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Abstract

This article summarizes some propositions regarding economic dynamics and implications of two-way altruism, on the basis of the human-capital-based OLG model of Ehrlich and Lui (1991) and Ehrlich and Kim (2007) with application of a modified, fertility-endogenized definition of linearly separable two-way altruism examined by Abel (1987) and Altig and Davis (1993). Some properties in both a transition process and a steady state, and the effect of unfunded social security on an equilibrium path are also discussed. My calibration results and analyses show that (1) the combination of altruism toward parents and children is crucial for determining a threshold level of initial human capital and productivity in a transition process (stagnant to growth or growth to stagnant), and the generation's attained utility, (2) dynamic consistency might not necessarily be the best choice to overpass the stumbling block against growth regime, (3) in this human-capital-based OLG model, a regular recursive induction approach might still cause inefficiency in terms of an ex-post Pareto optimality criterion as of two periods later, even if strategic effects for after children (two generations later) are appropriately taken account of, and (4) unfunded social security tax, which involves actuarially fair insurance as well as certainty premium transfer, does affect critical values for a regime change as well as dynamic equilibrium paths and corresponding subsequent life strategies, even in two-way altruistic economy.

1. Introduction

In this article, I try to extract some economic implications of "two-way altruism" in the context of intergenerational linkage on a human-capital-based, overlapping generation growth model. The primary reason why I am interested in this topic is that altruism should be closely related with some other forms of collaboration, teamwork or communication. Team work is clearly one pattern of collusive behavior, possibly a cause for market distortion in microeconomic context, however, can be surely a growth engine in a dynamic context by means of caring about marginal effects for others' physical/human capital accumulation. Communication, in often cases, is discussed from the viewpoint of Bayesian inference as a spontaneous process of transporting information, a surer predictor (posterior distribution) regarding uncertain productivity of technology. In these regards, "altruism", in either static or dynamic, certain or uncertain framework, seems to give a full incentive for these collaborations, by offering an economic motivation to internalize any consumption/capital externalities, if exist, from other families and to overpass a "stumbling block" for a more efficient allocation over a horizontal/vertical time horizon.

Turning our eyes to human capital perspectives, "human capital is knowledge embodied in people" as Becker, Murphy and Tamura (1990) already point out, and the transfer of knowledge can be regarded genuinely as one of various forms of communication. On the other hand, human capital (knowledge) holds clearly different economic attributes from ordinary physical capital, for example "a rising rate of return (Becker et al. (1990))", irreversibility as seen in most other aspects of "information", absence of intra-family market & pricing (although inter-family market & pricing may exist as an education/information industry), inter/intra family investment (transfer/accumulation) only through time allocation, trade-off between quality (human capital) and quantity (fertility), or etc. In addition, in a physical-capital-based dynasty model, it is rather clear, simply by comparing F. O. C.'s, that holding altruism toward parents as well as one toward children is, if appropriately taken, helpful for improving dynamic efficiency. Therefore I focus on the role of two-way altruism in a typical human-capital-based OLG model especially as an internalization of various externalities, which all the above peculiar attributes of human capital bring about.

To see the reasons why two-way altruism assumption is crucial in a typical OLG model, assume the OLG model consisting of three life stages (childhood, young adulthood (middle age) and old adulthood (retirement stage)). One generation, in order to determine its own life strategies, including number of children and human capital investment per child, in a recursive induction at the beginning of young adulthood (period t), needs to specify the way for linkage with at least one of its adjacent generations, forward or backward, which determines the amount of intergenerational transfer. However, in general, these linkages (λ and λ' as defined later) *cannot* be left for a spontaneous bargaining mechanism between these two generations, simply because a positive utility gain from a positive transfer for one generation implies a negative utility gain for another generation. Even if some intergenerational risk sharing mechanism with no social/market insurance available is considered between young and old adulthood, in which bequest (from parents to children) and compensation (vice versa) correspond with each revelation of mortality/income risks, the mortality risk of parental generation generates a peculiar shape of its indifference curve, and this shape makes it for both parents and children impossible to set initially some value for the state contingent claim between two states of mortality risk, or equivalently to set the initial relative price between bequest and compensation. This is a totally different point from Arrow-Debreu state-contingent exchange economy, in which state contingent claim (or state price) enables them to arrive at a market-clearing and Pareto optimal equilibrium. As a consequence, in this short-run bargaining frame work, an automatic price adjustment process to a unique equilibrium point cannot be expected, as far as any additional restrictions (e.g., regarding the marginal rate of substitution between bequest and compensation, or the proportion in the marginal utility of transfer) are not introduced, or a fixed level of (almost actuarially fair) social/market insurance is not available for old parents.[1]

Therefore I assume that fixed *altruistic* weights (proportions) of utilities between neighbor generations are exogenously given as a kind of social norm, although they may be variable in the long run. Consider the following forward altruistic utility of generation (vintage) t:

$$\begin{aligned} V_t &= u_t + \lambda u_{t+1} \\ &= \left\{u_t^{(y)} + \delta u_{t+1}^{(o)}\right\} + \lambda\left\{u_{t+1}^{(y)} + \delta u_{t+2}^{(o)}\right\} \end{aligned} \qquad (1.1)$$

Here $u_t \equiv \left\{u_t^{(y)} + \delta u_{t+1}^{(o)}\right\}$, $u_t^{(y)}$ and $u_{t+1}^{(o)}$ are the whole life, young and old adulthood utility of generation t, respectively. δ is a time preference discount factor for old (retirement) stage, and λ is a *weight of altruism* toward children (next generation $t+1$). Then generation t decides its life strategies at the beginning of period t, so that they might maximize its (altruistic) utility, while consequently the next generation $t+1$ faces an implicit restriction parental generation imposed regarding the ratio in marginal utility of intergenerational transfer (bequest/compensation) between $u_{t+1}^{(o)}$ and $u_{t+1}^{(y)}$. In other words, the generation holds, in general, two chances of intergenerational linkage, firstly through fertility and human capital investment decision during young adulthood, and secondly through transfer (compensation/bequest) during old stage. Therefore, next generation $t+1$ is required, at the beginning of M (period $t+1$), to take this implicit restriction as given, or to internalize it by adding an old adulthood utility part $u_{t+1}^{(o)}$ of parental generation t weighted by some altruistic coefficient. Thus two-way altruistic utility of the following form makes sense, so that it enables the recursive computation of value function without

[1] These are typical economic features of intergenerational risk sharing, as described in Aoki (2007).

intergenerational bargaining confliction:

$$V_t = \lambda' u_t^{(o)} + u_t + \lambda u_{t+1} = \lambda' u_t^{(o)} + \{u_t^{(y)} + \delta u_{t+1}^{(o)}\} + \lambda u_{t+1} \tag{1.2}$$

Since, as a natural result, this form, which ranges at least two (actually three in this case) adjacent periods, contains a utility part of next generation $u_{t+1} = u_{t+1}^{(y)} + \delta u_{t+2}^{(o)}$ and its own old adulthood utility part $u_{t+1}^{(o)}$, the generation maximizes (1.2), taking account of subsequent effects its own life strategies brings on the next generation. While this (or a similar) form of two-way altruism is examined by Abel (1985), Kimball (1987) or Altig and Steve (1991) basically from the viewpoints of steady states, they do not pay much consideration to the timing of determination of life strategies, as well as a rebound and indirect effect on its old age utility through next generation's strategy setting. In addition, if the generation is *dynamically consistent* in the sense that it behaves as parents expected in terms of transfer motive during old (retirement) stage, it must hold that $\lambda' = \delta / \lambda$, which implies that the generation, who holds less altruism toward children, should be more altruistic toward parents. However, in general, there is no guarantee that each generation holds a dynamically consistent altruism toward its parental generation, or that above all dynamic consistency leads to dynamic efficiency, especially where the economy is not in a steady state. On the other hand, from socio-biological point of view, it is quite natural to assume $\lambda' = \lambda\delta$ instead, because an *egoistic* person is normally egoistic both for parents and children, therefore tends to hold smaller λ' as smaller λ is. Thus the dynamic consistency is a concept somewhat contrary to our intuition in socio-biological context. Therefore, in this article, I am going to summarize some propositions and features regarding economic dynamics and implications of two-way altruism, being careful enough for these crucial aspects, specification of intergenerational linkage, timing of strategy determination and indirect strategic effects, effects of dynamic consistency/inconsistency and steady/unsteady states, and in addition, roles of mandatory intergenerational transfer between old parents and young adulthood as implemented in the form of unfunded social security. This is exactly the objective of this article.

2. Literature Review

Becker (1976) is the first article, which developed a powerful analysis by joining the individual rationality of the economist to the group rationality of the socio-biologist, and showed that the models of genetic group selection are unnecessary, since altruistic behavior can be selected as a consequence of individual rationality. Becker and Murphy (1988) try to understand the widespread intervention by governments in families, concluding that many public actions achieve more efficient arrangements between parents and children. Becker, Murphy and Tamura (1990) analyze a growth model which assumes endogenous fertility and a rising rate of return on human capital, and derive two stable steady states, of which one has large families and little human capital, and the other has small families and perhaps growing human and physical capital. Becker and Barro (1988) develops an economic analysis of the linkages in fertility rates and capital accumulation across generations, considering the determination of fertility and capital accumulation in each generation when wage rates and interest rates are parameters to each family and to open econonmies. Ehrlich and Lui (1991) develop an overlapping generation model in which human capital is an endogenous engine of growth and the generations are linked through material interdependency as an implicit contract, as well as through emotional "companionship". They succeed in extracting some typical economic features, which might be possibly observed in a real OLG economy both in a steady state and in a (demographic) transition process, for example, showing that an increase in young-age longevity is likely to produce a greater increase in the growth rate and a reduction in the fertility rate, while population aging may also raise the growth rate. Ehrlich and Kim (2005) develop a dynamic model of endogenous fertility, longevity, and human capital formation within a Malthusian framework that allows for diminishing returns to labor but also for the role of human capital as an engine of growth. Their model accounts for economic stagnation with high fertility and mortality and constant population and income, as predicted by Malthus, but also for takeoffs to a growth regime and a demographic transition toward low fertility and mortality rates, and a persistent growth in per-capita income. Ehrlich (1990) and Ehrlich and Lui (1997) respectively offer thorough and sophisticated surveys of population and growth literatures as of the time the article was written. Lucas (1988) considers the prospects for constructing a neoclassical theory of growth and international trade that is consistent with some of the main features of economic development. Three models are considered and compared to evidence: a model emphasizing physical capital accumulation and technological change, a model emphasizing human capital accumulation through schooling, and a model emphasizing specialized human capital accumulation through learning-by-doing.

Cigno and Rosati (1996) derive some comparative-statistics predictions of models of the joint determination of household saving and fertility under various hypotheses (self-interest, altruism of parents towards children, altruism of children towards parents, etc.) and compare them with those of models which determine saving under the assumption of exogenous fertility. Boldrin, De Nardi and Jones (2005) explore the type of model, which is consistent with the data showing that an increase in government provided old-age pensions is strongly correlated with a reduction in fertility, using the one by Barro and Becker (1988) and the other inspired by Caldwell and developed by Boldrin and Jones (2002). Abel (1987) determines conditions under which each of the intergenerational transfer motives is operative if individual consumers have two-sided transfer motives. Altig and Steve (1991) develop the implications of borrowing constraints and two sided altruism in an overlapping generations framework with agents who lived three periods. Six equilibrium patterns of inter-temporal and intergenerational linkage in the no-loan economy, one of which corresponds to the traditional life-cycle model, and one of which corresponds to Barro's dynastic model. Novel linkage patterns involve parent-to-child transfers early in the life cycle, child-to-parent gifts late in the life cycle, or both. Kimball (1987) analyzes recursive dependence of altruistic utility on the utility of both children and parents, and shows that dynamic inefficiency cannot be ruled out even in the presence of two-sided altruism, especially in the unsteady state. Obstfeld and Rogoff (1996) offer a standard textbook of international macroeconomics, which contains the description of typical overlapping generation models, and learning-by-doing externality in AK model. Raut (2006) extends the Samuelsonian overlapping generation general equilibrium framework to encompass a variety of altruistic preferences by recasting it into a Lindahl equilibrium framework. Here a complete characterization of Pareto optimal allocation is provided using the Lindahl equilibrium prices. Zhang, Zhang and Lee (2001) examine the impacts of mortality decline on long-run growth in a dynastic family, two-sector growth model with social security. A rise in longevity has direct effects on fertility, human capital investment, and growth, as well as indirect effects through unfunded social security contributions. Blackburn and Cipriani (2005) present an analysis of demographic transition based on the endogenous evolution of intergenerational transfers along an economy's endogenous path of development.

Aoki (2007) describes, within a myopic intergenerational bargaining framework incorporating two discrete periods and binary states of risks, some new aspects regarding the mixture of intergenerational risk sharing and social security. Here, state-dependent utility under mortality risk proves to generate parents' peculiar indifference curve regarding insurance contract, and self-insurance is shown to play a crucial role on the decision regarding social security holding and intergenerational transfer contract. Abel (1985) develops a general equilibrium model of precautionary saving and accidental bequests to analyze the implications of individual lifetime uncertainty for aggregate consumption and capital accumulation, and shows that, in the absence of a private annuity market, the introduction of actuarially fair Social Security crowds out private wealth by more than one for one, thereby reducing national wealth. Kohara and Ohtake (2006) analyze what adult children would do for their parents were they frail and in need of long-term care. They show that children provide parental care when their parents are wealthy enough to meet the costs of nursing, and that this parental care is not motivated entirely by altruism. Iwamoto (2006) examines, both in detail and with sophistication, the scheme of social security financing in Japan that will be sustainable under the coming population aging process, in which he claims that public pension should be a mixture of the pay-as-you-go system and the fully funded system. Nishimura and Zhang (1995) analyze two largely hypothetical social security systems that condition payments on individual fertility, as well as a conventional system that does not condition payments on fertility, and prove that, under the social security systems that relate payments to individual fertility, and equilibrium will converge to a sustainable steady state. Nishimura and Zhang (1992) show, in the generalized Veall's model of public pension, that gifts to the old, which can be viewed as social security contributions, are always positive in the steady states, that an optimal allocation, not sustainable in general, is sustainable if savings are zero and fertility is exogenous, and that, if a government enforces a social security plan setting the pension level at the optimal gifts and individuals optimize under the pension constraint, the resulting sustainable outcome is in general different from either the optimal or Nash outcome. Benhabib and Nishimura (1989) analyze one of the interesting hypothesis of population growth suggesting the possibility of self-generating fluctuations in population growth, using the Barro-Becker model, and show that under a broad class of preferences, fertility and per-capita incomes not only move together but endogenously oscillate.

Among the above literatures, the first serious economic study on altruism and on human-capital based overlapping generation model originate in Becker (1976) and Ehrlich and Lui (1991), respectively. Some arguments regarding dynamic consistency, intergenerational transfer motives and funded/unfunded social security under two-sided altruism appear, for example, in Abel (1987) or Altig and Steve (1991).

3. Model & Methodology

In principle, I adopt the same model settings and the same notations, as constructed in Ehrlich and Lui (1991), and Ehrlich and Kim (2005). The model assumes three stages in one generation's whole life, childhood, young and old adulthood, and adjacent generations (parents and children) sharing the same period t within adjacent life stages respectively. Young adult determines, at the beginning, some life strategies, fertility (the number of children per parent) n_t, human capital investment for each child h_t, and compensation rate ω_{t+1}, which is to be given by their children in the next period as old age support, and saving rate s_t. Here I examine the simplest case in which (1) there exists only one representative (identical) family, (2) there do not exist any inter/intra-generational human-capital externalities from outside families, and (3) there is no weight of companionship toward children:

$$y_t = [(1 - vn_t - \varsigma h_t n_t)(H_t + \overline{H})]^{\gamma} \qquad \gamma = 1 \tag{3.1}$$

$$H_{t+1} = A(H_t + \overline{H})h_t \text{ (3.2-a)}, \quad \widetilde{S}_t = D[H_t + \overline{H}](s_t)^m \text{ (3.2-b)} \qquad \text{(Law of motion)}$$

$$c_{1,t} = (1 - vn_t - \varsigma h_t n_t - \rho s_t)\eta_t(H_t + \overline{H}) - \omega_t \pi_2 \eta_t (H_t + \overline{H}) \qquad \text{(3.3-a)}$$ [2]

$$c_{2,t+1} = \omega_{t+1}\pi_1 n_t [\eta_{t+1}(H_{t+1} + \overline{H})] + D(H_t + \overline{H})(s_t)^m \qquad \text{(3.3-b)}$$

$$u_t^{(y)} \equiv u(c_{1,t}) = \left(\frac{1}{1-\sigma}\right) c_{1,t}^{\;1-\sigma}, \quad u_{t+1}^{(o)} \equiv u(c_{2,t+1}) = \left(\frac{1}{1-\sigma}\right) c_{2,t+1}^{\;1-\sigma} \qquad (3.4)$$

$$N_{t+1} = N_t n_t \qquad (3.5)$$ [3]

with constraints: $n_t \geq 0$, $h_t \geq 0$, $s_t \geq 0$, $y_t \geq 0$ and $c_{1,t} \geq 0$, $c_{2,t+1} \geq 0$.[4]

$u(c) \equiv (1/(1-\sigma))c^{1-\sigma}$: a utility where σ represents the inverse value of the inter-temporal elasticity of substitution or the constant relative risk aversion coefficient. t: Period t. y, o: young/old adulthood, children. N_t: Number of couple (young adult) at period t. n_t: Number of children (fertility) the generation t bears per parent. y_t: Production by generation t. H_t: Acquired human capital. $\overline{H}$: Raw labor capital. (Therefore, $H_t + \overline{H}$ represents total production capacity.) $\widetilde{S}_{t-1}$: Total saving. h_t: Time allocation a parent (young adult) devotes to educate each child. v: Allocation of labor time a parent (young adult) devotes to raise each child. s_t : A fraction of income $\eta_t(H_t + \overline{H})$ allocated to saving during young adulthood. ς, ρ : Time efficiency for

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[2] (3.3-a) can be rewritten as $c_{1,t} = (1 - vn_t - \rho s_t - \eta_t \pi_2)(H_t + \overline{H}) - \frac{\varsigma}{A} H_{t+1} n_t$ (3.3-b).

[3] From (3.5), $n_t = N_{t+1}/N_t$, $n_{t+1} = N_{t+2}/N_{t+1}$ and:

$$\frac{d}{dN_{t+1}} = \left(\frac{1}{N_t}\right)\frac{\partial}{\partial n_t} - \left(\frac{N_{t+2}}{N_{t+1}^{\;2}}\right)\frac{\partial}{\partial n_{t+1}} = \left(\frac{1}{N_t}\right)\left(\frac{\partial}{\partial n_t} - \frac{n_{t+1}}{n_t}\frac{\partial}{\partial n_{t+1}}\right)$$

[4] Since there is no restriction imposed in ω_{t+1} here, ω_{t+1} can be negative, in other words, parents could transfer a positive amount of bequests to children, instead of receiving compensation (gift) from them. This case matters especially under the assumption of saving habit ($s_t > 0$).

intra-family education and saving. ω_t: Compensation rate from a young adult (a child) to an old adult (a parent). η_t: Wage for a unit of human capital (production capacity).[5] $c_{1,t}$: Young adulthood at period t. $c_{2,t+1}$: Old adulthood consumption at period $t+1$. u_t: Whole life utility (of both young and old adulthood) of generation t. $u_t^{(y)}$: Young-adulthood utility of the generation at period t. $u_t^{(o)}$: Old-adulthood utility of parental generation (vintage $t-1$). π_1, π_2: Survival rate to young and old adulthood, respectively. B: Weight of companionship toward children. D: Rate of return in saving.

To make the analyses the simplest possible, I adopt in this article the following *additively separable* form of two-way altruism, which incorporates fertility decisions and covers all the relevant utilities of two adjacent generations during two adjacent periods:

$$\begin{aligned} V_t &= \hat{\lambda}'(n_{t-1})u_t^{(o)} + u_t + \hat{\lambda}(n_t)u_{t+1}^{(y)} \\ &= \hat{\lambda}'(n_{t-1})u(c_{2,t}) + \{u(c_{1,t}) + \delta\pi_2 u(c_{2,t+1})\} + \hat{\lambda}(n_t)u(c_{1,t+1}) \end{aligned} \quad (3.6)$$ [6]

Here $u_t^{(y)} \equiv u(c_{1,t})$, $u_{t+1}^{(o)} \equiv u(c_{2,t+1})$, $u_t \equiv u_t^{(y)} + \delta\pi_2 u_{t+1}^{(o)}$, $\hat{\lambda}(n_t) \equiv \lambda\pi_1 a(n_t)n_t$, $\hat{\lambda}'(n_{t-1}) \equiv \dfrac{\lambda'\delta\pi_2}{\pi_1 a(n_{t-1})n_{t-1}}$, and $a(n) \equiv n^{-\varepsilon}$ $(0 \le \varepsilon \le 1)$.[7] $\lambda(>0)$, $\lambda'(>0)$ and ε respectively denote the degree of pure altruism toward children, toward parents, and the constant elasticity of altruism per child as their number increases.[8] *Dynamic consistency* requires the condition $\lambda\lambda'=1$, while *socio-biological* consistency does $\lambda=\lambda'$, but I do not

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[5] With linear production function ($\gamma = 1$), $\eta_t = 1.0$.

[6] Another candidate is, for example, $V_t = \hat{\lambda}'(n_{t-1})u_t^{(o)} + u_t + \hat{\lambda}(n_t)u_{t+1}$ (3.6a).

