma A_0}{\delta_G + \lambda} - \left(\frac{\gamma A_0}{\delta_G + \lambda} - G_0^{1-\kappa} \right) e^{-(1-\kappa)(\delta_G+\lambda)t} \right]^{\frac{\kappa}{1-\kappa}} \frac{Z_0^{\alpha-1}}{N_0^{1-\sigma}} l_0 L_0 e^{\lambda t},$$

which shows that an increase in γ shifts up the entire time path of firm size.

When firm size grows sufficiently large, horizontal innovation begins and the economy enters the first stage of the industrial era. The growth rate of product variety is

$$n_t = \frac{1}{\beta}\left(\mu - 1 - \frac{\phi}{x_t l^*}\right) + \lambda - \rho, \tag{54}$$

which is positive if and only if

$$x_t > x_N \equiv \frac{\phi}{l^* \left[\mu - 1 - \beta(\rho - \lambda)\right]}. \tag{55}$$

Given the closed-form solution for the dynamics of public capital per capita, this threshold yields

$$T_N \equiv \arg \underset{t}{\text{solve}} \left\{ \frac{x_N}{x_0} = \left[\frac{\gamma A_0}{(\delta_G + \lambda) G_0^{1-\kappa}} \left[1 - e^{-(1-\kappa)(\delta_G+\lambda)t}\right] + e^{-(1-\kappa)(\delta_G+\lambda)t} \right]^{\frac{\kappa}{1-\kappa}} e^{\lambda t} \right\}.$$

This expression generalizes the notion of industrial takeoff time that we developed in the baseline model of flow public spending.

To appreciate better the novel insight developed by this result, consider the case of zero population growth ($\lambda = 0$). In the baseline model of flow public spending, this would cause permanent stagnation since equation (32) says that $T_N \to \infty$. Here, instead, the expression for the industrial takeoff time simplifies to

$$T_N = \frac{1}{(1-\kappa)\delta_G} \ln \left(\frac{\frac{\gamma A_0}{\delta_G} - G_0^{1-\kappa}}{\frac{\gamma A_0}{\delta_G} - G_0^{1-\kappa} \left(\frac{x_N}{x_0}\right)^{\frac{1-\kappa}{\kappa}}} \right),$$

this value is finite if

$$\frac{\gamma A_0}{\delta_G} > G_0^{1-\kappa} \left(\frac{x_N}{x_0} \right)^{\frac{1-\kappa}{\kappa}}.$$

If this inequality does not hold, there is permanent stagnation because $T_N \to \infty$. This special case offers a useful benchmark for understanding the insight that this section's extended model generates.

First, the accumulation of public capital is the sole endogenous mechanism triggering the Industrial Revolution (IR). Once the IR occurs, the accumulation of public capital interacts with technological innovation because the latter generates growth in public revenue. In this sense, the extended model provides one of the most sophisticated statement of the idea that public capital accumulation is a catalyst for industrialization. Second, an increase in the public spending ratio hastens industrialization with non-distortionary taxation because T_N is decreasing in γ, while it has an ambiguous effect on the timing of the IR because with distortionary taxation x_N is increasing in γ. Third, if the condition for permanent stagnation holds, output per capita converges to a constant level. Thus, while the public capital component of the model functions as a traditional capital-deepening mechanism, which by itself cannot sustain long-run growth, it nevertheless is a catalyst for the activation of the engine of long-run growth.

6.3 Industrial-era dynamics

In the first stage of the industrial era, the growth rate of output per capita is

$$g_t = \sigma n_t + \kappa(g_{G,t} - \lambda), \tag{56}$$

where the growth rate of product variety is given by equation (54). When firm size grows sufficiently large, the economy enters the second stage of the industrial era in which horizontal innovation and vertical innovation occur simultaneously. The growth rate of output per capita is

$$g_t = \alpha \left[(\mu - 1) x_t l^* - \phi \right] - \rho. \tag{57}$$

The growth rates of product variety and product quality are:

$$n_t \equiv n(x_t, g_{G,t}) = \frac{\rho + \kappa(g_{G,t} - \lambda) - \alpha\beta(\rho - \lambda)x_t l^*}{\alpha\beta x_t l^* - \sigma}; \tag{58}$$

$$z_t \equiv z(x_t, g_{G,t}) = \frac{\left[(\mu - 1)x_t l^* - \phi \right] \left(\alpha - \frac{\sigma}{\beta x_t l^*} \right) - \rho + \sigma\left(\rho - \lambda\right) - \kappa(g_{G,t} - \lambda)}{\alpha - \frac{\sigma}{\beta x_t l^*}}. \tag{59}$$

Quality growth is positive if and only if

$$g_{G,t} < \frac{1}{\kappa} \left\{ \left[(\mu - 1)x_t l^* - \phi \right] \left(\alpha - \frac{\sigma}{\beta x_t l^*} \right) - \rho + \sigma\left(\rho - \lambda\right) \right\} + \lambda.$$

This inequality describes a boundary in $(x_t, g_{G,t})$ space.

The equilibrium laws of motion of x_t and $g_{G,t}$ in the first stage are:

$$\dot{x}_t = \frac{(1-\sigma)\,\phi}{\beta l^*} - \left\{ \frac{1-\sigma}{\beta} \left[\mu - 1 - \beta(\rho - \lambda)\right] - \lambda - \kappa(g_{G,t} - \lambda) \right\} x_t; \tag{60}$$

$$\dot{g}_{G,t} = \left[\sigma n_t - (1-\kappa)(g_{G,t} - \lambda)\right](g_{G,t} + \delta_G). \tag{61}$$

The equilibrium laws of motion of $g_{G,t}$ and x_t in the second stage are:

$$\dot{x}_t = \frac{a_1 \left(x_t l^*\right)^2 - a_2(g_{G,t}) x_t l^* + a_3(g_{G,t})}{\alpha\beta l^* - \sigma/x_t}. \tag{62}$$

$$\dot{g}_{G,t} = \left\{\alpha\left[(\mu-1)x_t l^* - \phi\right] - \rho - (g_{G,t} - \lambda)\right\}(g_{G,t} + \delta_G); \tag{63}$$

In the first equation the composite coefficients are:

$$a_1 \equiv \alpha\beta(\mu-1)(\alpha-1);$$

$$a_2(g_{G,t}) \equiv (\alpha-1)[(\mu-1)\sigma + \alpha\beta\phi] - \beta[(\rho-\lambda)(1-\sigma) + (1-\kappa)\lambda] - \kappa\beta g_{G,t};$$

$$a_3(g_{G,t}) \equiv (\alpha-1)\phi\sigma - [(\rho-\lambda)(1-\sigma) + (1-\kappa)\lambda] - \kappa g_{G,t}.$$

This is our two-dimensional dynamical system for the extended model. It produces a steady state that has the same properties as the baseline model.