[7] The elasticity of altruism per child (ε) was introduced by Becker et al. (1990). $\varepsilon = 1.0$ denotes perfect inelasticity of altruism with number of child.

[8] Unlike ordinary time preference, I do not exclude the possibility of $\lambda \ge 1$, because parents are, sometimes as an important case, very likely to put more weight in children than in themselves.

impose these conditions as a general restriction. Also, define an "altruistic" economy as $\lambda\lambda' > 1$ and an "egoistic" one as $\lambda\lambda' < 1$. Assume that, throughout this article, time is finite and ends at $t = T$. Then generation t solves:

$$\widetilde{V}_t(H_t, n_{t-1}, \widetilde{S}_{t-1}) = \max_{\{h_t, n_t, \omega_t, s_t\}} V_t = \max_{\{h_t, n_t, \omega_t, s_t\}} \left(\hat{\lambda}'(n_{t-1})u_t^{(o)} + u_t + \hat{\lambda}(n_t)u_{t+1}^{(y)}\right) \quad (3.7)$$

Steady states can be defined as a set of life strategies $\{h_t, n_t, s_t, \omega_t\}_{t=-\infty}^{+\infty}$, which satisfies the equilibrium conditions, and in addition:

$$h_t = h \geq 0,\ n_t = n \geq 0,\ s_t = s \geq 0 \text{ and } \omega_t = \omega \text{ for all } t.$$

Here there exist two types of steady states, a *stagnant* equilibrium for $1/A > h \geq 0$ and a *growth* equilibrium for $h \geq 1/A$, as analyzed in Becker et al. (1988) and Ehrlich et al. (1991). In a stagnant equilibrium, $c_{1,t} = c_{1,t+1} \equiv c_1$, $c_{2,t} = c_{2,t+1} \equiv c_2$ and $H_t = H_{t+1} = Ah\overline{H}/(1-Ah) \equiv H$.[9] On the other hand, in a *growth* equilibrium, $H_{t+1} = AhH_t$ ($H_t >> \overline{H}$), $c_{1,t+1} = Ahc_{1,t} \equiv Ahc_1$, $c_{2,t+1} = Ahc_{2,t} \equiv Ahc_2$. At first I assume, for simplicity, condition (x): $\pi_1 = \pi_2 = 1$, $\rho = \varsigma = 1$, $\gamma = 1$ (linear technology) and $0 \leq \varepsilon \leq 1$.

4. Case A-Saving but no compensation economy

At first, I consider the benchmark case (case A hereafter), in which an old adulthood generation can make use of only its own saving, not compensation from the next generation. In this case, it is easily seen that each generation keeps an intergenerational linkage with subsequent generations only in its young adulthood only through human capital investment and fertility decision for children, not through any other channels, therefore the problem of

[9] Therefore, $H + \overline{H} = \overline{H}/(1-Ah)$.

dynamic inconsistency does not occur. Specifically generation t solves the following problem.

$$\tilde{V}_t(H_t, n_{t-1}, \tilde{S}_{t-1}) = \max_{\{h_t, n_t, s_t\}} V_t$$
$$= \max_{\{h_t, n_t, s_t\}} \left(\frac{(\delta\lambda')}{(n_{t-1})^{1-\varepsilon}} u(c_{2,t}) + [u(c_{1,t}) + \delta u(c_{2,t+1})] + \lambda(n_t)^{1-\varepsilon} u(c_{1,t+1}) \right)$$
$$= \max_{\{h_t, n_t, s_t\}} \left(\begin{array}{l} \frac{(\delta\lambda')}{(n_{t-1})^{1-\varepsilon}} \frac{\left[D(s_{t-1})^m (H_{t-1} + \overline{H})\right]^{1-\sigma}}{(1-\sigma)} \\ + \frac{\left[(1 - vn_t - s_t)(H_t + \overline{H}) - H_{t+1} n_t / A\right]^{1-\sigma}}{(1-\sigma)} \\ + \delta \frac{\left[D(s_t)^m (H_t + \overline{H})\right]^{1-\sigma}}{(1-\sigma)} \\ + \lambda(n_t)^{1-\varepsilon} \frac{\left[(1 - vn_{t+1} - s_{t+1})(H_{t+1} + \overline{H}) - H_{t+2} n_{t+1} / A\right]^{1-\sigma}}{(1-\sigma)} \end{array} \right) \quad (4.1)$$

s.t. $H_{t+1} = A[H_t + \overline{H}]h_t$, $\tilde{S}_{t-1} = D[H_{t-1} + \overline{H}](s_{t-1})^m$

and $h_{t+1} = h_{t+1}(H_{t+1})$, $n_{t+1} = n_{t+1}(H_{t+1})$, $s_{t+1} = s_{t+1}(H_{t+1})$

Clearly this value function can be divided into two additively separable parts:

$$\tilde{V}_t(H_t, n_{t-1}, \tilde{S}_{t-1}) = \breve{V}_t(H_t) + f(n_{t-1}, \tilde{S}_{t-1}) \quad (4.2)$$

As a consequence, the life strategies (h_t, n_t, s_t) of generation t depend at most only on its own human capital H_t, not on fertility or saving decision (n_{t-1} or $\tilde{S}_{t-1}$) of parental generation $t-1$, so that $h_t = h_t(H_t)$, $n_t = n_t(H_t)$, and $s_t = s_t(H_t)$, and the above maximization problem involves taking account of strategic effects on next generations' life strategies, $h_{t+1} = h_{t+1}(H_{t+1})$, $n_{t+1} = n_{t+1}(H_{t+1})$ and $s_{t+1} = s_{t+1}(H_{t+1})$. Then the first order conditions are:

$$s_t: \quad \begin{array}{l} \left[(1 - vn_t - h_t n_t - s_t)(H_t + \overline{H})\right]^{-\sigma} \left\{-(H_t + \overline{H})\right\} \\ + \delta\left[D(s_t)^m (H_t + \overline{H})\right]^{-\sigma} \left\{Dm(s_t)^{m-1}(H_t + \overline{H})\right\} = 0 \end{array}$$

$$
\begin{aligned}
H_{t+1}: \quad & \left[(1-vn_t-h_tn_t-s_t)(H_t+\overline{H})\right]^{-\sigma}\left\{-\frac{n_t}{A}\right\} \\
& +\lambda(n_t)^{1-\varepsilon}\left[(1-vn_{t+1}-h_{t+1}n_{t+1}-s_{t+1})(H_{t+1}+\overline{H})\right]^{-\sigma}\left\{1-vn_{t+1}-s_{t+1}\right\} \\
& +\frac{\partial V_t}{\partial s_{t+1}}\frac{\partial s_{t+1}}{\partial H_{t+1}}+\frac{\partial V_t}{\partial n_{t+1}}\frac{\partial n_{t+1}}{\partial H_{t+1}}+\frac{\partial V_t}{\partial H_{t+2}}\frac{\partial H_{t+2}}{\partial H_{t+1}}\leq 0
\end{aligned}
$$

with equality if $H_{t+1}>0$ ($h_t>0$).

$$
\begin{aligned}
n_t: \quad & \left[(1-vn_t-h_tn_t-s_t)(H_t+\overline{H})\right]^{-\sigma}\left\{-v(H_t+\overline{H})-\frac{H_{t+1}}{A}\right\} \\
& +\lambda(1-\varepsilon)(n_t)^{-\varepsilon}\left[(1-vn_{t+1}-h_{t+1}n_{t+1}-s_{t+1})(H_{t+1}+\overline{H})\right]^{1-\sigma}/(1-\sigma)\leq 0
\end{aligned}
$$

with equality if $n_t>0$.

(4.3)

Here we have $\partial V_t/\partial s_{t+1}<0$, $\partial V_t/\partial n_{t+1}<0$, $\partial V_t/\partial H_{t+2}<0$. Equilibrium conditions in stagnant/growth steady states for case A, as well as for case B, are described in Appendix 1.

Proposition 4-1: The degree of backward altruism toward its parental generation λ' does not affect the generation's own life strategies, whether it is dynamically consistent or not.

Proposition 4-2: Assume $\varepsilon=1.0$, in which the generation holds perfectly inelastic altruism with number of child. Then the fertility is always at the minimum level possible ($n_t=n_{\min}$, say), regardless of $\lambda>1$, $\lambda=1$ or $\lambda<1$.

It is noteworthy that, in a dynastic framework, fertility has an interior solution for $\lambda<1$ even with $\varepsilon=1.0$, while $\lambda>1$ is shown to may have a corner solution for fertility decision, either $n_{\min}$ or $n_{\max}$. [10] In saving but no-compensation economy, dynamic consistency/inconsistency does not matter and there do *not* exist any internalizable (positive/negative) externalities from the economic environment generated by parental

[10] This is derived from (A2.4-c) and (A2.4-e).

generation, relative population $1/n_{t-1}$, human capital H_{t-1} and saving $\widetilde{S}_{t-1}$.

Proposition 4-3: Life strategies of one generation t (n_t, h_t, s_t) depend only on its own human capital H_t. Especially, in a growth regime ($H_t >> \overline{H}$, $h_t > 1/A$), H_t (or any small shock in H_t) does not affect n_t, h_t or s_t. In other words, the economy is always in (growth) equilibrium.[11]

Now I show computation results for some parameter values, using programs for recursive dynamic computation. Set the values at $\sigma = 0.5$, $\varepsilon = 0.5$, $\delta = 0.5$, $D = 2.0$, $\nu = 0.1$, $\overline{H} = 1.0$ (Raw human capital).[12] Also hereafter, for the convenience of calibration, I limit the values of life strategies chosen, as following: [13] [14]

$$h_{\min} = 0 \le h_t \le 1.0 = h_{\max},\ s_{\min} = 0 \le s_t \le 0.5 = s_{\max},\ n_{\min} = 0.5 \le n_t \le 2 = n_{\max}$$ [15]

and $\omega_{\min} = -0.5 \le \omega_t \le 0.5 = \omega_{\max}$ (only in case B and C) (4.4)

In order to compare the attained welfare for different λ's, I define a normalized value function as $\widetilde{\widetilde{V}}_t(H_t) \equiv \widetilde{V}_t(H_t)/(1+\lambda)$. Assume that the generation is now at $t = 1$, and that time ends at $t = T$.[16] Table 1 plots, with $T = 3$, the transition paths of life strategies for case (i) $\lambda = 0.5$ and (ii) $\lambda = 2.0$, for some combinations of A and H_1, (a) $A = 5$ and H_1

---

[11] Therefore $\partial s_t / \partial H_t = 0$, $\partial n_t / \partial H_t = 0$ and $\partial h_t / \partial H_t = 0$ ($\partial H_{t+1} / \partial H_t \cong A h_t \neq 0$).

[12] $\delta D = 1.0$ implies a constant consumption habit under a regular representative agents' utility.

[13] $n_t = 1$ implies that the generation bears 2 children per couple.

[14] Throughout this paper I adopt the weighted calibration windows for H_t and $\widetilde{S}_{t-1}$, in which they are partitioned equally at the same interval for the low level around $\overline{H}$, but with smoothly broader intervals for the higher region ($H_t, \widetilde{S}_{t-1} >> \overline{H}$). The calibration results described in Table 1-4 are trivially different from those in the previous version, partly because of these modified window settings and partly because of some input error in the candidate values of fertility choice.

[15] In reality n_t takes some discrete values, 0, 0.5, 1.0, 1.5 and 2.0.

[16] This assumption is set in order to make the calibration possible and the economy fit well in reality. For convenience, I set $h_{T+1} = h_T$, $n_{T+1} = n_T$, $s_{T+1} = s_T$ (and $\omega_{T+1} = \omega_T$).

is moving at 0.5, 1.5 and 2.5, (b) $H_1 = 0$ and A is moving at 2.5, 5.0, 7.5 and 10.0, and (c) $H_1 = 10$ and A is moving at 1.0, 2.0, 3.0 and 4.0.

Also I search out the following critical values, important indices of transition process for another terminal period $T = 3$.

$H^{\uparrow}_{1\,A=5}$: Critical value of initial human capital H_1, which makes the economy pushed up into a growth state, when the productivity coefficient is at $A = 5$ ($A^{\downarrow}_{H_1=10} < A < A^{\uparrow}_{H_1=0}$).[17]

$A^{\uparrow}_{H_1=0}$: Critical value of productivity coefficient A, which makes the economy pushed up into a growth state, when the initial human capital is the lowest at $H_1 = 0 << \overline{H} = 1$.

$A^{\downarrow}_{H_1=10}$: Critical value of A, which makes the economy trapped into a stagnant state, when the initial human capital is relatively high at $H_1 = 10 >> \overline{H} = 1$.

Here I assume that the economy is eventually in a growth regime, when $H_{T(=3)} \geq \max\{H_1, 5\overline{H}\}$. These critical values are shown in Table 3, together those with case B. It is intuitive that more altruism toward children ((ii) $\lambda = 2 > 1$) generates a lower threshold in either of these three criteria ($H^{\uparrow}_{1\,A=5}$, $A^{\uparrow}_{H_1=0}$ and $A^{\downarrow}_{H_1=10}$). On the other hand, we observe that $A^{\downarrow}_{H_1=10} < A^{\uparrow}_{H_1=0}$, that is, the threshold level of productivity from stagnant to growth regime is higher than that for keeping growth regime, which is a typical *hysteresis* aspect of transition process.[18] [19] Also the existence of $H^{\uparrow}_{1\,A=5}$ for (i) ($\lambda = 0.5 < 1$)

---

[17] $H^{\uparrow}_{1\,A=5} = H^{\downarrow}_{1\,A=5}$. $A = 5$ is taken here, because this value is situated in an intermediate region between unconditional growth regime and unconditional stagnant regime.

[18] This hysteresis aspect can be observed also for $T = 10$. Clearly it is a more conspicuous phenomenon in this two-way altruistic framework than in (altruistic) dynasty model.

[19] One intuitive explanation is as follows. Defining total human capital $\widetilde{H}_t \equiv H_t + \overline{H}$, the effective

is one typical feature of "Malthusian trap", in which the lower initial human capital relative to raw human capital $\overline{H}$ could be a main stumbling block into a growth regime even with the same productivity coefficient. In Table 1, we see that, in less altruism toward children ((i) $\lambda = 0.5$), a regime change from stagnant to growth state occurs more drastically, as initial human capital or productivity coefficient increase, while, in more altruism toward children ((ii) $\lambda = 2 > 1$), this regime change happens more gradually, but more steadily. In (ii) with an intermediate productivity ((a) $A = 5$) and lower initial human capital, human capital investment h_t is small at first and increasing as time passes, if T is finite and small. (For example, $h_1 = 0.02$, $h_2 = 0.3$ and $h_3 = 0.74$ for $T = 3$ and $H_1 = 0.5$.) In addition, with relatively lower initial human capital and intermediate productivity coefficient in a quasi-growth regime, more altruism toward children (ii) proves to be an *accelerating* factor of growth in comparison with (i). The calibration results in Table 1 confirm that in a growth regime fertility is low, while in a stagnant regime it is high, and this aspect also holds in either Case B (Table 2) or C. As far as the attained utility are concerned, (ii) is better than (i) in value function $\widetilde{V}_t(H_t)$, but worse in normalized value function $\widetilde{\widetilde{V}}_t(H_t)$ for all the parameter settings ((a), (b) and (c)).

In a growth regime where $H_{t+1} >> \overline{H}$ and $h_{t+1} > 1/A$, it holds that $\partial s_{t+1} / \partial H_{t+1} = 0$, $\partial n_{t+1} / \partial H_{t+1} = 0$, $\partial H_{t+2} / \partial H_{t+1} \cong Ah_{t+1} \neq 0$ (Proposition 4-3), and $\partial V_t / \partial H_{t+2} = \lambda (n_t)^{1-\varepsilon} u'(c_{1,t+1}) \{-n_{t+1} / A\}$. Therefore:

return in human capital investment is $\widetilde{H}_{t+1} / \widetilde{H}_t = Ah_t + \overline{H} / (H_t + \overline{H})$. Then $\widetilde{H}_{t+1} / \widetilde{H}_t \cong Ah_t$ for $H_t >> \overline{H}$, bur $\widetilde{H}_{t+1} / \widetilde{H}_t \cong Ah_t + \overline{H} / (H_t + \overline{H}) > Ah_t$ for $H_t \cong \overline{H}$. For example, $\widetilde{H}_{t+1} / \widetilde{H}_t \cong 1$ even for $h_t = 0$, so even if productivity coefficient A is relatively high, $h_t = 0$ might be still possibly optimal in a stagnant state.

$$\frac{u'(c_{2,t+1})}{u'(c_{1,t+1})} \cong \frac{A\lambda(n_t)^{-\varepsilon}\{1 - vn_{t+1} - h_{t+1}n_{t+1} - s_{t+1}\}}{\delta Dm(s_t)^{m-1}} \tag{4.5}$$

Then effective altruism toward parents is defined as:[20]

$$\tilde{\lambda}' \equiv \frac{(n_{t-1})^{-\varepsilon}}{\delta}\frac{u'(c_{1,t})}{u'(c_{2,t})} \cong \frac{Dm(s_{t-1})^{m-1}}{A\lambda\{1 - vn_t - h_t n_t - s_t\}} \tag{4.6}$$

In case A, the economy happens to be *effectively* in dynamic consistency, only if:

$$\frac{Dm(s_{t-1})^{m-1}}{A\{1 - vn_t - h_t n_t - s_t\}} \cong 1 \tag{4.7}$$

5. Case B-Compensation but no saving economy

Next I consider the alternative case (case B hereafter), in which an old adulthood generation can make use of only compensation (old-age support) from the next generation. In this case, each generation not only keeps one intergenerational linkage with subsequent generations in its young adulthood through fertility decision and human capital investment for children, but also another in its old adulthood through implicit compensation contract, therefore there *does* exist a possibility of dynamic inconsistency. Specifically generation t solves the following problem:

[20] The definition of $\tilde{\lambda}'$, $\tilde{\lambda}' \equiv ((n_{t-1})^{-\varepsilon} u'(c_{1,t}))/(\delta u'(c_{2,t}))$, is derived from the first order condition of compensation scheme (case B). See (5.2).

$$\tilde{V}_t(H_t, n_{t-1}) = \max_{\{h_t, n_t, \omega_t\}} V_t$$

$$= \max_{\{h_t, n_t, \omega_t\}} \left(\frac{(\delta\lambda')}{(n_{t-1})^{1-\varepsilon}} u(c_{2,t}) + [u(c_{1,t}) + \delta u(c_{2,t+1})] + \lambda (n_t)^{1-\varepsilon} u(c_{1,t+1}) \right)$$

$$= \max_{\{h_t, n_t, \omega_t\}} \left(\begin{array}{l} \frac{(\delta\lambda')}{(n_{t-1})^{1-\varepsilon}} \frac{\left[\omega_t n_{t-1}(H_t + \overline{H})\right]^{1-\sigma}}{(1-\sigma)} \\ + \frac{\left[(1 - vn_t - \omega_t)(H_t + \overline{H}) - H_{t+1} n_t / A\right]^{1-\sigma}}{(1-\sigma)} \\ + \delta \frac{\left[\omega_{t+1} n_t (H_{t+1} + \overline{H})\right]^{1-\sigma}}{(1-\sigma)} \\ + \lambda (n_t)^{1-\varepsilon} \frac{\left[(1 - vn_{t+1} - \omega_{t+1})(H_{t+1} + \overline{H}) - H_{t+2} n_{t+1} / A\right]^{1-\sigma}}{(1-\sigma)} \end{array} \right) \tag{5.1}$$

s.t.

$$H_{t+1} = A[H_t + \overline{H}]h_t,\ \ h_{t+1} = h_{t+1}(H_{t+1}, n_t),\ \ n_{t+1} = n_{t+1}(H_{t+1}, n_t),\ \ \omega_{t+1} = \omega_{t+1}(H_{t+1}, n_t)$$

Also, in order to compare the attained welfare for different λ's and λ''s, I define a normalized value function as $\tilde{\tilde{V}}_t(H_t, n_{t-1}) \equiv \tilde{V}_t(H_t, n_{t-1})/(1 + \lambda + \lambda')$. Then the first order conditions are:

$$\omega_t: \quad \begin{array}{l} \frac{\delta\lambda'}{(n_{t-1})^{1-\varepsilon}} \left[\omega_t n_{t-1}(H_t + \overline{H})\right]^{-\sigma} n_{t-1}(H_t + \overline{H}) \\ - \left[(1 - vn_t - h_t n_t - \omega_t)(H_t + \overline{H})\right]^{-\sigma} (H_t + \overline{H}) = 0 \end{array}$$

$$H_{t+1}: \quad \begin{array}{l} \left[(1 - vn_t - h_t n_t - \omega_t)(H_t + \overline{H})\right]^{-\sigma} \left\{-\frac{n_t}{A}\right\} \\ + \delta \left[\omega_{t+1} n_t (H_{t+1} + \overline{H})\right]^{-\sigma} \{\omega_{t+1} n_t\} \\ + \lambda (n_t)^{1-\varepsilon} \left[(1 - vn_{t+1} - h_{t+1} n_{t+1} - \omega_{t+1})(H_{t+1} + \overline{H})\right]^{-\sigma} \{1 - vn_{t+1} - \omega_{t+1}\} \\ + \frac{\partial V_t}{\partial \omega_{t+1}} \frac{\partial \omega_{t+1}}{\partial H_{t+1}} + \frac{\partial V_t}{\partial n_{t+1}} \frac{\partial n_{t+1}}{\partial H_{t+1}} + \frac{\partial V_t}{\partial H_{t+2}} \frac{\partial H_{t+2}}{\partial H_{t+1}} \le 0 \end{array}$$

with equality if $H_{t+1} > 0$ ($h_t > 0$).

$$n_t: \quad \begin{aligned} &\left[(1-vn_t-h_tn_t-\omega_t)(H_t+\overline{H})\right]^{-\sigma}\left\{-v(H_t+\overline{H})-\frac{H_{t+1}}{A}\right\} \\ &+\delta\left[\omega_{t+1}n_t(H_{t+1}+\overline{H})\right]^{-\sigma}\left\{\omega_{t+1}(H_{t+1}+\overline{H})\right\} \\ &+\lambda(1-\varepsilon)(n_t)^{-\varepsilon}\left[(1-vn_{t+1}-h_{t+1}n_{t+1}-\omega_{t+1})(H_{t+1}+\overline{H})\right]^{1-\sigma}/(1-\sigma) \\ &+\frac{\partial V_t}{\partial \omega_{t+1}}\frac{\partial \omega_{t+1}}{\partial n_t}+\frac{\partial V_t}{\partial n_{t+1}}\frac{\partial n_{t+1}}{\partial n_t}+\frac{\partial V_t}{\partial H_{t+2}}\frac{\partial H_{t+2}}{\partial n_t}\le 0 \end{aligned}$$

$$\text{with equality if } n_t > 0. \qquad (5.2)$$

Here we have $\partial V_t / \partial \omega_{t+1} > 0, < 0, = 0$, if $\lambda\lambda' < 1$, > 1, $= 1$, respectively. Furthermore, $\partial V_t / \partial n_{t+1} < 0$ and $\partial V_t / \partial H_{t+2} < 0$. h_t, n_t and ω_t are not affected, unlike in case A, by the value of H_t, but are the functions only of n_{t-1}, if $H_t >> \overline{H}$, probably in a growth state, while H_t matters, if $H_t \cong \overline{H}$ or $H_t \le \overline{H}$, probably in a stagnant state. In the latter case, the relative value of H_t on $\overline{H}(=1)$ is a crucial determinant, in the forthcoming transition process, to overpass a stumbling block to a growth state, or to go down into a Malthusian trap. In order to enable the analysis with a clear contrast, I consider the following four combinations of coefficients of altruism toward children and parents, (λ, λ').

Case I: $(\lambda, \lambda') = (0.5, 2.0)$ ($\lambda\lambda' = 1.0$. Less altruistic toward children, but more altruistic toward parents. Dynamically consistent.)

Case II: $(\lambda, \lambda') = (0.5, 0.5)$ ($\lambda\lambda' = 0.25 < 1$. (Egoistic economy) Less altruistic toward both children and parents. Dynamically inconsistent.)

Case III: $(\lambda, \lambda') = (2.0, 0.5)$ ($\lambda\lambda' = 1.0$. Less altruistic toward children, but more altruistic toward parents. Dynamically consistent.)

Case IV: $(\lambda, \lambda') = (2.0, 2.0)$ ($\lambda\lambda' = 4 > 1$. (Altruistic economy) More altruistic toward both children and parents. Dynamically inconsistent.)

I set the same parameter values and calibration conditions, and in addition, $n_0 = 1$. The generation is now at $t = 1$, and time ends at $t = T = 10$. Table 2 plots the transition paths of life strategies for case I, II, III and IV, for the same combinations of A and H_1 as adopted in case A, (a) $A = 5$ and H_1 is moving at 0.5, 1.5 and 2.5, (b) $H_1 = 0$ and A is moving at 2.5, 5.0, 7.5 and 10.0, and (c) $H_1 = 10$ and A is moving at 1.0, 2.0, 3.0 and 4.0. In I, II, III and IV of case B of Table 3, we observe, as in case A, that $A^{\downarrow}_{H_1=10} < A^{\uparrow}_{H_1=0}$, a *hysteresis* aspect of transition process, and the existence of $H^{\uparrow}_{1\,A=5} (> 0)$ for $\lambda < 1$, one feature of "Malthusian trap". Other features are summarized as follows. (1) More altruism toward children (III or IV, $\lambda > 1$) generates a lower threshold of initial human capital from stagnant to growth state, than less altruism toward children (I or II, $\lambda < 1$). (2) In $\lambda < 1$ (I or II), dynamic consistency (I) generates a lower threshold of initial human capital, but a higher threshold of productivity from stagnant to growth state, than dynamic inconsistency & egoism (II) (3) dynamic consistency (I or III) generates a higher threshold of productivity for keeping growth state, than dynamic inconsistency (II or IV), (4) Less altruism toward parents (II or III, $\lambda' < 1$) generates a higher threshold of productivity, from stagnant to growth state, than more altruism toward parents (I or IV, $\lambda' > 1$). From Table 2, it is not so straightforward to conclude which combination (I, II, III or Iv) is better or worse, but roughly speaking, the following features are summarized: (1) Dynamic consistency (I or III, $\lambda\lambda' = 1$) does not necessarily attain a better growth path than inconsistent case in every parameter setting of initial human capital H_1 and productivity A. (2) The egoistic economy (II, $\lambda = \lambda' = 0.5$) exhibits, in comparison with case I (dynamically consistent, $\lambda = 0.5, \lambda' = 2.0$), a drastic regime change from stagnant to growth state as initial human capital or productivity are increasing, while, in the altruistic economy (IV, $\lambda = \lambda' = 2$), this regime change happens very gradually and steadily. (3) In more altruism toward children (III or IV, $\lambda = 2 > 1$) under a quasi-growth regime, especially in the altruistic economy (IV), human capital investment h_t is non zero but relatively low at first ($t = 1$), and gradually increasing as the terminal period T comes near.

From some of the above results:

$$H^{\uparrow}_{1\,A=5}(I) > H^{\uparrow}_{1\,A=5}(III),\ \ A^{\uparrow}_{H_1=0}(I) > A^{\uparrow}_{H_1=0}(III),\ \ A^{\downarrow}_{H_1=10}(I) < A^{\downarrow}_{H_1=10}(III) \quad \text{(Case B)}$$

Note that both cases, (I) and (III), are dynamically consistent $(\lambda\lambda' = 1)$. From these results I claim the following proposition.

Proposition 5-1: In case B with dynamic consistency $(\lambda\lambda' = 1)$, the economy less altruistic toward children (I) holds the higher thresholds for initial human capital or productivity from stagnant to growth state, than the more altruistic economy (III), but a lower threshold productivity from growth to stagnant state.

The implication of this proposition is that, for the pure purpose of *eventually arriving at or maintaining a growth regime*, the altruistic attitude toward children is preferred in an initially stagnant economy $(H_1 < \overline{H})$, but the egoistic attitude is rather preferred in an initially in-growth economy $(H_1 >> \overline{H})$.

Effect of initial fertility level n_0 and initial human capital H_1

The Effect of initial fertility level n_0 depends on the values of σ and ε. If $\varepsilon > \sigma$, lower initial fertility (small n_0) induces lower compensation rate ω_1, and through a positive income effect induces larger investment for human capital and fertility (h_1 and n_1), being an accelerating factor of growth, while larger initial fertility might be an impediment against growth. And vice versa if $\varepsilon < \sigma$. Instead, if $\varepsilon = \sigma$, then any shock in n_0 is neutralized (canceled off) and does not affect the life strategies of generation 1 (h_1, n_1 and ω_1). As for the effect of initial human capital H_1, the same property holds, in which it does not affect the subsequent life strategies under a growth regime. Summarizing the above, I have the following proposition as an analogy of proposition 4-3.

Proposition 5-2:

If $\varepsilon = \sigma$, then $\partial\omega_t / \partial n_{t-1} = 0$, $\partial n_t / \partial n_{t-1} = 0$ and $\partial h_t / \partial n_{t-1} = 0$ ($\partial H_{t+1} / \partial n_{t-1} = 0$). If $H_t >> \overline{H}$ and $h_t > 1/A$, then $\partial\omega_t / \partial H_t = 0$, $\partial n_t / \partial H_t = 0$ and $\partial h_t / \partial H_t = 0$ ($\partial H_{t+1} / \partial H_t \cong Ah_t \neq 0$). The economy is always in a steady state if $\varepsilon = \sigma$ (Relative risk aversion coefficient is equal to elasticity of altruism per child.) and $H_t >> \overline{H}$ (a growth regime).

Effect of dynamic inconsistency ($\lambda\lambda' \neq 1$)

As a matter of fact, dynamic inconsistency is not any problem as far as generation t predicts precisely the reaction functions of next generation, $h_{t+1} = h_{t+1}(H_{t+1}, n_t)$, $n_{t+1} = n_{t+1}(H_{t+1}, n_t)$ and $\omega_{t+1} = \omega_{t+1}(H_{t+1}, n_t)$, and the *ex-ante* efficiency is kept in the sense that the generation maximizes its own two-way altruistic utility, given initial human capital H_t, initial fertility n_{t-1}, and the above strategic effects on next generation. However, some points are still to be noted. At first, as seen in (5.2), the first order condition of (5.1) in fertility is done for n_t, not for N_{t+1}. This implies that (5.2) does not take account of the effect of its fertility decision on the welfare after $t+1$. Next, see that (5.1) is equivalent with the following problem:

$$\tilde{V}_t(H_t, n_{t-1})$$
$$= \max_{\{h_t, n_t, \omega_t\}} \begin{pmatrix} \hat{\lambda}'(n_{t-1})u_t^{(o)} + u_t^{(y)} + (1-\lambda\lambda')\delta\pi_2 u_{t+1}^{(o)} \\ + \hat{\lambda}(n_t)\left\{\tilde{V}_{t+1}(H_{t+1}, n_t) - \delta\pi_2 u_{t+2}^{(o)} - \hat{\lambda}(n_{t+1})u_{t+2}^{(y)}\right\} \end{pmatrix} \quad (5.1a)$$

s.t. $h_{t'} = h_{t'}(H_{t'}, n_{t'-1})$, $n_{t'} = n_{t'}(H_{t'}, n_{t'-1})$, $\omega_{t'} = \omega_{t'}(H_{t'}, n_{t'-1})$, $H_{t'+1} = A[H_{t'} + \overline{H}]h_{t'}$

$(t' = t+1, t+2)$

This is a Bellman-equation-like-version of (5.1). This form involves the utility parts (parents/children) and reaction functions until period $t+2$. So the internalization of the first order conditions within this recursive computation framework might be possibly incomplete whether the economy is in a steady state (or is in a transition process) or not, and this aspect causes somewhat inefficiency. On the other hand, from (5.1-a), it is seen how altruism toward parents λ' affects value function $\tilde{V}_t(H_t, n_{t-1})$, in which more λ' contributes in the utility part $u_t^{(o)}$, but hurts the part $u_{t+1}^{(o)}$. As far as the above calibration results ((a) , (b) and (c) for Case I to IV) are concerned, more altruism either toward parents (larger λ') or toward children (larger λ) increases $\tilde{V}_t(H_t, n_{t-1})$, but decreases normalized value function $\tilde{\tilde{V}}_t(H_t, n_{t-1})$.[21] Between I and III (both $\lambda\lambda' = 1$), III ($(\lambda, \lambda') = (2.0, 0.5)$) is better than I ($(\lambda, \lambda') = (0.5, 2.0)$) both in $\tilde{V}_t(H_t, n_{t-1})$ and $\tilde{\tilde{V}}_t(H_t, n_{t-1})$.

6. Specification and implications of intergenerational linkage- miscellaneous issues

Case-C: Both saving and compensation economy

In Figure 1 I show the specification of intergenerational linkage in Case C-Both saving and compensation economy, as well as Case A and B. In this section, I just write down

[21] $\tilde{\tilde{V}}_t(H_t, n_{t-1}) \equiv \tilde{V}_t(H_t, n_{t-1}) / (1 + \lambda + \lambda')$

a few comments about case C. In case C, generation t solves the following problem:

$$
\begin{aligned}
&\widetilde{V}_t(H_t, n_{t-1}, \widetilde{S}_{t-1}) = \max_{\{h_t, n_t, \omega_t, s_t\}} V_t \\
&= \max_{\{h_t, n_t, \omega_t, s_t\}} \left(\frac{(\delta\lambda')}{(n_{t-1})^{1-\varepsilon}} u(c_{2,t}) + [u(c_{1,t}) + \delta u(c_{2,t+1})] + \lambda (n_t)^{1-\varepsilon} u(c_{1,t+1}) \right) \\
&= \max_{\{h_t, n_t, \omega_t, s_t\}} \left(\begin{aligned}
&\frac{(\delta\lambda')}{(n_{t-1})^{1-\varepsilon}} \frac{\left[\omega_t n_{t-1}(H_t + \overline{H}) + \widetilde{S}_{t-1}\right]^{1-\sigma}}{(1-\sigma)} \\
&+ \frac{\left[(1 - v n_t - s_t - \omega_t)(H_t + \overline{H}) - H_{t+1} n_t / A\right]^{1-\sigma}}{(1-\sigma)} \\
&+ \delta \frac{\left[\omega_{t+1} n_t (H_{t+1} + \overline{H}) + \widetilde{S}_t\right]^{1-\sigma}}{(1-\sigma)} \\
&+ \lambda (n_t)^{1-\varepsilon} \frac{\left[(1 - v n_{t+1} - s_{t+1} - \omega_{t+1})(H_{t+1} + \overline{H}) - H_{t+2} n_{t+1} / A\right]^{1-\sigma}}{(1-\sigma)}
\end{aligned} \right)
\end{aligned} \tag{6.1}
$$

s.t. $\quad H_{t+1} = A[H_t + \overline{H}]h_t,\ \widetilde{S}_t = D[H_t + \overline{H}](s_t)^m,\ h_{t+1} = h_{t+1}(H_{t+1}, n_t, \widetilde{S}_t)$,

$$n_{t+1} = n_{t+1}(H_{t+1}, n_t, \widetilde{S}_t),\ \omega_{t+1} = \omega_{t+1}(H_{t+1}, n_t, \widetilde{S}_t),\ s_{t+1} = s_{t+1}(H_{t+1}, n_t, \widetilde{S}_t)$$

Here state variables are $H_t, n_{t-1}, \widetilde{S}_{t-1}$ and control variables are n_t, h_t, ω_t, s_t. First, in this case, the determination of saving rate s_t crucially depends on altruism toward parents λ', as well as altruism toward children λ, time preference δ, interest rate D, and other parameters like σ, ε, v, $\overline{H}$. This is a different point from Case A, in which the old age utility of parents and the middle age utility of children are not directly altruistically linked.

One important aspect of case C is that compensation ω_t might not necessarily be positive but negative (i.e., a positive bequest) even if λ' is positive, under the existence of a high parental saving $\widetilde{S}_{t-1}$. This tendency is strengthened especially when the old-age mortality risk exists and the intergenerational transfer involves a corresponding risk sharing between the two adjacent generations (old and young adulthood). Reversely speaking, the negative expected (mean) compensation does not necessarily imply the non-existence of positive altruism. On the other hand, the absence of well-functioning risk sharing of old-age life uncertainty with social/market insurance induces a precautionary saving in middle age saving $\widetilde{S}_{t-1}$, because of the necessity of self-insurance. Therefore it is a delicate task to estimate econometrically λ' with a potential mixture of social/market insurance under continuous time mortality risk.[22]

If $\widetilde{\lambda}'$ is far different from λ', then clearly s_t deviates from in case A.[23] [24] Next, the direct linkage between parents' old age and children's middle age through setting λ' enables the internalization of the externality from unexpected change in parents' saving to some extent, and might lead to more efficiency than in case A, as far as some of these intergenerational linkage are not binding. However, this kind of internalization might still end in insufficiency, exactly because of the same reason stated in section 5 (case B). Thus an empirical investigation of (λ, λ') with other parameters is expected to offer useful information for the construction of well balanced funded/unfunded social security as well as for the robustness against a drastic transition process.

Welfare analysis-What does "efficiency" imply in this dynamic programming?

As already stated, each generation, who solves problems (4.1) for case A, (5.1) for case B, and (6.1) for case C, behaves "efficiently" as far as it precisely predicts the reaction functions of subsequent generations, whether the life strategies are in a steady state or in a transition

[22] In this regard, Kohara and Ohtake (2006) state that the parental care supplied by Japanese middle age is not motivated entirely by altruism. This, if true, implies that, as far as home health care service for frail or ill parents is concerned, a competitive bargaining between the two generation rather than altruism toward old parents prevails, and as a consequence, the weight of altruism λ' does not matter. Even in this case, the estimated λ', substantially replaced by $\widetilde{\lambda}'$, may not be zero but a positive value.

[23] If in case C $\widetilde{\lambda}' = \lambda'$ by chance, then $\omega_t = 0$.

[24] For example, a drastically less altruism toward parents (small λ') causes precautionary saving, especially under the old-age mortality risk, because of the necessity of self-insurance.

process. (*Ex-ante optimality of generation* t *as of period* t *in terms of two-way altruistic utility*) However, since the objective function involves the utilities of two adjacent generations over just two corresponding periods, the following two welfare criteria could be considered.

Criterion 1 (case C)-Ex-post Pareto optimality between generation t and t+1 as of period t+2:

Assume that the period t state variables $H_t, n_{t-1}, \widetilde{S}_{t-1}$ and the period $t+2$ state variables $H_{t+2}, n_{t+1}, \widetilde{S}_{t+1}$ are exogenously given, and that the period $t+2$ state variables $H_{t+2}, n_{t+1}, \widetilde{S}_{t+1}$ and life strategies $\{n_{t'}, h_{t'}, \omega_{t'}, s_{t'}\}_{t'=t}^{+\infty}$ solve the problem (5.1), given the period t state variables $H_t, n_{t-1}, \widetilde{S}_{t-1}$. Then is the allocation, which is determined between generation t and $t+1$, as generation t control variables $\{n_t, h_t, s_t\}$ and $t+1$ control variables $\{\omega_{t+1}\}$, Pareto-efficient for generation t's per-capita utility (u) and $t+1$'s altruism-adjusted aggregate utility ($\ddot{u}_{t+1} \equiv (n_t)^{1-\varepsilon} u_{t+1}$)? [25] [26] [27]

Criterion 2 (case C)-Ex-post efficiency of generation t as of period t+1:

Assume that the period t state variables $H_t, n_{t-1}, \widetilde{S}_{t-1}$ are exogenously given, and that the period $t+1$ state variables $H_{t+1}, n_t, \widetilde{S}_t$ and life strategies $\{n_{t'}, h_{t'}, \omega_{t'}, s_{t'}\}_{t'=t}^{+\infty}$ solve the problem (5.1), given the period t state variables $H_t, n_{t-1}, \widetilde{S}_{t-1}$. Then is the allocation, which

---

[25] $u_t \equiv u_t^{(c)} + \delta\pi_2 u_{t+1}^{(c)}$ is the *per-capita* whole-life utility of generation t. The *altruism-adjusted aggregate* utility, $\ddot{u}_{t+1} \equiv (n_t)^{1-\varepsilon} u_{t+1}$, could be replaced with a per-capita whole-life utility u_{t+1} or a simple *aggregate* utility $\ddot{u}_{t+1} \equiv n_t u_{t+1}$.

[26] Clearly $\ddot{u}_{t+1} \equiv (n_t)^{1-\varepsilon} u_{t+1}$ is increasing and concave with respect to n_t. In case B, u_t is concave with respect to n_t, increasing for $n_t < \hat{n}$, but decreasing for $\hat{n} < n_t$.

[27] In case A, state variables are H_t and control variables are n_t, h_t, s_t. In case B, state variables are H_t, n_{t-1} and control variables are n_t, h_t, ω_t.

generation t implements as control variables $\{n_t, h_t, \omega_t, s_t\}$, the best for generation t's per-capita two-way altruistic utility (V_t), keeping $t+1$'s altruism-adjusted aggregate two-way altruistic utility ($\ddot{V}_{t+1} = (n_t)^{1-\varepsilon} \widetilde{V}_{t+1}(H_{t+1}, n_t, \widetilde{S}_t)$) constant?