6.4 Global dynamics

We can sketch the model's global dynamics in the phase diagram in Figure 3 using two pairs of loci. In the first stage, we have:

$$\dot{g}_{G,t} \geq 0 \Longrightarrow g_{G,t} \leq \frac{\sigma\left[(\mu-1)x_t l^* - \phi\right]}{\beta(1-\kappa)x_t l^*} + \frac{\lambda(1-\kappa) - \sigma(\rho-\lambda)}{1-\kappa};$$

$$\dot{x}_t \geq 0 \Longrightarrow g_{G,t} \geq \frac{1-\sigma}{\kappa\beta}\left[\mu - 1 - \beta(\rho-\lambda)\right] - \frac{(1-\kappa)\lambda}{\kappa} - \frac{(1-\sigma)\,\phi}{\kappa\beta x_t l^*}.$$

In the second stage, we have:

$$\dot{g}_{G,t} \geq 0 \Longrightarrow g_{G,t} \leq \alpha\left[(\mu-1)x_t l^* - \phi\right] - \rho + \lambda;$$

$$\dot{x}_t \geq 0 \Longrightarrow g_{G,t} \geq \frac{b_1\left(x_t l^*\right)^2 - b_2 x_t l^* + b_3}{\kappa(1-\beta x_t l^*)}.$$

In the second locus the composite parameters are:

$$b_1 \equiv \alpha\beta(\mu-1)(\alpha-1);$$

$$b_2 \equiv (\alpha-1)[(\mu-1)\sigma + \alpha\beta\phi] - \beta[(\rho-\lambda)(1-\sigma) + (1-\kappa)\lambda];$$

$$b_3 \equiv (\alpha-1)\phi\sigma - [(\rho-\lambda)(1-\sigma) + (1-\kappa)\lambda].$$

The dynamics are a nice generalization of the baseline one-dimensional model.

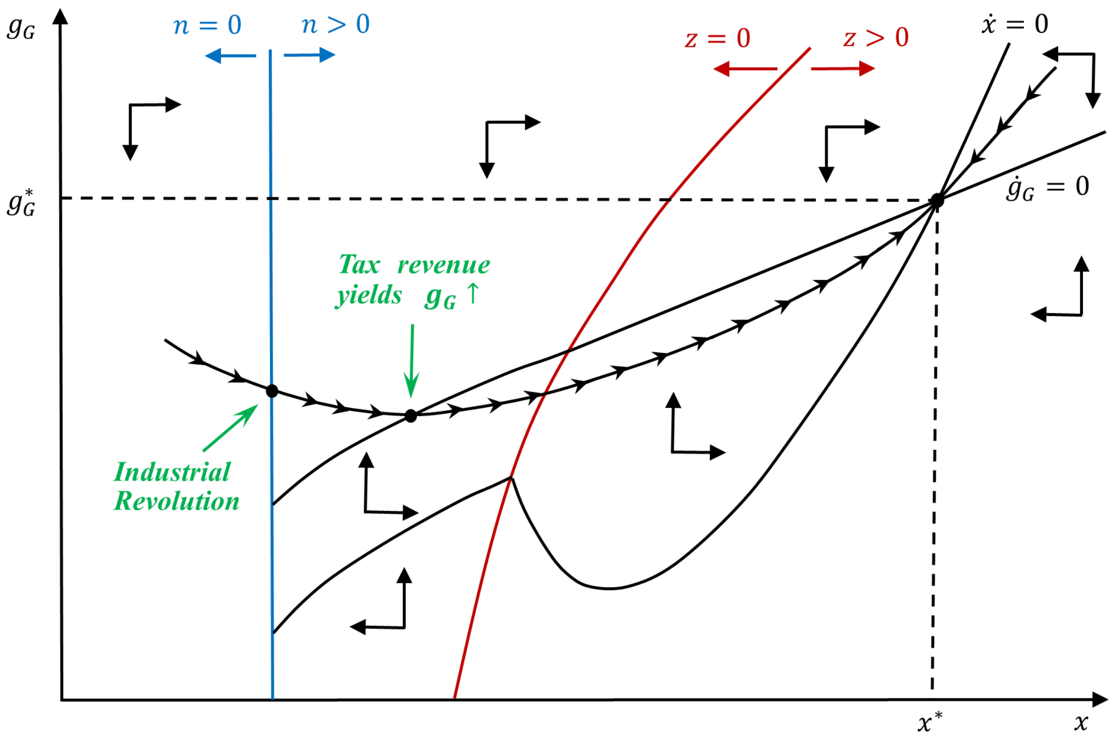

Figure 3: Public capital accumulation, phase diagram.

In steady state the growth rate of public capital, $g_{G,t} = \gamma\,(Y_t/G_t) - \delta_G$, is constant because the average product of public capital, Y_t/G_t, is constant. This yields

$$g_G^* = g_Y^* = \lambda + g^*,$$

where $g^* = \sigma n^* + \alpha z^* + \kappa(g_G^* - \lambda)$. We can simplify this expression for the steady-state growth rate of output to $g^* = (\sigma n^* + \alpha z^*)/(1-\kappa)$. Using the steady-state condition $g_G^* = \gamma\,(Y/G)^* - \delta_G$ and $g_G^* = \lambda + g^*$ yields the steady-state average product of public capital

$$\left(\frac{Y}{G}\right)^* = \frac{\delta_G + \lambda + g^*}{\gamma}.$$

The steady-state growth rate of output per capita is

$$g^* = \alpha\left[(\mu - 1)x^* l^* - \phi\right] - \rho. \tag{64}$$

Using $\dot{x}_t = 0$ and $g_G^* = \lambda + g^*$, we solve for steady-state firm size

$$x^* l^* = \frac{a_2^* - \sqrt{(a_2^*)^2 - 4a_1^* a_3^*}}{2a_1^*}, \tag{65}$$

where:

$$a_1^* = \alpha\beta(\mu - 1)(\alpha - 1);$$

$$a_2^* = (\alpha - 1)[(\mu - 1)\sigma + \alpha\beta\phi] - \beta[\rho - \sigma(\rho - \lambda)] - \kappa\beta g^*;$$

$$a_3^* = (\alpha - 1)\phi\sigma - [\rho - \sigma(\rho - \lambda)] - \kappa g^*.$$

This solution says that firm size is a function of the growth rate via the coefficients a_2^* and a_3^*. Since the growth rate is a function of firm size, (64) and (65) jointly determine the steady-state values of firm size, $x^* l^*$, and the growth rate, g^*. These values are independent of the public spending ratio due to the scale invariance property of the model.