As a matter of fact, criterion 1 is the static analysis on an *ex-post* Pareto efficiency criterion as of period $t+2$, in terms of the individual whole life utilities of two adjacent generations t and $t+1$. Then it is possible to consider some virtual and symmetric bargaining between u_t and $\ddot{u}_{t+1}$ in the ex-post context as of period $t+2$, as if fertility n_t and human capital investment h_t (and compensation ω_t) are traded commodities. In general, these recursive dynamic equations ((4.1), (5.1) and (6.1)) do not succeed in internalizing the marginal effects of generation $t+1$'s human capital H_{t+1} on $t+2$'s human capital H_{t+2}, within generation $t+1$'s two-way altruistic utility V_{t+1}, therefore an envelope theorem $d\widetilde{V}_{t+1}(H_{t+1}, \cdots)/dH_{t+1} = \partial V_{t+1}/\partial H_{t+1}$ does not hold. In addition, since we have $\partial H_{t+2}/\partial H_{t+1} \cong Ah_{t+1} \neq 0$ even under a growth regime, the failure in this envelope theorem could be a potential cause for the ex-post distortion (misallocation) in $\{n_t, h_t, \ldots\}$ between generation t and $t+1$. As a consequence, the next equations for some positive coefficient κ might *not* be satisfied for equating the marginal rate of substitution between u_t and $\ddot{u}_{t+1}$ among traded commodities, n_t, $H_{t+1}(h_t)$ (and ω_{t+1}):

$\partial u_t/\partial n_t + \kappa_n \partial \ddot{u}_{t+1}/\partial n_t = 0$, $\partial u_t/\partial H_{t+1} + \kappa_h \partial \ddot{u}_{t+1}/\partial H_{t+1} = 0$,

(and $\partial u_t/\partial \omega_{t+1} + \kappa_\omega \partial \ddot{u}_{t+1}/\partial \omega_{t+1} = 0$ in case B and C)

and $\kappa = \kappa_n = \kappa_h (= \kappa_\omega)$ (6.2)

In case A, other state variables, $n_t, \widetilde{S}_t$, do not affect the life strategies $\{n_{t+1}, h_{t+1}, s_{t+1}\}$, so the corresponding envelope theorems hold.[28] Then some distortionary effect in ex-post sense

[28] That is, $d\widetilde{V}_{t+1}(H_{t+1}, n_t, \widetilde{S}_t)/dn_t = \partial V_{t+1}/\partial n_t$ and $d\widetilde{V}_{t+1}(H_{t+1}, n_t, \widetilde{S}_t)/d\widetilde{S}_t = \partial V_{t+1}/\partial \widetilde{S}_t$.

essentially remains even under a growth regime, where $\kappa_n = \lambda > \kappa_h$.[29] In case B, under again a growth regime, $1/\lambda' = \kappa_\eta$ and $\lambda > \kappa_h$. So setting $\lambda\lambda' > 1$ ($\kappa_n \to \kappa_h$) enables the marginal rates of substitution between η_t and h_t to be closer across the two generations. Also we have $\partial\omega_{t+1}/\partial n_t >,=,< 0$, if $\varepsilon >,=,< \sigma$, respectively, and assume that $\partial n_{t+1}/\partial n_t$ and $\partial H_{t+2}/\partial n_t$ are negligibly small. Then the marginal rates of substitution between n_t and h_t are more closely equated, if $\lambda\lambda' > 1$ for $\varepsilon < \sigma$ or $\lambda\lambda' < 1$ for $\varepsilon > \sigma$. In case C, the general inefficiency in terms of the above criterion arises from one simple aspect that the old-age utility of parental generation, $u_t^{(o)}(= u(c_{2,t}))$, is not additively separable in terms of saving part $\widetilde{S}_{t-1} = D[H_{t-1} + \overline{H}](s_{t-1})^m$ and compensation part $\omega_t n_{t-1}(H_t + \overline{H})$, hence other state variables, $n_{t-1}, \widetilde{S}_{t-1}$, as well as H_t, *do* affect the life strategies $\{n_t, h_t, \omega_t, s_t\}$. Briefly one important implication of the above analysis is that:

Either in case A, B or C, or either in a growth or stagnant regime, dynamic consistency ($\lambda\lambda' = 1$) might not necessarily ensure the ex-post Pareto efficiency.

In spite of a perfect foresight with perfect certainty assumed in these economies, the ex-post efficiency (as of period $t+2$) is not achieved in general except for special cases, while the ex-ante efficiency (as of period t) always holds. As a matter of course, the above virtual thought experiment, using ex-post Pareto optimality criterion, contains some essential limitations that such symmetric bargaining between generation t and $t+1$ (in infancy) is actually unrealizable, and that searching for Pareto improving parameters including $\lambda, \lambda', \varepsilon, \sigma$ itself necessarily moves the values of the period $t+2$ state variables $H_{t+2}, n_{t+1}, \widetilde{S}_{t+1}$, which are assumed to be exogenously given.

---

[29] Under a growth regime (case A), the ex-post optimal allocation $\{n_{opt}, h_{opt}\}$ proves to satisfy $n_{opt} < n_t$ and $h_{opt} > h_t$, from $(\partial V_t/\partial H_{t+2})(\partial H_{t+2}/\partial H_{t+1}) < 0$.

Other miscellaneous issues regarding two-way altruism are now stated.

Does dynamic inconsistency imply dynamic inefficiency? Is it actually costly?

Not necessarily. In case A, the inconsistency does not matter. In case B, I showed that the combination of altruism toward parents and children (λ, λ') is crucial for determining a threshold level of initial human capital and productivity in a transition process (stagnant to growth or growth to stagnant), but dynamic consistency (I and III, $(\lambda\lambda' = 1)$) might not necessarily attain the lowest threshold into a growth regime, or the largest growth rate, or the best two-way altruistic utility of the generation. Rather, in the presented results, dynamic consistency holds a higher threshold of productivity for keeping a growth regime than inconsistent case (II or IV). In an egoistic economy (II, $(\lambda = 0.5, \lambda' = 0.5)$), once growth happens, then it attains similar or larger investment on human capital, as compared with a consistent economy (I, $(\lambda = 0.5, \lambda' = 2.0)$). With more altruism toward parents (III or IV, $(\lambda = 2)$), even under low productivity, the transition path toward growth is "patient", in the sense that the human capital investment h_t is slowly increasing as time passes. In terms of the attained utility, the value function increases with more altruism either toward parents or children, but the normalized value function decreases.

Formulation of altruism, vertical vs. horizontal altruism

In this article, altruism is defined as a linearly-weighted and additively separable utility, which is an implicit restriction, exhibiting a decreasing return to scale of economy. This assumption is also the object for future econometric test. Theoretically, at least one-sided altruism (forward or backward) is essential and sufficient for the derivation of life strategies in a typical overlapping generation model, if dynamic consistency is implicitly assumed. Further positive implication of altruism can be found, for example, where there exists some positive production externality in a small community (firms, families, neighborhood or etc.) and this externality could be internalized, to some extent, within the altruistic utility framework. In this situation, especially in "horizontal" altruism, a member is obliged to take care of the marginal effects of other members' human/physical capital accumulation, and consequently the price of capital becomes closer to a socially optimal level and for some condition the economy might be shifted from stagnant to growth regime.[30] Here one point, which should be paid careful attention for, is that the existence of consumption externality is not a sufficient condition for enabling this mechanism, but the way to control the consumption of other members (at intra-family, intra-firm or intra-community) through some investment/distributive channels needs to be precisely specified.[31] The model of this article, with two-way vertical altruism, satisfies it between two neighbor generations (young/old adulthood) within one identical family, while another important factor, specification of old-age capital investment & production, is missing here. Therefore, the additional incorporation of old-age working activities is essential to examine the degree of intergenerational economic synergies caused by the internalization of intergenerational production externalities, and is left for further work.[32]

Fluctuation and patience

Nishimura and Benhabib (1993, 1989) analyze the mechanism for endogenous fertility fluctuation in a Barro-Becker model. Their results basically apply also to the

[30] This is clear, for example, from the explanation by Obstfeld and Rogoff (1996) (section 7.3.1), in which the internalization of production externality under the AK model is described.

[31] In other words, without this specification the inter/intra-generational consumption externality remains to be just an externality.

[32] As a matter of fact, this kind of intra-family and inter-generational (old-middle age) interaction should not be over-valued, simply because it is rather a rare case that children inherit the same (or similar) profession as parents', and this interaction might be limited to household activities, for example, bringing up grandchildren.

two-way altruistic model in this article, even for the case of adjusted-fertility neutralization, $\varepsilon = \sigma$. The phenomenon of "patience" appears in a transition process with a relatively low (mediocre but not too low) productivity A and initial human capital H_1, and a high λ, when the time horizon is finite and the reaction (policy) functions gradually change as the expectation that the day the next generation will be sure to take the same life strategies is coming soon. See, for example, Lengwiler (2005), Kim, Kim and Levin (2003) or Gong (2006).

Construction of old-age pension scheme

The old-age pension scheme can be represented by a set (s_t^I, ω_t^I), where $s_t = s_t^P + s_t^I$ and $\omega_t = \omega_t^P + \omega_t^I$. Here superscript P and I denote a private, and a social/market insurance part respectively, therefore s_t^I, ω_t^I and ω_t^P represent funded pension, unfunded pension and intra-family intergenerational transfer, respectively. ω_t^I and ω_t^P can be divided into two parts, premium (certainty) part $\overline{\omega}_t^I$ and $\overline{\omega}_t^P$, and actuarially fair risk sharing (insurance) part $\widetilde{\omega}_t^I$ and $\widetilde{\omega}_t^P$ ($\omega_t^I = \overline{\omega}_t^I + \widetilde{\omega}_t^I$ and $\omega_t^P = \overline{\omega}_t^P + \widetilde{\omega}_t^P$).[33] As is clear from the discussion so far, (s_t^I, ω_t^I) and (s_t^P, ω_t^P) do depend on the combination of altruism (λ, λ'), as well as the externalities from parental generation, $n_{t-1}, \widetilde{S}_{t-1}$.[34]

7. Comparison with dynastic framework

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[33] $E\widetilde{\omega}_t^I = E\widetilde{\omega}_t^P = 0$.

[34] Iwamoto (2006) suggests the importance of the social/market compensation scheme in its insurance part $\widetilde{\omega}_t^I$, as well as in its premium part $\overline{\omega}_t^I$.

The definitions, equilibrium conditions and corresponding properties of a simple dynastic utility maximizer are summarized in Appendix 2. Here I consider *the backward induction of a dynastic utility incorporating intergenerational linkage with parents (backward altruism)*. A simple dynastic utility is defined as:

$$
\begin{aligned}
\dot{V}_t &\equiv u_t + \hat{\lambda}(n_t)u_{t+1} + \hat{\lambda}(n_t)\hat{\lambda}(n_{t+1})u_{t+2} + \cdots \\
&= u_t + \lambda\pi_1 a(n_t)n_t u_{t+1} + \lambda^2 {\pi_1}^2 a(n_t)n_t a(n_{t+1})n_{t+1}u_{t+2} + \cdots
\end{aligned}
\tag{7.1}
$$

Compensation rate ω_t is determined at period $t-1$ as an implicit contract between the generation (vintage t) and its parental generation (vintage $t-1$). In order to internalize this intergenerational linkage with parental generation, define a "two-way-altruistic" associated dynastic utility:

$$
\begin{aligned}
\hat{V}_t &\equiv \hat{\lambda}'(n_{t-1})u_t^{(o)} + u_t + \lambda\pi_1 a(n_t)n_t u_{t+1} + \lambda^2 {\pi_1}^2 a(n_t)n_t a(n_{t+1})n_{t+1}u_{t+2} + \cdots \\
&= \hat{\lambda}'(n_{t-1})u_t^{(o)} + \dot{V}_t \\
&= \hat{\lambda}'(n_{t-1})u_t^{(o)} + u_t + \lambda\pi_1 a(n_t)n_t \dot{V}_{t+1} \\
&= \hat{\lambda}'(n_{t-1})u_t^{(o)} + u_t^{(y)} + (1-\lambda\lambda')\delta\pi_2 u_{t+1}^{(o)} + \lambda\pi_1 a(n_t)n_t \hat{V}_{t+1}
\end{aligned}
\tag{7.2}
$$

where $\hat{\lambda}'(n_{t-1}) \equiv \dfrac{\lambda'\delta\pi_2}{\pi_1 a(n_{t-1})n_{t-1}}$ is the degree of altruism toward parent. Assuming that the generation holds a bargaining power for determining compensation rate ω_t, and determines its life strategies in a recursive and backward induction, the value function, in which ω_t is internalized within altruistic dynastic utility $\hat{V}_t$, becomes:

$$
V_t^*(H_t, n_{t-1}, \widetilde{S}_{t-1}) = \max_{\{h_t, n_t, \omega_t, s_t\}} \begin{pmatrix} \hat{\lambda}'(n_{t-1})u_t^{(o)} + u_t^{(y)} + (1-\lambda\lambda')\delta\pi_2 u_{t+1}^{(o)} \\ + \lambda\pi_1 a(n_t)n_t V_{t+1}^*(H_{t+1}, n_t, \widetilde{S}_t) \end{pmatrix}
\tag{7.3}
$$

Here the objective function is $\hat{V}_t$. In (7.3), $H_t, n_{t-1}, \widetilde{S}_{t-1}$ are state variables, and h_t, n_t, ω_t, s_t are control variables. Of course, generation t takes account of the strategic effects of its own decision on subsequent generations.

Proposition 7-1: Assume that the economy is *not* in steady state. Then, (7.3) are not sufficient for internalizing the equilibrium conditions for H_{t+1} (A2.3) and N_{t+1} (A2.4). In addition, assume $D = 0$ (no saving incentive). Then (A2.3) *could* be internalized.

Proposition 7-2: In problem (7.3), dynamic inconsistency ($\lambda\lambda' \neq 1$) implies dynamic inefficiency in terms of dynastic utility.

Proposition 7-3: Assume that the economy is both in steady state ($\{h_t, n_t, \omega_t, s_t\} = \{h, n, \omega, s\}$), and in dynamic consistency ($\lambda\lambda' = 1$). Then the dynamically efficient solution in terms of dynastic utility, which satisfies (A2.2-5), solves problem (7.3).

In case that the economy is *not* in steady state ($\{h_t, n_t, \omega_t, s_t\} \neq \{h, n, \omega, s\}$), (7.3) does not necessarily ensure, even under dynamic consistency, that equilibrium conditions, especially those for H_{t+1} and N_{t+1}, are automatically satisfied, because they range, as seen in (A2.3) and (A2.4), over three periods, therefore the envelope theorem, as expected in a representative agent's utility, does not hold.

Consider the following (altruistic) dynastic utility maximizer:

$$\{h'_{t'}, n'_{t'}, \omega'_{t'}, s'_{t'}\}_{t'=t}^{\infty} = \underset{\{h_{t'}, n_{t'}, \omega_{t'}, s_{t'}\}_{t'=t}^{\infty}}{\arg\max} \hat{V}_t$$

$$\{h''_{t'}, n''_{t'}, \omega''_{t'}, s''_{t'}\}_{t'=t+1}^{\infty} = \underset{\{h_{t'}, n_{t'}, \omega_{t'}, s_{t'}\}_{t'=t+1}^{\infty}}{\arg\max} \hat{V}_{t+1}$$

$$\text{where } H'_{t+1} = A(H_t + \overline{H})h'_t \text{ and } \widetilde{S}'_t \equiv D(H_t + \overline{H})(s'_t)^m$$

$\{h'_{t'}, n'_{t'}, \omega'_{t'}, s'_{t'}\}_{t'=t}^{\infty}$ is a set of subsequent life strategies, which maximizes the altruistic dynastic utility of generation t on the condition of current environment, $H_t, n_{t-1}, \widetilde{S}_{t-1}$, and $\{h''_{t'}, n''_{t'}, \omega''_{t'}, s''_{t'}\}_{t'=t+1}^{\infty}$ is a set of subsequent life strategies, which maximizes the altruistic dynastic utility of generation $t+1$ on the condition of next period environment, $H'_{t+1}, n'_t, \widetilde{S}'_t$. Although, under dynamic consistency $(\lambda\lambda' = 1)$, $\{h'_{t'}, n'_{t'}, \omega'_{t'}, s'_{t'}\}_{t'=t}^{\infty}$ satisfies the dynastic equilibrium conditions (A2.2-5), in general, $\{h'_t, n'_t, \omega'_t, s'_t\}$ does not necessarily coincide with the solution of (7.3). Also, there is *no* guarantee, in general, that next relations are automatically satisfied:

$$\{h'_{t'}, n'_{t'}, \omega'_{t'}, s'_{t'}\}_{t'=t+1}^{\infty} = \{h''_{t'}, n''_{t'}, \omega''_{t'}, s''_{t'}\}_{t'=t+1}^{\infty}$$

or $$\lim_{t'\to\infty}\{h'_{t'}, n'_{t'}, \omega'_{t'}, s'_{t'}\} = \lim_{t'\to\infty}\{h''_{t'}, n''_{t'}, \omega''_{t'}, s''_{t'}\}$$

This implies that, in case that parental life strategies remarkably deviate from a steady state (that is, $\{h_t, n_t, \omega_t, s_t\} = \{h+\Delta h, n+\Delta n, \omega+\Delta\omega, s+\Delta s\}$), the next generation might not necessarily trace the same dynamic path as their parents expected, wherein there might still be a cause of some *ex-ante* dynamic inefficiency and perturbation.

Lastly I come back again to the recursive induction problem of altruistic dynastic utility (7.3). I compare the transition paths of h_t, n_t and ω_t under (7.3) with those in Case B of two-way altruistic utility, using the same set of parameters as (b) ($H_1 = 0$ and A is moving at 2.5, 5.0 and 7.5). See Table 4. One distinct feature in dynastic utility approach, from two-way altruistic utility, is that even in IV (more altruism both toward parents and children) compensation rate ω_t is relatively low, because the larger altruism toward children λ, working as a multiplier and prevailing over the one toward parents λ', is a major determinant for a long-run compensation habit.[35] Other remarkable characteristics are as follows: (1) A threshold of productivity from stagnant to growth regime is lower than in two-way altruistic approach. (2) Under more altruism toward children ($\lambda = 2 > 1$, III and IV), human capital investment h_t is relatively high from the very beginning ($t = 1$), even with low productivity ($A = 2.5$). (3) Under less altruism toward children ($\lambda = 0.5 < 1$, I and II), compensation rate ω_t drastically decreases as productivity A increases, while, under

[35] In two-way altruistic approach, λ' is rather a major determinant factor for the compensation rate ω_t.

more altruism toward children ($\lambda = 2 > 1$, III and IV), ω_t is kept negligibly small.

8. Comparative statistics of equilibrium paths

Consider case B (compensation but no saving economy). The first order condition (5.2) can be compactly rewritten in the following equations.

$$\omega_t: \quad \frac{\partial V_t}{\partial \omega_t} \equiv V_t^{\omega_t} = 0$$

$$H_{t+1}: \quad \frac{\partial V_t}{\partial H_{t+1}} + \frac{\partial V_t}{\partial \omega_{t+1}}\frac{\partial \omega_{t+1}}{\partial H_{t+1}} + \frac{\partial V_t}{\partial n_{t+1}}\frac{\partial n_{t+1}}{\partial H_{t+1}} + \frac{\partial V_t}{\partial H_{t+2}}\frac{\partial H_{t+2}}{\partial H_{t+1}} \equiv V_t^{H_{t+1}} \le 0$$

with equality if $H_{t+1} > 0$ ($h_t > 0$).

$$n_t: \quad \frac{\partial V_t}{\partial n_t} + \frac{\partial V_t}{\partial \omega_{t+1}}\frac{\partial \omega_{t+1}}{\partial n_t} + \frac{\partial V_t}{\partial n_{t+1}}\frac{\partial n_{t+1}}{\partial n_t} + \frac{\partial V_t}{\partial H_{t+2}}\frac{\partial H_{t+2}}{\partial n_t} \equiv V_t^{n_t} \le 0 \qquad (5.2a)$$

with equality if $n_t > 0$.

Assume that the second and third equations in (5.2a) hold with equality, that is, the non-negativity constraints in human capital and fertility are not binding. With a slight change in some parameter represented as B_t, the first order total differentiation of (5.2a) becomes:

$$dV_t^{\omega_t} = \frac{\partial V_t^{\omega_t}}{\partial \omega_t} d\omega_t + \frac{\partial V_t^{\eta_t}}{\partial H_{t+1}} A[H_t + \overline{H}]dh_t + \frac{\partial V_t^{\omega_t}}{\partial n_t} dn_t + \frac{\partial V_t^{\omega_t}}{\partial B_t} dB_t = 0$$

$$dV_t^{H_{t+1}} = \frac{\partial V_t^{H_{t+1}}}{\partial \omega_t} d\omega_t + \frac{\partial V_t^{H_{t+1}}}{\partial H_{t+1}} A[H_t + \overline{H}]dh_t + \frac{\partial V_t^{H_{t+1}}}{\partial n_t} dn_t + \frac{\partial V_t^{H_{t+1}}}{\partial B_t} dB_t = 0$$

$$dV_t^{n_t} = \frac{\partial V_t^{n_t}}{\partial \omega_t} d\omega_t + \frac{\partial V_t^{n_t}}{\partial H_{t+1}} A[H_t + \overline{H}]dh_t + \frac{\partial V_t^{n_t}}{\partial n_t} dn_t + \frac{\partial V_t^{n_t}}{\partial B_t} dB_t = 0 \qquad (8.1)$$

In a matrix form:

$$X\begin{pmatrix} d\omega_t \\ dh_t \\ dn_t \end{pmatrix} = -\begin{pmatrix} \frac{\partial V_t^{\omega_t}}{\partial B_t} \\ \frac{\partial V_t^{H_{t+1}}}{\partial B_t} \\ \frac{\partial V_t^{n_t}}{\partial B_t} \end{pmatrix} dB_t \quad \text{or} \quad \begin{pmatrix} d\omega_t \\ dh_t \\ dn_t \end{pmatrix} = -X^{-1}\begin{pmatrix} \frac{\partial V_t^{\omega_t}}{\partial B_t} \\ \frac{\partial V_t^{H_{t+1}}}{\partial B_t} \\ \frac{\partial V_t^{n_t}}{\partial B_t} \end{pmatrix} dB_t$$

$$\text{where } X \equiv (X_{ij}) \equiv \begin{pmatrix} \frac{\partial V_t^{\omega_t}}{\partial \omega_t} & \frac{\partial V_t^{\omega_t}}{\partial H_{t+1}} A[H_t + \overline{H}] & \frac{\partial V_t^{\omega_t}}{\partial n_t} \\ \frac{\partial V_t^{H_{t+1}}}{\partial \omega_t} & \frac{\partial V_t^{H_{t+1}}}{\partial H_{t+1}} A[H_t + \overline{H}] & \frac{\partial V_t^{H_{t+1}}}{\partial n_t} \\ \frac{\partial V_t^{n_t}}{\partial \omega_t} & \frac{\partial V_t^{n_t}}{\partial H_{t+1}} A[H_t + \overline{H}] & \frac{\partial V_t^{n_t}}{\partial n_t} \end{pmatrix}$$

and $X^{-1} = adjX / \det X$ (8.2)[36]

If B_t is some parameter or state variable contained in old-age parental utility part $\hat{\lambda}'(n_{t-1})u(c_{2,t})$, for example, if $B_t = \lambda'$ or $B_t = n_{t-1}$, then $\partial V_t^{H_{t+1}} / \partial B_t = 0$ and $\partial V_t^{n_t} / \partial B_t = 0$. Therefore:

$d\omega_t = -(C_{11} / \det X)(\partial V_t^{\omega_t} / \partial B_t) dB_t = 0$, $dh_t = -(C_{12} / \det X)(\partial V_t^{H_{t+1}} / \partial B_t) dB_t = 0$ and

$dn_t = -(C_{13} / \det X)(\partial V_t^{n_t} / \partial B_t) dB_t = 0$, where C_{ij} is the cofactor of X. (8.3)

Clearly $d\omega_t$, dh_t and dn_t are the functions of λ as well as λ' and δ.[37] Now the directions of marginal effects of λ' and n_{t-1} on life strategies ω_t, h_t and n_t are summarized as follows.

Proposition 8-1:

$\text{sgn}(d\omega_t / d\lambda') = \text{sgn}(-C_{11} / \det X)$, $\text{sgn}(dh_t / d\lambda') = \text{sgn}(-C_{12} / \det X)$,

$\text{sgn}(dh_t / d\lambda') = \text{sgn}(-C_{13} / \det X)$

If $\varepsilon > \sigma$, then $\text{sgn}(d\omega_t / dn_{t-1}) = \text{sgn}(-C_{11} / \det X)$, $\text{sgn}(dh_t / dn_{t-1}) = \text{sgn}(-C_{12} / \det X)$,

and $\text{sgn}(dn_t / dn_{t-1}) = \text{sgn}(-C_{13} / \det X)$.

And if $\varepsilon < \sigma$, $\text{sgn}(d\omega_t / dn_{t-1}) = \text{sgn}(C_{11} / \det X)$, $\text{sgn}(dh_t / dn_{t-1}) = \text{sgn}(C_{12} / \det X)$,

and $\text{sgn}(dn_t / dn_{t-1}) = \text{sgn}(C_{13} / \det X)$.

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36 adj and $\det$ denote the adjoint and determinant of the matrix, respectively.

37 $\det X$, C_{11}, C_{12} and C_{13} are quadratic functions of λ as well as linear ones of λ'.

These results suggest the possibility of dynamic fluctuations in a quasi-growth state, especially where the non-negativity constraints in h_t and n_t are not binding. However, since h_t and n_t are substitutes, two other possible equilibria are one where $h_t = 0$ and n_t is large, and another where $n_t = n_{\min}$ and h_t is large. One intuitive explanation regarding the indirect strategic (feed-back) effect of backward altruism λ' is as follows. From the F.O.C in ω_t (5.2), we have $\omega_t = K(1 - vn_t - h_t n_t) \equiv \overline{\omega}_t$, where $K = 1/\{1 + \delta\lambda'(n_{t-1})^{\varepsilon-\sigma}\}^{-1/\sigma}$. Plugging this into young adult consumption term, $c_{2,t} = \left[(1-K)(1 - vn_t - h_t n_t)(H_t + \overline{H})\right]^{1-\sigma} /(1-\sigma)$, where $(1-K)$ is a negative income effect. On the other hand, the maximization problem of two-way altruistic utility (5.1) can be written as a problem of one-way forward altruism, in which the discount rate δ and forward altruism coefficient λ are effectively replaced with $\tilde{\delta} \equiv \delta/(1-K)^{1-\sigma}$ and $\tilde{\lambda} \equiv \lambda/(1-K)^{1-\sigma}$, respectively. In effect, $\tilde{\delta}(>\delta)$ and $\tilde{\lambda}(>\lambda)$ operate as an accelerator of human capital investment. It is quite ambiguous which effect prevails, the negative income effect or the positive forward pushing effect, and is one main purpose of calibration.

9. The impact of unfunded (PAYG) social security on the equilibrium paths

In this section, I analyze the impact of a defined benefit type unfunded social security (PAYG) system on equilibrium paths and regime change, on the basis of analysis by Ehrlich and Lui (1998) and Ehrlich and Kim (2007). As for this issue under two-sided altruism, we also have, for example, Altig and Davis (1993) or Cigno and Rosati (1996). Assuming that the altruism coefficients toward parents and children (λ, λ') are fixed in values, I consider the following three cases.[38] [39]

[38] Cigno and Rosati (1996) examine the three cases, self-interest, forward and backward altruism,

(1) Two-way altruism $\lambda = \lambda' = 1 > 0$

(2) Forward one-way altruism $\lambda = 1 > 0$, $\lambda' \cong 0$

(3) Backward one-way altruism $\lambda \cong 0$, $\lambda' = 1 > 0$

As in Ehrlich and Lui (1998), I define the unfunded social security tax as a mandatory transfer from the generation of young adult-hood children to old parents $T_t \equiv \theta H_t (= \overline{\omega}_t^{\,I})$, where θ is the (expected) proportional social security tax levied on the middle age income part earned only by the acquired human capital H_t. Then the young/adult-hood consumptions of generation t are:

$$c_{1,t} = (1 - vn_t - h_t n_t - s_t)(H_t + \overline{H}) - \omega_t \pi_2 (H_t + \overline{H}) - T_t$$

$$c_{2,t+1} = \omega_{t+1} \pi_1 n_t (H_{t+1} + \overline{H}) + D(H_t + \overline{H})(s_t)^m + (\pi_1 / \pi_2) n_t T_{t+1} \qquad (9.1)$$

Also assume $\pi_1 \cong 1$ and $\pi_2 < 1$. The *defined expected benefit* which is transferred from children to parents becomes $(\pi_1 / \pi_2) n_t T_{t+1}$, in which fully actuarially fair insurance is also implicitly assumed. Here the compensation rate ω_t, transferred as a conditional payment on parents' survival, is not determined through an actuarially fair condition or other restrictions, but only through altruism coefficients toward parents and children (λ' and λ), in which this spontaneous intra-family intergenerational transfer might not be necessarily actuarially fair.[40] [41] Table 5 shows the calibration results for $T = 5$ regarding the effects of

assuming the non-separable utility functions, if with altruism, incorporating individual utilities of parents or children.

[39] Case (1) exhibits symmetric altruism as well as dynamic consistency.

[40] Also the spontaneous transfer might not necessarily be a compensation ($E\omega_t > 0$), but might be rather a bequest ($E\omega_t < 0$).

[41] These settings assume that the intra-family bargaining between old-age parents and young adult-hood children is motivated on altruism (λ') as a state-contingent claim only on the survival of old parents, and the social security tax rate (θ) involves both certainty premium transfer, and actuarially fair (partial) insurance proportional to the amount of transfer. Under fully actuarially fair insurance instead, it proves that the intra-family bargaining involves only certainty premium re-transfer as a perfect substitute. In either way (with partial or full insurance), the existence of unfunded social security tax affects subsequent equilibrium paths.

social security tax rate θ on the growth rate and fertility, with $\sigma = 0.5$, $\varepsilon = 0.5$, $\delta = 0.5$, $D = 2.0$, $v = 0.1$, $m = 1.0$, $\overline{H} = 1.0$ and $\widetilde{S}_0 = 0.5(H_1 + \overline{H})$, which plots the transition paths of life strategies for case (1) (2) and (3) in case C (both saving and compensation economy), (a) $A = 5$ and H_1 is moving at 0.0, 2.5 and 5.0, (b) $H_1 = 0$ and A is moving at 2.0, 8.0, and (c) $H_1 = 10$ and A is moving at 1.0 and 3.0.[42] [43]

Since $m = 1.0$ exhibits a constant return to scale in saving, and so the compensation is likely to be especially in a steady state, either constraints in saving or compensation are likely to be binding as the calibration results show. The effects of the increase in unfunded social security tax rate θ on human capital investment and fertility are summarized in the next proposition.

Proposition 9-1: *Social security tax rate θ, which incorporates actuarially fair (partially or fully) insurance of mortality risk, as well as certainty premium transfer, does affect dynamic equilibrium paths, even if the intra-family bargaining between old-age parents and young adult-hood children is motivated on altruism (λ'). As far as calibration results are concerned, the larger θ tends to induce the decrease in fertility and the increase in human capital investment under two-way altruism ((1) $\lambda = \lambda' = 1 > 0$) or forward altruism ((2) $\lambda = 1 > 0$, $\lambda' \cong 0$), with an intermediate (mediocre but not too low) productivity A and a low initial human capital H_1.*

Introduction of mortality risk

The above analysis could be meaningfully extended to the case where the mortality risk for

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[42] The same calibration window as in (4.4) is used. The intervals are 0.1 in h, s, ω and 0.5 in n.

[43] In this calibration, I set $\lambda' \cong 0$ in (2) and $\lambda \cong 0$ in (3), in order to represent an egoistic (completely indifferent) attitude for the neighbor generation. However, since these assumptions easily bring about the constraints' in compensation, non-negative saving, or non-negative consumption, being binding, these values might not necessarily be appropriate to observe carefully the impact of social security on fertility and human capital investment decisions.

old adult-hood exists and the mechanisms of social/intra-family risk sharing, as well as premium transfer, are considered. Here the unfunded social security tax is rewritten as $T_t = \omega_t^I = \overline{\omega}_t^I + \widetilde{\omega}_t^I = \theta H_t + \widetilde{\omega}_t^I$ ($\overline{\omega}_t^I = \theta H_t$ and $E\widetilde{\omega}_t^I = 0$), and the intra-family transfer contract between old-age parents and young adulthood children is defined as $\omega_t^P = \overline{\omega}_t^P + \widetilde{\omega}_t^P$. The short-run aspects of intergenerational risk sharing, social/market insurance and self-insurance, are described in Aoki (2007), and can be applied to the arguments on dynamic paths. At first, some basic implications immediately extracted from Aoki (2007) are summarized as follows. See Appendix 3 and Figure 3 for complementary explanation. The "participation constraints" are now considered, in which both old-age parents and young adulthood children are, on the individual (not altruistic) utility of each generation, willing to accept the proposed intra-family transfer contract.

Proposition 9-2: Assume that the intra-family bargaining between old-age parents and young adult-hood children is motivated on altruism (λ'), and the social security tax rate (θ) involves only certainty premium transfer $\overline{\omega}_t^I (= \theta H_t)$, not actuarially fair insurance $\widetilde{\omega}_t^I$, and the participation constraint of each generation regarding intra-family bargaining is not binding. Then, θ does *not* affect any subsequent equilibrium life strategies. In other words, Recardian equivalence does hold.[44]

Other short-run implications under different assumptions are described as follows.[45]

Corollary 9-1: Assume again that the intra-family bargaining between old-age parents and young adult-hood children is motivated on altruism (λ'), and the social security tax rate (θ)

[44] This proposition is presumably equivalent with Proposition 5 in Altig and Davis (1993), but holds only with absence of actuarially fair insurance of social security.

[45] Corollary 9-2, 3 and 5 are straightforwardly derived from Lemma 1 and 2 in Aoki (2007).

involves only certainty premium transfer, not actuarially fair insurance, but that the participation constraint of young adult-hood children is now binding.[46] Then the larger θ induces the larger private compensation scheme ω_t^P in actuarially fair insurance part $\widetilde{\omega}_t^P$ with $Var\widetilde{\omega}_t^P (= Var\omega_t^P)$.

Corollary 9-2: Assume that the intra-family bargaining between old-age parents and young adult-hood children is not available, and the social security tax rate (θ) involves both certainty premium transfer, and fully actuarially fair insurance. Then the larger θ induces the larger insurance compensation scheme ω_t^I in both certainty premium $\overline{\omega}_t^I (= E\omega_t^I)$ and actuarially fair insurance $\widetilde{\omega}_t^I$ with $Var\widetilde{\omega}_t^I (= Var\omega_t^I)$.

Corollary 9-3: Assume that the intra-family bargaining between old-age parents and young adult-hood children is not available, and the social security tax rate (θ) involves only certainty premium transfer, not actuarially fair insurance. Then the larger θ induces the larger self-insurance cost.

Corollary 9-4: Assume that the intra-family bargaining between old-age parents and young adult-hood adult children is not available. Then one unit of certainty premium transfer from young adulthood generation to old adulthood generation is equivalent with the actuarially fair insurance of old-age mortality risk with variance $1/(\pi_2(1-\pi_2))$.

Corollary 9-5: Assume that the intra-family bargaining between old-age parents and young adult-hood children is competitive, and the social security tax rate (θ) involves both certainty premium transfer $\overline{\omega}_t^I (= \theta H_t)$ and the fixed level of actuarially fair insurance.[47]

[46] In Figure 3 (or figures (1-4) in Aoki (2007)), the only Pareto optimal contract, in which the participation constraint for children is binding with no social security available, is represented as point G.

[47] The actuarially fair insurance, which is more than a fixed level in variance, is necessary for enabling Arrow-Debreu competitive (state-contingent exchange) economy. (Aoki (2007))

Then the larger θ induces the larger private compensation scheme ω_t^P in actuarially fair insurance part $\widetilde{\omega}_t^P$ with $Var\widetilde{\omega}_t^P (= Var\omega_t^P)$.

Obviously under the assumptions of corollary 9-1 to 4, Recardian equivalence does not necessarily hold in a dynamic context. As a consequence, social security tax rate (θ) do affect not only the subsequent life strategies including fertility, but also the critical values of initial state variables for the regime (stagnant to growth, growth to stagnant) change. In general, these settings could be potentially, contrary to proposition 9-1, a trigger into a Malthusian trap because of a resultant negative income effect, which is a trade-off with a forward-pushing effect based on backward altruism.

10. Final remarks

Thus this article has just tried to pursuit the implications of simultaneous two-way altruism (both forward and backward altruism) in a human capital/fertility endogenized overlapping generation model. Here I have been trying to calculate out the two-way-altruism-based value function with steady/unsteady state analysis on a two-sided altruistic framework as well as a dynastic one. The concept of "two-sided altruism" originates in a series of related literatures, for example, Abel (1987), Altig and Davis (1993) or Kimball (1987). Also Blackburn and Cipriani (2005) examine two-sided altruism as well as forward altruism, in their fertility endogenous growth model. Especially Abel (1987) examines the implication of two-sided altruism rather essentially, basically from viewpoints of transfer motives and steady states. In my article, the *fertility-endogenized* and *human-capital-based* two-way altruistic utility and corresponding value function are defined as in (3.6) and (3.7), in which strategic effects for the offspring are also taken account of. It is seen that (3.6) covers all the relevant utilities for the adjacent two periods (t and $t+1$) with well capturing the generation's altruism for both their parents and children. This form does not only fit well in reality, but also enables the recursive computation even for the case $\lambda\pi_1 a(n)n >> 1$ without the possibility of value function's divergence to infinity, where the generation cares, as is often the case, far more about children than about themselves. In addition, any form of one-way (forward/backward) altruism is not exclusive with two-way altruism at all, but rather can be considered to be just one special case of this two-way altruistic setting. For example, the assumptions of Ehrlich & Lui (1991) are equivalent with the case where $\lambda >> 1$, $a(n) = 1/n$ (inelastic altruism per child), $\lambda\lambda' = 1$ (dynamic consistency), and in addition, an implicit contract regarding compensation rate, patents conclude with children during young adulthood, is now internalized within children's maximization problem in the next period. Or, "self interest" holds for the case where $\lambda \cong 0$ and $\lambda' \cong 0$, or "backward altruism" holds for the case where $\lambda \cong 0$ and $\lambda' >> 0$.

Therefore, two-way altruism is, more or less altruistic, just a convenient generalization as well as a technical simplification for enabling recursive programming, in which the determination of all the relevant life strategies for forthcoming two adjacent periods can be incorporated within the decision making problem of corresponding vintage generation, without any intergenerational bargaining confliction. Concisely speaking, two-way altruism might still make distinct implications from one-sided (forward/backward) altruism especially when:

1. Parental (forward) altruism toward offspring is *not* dynamically consistent with children's (backward) altruism toward parents, in terms of the ratio in intergenerational marginal rate of transfer (bequest/compensation). (That is, $\lambda\lambda' \neq 1$.)

2. The economy is *not* in a growth/stagnant *steady* state. (Kimball (1987) comments a similar implication from a different context, in which "dynamic inefficiency" arises whenever the initial life strategies deviate from Golden Rule.) This case includes a transition process from a stagnant/growth state to growth/stagnant one.

3. Steady states in two-way altruism are different from those in dynastic utility because of a strategic rebound effect from the next generation.

4. There exist some unexpected shocks in initial state variables $H_1, n_0, \widetilde{S}_0$, life strategies $\{h, n, \omega, s\}$, or productivity or etc, which the next generation might (or might not) fully insure within its own decision making in a two-way altruistic framework.

5. Unfunded social security system involves mandatory inter-generational transfer in terms of either certainty premium or actuarially fair insurance.

On the other hand, this article does not assume the old-age working activity, and the intra-family production externality from parental generation does not exist here. Therefore, by means of specifying the altruism toward parents, the Pareto improvement is achieved by young adulthood only through more efficient allocation with old parents, while the altruism toward children is directly a driving force for human capital/fertility investment, a motivation for future return (including compensation) from children hopefully in a growth regime. Thus the role of altruism is substantially distinct in each direction (toward parents or children).[48]

Within some limited focus stated above, I have just summarized, under this generalized two-way altruistic framework, propositions and implications regarding

[48] As a matter of fact, the larger λ' (more altruism toward parents) could be a negative direct effect for future (human capital/fertility) investment toward children through a negative income effect for the generation, while it could be, as a social norm in the long run, an incentive (positive indirect feed-back effect) for future return. Since these two direct/indirect effects, involving complicated first order conditions, cannot be analytically solved in a rigorous closed form, I calculate out their effects by means of computational calibration.

dynamics, and calibration results, especially focusing on 1 and 2 among the above points. The results also support the existence of two equilibria (one with larger growth and low fertility, and the other with low growth and large fertility), claimed by Becker et al. (1990) and Ehrlich et al. (1991), even under two-way altruism.

The implications obtained in this article suggest a strong necessity to implement empirical works involving two-way altruism.[49] For example, it may be meaningful to construct the model incorporating two-way altruism and human capital investment, as well as endogenized fertility and saving, in addition, allowing for a continuous-time old age with mortality risk and a non-separable utility form.[50] For one simple example, we may try an econometric test for the hypothesis that this OLG is dynamically consistent, that is $\hat{\lambda}'(n_{t-1}) = \delta\pi_2 / \{\lambda\pi_1 a(n_{t-1})n_{t-1}\}$. At least, estimating separately λ' and λ (altruism toward parents and children) as well as other parameters, especially in Japan, would be a useful information for the construction of appropriate funded/unfunded social security programs.

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References

[49] A series of Cigno & Rosati papers, especially EER (1996) paper offers a nice example in terms of both flexible model construction and implementation of econometric test.

[50] The reason why I am going to consider continuous-time (or finely divided discrete period) old-age utility with mortality (longevity) risk is articulated in Aoki (2007) and as follows. The existence of mortality risk generates a peculiar shape of indifference curve of old-age indirect utility, in which *self-insurance* plays a crucial role on the decision regarding the mixture of social security and intergenerational transfer (compensation/bequest). Therefore, using a regular utility form instead of this old-age specific indirect utility might lead to a wrong-estimation of the weight of altruism toward parents (λ'), therefore a wrong statistic result regarding dynamic consistency.