The phase diagram in Figure 3 shows two boundaries. In blue is the IR boundary where product variety growth, n_t, becomes positive. In red is the boundary where quality growth, z_t, becomes positive. The diagram tells the following story. Initially, the accumulation of public capital fuels the growth of the economy. This process is subject to decreasing returns to scale (DRS) to public capital, G_t, and slows down as long as technology remains constant. We mark in green two key events. The IR, where n_t becomes positive, and the minimum of the U-shaped trajectory that the economy follows. That minimum point marks the moment when the ratio $I_{G,t}/G_t = \gamma Y_t/G_t$ stops falling and starts growing due to the fact that technological innovation raises output and thereby tax revenue so fast that it compensates for the falling average product of public capital, Y_t/G_t. The trajectory, then, tells a story of the secular rise of $I_{G,t}$ due to the secular rise of tax revenues, which in turn is due to the acceleration of the rate of technological change.

7 Conclusion

In this study, we developed a Schumpeterian growth model with productive government spending and endogenous takeoff. We took the seminal study by Barro (1990) as our point of departure and focused on the government's provision of productive public services to the private sector. We then took the analysis in several new directions. First, we investigated the role of government spending in a scale-invariant Schumpeterian model of endogenous innovation. Second, we investigated the role of public spending as the catalyst of the takeoff of the economy. Third, we postulated a production structure that violates the condition for endogenous growth stressed in Barro (1990), namely, that the economy's reduced-form production function must be linear in the accumulated factors. Fourth, we extended the model to public capital, uncovering novel properties of the dynamics. Consequently, our results expand significantly our understanding of the conditions under which general equilibrium models produce industrial takeoff and convergence to constant exponential growth in steady state, in particular innovation-driven growth that starts at a specific date, accelerates throughout the secular transition, and in steady state is scale invariant and subject to policy action.

Our main results can be summarized as follows. When public spending is financed with a distortionary labor income tax, there is a value of productive government spending that yields the earliest industrial takeoff and also maximizes the transitional growth rate. This theoretical prediction of a hump-shaped effect of productive government spending on economic growth is consistent with the stylized facts reported in the literature that we revisited here using cross-country panel data. Calibrating the model to the US, we found that raising productive government spending to its growth-maximizing value causes the industrial takeoff to occur earlier by about two decades and the long-run level of output to increase by roughly 40%, confirming the importance of productive public spending for industrialization and innovation. From a policy perspective, our growth-theoretic framework provides support for the importance of the United Nations Sustainable Development Goals, which specifically mention building resilient infrastructure, promoting sustainable industrialization and fostering innovation as necessary ingredients to success. Finally, it is worth noting that we focus on productive government spending in this study, but one can also introduce to the model an extra parameter that represents a wedge between overall government spending and productive government spending to capture a notion of unproductive government spending. We leave this extension to future research.

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Appendix A: Proofs

Dynamic optimization of the monopolistic firm. The current-value Hamiltonian for monopolistic firm i is

$$H_t(i) = \Pi_t(i) - I_t(i) + \zeta_t(i)\dot{Z}_t(i) + \xi_t(i)\left[\mu - P_t(i)\right], \tag{A1}$$

where $\zeta_t(i)$ is the co-state variable on $\dot{Z}_t(i)$ and $\xi_t(i)$ is the multiplier on $P_t(i) \leq \mu$. We substitute (7), (8) and (9) into (A1) and derive

$$\frac{\partial H_t(i)}{\partial P_t(i)} = 0 \Rightarrow \frac{\partial \Pi_t(i)}{\partial P_t(i)} = \xi_t(i), \tag{A2}$$

$$\frac{\partial H_t(i)}{\partial I_t(i)} = 0 \Rightarrow \zeta_t(i) = 1, \tag{A3}$$

$$\begin{aligned} \frac{\partial H_t(i)}{\partial Z_t(i)} &= \alpha\left\{\left[P_t(i) - 1\right]\left[\frac{\theta}{P_t(i)}\right]^{1/(1-\theta)} Z_t^{\alpha-1}(i)\left(\frac{G_t}{L_t}\right)^{\kappa}\frac{L_{y,t}}{N_t^{1-\sigma}} - \phi\left(\frac{Z_t}{Z_t(i)}\right)^{1-\alpha}\right\} \\ &= r_t\zeta_t(i) - \dot{\zeta}_t(i). \end{aligned} \tag{A4}$$

When $P_t(i) < \mu$, $\xi_t(i) = 0$, which implies $\partial \Pi_t(i)/\partial P_t(i) = 0$ such that $P_t(i) = 1/\theta$. When the constraint is binding, i.e., $P_t(i) = \mu$, $\xi_t(i) > 0$. Thus, we have proven (11). The assumption $\mu < 1/\theta$ yields $P_t(i) = \mu$. Using (A3), (19), $P_t(i) = \mu$ and symmetry in (A4) yields (20). ■

Monopolistic profit in the pre-industrial era. In the pre-industrial era, firm size is not large enough for monopolistic firms with increasing-returns technology to earn positive profit, i.e.,

$$(\gamma^{\kappa} l)^{1/(1-\kappa)} x_t < \phi/(\mu - 1) \Leftrightarrow \Pi_t < 0,$$

where l is given in (26). As a result, competitive firms produce existing N_0 intermediate goods and make zero profit. When $(\gamma^{\kappa} l)^{1/(1-\kappa)} x_t \geq \phi/(\mu - 1)$, we assume that agents do not deploy increasing-returns technology until $x_t \geq x_N$. ■