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Case A (Saving but no compensation economy)
(a) A=5

(i) lamda=0.5

h	H1=0.5	H1=1.5	H1=2.5
t=1	0.02	0.48	0.4
t=2	0	0.5	0.5
t=3	0	0.52	0.52
t=4	0	0.52	0.52
t=5	0	0.52	0.52
t=6	0	0.52	0.52
t=7	0	0.54	0.54
t=8	0	0.5	0.5
t=9	0.02	0.64	0.64
t=10	0	0.42	0.42
t=11	–	–	–

n	H1=0.5	H1=1.5	H1=2.5
t=1	2	0.5	0.5
t=2	2	0.5	0.5
t=3	2	0.5	0.5
t=4	2	0.5	0.5
t=5	2	0.5	0.5
t=6	2	0.5	0.5
t=7	2	0.5	0.5
t=8	2	0.5	0.5
t=9	2	0.5	0.5
t=10	2	0.5	0.5
t=11	–	–	–

s	H1=0.5	H1=1.5	H1=2.5
t=1	0.25	0.24	0.25
t=2	0.27	0.23	0.23
t=3	0.27	0.23	0.23
t=4	0.27	0.23	0.23
t=5	0.27	0.23	0.23
t=6	0.27	0.23	0.23
t=7	0.27	0.23	0.23
t=8	0.27	0.23	0.23
t=9	0.25	0.21	0.21
t=10	0.12	0.13	0.13
t=11	–	–	–

(ii) lamda=2

h	H1=0.5	H1=1.5	H1=2.5
t=1	0.16	0.7	0.7
t=2	0.04	1	1
t=3	0.14	0.72	0.72
t=4	0.52	1	1
t=5	0.4	0.72	0.72
t=6	0.02	1	1
t=7	0.12	0.72	0.72
t=8	0.02	0.98	0.98
t=9	0.28	0.78	0.78
t=10	0.74	0.78	0.78
t=11	–	–	–

n	H1=0.5	H1=1.5	H1=2.5
t=1	2	1	1
t=2	2	0.5	0.5
t=3	2	1	1
t=4	1	0.5	0.5
t=5	1.5	1	1
t=6	2	0.5	0.5
t=7	2	1	1
t=8	2	0.5	0.5
t=9	2	1	1
t=10	0.5	0.5	0.5
t=11	–	–	–

s	H1=0.5	H1=1.5	H1=2.5
t=1	0.16	0.06	0.06
t=2	0.24	0.16	0.16
t=3	0.18	0.06	0.06
t=4	0.12	0.16	0.16
t=5	0.08	0.06	0.06
t=6	0.26	0.16	0.16
t=7	0.18	0.06	0.06
t=8	0.26	0.16	0.16
t=9	0.08	0.04	0.04
t=10	0.02	0.02	0.02
t=11	–	–	–

Table 1-1 Transition Paths in Case A

Case A (Saving but no compensation economy)
(b) H1=0.0

(i) lamda=0.5

h	A=2.5	A=5.0	A=7.5	A=10.0
t=1	0	0	0	0.1
t=2	0	0	0.02	0.46
t=3	0	0	0.04	0.62
t=4	0	0	0.04	0.72
t=5	0	0	0.02	0.74
t=6	0	0	0	0.74
t=7	0	0	0	0.78
t=8	0	0	0.02	0.66
t=9	0	0.02	0.04	0.94
t=10	0	0	0.02	0.52
t=11	–	–	–	–

n	A=2.5	A=5.0	A=7.5	A=10.0
t=1	2	2	2	2
t=2	2	2	2	0.5
t=3	2	2	2	0.5
t=4	2	2	2	0.5
t=5	2	2	2	0.5
t=6	2	2	2	0.5
t=7	2	2	2	0.5
t=8	2	2	2	0.5
t=9	2	2	2	0.5
t=10	2	2	2	0.5
t=11	–	–	–	–

s	A=2.5	A=5.0	A=7.5	A=10.0
t=1	0.27	0.27	0.27	0.2
t=2	0.27	0.27	0.25	0.24
t=3	0.27	0.27	0.24	0.22
t=4	0.27	0.27	0.24	0.2
t=5	0.27	0.27	0.25	0.2
t=6	0.27	0.27	0.27	0.2
t=7	0.27	0.27	0.27	0.18
t=8	0.27	0.27	0.25	0.2
t=9	0.27	0.25	0.24	0.16
t=10	0.12	0.12	0.11	0.1
t=11	–	–	–	–

(ii) lamda=2

h	A=2.5	A=5.0	A=7.5	A=10.0
t=1	0.19	0.32	0.86	0.32
t=2	0.16	0.04	0.06	0.02
t=3	0.11	0.06	0.18	0.12
t=4	0.04	1	0.04	0.02
t=5	0.03	0.18	0.14	0.12
t=6	0.14	0.04	0.04	0.04
t=7	0.11	0.1	0.18	0.7
t=8	0.09	0.02	0.1	0.96
t=9	0.15	0.28	0.74	0.84
t=10	0	0.74	0.8	0.82
t=11	–	–	–	–

n	A=2.5	A=5.0	A=7.5	A=10.0
t=1	2	2	1	2
t=2	2	2	2	2
t=3	2	2	2	2
t=4	2	0.5	2	2
t=5	2	2	2	2
t=6	2	2	2	2
t=7	2	2	2	1
t=8	2	2	2	0.5
t=9	2	2	1	1
t=10	2	0.5	0.5	0.5
t=11	–	–	–	–

s	A=2.5	A=5.0	A=7.5	A=10.0
t=1	0.14	0.06	0.02	0.06
t=2	0.16	0.24	0.22	0.26
t=3	0.19	0.22	0.14	0.18
t=4	0.2	0.16	0.24	0.26
t=5	0.2	0.14	0.18	0.18
t=6	0.17	0.24	0.24	0.24
t=7	0.19	0.2	0.14	0.06
t=8	0.2	0.26	0.2	0.16
t=9	0.17	0.08	0.06	0.02
t=10	0.03	0.02	0.02	0.02
t=11	–	–	–	–

Table 1-2 Transition Paths in Case A

Case A (Saving but no compensation economy)
(c) H1=10.0

(i) lamda=0.5

h	A=1	A=2	A=3	A=4
t=1	0.08	0.36	0.46	0.44
t=2	0	0.2	0.3	0.46
t=3	0	0	0.24	0.46
t=4	0	0	0.22	0.46
t=5	0	0	0.22	0.46
t=6	0	0	0.24	0.46
t=7	0	0	0.24	0.46
t=8	0	0	0.22	0.44
t=9	0	0	0.32	0.56
t=10	0	0	0.26	0.36

n	A=1	A=2	A=3	A=4
t=1	0.5	0.5	0.5	0.5
t=2	2	0.5	0.5	0.5
t=3	2	1.5	0.5	0.5
t=4	2	2	0.5	0.5
t=5	2	2	0.5	0.5
t=6	2	2	0.5	0.5
t=7	2	2	0.5	0.5
t=8	2	2	0.5	0.5
t=9	2	2	0.5	0.5
t=10	2	2	0.5	0.5

s	A=2.5	A=5.0	A=7.5	A=10.0
t=1	0.3	0.26	0.24	0.24
t=2	0.26	0.28	0.26	0.24
t=3	0.26	0.28	0.28	0.24
t=4	0.26	0.26	0.28	0.24
t=5	0.26	0.26	0.28	0.24
t=6	0.26	0.26	0.28	0.24
t=7	0.26	0.26	0.28	0.24
t=8	0.26	0.26	0.28	0.24
t=9	0.26	0.26	0.26	0.22
t=10	0.12	0.12	0.18	0.16

(ii) lamda=2

h	A=1	A=2	A=3	A=4
t=1	0.08	0.08	0.12	0.8
t=2	0.01	0.02	0.06	0.18
t=3	0	0.18	0.04	1
t=4	0	0.02	1	0.12
t=5	0	0.1	0.12	0.16
t=6	0	0.12	0.08	0.06
t=7	0	0.14	0.2	0.12
t=8	0	0.08	0.02	0
t=9	0	0.2	0.16	0.22
t=10	0	0	0.02	0.08
t=11	−	−	−	−

n	A=1	A=2	A=3	A=4
t=1	2	2	2	1
t=2	2	2	2	2
t=3	2	2	2	0.5
t=4	2	2	0.5	2
t=5	2	2	2	2
t=6	2	2	2	2
t=7	2	2	2	2
t=8	2	2	2	2
t=9	2	2	2	2
t=10	2	2	2	2
t=11	−	−	−	−

s	A=2.5	A=5.0	A=7.5	A=10.0
t=1	0.21	0.2	0.18	0.04
t=2	0.26	0.2	0.22	0.14
t=3	0.27	0.15	0.24	0.16
t=4	0.27	0.2	0.16	0.18
t=5	0.27	0.2	0.18	0.16
t=6	0.27	0.19	0.22	0.22
t=7	0.27	0.17	0.14	0.18
t=8	0.27	0.2	0.26	0.26
t=9	0.27	0.13	0.16	0.12
t=10	0.03	0.03	0.04	0.02
t=11	−	−	−	−

Table 1-3 Transition Paths in Case A

Case B (Compensation but no saving economy)
(a) A=5

I. lamda=0.5, lamdaprime=2.0

h	H1=0.5	H1=1.5	H1=2.5
t=1	0.06	0.54	0.56
t=2	0.01	0.58	0.6
t=3	0.02	0.6	0.6
t=4	0.01	0.6	0.6
t=5	0.02	0.6	0.6
t=6	0.01	0.6	0.6
t=7	0.02	0.6	0.6
t=8	0.01	0.56	0.56
t=9	0	0.68	0.68
t=10	0	0.42	0.42
t=11	–	–	–

n	H1=0.5	H1=1.5	H1=2.5
t=1	2	0.5	0.5
t=2	2	0.5	0.5
t=3	2	0.5	0.5
t=4	2	0.5	0.5
t=5	2	0.5	0.5
t=6	2	0.5	0.5
t=7	2	0.5	0.5
t=8	2	0.5	0.5
t=9	2	0.5	0.5
t=10	2	0.5	0.5
t=11	–	–	–

ω	H1=0.5	H1=1.5	H1=2.5
t=1	0.34	0.34	0.34
t=2	0.38	0.32	0.32
t=3	0.38	0.32	0.32
t=4	0.38	0.32	0.32
t=5	0.38	0.32	0.32
t=6	0.38	0.32	0.32
t=7	0.38	0.32	0.32
t=8	0.38	0.34	0.34
t=9	0.4	0.3	0.3
t=10	0.3	0.3	0.3
t=11	–	–	–

II. lamda=0.5, lamdaprime=0.5

h	H1=0.5	H1=1.5	H1=2.5
t=1	0.18	0.18	0.68
t=2	0.06	0.02	0.8
t=3	0.04	0	0.82
t=4	0.12	0.1	0.82
t=5	0.04	0.06	0.82
t=6	0	0	0.82
t=7	0.1	0.1	0.84
t=8	0.04	0.02	0.76
t=9	0.04	0.04	0.94
t=10	0	0	0.52
t=11	–	–	–

n	H1=0.5	H1=1.5	H1=2.5
t=1	2	2	0.5
t=2	2	2	0.5
t=3	2	2	0.5
t=4	2	2	0.5
t=5	2	2	0.5
t=6	2	2	0.5
t=7	2	2	0.5
t=8	2	2	0.5
t=9	2	2	0.5
t=10	2	2	0.5
t=11	–	–	–

ω	H1=0.5	H1=1.5	H1=2.5
t=1	0.03	0.03	0.04
t=2	0.04	0.04	0.03
t=3	0.04	0.05	0.03
t=4	0.03	0.04	0.03
t=5	0.04	0.04	0.03
t=6	0.05	0.05	0.03
t=7	0.04	0.04	0.03
t=8	0.04	0.04	0.03
t=9	0.04	0.04	0.03
t=10	0.09	0.09	0.07
t=11	–	–	–

III. lamda=2.0, lamdaprime=0.5

h	H1=0.5	H1=1.5	H1=2.5
t=1	0.24	0.88	0.88
t=2	1	0.2	0.16
t=3	0.2	0.24	0.24
t=4	0.04	0.04	0.04
t=5	0.16	0.16	0.16
t=6	0.04	0.04	0.04
t=7	0.12	0.12	0.12
t=8	0.08	0.08	0.08
t=9	0.32	0.32	0.32
t=10	0.76	0.76	0.76
t=11	–	–	–

n	H1=0.5	H1=1.5	H1=2.5
t=1	2	1	1
t=2	0.5	2	2
t=3	2	2	2
t=4	2	2	2
t=5	2	2	2
t=6	2	2	2
t=7	2	2	2
t=8	2	2	2
t=9	2	2	2
t=10	0.5	0.5	0.5
t=11	–	–	–

ω	H1=0.5	H1=1.5	H1=2.5
t=1	0.02	0	0
t=2	0.03	0.02	0.03
t=3	0.02	0.02	0.02
t=4	0.04	0.04	0.04
t=5	0.03	0.03	0.03
t=6	0.04	0.04	0.04
t=7	0.03	0.03	0.03
t=8	0.04	0.04	0.04
t=9	0.01	0.01	0.01
t=10	0.01	0.01	0.01
t=11	–	–	–

IV. lamda=2.0, lamdaprime=2.0

h	H1=0.5	H1=1.5	H1=2.5
t=1	0.08	0.64	0.4
t=2	0	1	1
t=3	0.04	1	1
t=4	1	1	1
t=5	0.16	1	1
t=6	0.08	1	1
t=7	0.12	1	1
t=8	0.04	0.96	0.96
t=9	0.28	0.76	0.76
t=10	0.72	0.76	0.76

n	H1=0.5	H1=1.5	H1=2.5
t=1	2	1	1.5
t=2	2	0.5	0.5
t=3	2	0.5	0.5
t=4	0.5	0.5	0.5
t=5	2	0.5	0.5
t=6	2	0.5	0.5
t=7	2	0.5	0.5
t=8	2	0.5	0.5
t=9	2	1	1
t=10	0.5	0.5	0.5

ω	H1=0.5	H1=1.5	H1=2.5
t=1	0.32	0.12	0.12
t=2	0.4	0.22	0.22
t=3	0.36	0.22	0.22
t=4	0.22	0.22	0.22
t=5	0.24	0.22	0.22
t=6	0.32	0.22	0.22
t=7	0.28	0.22	0.22
t=8	0.36	0.24	0.24
t=9	0.12	0.06	0.06
t=10	0.06	0.06	0.06

Table 2-1 Transition Paths in Case B

Case B (Compensation but no saving economy)
(b) H1=0.0

I. lamda=0.5, lamdaprime=2.0

h	A=2.5	A=5.0	A=7.5	A=10.0
t=1	0	0	0.04	0.04
t=2	0	0.01	0.04	0.6
t=3	0	0.02	0.08	0.8
t=4	0	0.01	0.08	0.8
t=5	0	0.02	0.04	0.84
t=6	0	0	0	0.8
t=7	0	0.01	0	0.84
t=8	0	0.01	0.04	0.76
t=9	0	0	0.12	0.96
t=10	0	0	0.44	0.52
t=11	−	−	−	−

n	A=2.5	A=5.0	A=7.5	A=10.0
t=1	2	2	2	2
t=2	2	2	2	0.5
t=3	2	2	2	0.5
t=4	2	2	2	0.5
t=5	2	2	2	0.5
t=6	2	2	2	0.5
t=7	2	2	2	0.5
t=8	2	2	2	0.5
t=9	2	2	2	0.5
t=10	2	2	0.5	0.5
t=11	−	−	−	−

ω	A=2.5	A=5.0	A=7.5	A=10.0
t=1	0.4	0.4	0.36	0.36
t=2	0.4	0.38	0.36	0.32
t=3	0.4	0.38	0.32	0.28
t=4	0.4	0.38	0.32	0.28
t=5	0.4	0.38	0.36	0.26
t=6	0.4	0.4	0.4	0.28
t=7	0.4	0.38	0.4	0.26
t=8	0.4	0.38	0.36	0.28
t=9	0.4	0.4	0.28	0.24
t=10	0.3	0.3	0.28	0.26
t=11	−	−	−	−

II. lamda=0.5, lamdaprime=0.5

h	A=2.5	A=5.0	A=7.5	A=10.0
t=1	0	0.02	0.2	1
t=2	0	0	0.72	1
t=3	0	0.08	0.92	1
t=4	0	0.08	0.96	1
t=5	0	0.06	0.96	1
t=6	0	0.08	0.96	1
t=7	0	0.09	0.96	1
t=8	0	0.03	0.92	1
t=9	0	0.04	1	1
t=10	0	0	0.6	0.64
t=11	−	−	−	−

n	A=2.5	A=5.0	A=7.5	A=10.0
t=1	2	2	2	0.5
t=2	2	2	0.5	0.5
t=3	2	2	0.5	0.5
t=4	2	2	0.5	0.5
t=5	2	2	0.5	0.5
t=6	2	2	0.5	0.5
t=7	2	2	0.5	0.5
t=8	2	2	0.5	0.5
t=9	2	2	0.5	0.5
t=10	2	2	0.5	0.5
t=11	−	−	−	−

ω	A=2.5	A=5.0	A=7.5	A=10.0
t=1	0.05	0.04	0.02	0.03
t=2	0.05	0.05	0.03	0.03
t=3	0.05	0.04	0.03	0.03
t=4	0.05	0.04	0.03	0.03
t=5	0.05	0.04	0.03	0.03
t=6	0.05	0.04	0.03	0.03
t=7	0.05	0.04	0.03	0.03
t=8	0.05	0.04	0.03	0.03
t=9	0.05	0.04	0.03	0.03
t=10	0.09	0.09	0.07	0.08
t=11	−	−	−	−

III. lamda=2.0, lamdaprime=0.5

h	A=2.5	A=5.0	A=7.5	A=10.0
t=1	0.64	0.28	0.36	0.84
t=2	0.08	1	1	1
t=3	1	0.2	0.84	0.84
t=4	0.2	0.04	1	1
t=5	1	0.16	0.84	0.84
t=6	0.16	0.04	1	1
t=7	1	0.12	0.84	0.84
t=8	0.04	0.08	1	1
t=9	0.2	0.32	0.84	0.84
t=10	0.04	0.76	0.84	0.84
t=11	−	−	−	−

n	A=2.5	A=5.0	A=7.5	A=10.0
t=1	1	2	2	1
t=2	2	0.5	0.5	0.5
t=3	0.5	2	1	1
t=4	2	2	0.5	0.5
t=5	0.5	2	1	1
t=6	2	2	0.5	0.5
t=7	0.5	2	1	1
t=8	2	2	0.5	0.5
t=9	2	2	1	1
t=10	2	0.5	0.5	0.5
t=11	−	−	−	−

ω	A=2.5	A=5.0	A=7.5	A=10.0
t=1	0.02	0.01	0.01	0.01
t=2	0.04	0.03	0.03	0.03
t=3	0.03	0.02	0.01	0.01
t=4	0.02	0.04	0.03	0.03
t=5	0.03	0.03	0.01	0.01
t=6	0.03	0.04	0.03	0.03
t=7	0.03	0.03	0.01	0.01
t=8	0.04	0.04	0.03	0.03
t=9	0.02	0.01	0.01	0.01
t=10	0.02	0.01	0.01	0.01
t=11	−	−	−	−

IV. lamda=2.0, lamdaprime=2.0

h	A=2.5	A=5.0	A=7.5	A=10.0
t=1	0.2	0.28	0.04	0.48
t=2	0.08	0.04	1	1
t=3	1	0	1	0.8
t=4	0.08	0.92	0.2	1
t=5	0.08	0.2	0.56	0.8
t=6	0.2	1	1	1
t=7	0.16	1	0.72	0.8
t=8	0.16	0.96	0.92	1
t=9	0.16	0.76	0.8	0.8
t=10	0	0.8	0.8	0.8
t=11	−	−	−	−

n	A=2.5	A=5.0	A=7.5	A=10.0
t=1	2	2	2	1.5
t=2	2	2	0.5	0.5
t=3	0.5	2	0.5	1
t=4	2	0.5	2	0.5
t=5	2	1.5	1	1
t=6	2	0.5	0.5	0.5
t=7	2	0.5	1	1
t=8	2	0.5	0.5	0.5
t=9	2	1	1	1
t=10	2	0.5	0.5	0.5
t=11	−	−	−	−

ω	A=2.5	A=5.0	A=7.5	A=10.0
t=1	0.2	0.1	0.35	0.05
t=2	0.3	0.35	0.2	0.2
t=3	0.2	0.4	0.2	0.05
t=4	0.3	0.25	0.2	0.2
t=5	0.3	0.25	0.15	0.05
t=6	0.2	0.2	0.2	0.2
t=7	0.25	0.2	0.1	0.05
t=8	0.25	0.25	0.25	0.2
t=9	0.25	0.05	0.05	0.05
t=10	0.1	0.05	0.05	0.05
t=11	−	−	−	−

Table 2-2 Transition Paths in Case B

Case B (Compensation but no saving economy)
(c) H1=10.0

I. lamda=0.5, lamdaprime=2.0

h	A=1	A=2	A=3	A=4
t=1	0.1	0.32	0.28	0.52
t=2	0	0.28	0.24	0.52
t=3	0	0.28	0.2	0.52
t=4	0	0	0.24	0.52
t=5	0	0	0.24	0.52
t=6	0	0	0.12	0.52
t=7	0	0	0	0.52
t=8	0	0	0	0.52
t=9	0	0	0	0.6
t=10	0	0	0	0.4
t=11	−	−	−	−

n	A=1	A=2	A=3	A=4
t=1	0.5	0.5	0.5	0.5
t=2	2	0.5	0.5	0.5
t=3	2	0.5	0.5	0.5
t=4	2	2	0.5	0.5
t=5	2	2	0.5	0.5
t=6	2	2	1	0.5
t=7	2	2	2	0.5
t=8	2	2	2	0.5
t=9	2	2	2	0.5
t=10	2	2	2	0.5
t=11	−	−	−	−

ω	A=2.5	A=5.0	A=7.5	A=10.0
t=1	0.44	0.4	0.4	0.34
t=2	0.4	0.4	0.42	0.34
t=3	0.4	0.4	0.42	0.34
t=4	0.4	0.4	0.42	0.34
t=5	0.4	0.4	0.42	0.34
t=6	0.4	0.4	0.38	0.34
t=7	0.4	0.4	0.4	0.34
t=8	0.4	0.4	0.4	0.34
t=9	0.4	0.4	0.4	0.32
t=10	0.3	0.3	0.3	0.32
t=11	−	−	−	−

II. lamda=0.5, lamdaprime=0.5

h	A=1	A=2	A=3	A=4
t=1	0.28	0.64	0.64	0.6
t=2	0	0.28	0.64	0.24
t=3	0	0.28	0.64	0.48
t=4	0	0.16	0.64	0.64
t=5	0	0	0.64	0.72
t=6	0	0	0.64	0.76
t=7	0	0	0.64	0.76
t=8	0	0	0.64	0.72
t=9	0	0	0.72	0.84
t=10	0	0	0.44	0.48
t=11	−	−	−	−

n	A=1	A=2	A=3	A=4
t=1	0.5	0.5	0.5	0.5
t=2	2	0.5	0.5	1
t=3	2	0.5	0.5	0.5
t=4	2	1	0.5	0.5
t=5	2	2	0.5	0.5
t=6	2	2	0.5	0.5
t=7	2	2	0.5	0.5
t=8	2	2	0.5	0.5
t=9	2	2	0.5	0.5
t=10	2	2	0.5	0.5
t=11	−	−	−	−

ω	A=2.5	A=5.0	A=7.5	A=10.0
t=1	0.05	0.04	0.04	0.04
t=2	0.05	0.05	0.04	0.04
t=3	0.05	0.05	0.04	0.04
t=4	0.05	0.04	0.04	0.04
t=5	0.05	0.05	0.04	0.03
t=6	0.05	0.05	0.04	0.03
t=7	0.05	0.05	0.04	0.03
t=8	0.05	0.05	0.04	0.03
t=9	0.05	0.05	0.03	0.03
t=10	0.09	0.09	0.07	0.07
t=11	−	−	−	−