Dynamics of the consumption-output ratio in the industrial era. The value of assets owned by each member of the household is

$$a_t = V_t N_t / L_t. \tag{A5}$$

If $n_t > 0$, then $V_t = \beta X_t$ in (13) holds. Substituting (13) and $\mu X_t N_t = \theta Y_t$ into (A5) yields

$$a_t = \beta X_t N_t / L_t = (\theta/\mu)\beta Y_t / L_t = (\theta/\mu)\beta y_t, \tag{A6}$$

which implies that a_t/y_t is constant. Substituting (A6), (3), (6) and (22) into (2) yields

$$\begin{aligned} \frac{\dot{y}_t}{y_t} = \frac{\dot{a}_t}{a_t} &= r_t - \lambda + \frac{(1-\tau_t) w_t l_t - c_t}{a_t} \\ &= \frac{\dot{c}_t}{c_t} + \rho - \lambda + \frac{(1-\theta-\gamma)\mu}{\beta\theta} - \frac{\mu}{\beta\theta}\frac{c_t}{y_t}, \end{aligned} \tag{A7}$$

Equation (A7) can be rearranged as

$$\frac{\dot{c}_t}{c_t} - \frac{\dot{y}_t}{y_t} = \frac{\mu}{\beta\theta}\frac{c_t}{y_t} - \frac{(1-\theta-\gamma)\,\mu}{\beta\theta} - \rho + \lambda, \tag{A8}$$

which implies that c_t/y_t jumps to its steady-state value in (28) whenever $n_t > 0$. Substituting (28) into (23) yields (29). ■

Convergence to the balanced growth path. Here, we discuss the differences of the two cases: $\alpha + \kappa > 1$ and $\alpha + \kappa < 1$.

Case 2. Consider $\alpha + \kappa > 1$, which yields $a_1 > 0$. Then, the condition for $x^* > 0$ is simply that x^* exists and is real, i.e., $a_2^2 > 4a_1a_3$. This is an inequality restriction on the parameters. We then add to this inequality the restrictions for $g^* > 0$ and $z^* > 0$ and the sufficient condition (40). This case exhibits the property that the steady-state entry rate, n^*, is increasing in the growth rate of quality, z^*. In other words, quality innovation is so effective that it creates room for variety growth faster than in the canonical case $\alpha+\kappa = 1$. To understand why the model generates constant growth under seemingly explosive conditions, we revisit equation (19) that defines the composite state variable x_t. That equation shows that the model's mechanics identifies the ratio

$$\ell_t \equiv L_t/N_t^{1-\frac{\sigma}{1-\kappa}}$$

as the key measure of labor input per intermediate good. The interpretation is that this ratio measures the flow of raw labor services effectively allocated to the typical intermediate good after we account for the two forms of congestion at work in the model. In the steady state, this measure of labor allocation has the following growth rate:

$$\frac{\dot{\ell}_t}{\ell_t} = \lambda - \left(1 - \frac{\sigma}{1-\kappa}\right) n^* = -\left(\frac{\alpha}{1-\kappa} - 1\right) z^* < 0.$$

The interpretation of this steady state, therefore, is that the economy exhibits constant endogenous growth because the mass of firms grows sufficiently faster than the population so that there is continuous dilution of labor services across firms. This dilution offsets the explosive pressure due to the property that production is convex in average knowledge, Z.

Case 3. Consider the case $\alpha + \kappa < 1$, which yields $a_1 < 0$. The condition for $x^* > 0$ then is $a_2 < \sqrt{a_2^2 - 4a_1a_3}$, where $a_2^2 > 4a_1a_3$ must hold for x^* to exist and be real. When $a_2 < 0$, the condition that we seek is simply $a_2^2 > 4a_1a_3$. When $a_2 \geq 0$, we obtain that x^* is positive when $a_2^2 < a_2^2 - 4a_1a_3$, which reduces to $4a_1a_3 < 0$. Since $a_1 < 0$, this holds for $a_3 > 0$. But $a_3 > 0$ and $a_1 < 0$ imply that the inequality $a_2^2 > 4a_1a_3$ always holds so that x^* surely exists and is real. Therefore, the condition that we seek is simply $a_3 > 0$ when $a_2 \geq 0$. These are again inequality restrictions on the parameters. As in the previous case, we add to these inequalities the restrictions for $g^* > 0$ and $z^* > 0$ and the sufficient condition (40). This case exhibits the property that n^* is decreasing in z^* because quality innovation is not sufficiently effective and the economy can sustain endogenous growth if and only if variety growth is slower than in the canonical case $\alpha + \kappa = 1$. As we saw in the previous case, this property has implications for the dynamics of the ratio ℓ_t. In this case, in particular, we have

$$\frac{\dot{\ell}_t}{\ell_t} = \lambda - \left(1 - \frac{\sigma}{1-\kappa}\right) n^* = -\left(\frac{\alpha}{1-\kappa} - 1\right) z^* > 0.$$

The interpretation of this steady state, therefore, is that the economy exhibits constant endogenous growth because the mass of firms grows sufficiently slower than the population so that there is continuous concentration of labor services across firms. This concentration offsets the implosive pressure due to the property that production is concave in average knowledge, Z. ■

Proof of Proposition 1. In the pre-industrial era, firm size is not large enough for horizontal innovation and vertical innovation to be viable such that $n_t = z_t = 0$. As a result, labor supply l is given by (26), government spending share is given by (22), and the state variable $x_t = (\theta/\mu)^{\frac{1-\kappa(1-\theta)}{(1-\kappa)(1-\theta)}} Z_0^{\frac{\alpha}{1-\kappa}-1} L_t/N_0^{1-\sigma/(1-\kappa)}$ increases at the exogenous population growth rate λ. Therefore, the dynamics of x_t in the pre-industrial era is given by

$$\dot{x}_t = \lambda x_t > 0. \tag{A9}$$

In the first phase of the industrial era, firm size becomes large enough for horizontal innovation (but not for vertical innovation) to be viable such that $n_t > 0$ and $z_t = 0$. The variety growth rate n_t is positive if and only if (31) holds. The dynamics of $x_t = (\theta/\mu)^{\frac{1-\kappa(1-\theta)}{(1-\kappa)(1-\theta)}} Z_0^{\frac{\alpha}{1-\kappa}-1} L_t/N_t^{1-\sigma/(1-\kappa)}$ is

$$\dot{x}_t = \left[\lambda - \left(1 - \frac{\sigma}{1-\kappa}\right) n_t\right] x_t = \frac{1-\sigma-\kappa}{\beta(1-\kappa)} \left\{ \frac{\phi}{(\gamma^\kappa l^*)^{1/(1-\kappa)}} - \left[\mu - 1 - \beta\left(\rho + \frac{\sigma\lambda}{1-\sigma-\kappa}\right)\right] x_t \right\}, \tag{A10}$$

which uses (30) for n_t.