III. lamda=2.0, lamdaprime=0.5

h	A=1	A=2	A=3	A=4
t=1	0.06	0.24	0.5	0.32
t=2	0.06	0.16	0.08	0.1
t=3	0.06	0.08	1	0.12
t=4	0	0.12	0.12	0.02
t=5	0.06	0.68	0.14	0
t=6	0	0.1	0.04	1
t=7	0.06	1	0.16	0.22
t=8	0	0.14	0.02	0.04
t=9	0.06	0.2	0.16	0.3
t=10	0	0	0.02	0.74
t=11	−	−	−	−

n	A=1	A=2	A=3	A=4
t=1	2	2	1.5	2
t=2	2	2	2	2
t=3	2	2	0.5	2
t=4	2	2	2	2
t=5	2	1	2	2
t=6	2	2	2	0.5
t=7	2	0.5	2	2
t=8	2	2	2	2
t=9	2	2	2	2
t=10	2	2	2	0.5
t=11	−	−	−	−

ω	A=2.5	A=5.0	A=7.5	A=10.0
t=1	0.04	0.02	0.01	0.01
t=2	0.04	0.03	0.04	0.04
t=3	0.04	0.04	0.03	0.03
t=4	0.05	0.03	0.03	0.04
t=5	0.04	0.01	0.03	0.05
t=6	0.05	0.04	0.04	0.03
t=7	0.04	0.03	0.03	0.02
t=8	0.05	0.03	0.04	0.04
t=9	0.04	0.02	0.03	0.01
t=10	0.02	0.02	0.02	0.01
t=11	−	−	−	−

IV. lamda=2.0, lamdaprime=2.0

h	A=1	A=2	A=3	A=4
t=1	0.1	0.85	1	1
t=2	0	0.7	1	1
t=3	0	0.7	1	1
t=4	0	0.35	1	1
t=5	0	0.2	1	1
t=6	0	0.1	1	1
t=7	0	0	1	1
t=8	0	0	0.8	0.8
t=9	0	0.2	0.7	0.75
t=10	0	0	0.75	0.75
t=11	−	−	−	−

n	A=1	A=2	A=3	A=4
t=1	2	0.5	0.5	0.5
t=2	2	0.5	0.5	0.5
t=3	2	0.5	0.5	0.5
t=4	2	1	0.5	0.5
t=5	2	1.5	0.5	0.5
t=6	2	2	0.5	0.5
t=7	2	2	0.5	0.5
t=8	2	2	0.5	0.5
t=9	2	2	1	1
t=10	2	2	0.5	0.5
t=11	−	−	−	−

ω	A=2.5	A=5.0	A=7.5	A=10.0
t=1	0.3	0.25	0.225	0.225
t=2	0.4	0.3	0.225	0.225
t=3	0.4	0.3	0.225	0.225
t=4	0.4	0.275	0.225	0.225
t=5	0.4	0.275	0.225	0.225
t=6	0.4	0.3	0.225	0.225
t=7	0.4	0.4	0.225	0.225
t=8	0.4	0.4	0.275	0.275
t=9	0.4	0.2	0.1	0.075
t=10	0.1	0.1	0.075	0.075
t=11	−	−	−	−

Table 2-3 Transition Paths in Case B

Case A	(i)	(ii)
$H_{1\,A=5}^{\uparrow}$	1.4	1.5
$A_{H_1=0}^{\uparrow}$	8.5	7.0
$A_{H_1=10}^{\downarrow}$	2.8	2.8

Case B	I	II	III	IV
$H_{1\,A=5}^{\uparrow}$	1.5	2.25	0.0	1.0
$A_{H_1=0}^{\uparrow}$	8.5	6.5	5	8.5
$A_{H_1=10}^{\downarrow}$	2.75	2.0	3.25	1.75

Table 3
Critical Values of Transition Process (T=3) in Case A and B

Altruistic Dynastic Utility Approach
(b) H1=0

I. lamda=0.5, lamdaprime=2.0

h	A=2.5	A=5.0	A=7.5
t=1	0	0.34	0.4
t=2	0	0.34	0.56
t=3	0	0.5	0.56
t=4	0	0.5	0.8
t=5	0	0.5	0.8
t=6	0	0.46	0.8
t=7	0	0.46	0.8
t=8	0	0.5	0.8
t=9	0	0.5	0.68
t=10	0	0.42	0.48
t=11	−	−	−

n	A=2.5	A=5.0	A=7.5
t=1	2	2	2
t=2	2	2	1.5
t=3	2	1.5	1.5
t=4	2	1.5	1
t=5	2	1.5	1
t=6	2	1.5	1
t=7	2	1.5	1
t=8	2	1	1
t=9	2	1	1
t=10	2	0.5	0.5
t=11	−	−	−

ω	A=2.5	A=5.0	A=7.5
t=1	0.4	0.04	0
t=2	0.4	0.04	0
t=3	0.4	0.04	0
t=4	0.4	0.04	0.04
t=5	0.4	0.04	0.04
t=6	0.4	0.08	0.04
t=7	0.4	0.08	0.04
t=8	0.4	0.2	0.04
t=9	0.4	0.2	0.12
t=10	0.3	0.32	0.28
t=11	−	−	−

II. lamda=0.5, lamdaprime=0.5

h	A=2.5	A=5.0	A=7.5
t=1	0	0.36	0.4
t=2	0	0.84	0.88
t=3	0	0.84	0.88
t=4	0	0.84	0.88
t=5	0	0.84	0.88
t=6	0	0.8	0.88
t=7	0	0.8	0.84
t=8	0	0.76	0.8
t=9	0	1	0.76
t=10	0	0.52	0.6
t=11	−	−	−

n	A=2.5	A=5.0	A=7.5
t=1	2	2	2
t=2	2	1	1
t=3	2	1	1
t=4	2	1	1
t=5	2	1	1
t=6	2	1	1
t=7	2	1	1
t=8	2	1	1
t=9	2	0.5	1
t=10	2	0.5	0.5
t=11	−	−	−

ω	A=2.5	A=5.0	A=7.5
t=1	0.05	0.01	0
t=2	0.05	0.01	0
t=3	0.05	0.01	0
t=4	0.05	0.01	0
t=5	0.05	0.01	0
t=6	0.05	0.01	0
t=7	0.05	0.01	0.01
t=8	0.05	0.01	0.01
t=9	0.05	0.03	0.01
t=10	0.09	0.07	0.07
t=11	−	−	−

III. lamda=2.0, lamdaprime=0.5

h	A=2.5	A=5.0	A=7.5
t=1	0.4	0.4	0.4
t=2	0.4	0.4	0.88
t=3	0.4	0.88	0.88
t=4	0.4	0.88	0.88
t=5	0.4	0.88	0.88
t=6	0.4	0.88	0.88
t=7	0.4	0.88	0.88
t=8	0.56	0.88	0.88
t=9	0.88	0.88	0.88
t=10	0.76	0.8	0.84
t=11	−	−	−

n	A=2.5	A=5.0	A=7.5
t=1	2	2	2
t=2	2	2	1
t=3	2	1	1
t=4	2	1	1
t=5	2	1	1
t=6	2	1	1
t=7	2	1	1
t=8	1.5	1	1
t=9	1	1	1
t=10	0.5	0.5	0.5
t=11	−	−	−

ω	A=2.5	A=5.0	A=7.5
t=1	0	0	0
t=2	0	0	0
t=3	0	0	0
t=4	0	0	0
t=5	0	0	0
t=6	0	0	0
t=7	0	0	0
t=8	0	0	0
t=9	0	0	0
t=10	0.02	0.01	0.01
t=11	−	−	−

IV. lamda=2.0, lamdaprime=2.0

h	A=2.5	A=5.0	A=7.5
t=1	0.4	0.4	0.4
t=2	0.4	0.4	0.88
t=3	0.4	0.88	0.88
t=4	0.4	0.88	0.88
t=5	0.4	0.88	0.88
t=6	0.4	0.88	0.88
t=7	0.88	0.88	0.88
t=8	0.88	0.88	0.88
t=9	0.88	0.88	0.88
t=10	0.68	0.76	0.8
t=11	−	−	−

n	A=2.5	A=5.0	A=7.5
t=1	2	2	2
t=2	2	2	1
t=3	2	1	1
t=4	2	1	1
t=5	2	1	1
t=6	2	1	1
t=7	1	1	1
t=8	1	1	1
t=9	1	1	1
t=10	0.5	0.5	0.5
t=11	−	−	−

ω	A=2.5	A=5.0	A=7.5
t=1	0	0	0
t=2	0	0	0.01
t=3	0	0.01	0.01
t=4	0	0.01	0.01
t=5	0	0.01	0.01
t=6	0	0.01	0.01
t=7	0.01	0.01	0.01
t=8	0.01	0.01	0.01
t=9	0.01	0.01	0.01
t=10	0.1	0.06	0.05
t=11	−	−	−

Table 4 Transition Paths in Altruistic Dynastic Utility
Case B (Compensation but no saving)

Effect of Social Security Tax
Case C (Both Saving and Conpensation Economy)
(a) A=5

(1) lamda=1.0, lamdaprime=1.0, θ=0.0

h	H1=0	H1=2.5	H1=5
t=1	0	0.7	0.8
t=2	0	1	1
t=3	0.1	1	1
t=4	0.2	1	1
t=5	0.7	0.7	0.7
t=6	–	–	–

n	H1=0	H1=2.5	H1=5
t=1	2	1	1
t=2	2	0.5	0.5
t=3	2	0.5	0.5
t=4	2	0.5	0.5
t=5	0.5	0.5	0.5
t=6	–	–	–

ω	H1=0	H1=2.5	H1=5
t=1	−0.3	−0.4	−0.4
t=2	0.2	0.1	0.1
t=3	0.1	0.1	0.1
t=4	0.1	0.1	0.1
t=5	0.1	0.1	0.1
t=6	–	–	–

s	H1=0	H1=2.5	H1=5
t=1	0	0	0
t=2	0	0	0
t=3	0	0	0
t=4	0	0	0
t=5	0	0	0
t=6	–	–	–

(1) lamda=1.0, lamdaprime=1.0, θ=0.4

h	H1=0	H1=2.5	H1=5
t=1	0.3	0.3	1
t=2	0.3	1	1
t=3	1	1	1
t=4	1	1	1
t=5	0.6	0.6	0.6
t=6	–	–	–

n	H1=0	H1=2.5	H1=5
t=1	2	1.5	0.5
t=2	1.5	0.5	0.5
t=3	0.5	0.5	0.5
t=4	0.5	0.5	0.5
t=5	0.5	0.5	0.5
t=6	–	–	–

ω	H1=0	H1=2.5	H1=5
t=1	−0.4	−0.5	−0.5
t=2	−0.4	−0.5	−0.5
t=3	−0.5	−0.5	−0.5
t=4	−0.5	−0.5	−0.5
t=5	−0.5	−0.5	−0.5
t=6	–	–	–

s	H1=0	H1=2.5	H1=5
t=1	0	0	0
t=2	0	0	0
t=3	0	0	0
t=4	0	0	0
t=5	0	0	0
t=6	–	–	–

(2) lamda=1.0, lamdaprime=0.0, θ=0.0

h	H1=0	H1=2.5	H1=5
t=1	0.7	0.5	0.5
t=2	0.9	1	1
t=3	0.6	0.6	0.6
t=4	0.7	0.7	0.7
t=5	0.7	0.7	0.7
t=6	–	–	–

n	H1=0	H1=2.5	H1=5
t=1	1	1.5	1.5
t=2	0.5	0.5	0.5
t=3	0.5	0.5	0.5
t=4	1	1	1
t=5	0.5	0.5	0.5
t=6	–	–	–

ω	H1=0	H1=2.5	H1=5
t=1	−0.5	−0.5	−0.5
t=2	0	0	0
t=3	0	0	0
t=4	−0.1	−0.1	−0.1
t=5	0.1	0.1	0.1
t=6	–	–	–

s	H1=0	H1=2.5	H1=5
t=1	0	0	0
t=2	0	0	0
t=3	0.1	0.1	0.1
t=4	0	0	0
t=5	0	0	0
t=6	–	–	–

Table 5-1 Effects of Social Security Tax
Case C (Both saving and compensation)

(2) lamda=1.0, lamdaprime=0.0, θ=0.4

h	H1=0	H1=2.5	H1=5
t=1	0	0.3	1
t=2	0.1	1	1
t=3	1	1	1
t=4	1	1	1
t=5	0.6	0.6	0.6
t=6	−	−	−

n	H1=0	H1=2.5	H1=5
t=1	2	1.5	0.5
t=2	2	0.5	0.5
t=3	0.5	0.5	0.5
t=4	0.5	0.5	0.5
t=5	0.5	0.5	0.5
t=6	−	−	−

ω	H1=0	H1=2.5	H1=5
t=1	−0.5	−0.5	−0.5
t=2	0	−0.5	−0.5
t=3	−0.2	−0.5	−0.5
t=4	−0.5	−0.5	−0.5
t=5	−0.5	−0.5	−0.5
t=6	−	−	−

s	H1=0	H1=2.5	H1=5
t=1	0	0	0
t=2	0	0	0
t=3	0	0	0
t=4	0	0	0
t=5	0	0	0
t=6	−	−	−

(3) lamda=0.0, lamdaprime=1.0, θ=0.0

h	H1=0	H1=2.5	H1=5
t=1	0	0	0
t=2	0	0	0
t=3	0	0	0
t=4	0	0	0
t=5	0	0	0
t=6	−	−	−

n	H1=0	H1=2.5	H1=5
t=1	0.5	0.5	0.5
t=2	0.5	0.5	0.5
t=3	0.5	0.5	0.5
t=4	0.5	0.5	0.5
t=5	2	2	2
t=6	−	−	−

ω	H1=0	H1=2.5	H1=5
t=1	−0.2	−0.2	−0.2
t=2	0.2	0.2	0.2
t=3	0.2	0.2	0.2
t=4	0.2	0.2	0.2
t=5	0.5	0.5	0.5
t=6	−	−	−

s	H1=0	H1=2.5	H1=5
t=1	0	0	0
t=2	0	0	0
t=3	0	0	0
t=4	0	0	0
t=5	0	0	0
t=6	−	−	−

(3) lamda=0.0, lamdaprime=1.0, θ=0.4

h	H1=0	H1=2.5	H1=5
t=1	0	0	0
t=2	0	0	0
t=3	0	0	0
t=4	0	0	0
t=5	0.4	0.4	0.4
t=6	−	−	−

n	H1=0	H1=2.5	H1=5
t=1	0.5	0.5	0.5
t=2	0.5	0.5	0.5
t=3	0.5	0.5	0.5
t=4	0.5	0.5	0.5
t=5	0.5	0.5	0.5
t=6	−	−	−

ω	H1=0	H1=2.5	H1=5
t=1	−0.2	−0.5	−0.5
t=2	0.2	0.2	0.2
t=3	0.2	0.2	0.2
t=4	0.2	0.2	0.2
t=5	0.4	0.4	0.4
t=6	−	−	−

s	H1=0	H1=2.5	H1=5
t=1	0	0	0
t=2	0	0	0
t=3	0	0	0
t=4	0	0	0
t=5	0	0	0
t=6	−	−	−

Table 5-2 Effects of Social Security Tax
Case C (Both saving and compensation)

Effect of Social Security Tax
Case C (Both Saving and Conpensation Economy)
(b) H1=0.0

(1) lamda=1.0, lamdaprime=1.0, θ =0.0

h	A=2	A=8
t=1	0	0.3
t=2	0	1
t=3	0	1
t=4	0	0.8
t=5	0	0.7
t=6	–	–

n	A=2	A=8
t=1	2	2
t=2	2	0.5
t=3	2	0.5
t=4	2	1
t=5	2	0.5
t=6	–	–

ω	A=2	A=8
t=1	−0.2	−0.3
t=2	0.2	0.1
t=3	0.2	0.1
t=4	0.2	−0.2
t=5	0.2	0.1
t=6	–	–

s	A=2	A=8
t=1	0	0
t=2	0	0
t=3	0	0.1
t=4	0	0
t=5	0	0
t=6	–	–

(1) lamda=1.0, lamdaprime=1.0, θ =0.4

h	A=2	A=8
t=1	0.1	0.7
t=2	0.5	1
t=3	0.2	1
t=4	0.9	1
t=5	0.5	0.7
t=6	–	–

n	A=2	A=8
t=1	2	1
t=2	1	0.5
t=3	1.5	0.5
t=4	0.5	0.5
t=5	0.5	0.5
t=6	–	–

ω	A=2	A=8
t=1	−0.2	−0.3
t=2	−0.4	−0.5
t=3	−0.5	−0.5
t=4	−0.5	−0.5
t=5	−0.5	−0.5
t=6	–	–

s	A=2	A=8
t=1	0	0
t=2	0	0
t=3	0	0
t=4	0	0
t=5	0	0
t=6	–	–

(2) lamda=1.0, lamdaprime=0.0, θ =0.0

h	A=2	A=8
t=1	0	0.3
t=2	0	0.7
t=3	0	1
t=4	0.1	0.7
t=5	0.6	0.7
t=6	–	–

n	A=2	A=8
t=1	2	2
t=2	2	1
t=3	2	0.5
t=4	2	1
t=5	0.5	0.5
t=6	–	–

ω	A=2	A=8
t=1	−0.4	−0.4
t=2	0	0
t=3	0	0
t=4	0	0
t=5	0.1	0.1
t=6	–	–

s	A=2	A=8
t=1	0	0
t=2	0	0
t=3	0	0
t=4	0	0
t=5	0	0
t=6	–	–

(2) lamda=1.0, lamdaprime=0.0, θ =0.4

h	A=2	A=8
t=1	0.1	0.8
t=2	0.6	1
t=3	0.2	1
t=4	0.9	1
t=5	0.5	0.7
t=6	–	–

n	A=2	A=8
t=1	2	1
t=2	1	0.5
t=3	1.5	0.5
t=4	0.5	0.5
t=5	0.5	0.5
t=6	–	–

ω	A=2	A=8
t=1	−0.4	−0.4
t=2	−0.5	−0.5
t=3	−0.5	−0.5
t=4	−0.5	−0.5
t=5	−0.5	−0.5
t=6	–	–

s	A=2	A=8
t=1	0	0
t=2	0	0
t=3	0	0
t=4	0	0
t=5	0	0
t=6	–	–

(3) lamda=0.0, lamdaprime=1.0, θ =0.0

h	A=2	A=8
t=1	0	0
t=2	0	0
t=3	0	0
t=4	0	0
t=5	0	0
t=6	–	–

n	A=2	A=8
t=1	0.5	0.5
t=2	0.5	0.5
t=3	0.5	0.5
t=4	0.5	0.5
t=5	2	2
t=6	–	–

ω	A=2	A=8
t=1	−0.1	−0.1
t=2	0.2	0.2
t=3	0.2	0.2
t=4	0.2	0.2
t=5	0.5	0.5
t=6	–	–

s	A=2	A=8
t=1	0	0
t=2	0	0
t=3	0	0
t=4	0	0
t=5	0	0
t=6	–	–

(3) lamda=0.0, lamdaprime=1.0, θ =0.4

h	A=2	A=8
t=1	0	0
t=2	0	0
t=3	0	0
t=4	0	0
t=5	0	0.5
t=6	–	–

n	A=2	A=8
t=1	0.5	0.5
t=2	0.5	0.5
t=3	0.5	0.5
t=4	0.5	0.5
t=5	2	0.5
t=6	–	–

ω	A=2	A=8
t=1	−0.1	−0.1
t=2	0.2	0.2
t=3	0.2	0.2
t=4	0.2	0.2
t=5	0.5	0.5
t=6	–	–

s	A=2	A=8
t=1	0	0
t=2	0	0
t=3	0	0
t=4	0	0
t=5	0	0
t=6	–	–

Table 5-3 Effects of Social Security Tax
Case C (Both saving and compensation)

Effect of Social Security Tax
Case C (Both Saving and Conpensation Economy)
(c) H1=10.0