In the second phase of the industrial era, firm size becomes large enough for both horizontal and vertical innovation to be viable such that $n_t > 0$ and $z_t > 0$. The quality growth rate z_t is positive if and only if (37) holds. We use (34), (35) and $z_t = [(1-\kappa)/\alpha]g_t - (\sigma/\alpha)n_t$ to derive n_t and the dynamics of $x_t = (\theta/\mu)^{\frac{1-\kappa(1-\theta)}{(1-\kappa)(1-\theta)}} Z_t^{\frac{\alpha}{1-\kappa}-1} L_t/N_t^{1-\sigma/(1-\kappa)}$ as

$$\dot{x}_t = \frac{a_1\left[(\gamma^\kappa l^*)^{1/(1-\kappa)} x_t\right]^2 - a_2(\gamma^\kappa l^*)^{1/(1-\kappa)} x_t + a_3}{(1-\kappa)\left[\alpha\beta(\gamma^\kappa l^*)^{1/(1-\kappa)} - \sigma/x_t\right]}, \tag{A11}$$

where

$$a_1 = \alpha\beta(\mu-1)(1-\kappa)(\alpha+\kappa-1),$$

$$a_2 = (\alpha+\kappa-1)\{[\mu-1-\beta(\rho-\lambda)]\sigma+\beta(1-\kappa)(\rho+\phi\alpha)\}+\alpha(1-\kappa-\sigma)[(\mu-1)\kappa-\beta(\rho-\lambda)]-\alpha\beta\lambda(1-\kappa),$$

$$a_3 = (\alpha+\kappa-1)\phi\sigma + (1-\kappa-\sigma)[\alpha\phi\kappa - (1-\kappa)\rho] - \sigma\lambda(1-\kappa).$$

Using $\dot{x}_t = 0$ we can derive the steady-state firm size

$$(\gamma^\kappa l^*)^{1/(1-\kappa)} x^* = \frac{a_2 \pm \sqrt{a_2^2 - 4a_1a_3}}{2a_1},$$

which is independent of γ because a_1, a_2 and a_3 are independent of γ.

Case 1. When $\alpha + \kappa = 1$, $a_1 = 0$, $a_2 = \alpha(1-\kappa-\sigma)\{\kappa(\mu-1) - \beta[\rho + \sigma\lambda/(1-\kappa-\sigma)]\}$, and $a_3 = \alpha(1-\kappa-\sigma)\{\kappa\phi - [\rho + \sigma\lambda/(1-\kappa-\sigma)]\}$. In this case, the dynamics of x_t in the pre-industrial and first phase of industrial eras are the same as in (A9) and (A10), and in the second phase of industrial era it becomes

$$\dot{x}_t = \frac{1-\kappa-\sigma}{(1-\kappa)\beta - \sigma/[(\gamma^\kappa l^*)^{1/(1-\kappa)}x_t]} \left\{ \left[\kappa\phi - \left(\rho + \frac{\sigma\lambda}{1-\kappa-\sigma}\right)\right] \frac{1}{(\gamma^\kappa l^*)^{1/(1-\kappa)}} - \left[\kappa(\mu-1) - \beta\left(\rho + \frac{\sigma\lambda}{1-\kappa-\sigma}\right)\right] x_t \right\}.$$

The differential equation $\dot{x}_t = 0$ only has one stable real root $x^* > 0$ under the parameter conditions in (43) and (44), which ensure $z^* > 0$.

Case 2. When $\alpha + \kappa > 1$, $a_1 > 0$. If $\Delta = a_2^2 - 4a_1a_3 > 0$, then we can obtain the following results for $\Phi[(\gamma^\kappa l^*)^{1/(1-\kappa)}x^*] = a_1 \left[(\gamma^\kappa l^*)^{1/(1-\kappa)}x^*\right]^2 - a_2(\gamma^\kappa l^*)^{1/(1-\kappa)}x^* + a_3 = 0$:

$$\begin{cases} \text{If } a_2 > 0, a_3 > 0, & \text{2 positive real roots: } \frac{a_2+\sqrt{\Delta}}{2a_1}(\text{unstable}),\ \frac{a_2-\sqrt{\Delta}}{2a_1}(\text{stable}); \\ \text{If } a_2 > 0, a_3 \leq 0, & \text{1 positive real root: } \frac{a_2+\sqrt{\Delta}}{2a_1}(\text{unstable}); \\ \text{If } a_2 \leq 0, a_3 \geq 0, & \text{no positive real root;} \\ \text{If } a_2 \leq 0, a_3 < 0, & \text{1 positive real roots: } \frac{a_2+\sqrt{\Delta}}{2a_1}(\text{unstable}). \end{cases}$$

If $\Delta \leq 0$, then $\Phi[(\gamma^\kappa l^*)^{1/(1-\kappa)}x^*] = 0$ does not have a positive stable real root. In summary, when $\alpha + \kappa > 1$, there exists a stable steady-state value $x^* > 0$ only if $\Delta > 0$, $a_2 > 0$ and $a_3 > 0$.

Case 3. When $\alpha + \kappa < 1$, $a_1 < 0$. If $\Delta > 0$, then we can obtain the following results for $\Phi(x^*) = 0$:

$$\begin{cases} \text{If } a_2 \geq 0, a_3 > 0, & \text{1 positive real root: } \frac{a_2-\sqrt{\Delta}}{2a_1}(\text{stable}); \\ \text{If } a_2 \geq 0, a_3 \leq 0, & \text{no positive real root;} \\ \text{If } a_2 < 0, a_3 \geq 0, & \text{1 positive real root: } \frac{a_2-\sqrt{\Delta}}{2a_1}(\text{stable}); \\ \text{If } a_2 < 0, a_3 < 0, & \text{2 positive real roots: } \frac{a_2+\sqrt{\Delta}}{2a_1}(\text{unstable}),\ \frac{a_2-\sqrt{\Delta}}{2a_1}(\text{stable}). \end{cases}$$

If $\Delta \leq 0$, then $\Phi[(\gamma^\kappa l^*)^{1/(1-\kappa)}x^*] = 0$ does not have a positive stable real root. In summary, when $\alpha + \kappa < 1$, there exists a stable steady-state value $x^* > 0$ only if $\Delta > 0$, $a_2 \geq 0$ and $a_3 > 0$, or $\Delta > 0$ and $a_2 < 0$.

In summary, given $x_0 < x_N < x_Z$ and the parameter conditions discussed above, the autonomous dynamics of x_t is stable and governed by (A9), (A10) and (A11). Given an initial value x_0, the state variable x_t increases according to (A9) until x_t reaches the first threshold x_N. Then, x_t increases according to (A10) until x_t reaches the second threshold x_Z. Finally, x_t increases according to (A11) until x_t converges to its steady-state value x^* in (45). ■

Proof of Proposition 2. Taking derivative for $\ln x_N$ with respect to γ yields

$$\frac{\partial \ln x_N}{\partial \gamma} = -\frac{\kappa}{1-\kappa}\frac{1}{\gamma} - \frac{1}{1-\kappa}\frac{\partial l^*/\partial \gamma}{l^*},$$

where

$$\frac{\partial l^*}{\partial \gamma} = -(l^*)^2 \frac{\eta\beta\theta(\rho-\lambda)}{\mu(1-\theta-\gamma)^2}.$$

Substituting $\partial l^*/\partial\gamma$ into $\partial \ln x_N/\partial\gamma$ yields

$$\frac{\partial \ln x_N}{\partial \gamma}=\frac{\eta\beta\theta(\rho-\lambda)\gamma-\kappa(1-\theta-\gamma)[(1+\eta)(1-\theta-\gamma)\mu+\eta\beta\theta(\rho-\lambda)]}{\gamma(1-\kappa)(1-\theta-\gamma)[(1+\eta)(1-\theta-\gamma)\mu+\eta\beta\theta(\rho-\lambda)]},$$

The sign of $\partial \ln x_N/\partial\gamma$ is determined by the numerator. It can be shown that the numerator is increasing in γ, and we denote the numerator as Λ.

$$\lim_{\gamma\to 0}\Lambda=-\kappa(1-\theta)[(1+\eta)(1-\theta)\mu+\eta\beta\theta(\rho-\lambda)]<0;$$

$$\lim_{\gamma\to 1-\theta}\Lambda=\eta\beta\theta(1-\theta)(\rho-\lambda)>0.$$

Therefore, there exists a threshold value $\tilde{\gamma}\in(0,1-\theta)$ such that $\partial \ln x_N/\partial\gamma<0$ for $\gamma\in(0,\tilde{\gamma})$ and $\partial \ln x_N/\partial\gamma>0$ for $\gamma\in(\tilde{\gamma},1)$. Therefore, for a relatively small (large) $\gamma<\tilde{\gamma}$ ($\gamma>\tilde{\gamma}$), an increase in γ leads to a smaller (larger) x_N, which causes an earlier (a delayed) takeoff.

From (30) and (34), for a given $x_t\in(x_N,x^*)$, an increase in γ increases (decreases) the equilibrium growth rate when $\gamma<\tilde{\gamma}$ ($\gamma>\tilde{\gamma}$). From (46), γ does not affect the steady-state growth rate due to the scale-invariant property of the model. ■

Dynamics of corporate income tax rate in the industrial era. The profit as a share of output is given by

$$\frac{N_t\left(\Pi_t-I_t\right)}{Y_t}=\frac{\theta}{\mu}\left[\mu-1-\frac{\phi+z_t}{(\gamma^{1-\alpha}l^*)^{1/\alpha}x_t}\right]=\frac{\beta\theta(n_t+\rho-\lambda)}{\mu(1-\tau_{\Pi,t})},$$

which uses the growth rate of variety given by

$$n_t=\frac{1-\tau_{\Pi,t}}{\beta}\left[\mu-1-\frac{\phi+z_t}{(\gamma^{1-\alpha}l^*)^{1/\alpha}x_t}\right]+\lambda-\rho.$$

Substituting the profit share into the government budget constraint yields

$$\tau_{\Pi,t}=\frac{\mu[\gamma-\tau\left(1-\theta\right)](1-\tau_{\Pi,t})}{\beta\theta(n_t+\rho-\lambda)}.$$

Solving for $\tau_{\Pi,t}$ yields the corporate income tax rate in Section 4.2. ■

Dynamics of x_t under corporate income tax. In the pre-industrial era, the dynamics of x_t is the same as before:

$$\dot{x}_t=\lambda x_t>0.$$

In the first phase of industrial era, the dynamics of x_t becomes

$$\dot{x}_t=\frac{\alpha-\sigma}{\alpha\beta}\left\{\frac{\phi(1-\tau_{\Pi,t})}{(\gamma^{1-\alpha}l^*)^{1/\alpha}}-\left[(1-\tau_{\Pi,t})(\mu-1)-\beta\left(\rho+\frac{\sigma\lambda}{\alpha-\sigma}\right)\right]x_t\right\}.$$

In the second phase of industrial era, the dynamics of x_t becomes

$$\dot{x}_t=\frac{\alpha-\sigma}{\alpha\beta}\left\{\left[(1-\alpha)\phi-\left(\rho+\frac{\sigma\lambda}{\alpha-\sigma}\right)-\frac{\dot{\tau}_{\Pi,t}}{1-\tau_{\Pi,t}}\right]\frac{1-\tau_{\Pi,t}}{(\gamma^{1-\alpha}l^*)^{1/\alpha}}\right.$$
$$\left.-\left[\left(1-\tau_{\Pi,t}\right)\left(\mu-1\right)\left(1-\alpha\right)-\beta\left(\rho+\frac{\sigma\lambda}{\alpha-\sigma}\right)\right]x_t\right\},$$

where we have used $(1-\tau_{\Pi,t})\sigma/[(\gamma^{1-\alpha}l^*)^{1/\alpha}x_t] \cong 0$. ■

Proof of Proposition 3. Taking derivative for $\ln x_N$ with respect to γ yields

$$\begin{aligned}\frac{\partial \ln x_N}{\partial \gamma} &= -\frac{1-\alpha}{\alpha}\frac{1}{\gamma} + \frac{\mu/\theta}{\mu - 1 - \beta(\rho-\lambda) - [\gamma-(1-\theta)\tau]\mu/\theta} \\ &= \frac{(\mu/\theta)\gamma - (1-\alpha)[\mu - 1 - \beta(\rho-\lambda) + (1-\theta)\tau\mu/\theta]}{\alpha\gamma\{\mu - 1 - \beta(\rho-\lambda) - [\gamma-(1-\theta)\tau]\mu/\theta\}},\end{aligned}$$

The sign of $\partial \ln x_N/\partial\gamma$ is determined by the numerator, because the denominator must be positive to ensure $x_N > 0$. It is useful to note that the numerator is increasing in γ, and we denote the numerator as Γ.

$$\lim_{\gamma\to(1-\theta)\tau} \Gamma = \alpha\tau\mu(1-\theta)/\theta - (1-\alpha)[\mu - 1 - \beta(\rho-\lambda)].$$

Then, $\partial \ln x_N/\partial\gamma = 0$ yields

$$\hat{\gamma} = (1-\alpha)\{[\mu - 1 - \beta(\rho-\lambda)]\theta/\mu + (1-\theta)\tau\}.$$

When $\hat{\gamma} > (1-\theta)\tau$, $\lim_{\gamma\to(1-\theta)\tau} \Gamma < 0$ implies that

$$\tau < \min\left\{\frac{(1-\alpha)\theta}{\alpha\mu(1-\theta)}[\mu - 1 - \beta(\rho-\lambda)], 1\right\}.$$

Therefore, if the labor income tax rate τ is small enough such that $\lim_{\gamma\to(1-\theta)\tau} \Gamma < 0$, then there exists a threshold value $\hat{\gamma}$ such that $\partial \ln x_N/\partial\gamma < 0$ for $\gamma < \hat{\gamma}$ and $\partial \ln x_N/\partial\gamma > 0$ for $\gamma > \hat{\gamma}$. Therefore, for a relatively small (large) $\gamma < \hat{\gamma}$ ($\gamma > \hat{\gamma}$), an increase in γ leads to an earlier (a delayed) takeoff. ■

Appendix B: Data

Table B1: Summary statistics

	obs	mean	std. dev.	min	max
Growth of real GDP	189	0.039	0.030	-0.037	0.169
Growth of real GDP per capita	189	0.034	0.029	-0.030	0.144
Growth of real GDP per worker	189	0.030	0.025	-0.021	0.106
Productive government spending	189	0.165	0.039	0.053	0.252
Log real GDP per capita	189	10.075	0.704	7.860	11.459
Log population	189	6.815	1.806	2.069	11.844
Log capital stock	189	13.934	1.933	8.361	17.963
Trade openness	189	0.908	0.746	0.135	5.080

Table B2: Productive government spending and growth: income levels

	Low-income group			High-income group		
	(1)	(2)	(3)	(4)	(5)	(6)
	GDP	GDP pc	GDP pw	GDP	GDP pc	GDP pw
γ_{it}	1.577***	1.282**	0.930**	0.396	0.600	0.923
	(0.520)	(0.504)	(0.370)	(1.101)	(1.047)	(1.865)
γ_{it}^2	-4.808***	-3.975**	-2.422**	-2.738	-3.153	-2.951
	(1.713)	(1.696)	(1.201)	(3.667)	(3.412)	(5.342)
y_{it-1}	-0.072***	-0.077***	-0.037***	-0.189***	-0.202***	-0.159***
	(0.025)	(0.024)	(0.006)	(0.029)	(0.022)	(0.028)
Country FE	Yes	Yes	Yes	Yes	Yes	Yes
Period FE	Yes	Yes	Yes	Yes	Yes	Yes
Observations	142	142	142	47	47	47
R-squared	0.712	0.706	0.686	0.894	0.874	0.770

Note: *** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$. Cluster-robust standard errors are in parentheses. The dependent variable in column (1) is the average annual growth rate of real GDP for the low-income group. The dependent variable in column (2) is the average annual growth rate of real GDP per capita for the low-income group. The dependent variable in column (3) is the average annual growth rate of real GDP per worker for the low-income group. The dependent variable in column (4) is the average annual growth rate of real GDP for the high-income group. The dependent variable in column (5) is the average annual growth rate of real GDP per capita for the high-income group. The dependent variable in column (6) is the average annual growth rate of real GDP per worker for the high-income group.

Table B3: Productive government spending and growth: income levels (with controls)

	Low-income group			High-income group		
	(1)	(2)	(3)	(4)	(5)	(6)
	GDP	GDP pc	GDP pw	GDP	GDP pc	GDP pw
γ_{it}	1.773***	1.464***	1.101***	0.575	0.512	0.992
	(0.524)	(0.494)	(0.352)	(2.418)	(1.793)	(1.879)
γ_{it}^2	-5.412***	-4.548**	-3.303***	-3.577	-3.414	-3.603
	(1.823)	(1.779)	(1.130)	(6.810)	(4.840)	(5.389)
y_{it-1}	-0.097***	-0.102***	-0.076***	-0.207*	-0.248**	-0.226***
	(0.019)	(0.019)	(0.017)	(0.105)	(0.088)	(0.078)
Controls	Yes	Yes	Yes	Yes	Yes	Yes
Country FE	Yes	Yes	Yes	Yes	Yes	Yes
Period FE	Yes	Yes	Yes	Yes	Yes	Yes
Observations	142	142	142	47	47	47
R-squared	0.788	0.785	0.778	0.908	0.910	0.857

Note: *** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$. Cluster-robust standard errors are in parentheses. The dependent variable in column (1) is the average annual growth rate of real GDP for the low-income group. The dependent variable in column (2) is the average annual growth rate of real GDP per capita for the low-income group. The dependent variable in column (3) is the average annual growth rate of real GDP per worker for the low-income group. The dependent variable in column (4) is the average annual growth rate of real GDP for the high-income group. The dependent variable in column (5) is the average annual growth rate of real GDP per capita for the high-income group. The dependent variable in column (6) is the average annual growth rate of real GDP per worker for the high-income group. The additional control variables are the log value of population size, the log value of capital stock, and the degree of trade openness.

Appendix C: Phase diagrams

Figure C1 presents the equilibrium with only variety growth for $\alpha + \kappa = 1$.

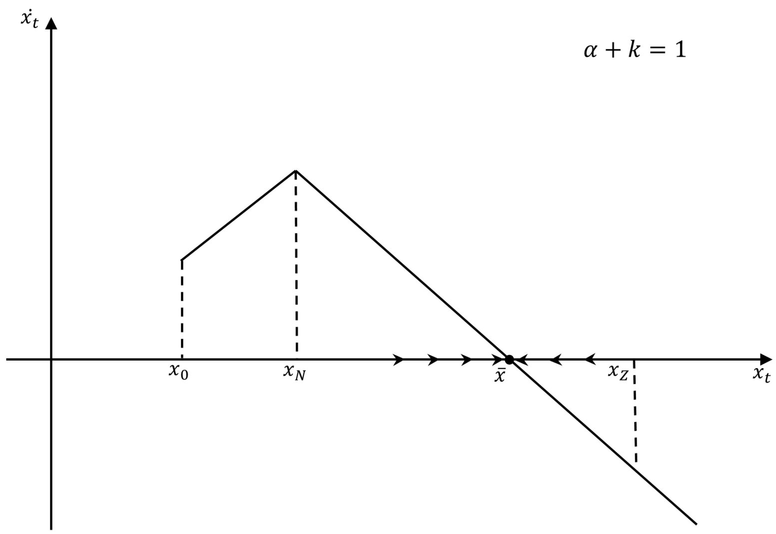

Figure C1: Variety growth only ($\alpha + \kappa = 1$)

Figure C2 presents the equilibrium with only variety growth for $\alpha + \kappa > 1$.

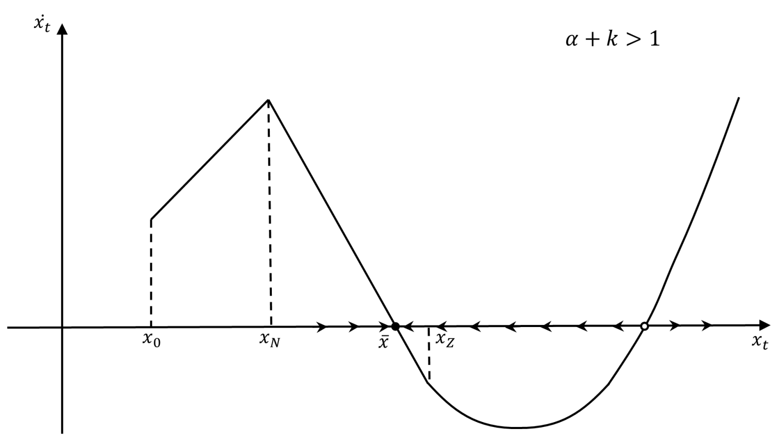

Figure C2: Variety growth only ($\alpha + \kappa > 1$)

Figure C3 presents the equilibrium with explosive growth for $\alpha + \kappa > 1$.

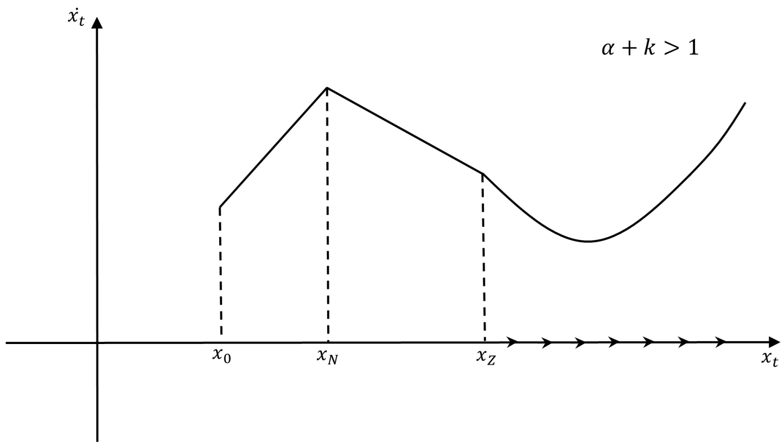

Figure C3: Explosive growth ($\alpha + \kappa > 1$)

Figure C4 presents the equilibrium with only variety growth for $\alpha + \kappa < 1$.

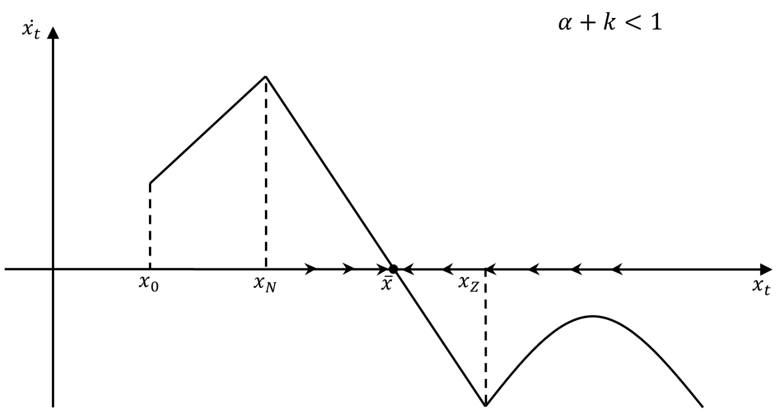

Figure C4: Variety growth only ($\alpha + \kappa < 1$)

Figure C5 presents the equilibrium with multiple equilibria for $\alpha + \kappa < 1$.

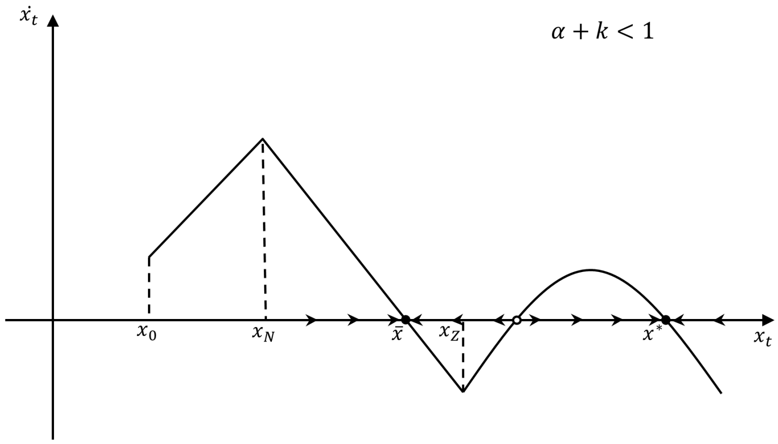

Figure C5: Multiple equilibria ($\alpha + \kappa < 1$)