(1) lamda=1.0, lamdaprime=1.0, θ=0.0

h	A=1	A=3
t=1	0.7	1
t=2	0	1
t=3	0	1
t=4	0	1
t=5	0	0.6
t=6	−	−

n	A=1	A=3
t=1	0.5	0.5
t=2	2	0.5
t=3	2	0.5
t=4	2	0.5
t=5	2	0.5
t=6	−	−

ω	A=1	A=3
t=1	−0.3	−0.3
t=2	−0.3	0.1
t=3	−0.5	0.1
t=4	0.2	0.1
t=5	0.2	0.1
t=6	−	−

s	A=1	A=3
t=1	0.1	0
t=2	0.1	0
t=3	0	0
t=4	0	0
t=5	0	0
t=6	−	−

(1) lamda=1.0, lamdaprime=1.0, θ=0.4

h	A=1	A=3
t=1	0.1	0.9
t=2	0	0.9
t=3	0	0.9
t=4	0	1
t=5	0	0.6
t=6	−	−

n	A=1	A=3
t=1	2	0.5
t=2	2	0.5
t=3	2	0.5
t=4	2	0.5
t=5	2	0.5
t=6	−	−

ω	A=1	A=3
t=1	−0.5	−0.5
t=2	−0.2	−0.5
t=3	0.2	−0.5
t=4	0.2	−0.5
t=5	0.2	−0.5
t=6	−	−

s	A=1	A=3
t=1	0	0
t=2	0	0
t=3	0	0
t=4	0	0
t=5	0	0
t=6	−	−

(2) lamda=1.0, lamdaprime=0.0, θ=0.0

h	A=1	A=3
t=1	0.6	1
t=2	0	1
t=3	0	1
t=4	0	1
t=5	0	0.6
t=6	−	−

n	A=1	A=3
t=1	0.5	0.5
t=2	2	0.5
t=3	2	0.5
t=4	2	0.5
t=5	2	0.5
t=6	−	−

ω	A=1	A=3
t=1	−0.5	−0.5
t=2	−0.5	−0.1
t=3	−0.5	0
t=4	0	0
t=5	0.1	0.1
t=6	−	−

s	A=1	A=3
t=1	0.1	0.1
t=2	0.1	0
t=3	0	0
t=4	0	0
t=5	0	0
t=6	−	−

(2) lamda=1.0, lamdaprime=0.0, θ=0.4

h	A=1	A=3
t=1	0	0.2
t=2	0	0.1
t=3	0	0
t=4	0	0.1
t=5	0	0
t=6	−	−

n	A=1	A=3
t=1	2	1.5
t=2	2	2
t=3	2	2
t=4	2	2
t=5	2	2
t=6	−	−

ω	A=1	A=3
t=1	−0.5	−0.5
t=2	−0.5	−0.5
t=3	0	−0.5
t=4	0	−0.3
t=5	0.1	0.1
t=6	−	−

s	A=1	A=3
t=1	0.1	0
t=2	0	0
t=3	0	0.1
t=4	0	0
t=5	0	0
t=6	−	−

(3) lamda=0.0, lamdaprime=1.0, θ=0.0

h	A=1	A=3
t=1	0	0
t=2	0	0
t=3	0	0
t=4	0	0
t=5	0	0
t=6	−	−

n	A=1	A=3
t=1	0.5	0.5
t=2	0.5	0.5
t=3	0.5	0.5
t=4	0.5	0.5
t=5	2	2
t=6	−	−

ω	A=1	A=3
t=1	−0.2	−0.2
t=2	0.2	0.2
t=3	0.2	0.2
t=4	0.2	0.2
t=5	0.5	0.5
t=6	−	−

s	A=1	A=3
t=1	0	0
t=2	0	0
t=3	0	0
t=4	0	0
t=5	0	0
t=6	−	−

(3) lamda=0.0, lamdaprime=1.0, θ=0.4

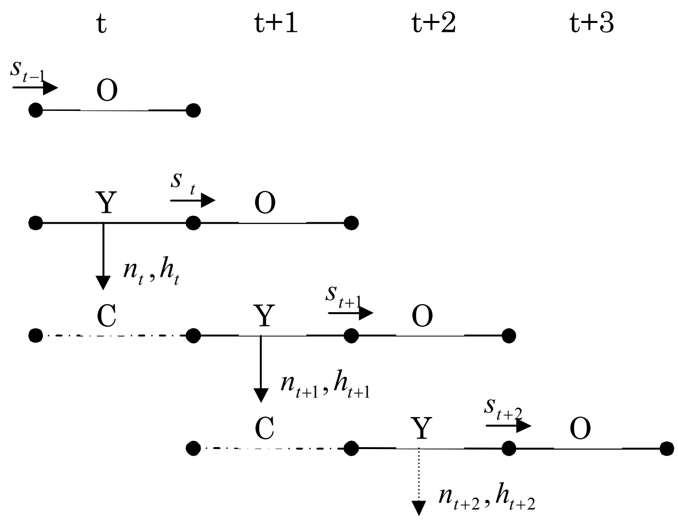
h	A=1	A=3
t=1	0	0
t=2	0	0
t=3	0	0
t=4	0	0
t=5	0	0
t=6	−	−

n	A=1	A=3
t=1	0.5	0.5
t=2	0.5	0.5
t=3	0.5	0.5
t=4	0.5	0.5
t=5	2	2
t=6	−	−

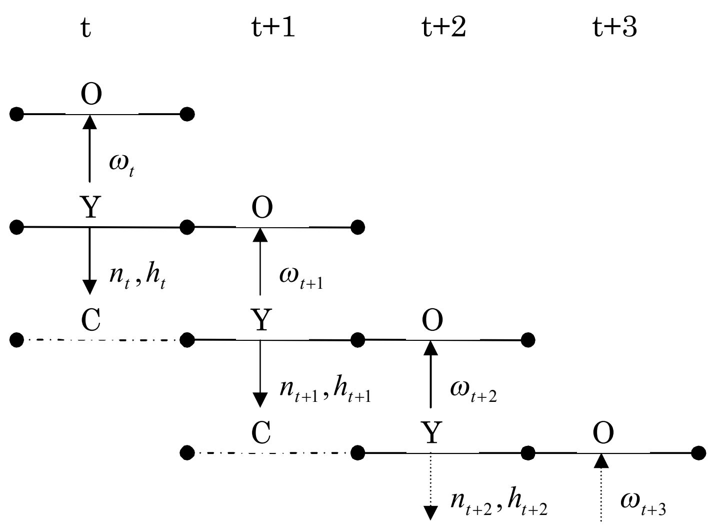
ω	A=1	A=3
t=1	−0.5	−0.5
t=2	0.2	0.2
t=3	0.2	0.2
t=4	0.2	0.2
t=5	0.5	0.5
t=6	−	−

s	A=1	A=3
t=1	0	0
t=2	0	0
t=3	0	0
t=4	0	0
t=5	0	0
t=6	−	−

Table 5-4 Effects of Social Security Tax
Case C (Both saving and compensation)

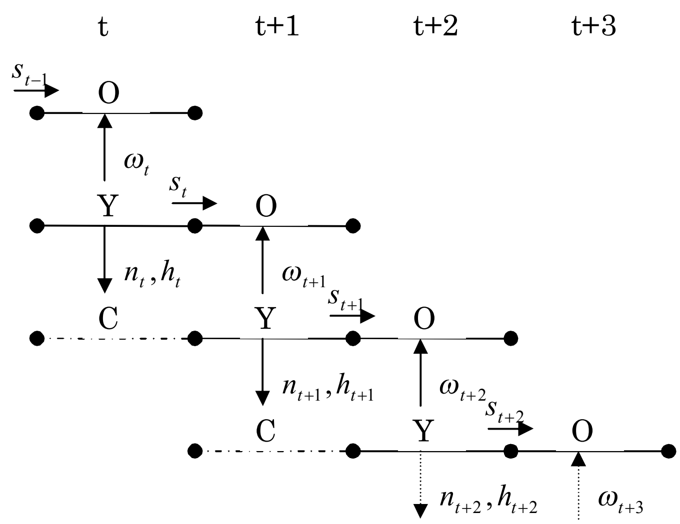


Case A



Case B

Figure 2 Specification of Intergenerational Linkage



Case C

Figure 2 Specification of Intergenerational Linkage (Cont'd)

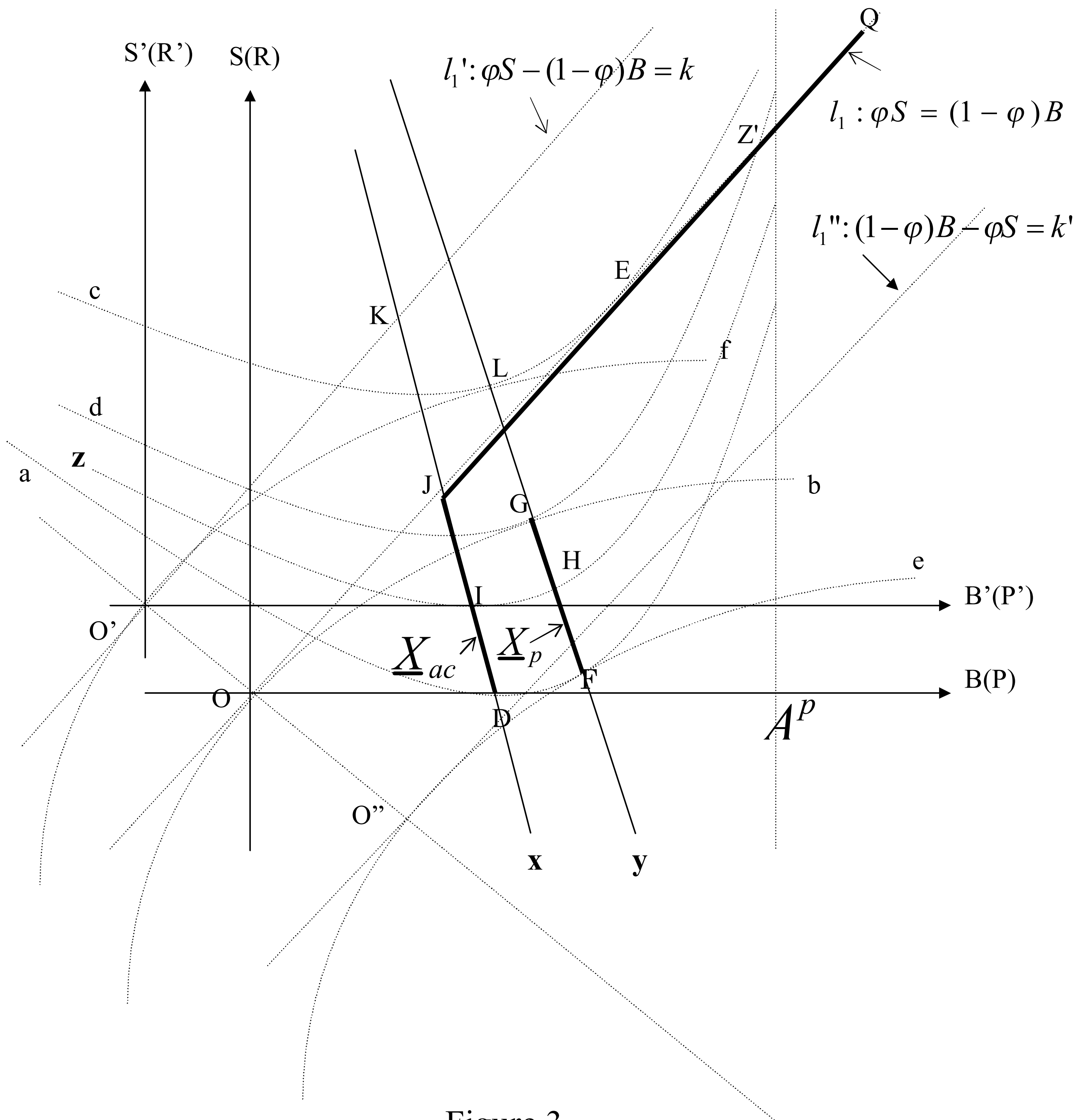


Figure 3
Intra-family & Inter-generational Transfer and Unfunded Social Security

$l_2 : S = -B$

Appendix 1- Equilibrium conditions in stagnant/growth steady states for case A and B

In these two states, I require just the following stationary conditions:

$V_t(\ )=V(\ )$, $H_{t+1}(\ )=H_{+1}(\ )$, $H_{t+2}(\ )=H_{+2}(\ )$, $s_t(\ )=s(\ )$, $n_t(\ )=n(\ )$, $\omega_t(\ )=\omega(\ )$, $s_{t+1}(\ )=s_{+1}(\ )$, $n_{t+1}(\ )=n_{+1}(\ )$, $\omega_{t+1}(\ )=\omega_{+1}(\ )$ etc.

<u>*Case A*</u>

Stagnant equilibrium

In a stagnant equilibrium, (4.3) can be rewritten as:

$$s: \quad [1-vn-hn-s] \le \left\{D^{\sigma-1}(\delta m)^{-1}(s)^{\sigma m-m+1}\right\}^{1/\sigma} \tag{A1.1a}$$

$$H: \quad \left[-\frac{n}{A}+\lambda n^{1-\varepsilon}\{1-vn-s\}\right] + \left(\frac{\partial V}{\partial s_{+1}}\frac{\partial s_{+1}}{\partial H_{+1}}+\frac{\partial V}{\partial n_{+1}}\frac{\partial n_{+1}}{\partial H_{+1}}+\frac{\partial V}{\partial H_{+2}}\frac{\partial H_{+2}}{\partial H_{+1}}\right)[1-vn-hn-s]^{\sigma} \le 0 \tag{A1.2a}$$

$$n: \quad \frac{\lambda(1-\varepsilon)n^{-\varepsilon}}{(1-\sigma)}[1-vn-hn-s] \le v+h \tag{A1.3a}$$

Growth equilibrium

On the other hand, assume that the economy is in a *growth* equilibrium. Then, under condition (x), (4.3) are reduced to:

$s:$ Same as stagnant eq. (A1.1a)

$$H: \quad \left[-\frac{n}{A}+\lambda n^{1-\varepsilon}(Ah)^{-\sigma}\{1-vn-s\}\right] + \left(\frac{\partial V}{\partial \eta_{+1}}\frac{\partial \eta_{+1}}{\partial H_{+1}}+\frac{\partial V}{\partial n_{+1}}\frac{\partial n_{+1}}{\partial H_{+1}}+\frac{\partial V}{\partial H_{+2}}\frac{\partial H_{+2}}{\partial H_{+1}}\right)[1-vn-hn-s]^{\sigma} \le 0 \tag{A1.2b}$$

$$n: \quad \frac{\lambda(1-\varepsilon)n^{-\varepsilon}}{(1-\sigma)}[1-vn-hn-s](Ah)^{1-\sigma} \le v+h \tag{A1.3b}$$

Case B

Stagnant equilibrium

Under condition (i), (5.2) can be rewritten as:

$$\omega: \quad [1 - vn - hn - \omega] \leq \left\{\delta\lambda' n^{\varepsilon}\right\}^{-1/\sigma} \omega n \qquad \text{(A1.4a)}$$

$$H: \quad \begin{aligned} &\left[-\frac{n}{A} + \lambda n^{1-\varepsilon}\left\{1 - vn - \omega\right\}\right](1 - vn - hn - \omega)^{-\sigma} \\ &+ \delta(\omega n)^{1-\sigma} \\ &+ \left(\frac{\partial V}{\partial \omega_{+1}}\frac{\partial \omega_{+1}}{\partial H_{+1}} + \frac{\partial V}{\partial n_{+1}}\frac{\partial n_{+1}}{\partial H_{+1}} + \frac{\partial V}{\partial H_{+2}}\frac{\partial H_{+2}}{\partial H_{+1}}\right) \leq 0 \end{aligned} \qquad \text{(A1.5a)}$$

$$n: \quad \begin{aligned} &\left\{-(v + h) + \lambda(1-\varepsilon)n^{-\varepsilon}\frac{[1 - vn - hn - \omega]}{(1-\sigma)}\right\}(1 - vn - hn - \omega)^{-\sigma} \\ &+ \delta\omega(\omega n)^{-\sigma} \\ &+ \left(\frac{\partial V}{\partial \omega_{+1}}\frac{\partial \omega_{+1}}{\partial n} + \frac{\partial V}{\partial n_{+1}}\frac{\partial n_{+1}}{\partial n} + \frac{\partial V}{\partial H_{+2}}\frac{\partial H_{+2}}{\partial n}\right) \leq 0 \end{aligned} \qquad \text{(A1.6a)}$$

Growth equilibrium

On the other hand, assume that the economy is in a *growth* equilibrium. Then, under condition (x), (5.2) are reduced to:

ω: Same as stagnant equilibrium. (A1.4a)

$$H: \quad \begin{aligned} &\left[-\frac{n}{A} + \lambda n^{1-\varepsilon}(Ah)^{-\sigma}\left\{1 - vn - \omega\right\}\right](1 - vn - hn - \omega)^{-\sigma} \\ &+ \delta(\omega n)^{1-\sigma}(Ah)^{-\sigma} \\ &+ \left(\frac{\partial V}{\partial \omega_{+1}}\frac{\partial \omega_{+1}}{\partial H_{+1}} + \frac{\partial V}{\partial n_{+1}}\frac{\partial n_{+1}}{\partial H_{+1}} + \frac{\partial V}{\partial H_{+2}}\frac{\partial H_{+2}}{\partial H_{+1}}\right) \leq 0 \end{aligned} \qquad \text{(A1.5b)}$$

n:

$$\left\{-(v+h)+\lambda(1-\varepsilon)n^{-\varepsilon}(Ah)^{1-\sigma}\frac{[1-vn-hn-\omega]}{(1-\sigma)}\right\}(1-vn-hn-\omega)^{-\sigma}$$
$$+\delta\omega(\omega n)^{-\sigma}(Ah)^{1-\sigma} \qquad \text{(A1.6b)}$$
$$+\left(\frac{\partial V}{\partial\omega_{+1}}\frac{\partial\omega_{+1}}{\partial n}+\frac{\partial V}{\partial n_{+1}}\frac{\partial n_{+1}}{\partial n}+\frac{\partial V}{\partial H_{+2}}\frac{\partial H_{+2}}{\partial n}\right)\leq 0$$

Appendix 2- Some aspects of dynastic utility approach

At first I construct a *simple* dynastic utility, according to Becker et al. (1990).

$$V_t^T = u_t + \hat{\lambda}(n_t)V_{t+1}^T$$
$$= u_t + \lambda\pi_1 a(n_t)n_t u_{t+1} + \lambda^2\pi_1{}^2 a(n_t)n_t a(n_{t+1})n_{t+1}u_{t+2} + \cdots + (\lambda\pi_1)^{T-t}\left(\prod_{u=t}^{T-1}a(n_u)n_u\right)u_T \qquad \text{(A2.1)}$$

Abbreviating notation T just for simplicity, V_t^T and V_{t+1}^T can be replaced with V_t and V_{t+1}, respectively. Here $a(n)\equiv n^{-\varepsilon}$ $(0\leq\varepsilon\leq 1)$.

Dynastic equilibrium conditions of a simple dynastic utility maximizer

Assuming that production function is linear ($\gamma=1$) with human capital, and human capital market is fully competitive ($w_t=1$), and the economy is in “dynamic consistency” ($\lambda\lambda'\neq 1$), a simple dynastic utility maximizer requires the following first order *equilibrium* conditions to hold for all t:[51]

$$\omega_{t+1}: \qquad \delta\pi_2 u'(c_{2,t+1})\pi_1 n_t H_{t+1} + \lambda\pi_1 a(n_t)n_t u'(c_{1,t+1})\{-\pi_2 H_{t+1}\} = 0 \qquad \text{(A2.2)}$$

[51] In this dynastic framework, the first order conditions are calculated for H_{t+1} and N_{t+1} (and ω_{t+1}, s_t), not h_t and n_t, because h_t and n_t have strategic effects in dynastic utility for infinitely descendant generations.

$$H_{t+1}: \quad \begin{aligned} & u'(c_{1,t})\left(-\frac{\varsigma}{A}n_t\right)+\delta\pi_2 u'(c_{2,t+1})\{\omega_{t+1}\pi_1 n_t\} \\ & +\lambda\pi_1 a(n_t)n_t\begin{bmatrix} u'(c_{1,t+1})\{(1-vn_{t+1}-\rho s_{t+1})-\omega_{t+1}\pi_2\} \\ +\delta\pi_2 u'(c_{2,t+2})D(s_{t+1})^m \end{bmatrix} \le 0 \end{aligned} \qquad \text{(A2.3)}$$

with equality if $H_{t+1} > 0$ $(h_t > 0)$

$$N_{t+1}: \quad \begin{aligned} & u'(c_{1,t})\left\{-v(H_t+\overline{H})-\frac{\varsigma}{A}H_{t+1}\right\}+\delta\pi_2 u'(c_{2,t+1})\omega_{t+1}\pi_1(H_{t+1}+\overline{H}) \\ & +\lambda\pi_1\begin{bmatrix} a(n_t)n_t u'(c_{1,t+1})\left\{-v(H_{t+1}+\overline{H})-\frac{\varsigma}{A}H_{t+2}\right\}\left(-\frac{n_{t+1}}{n_t}\right) \\ +u(c_{1,t+1})\{a(n_t)+a'(n_t)n_t\} \end{bmatrix} \\ & +\lambda\pi_1\delta\pi_2\begin{bmatrix} a(n_t)n_t u'(c_{2,t+2})\omega_{t+2}\pi_1(H_{t+2}+\overline{H})\left(-\frac{n_{t+1}}{n_t}\right) \\ +u(c_{2,t+2})\{a(n_t)+a'(n_t)n_t\} \end{bmatrix} \le 0 \end{aligned} \qquad \text{(A2.4)}$$

with equality if $N_{t+1} > 0$ $(n_t > 0)$

$$s_t: \quad u'(c_{1,t})\{-\rho(H_t+\overline{H})\}+\delta\pi_2 u'(c_{2,t+1})\{D(H_t+\overline{H})m(s_t)^{m-1}\} \le 0 \qquad \text{(A2.5)}$$

with equality if $s_t > 0$

Note that F.O.C. for H_{t+1} (human capital) and N_{t+1} (fertility) prevail over three adjacent periods, not two. Thus (A2.2), (A2.3) and (A2.5) can be rewritten as:

$$\omega_{t+1}: \quad \frac{u'(c_{2,t+1})}{u'(c_{1,t+1})} = \frac{\lambda a(n_t)}{\delta} \qquad \text{(A2.2-a)}$$

$$H_{t+1}: \frac{u'(c_{1,t+1})}{u'(c_{1,t})}[1-vn_{t+1}-\rho s_{t+1}]+\frac{u'(c_{2,t+2})}{u'(c_{1,t})}\delta\pi_2 D(s_{t+1})^m \le \frac{\varsigma}{A\lambda\pi_1 a(n_t)} \qquad \text{(A2.3-a)}$$

$$s_t: \quad \frac{u'(c_{2,t+1})}{u'(c_{1,t})} \le \frac{\rho}{\delta\pi_2 Dm(s_t)^{m-1}} \qquad \text{(A2.5-a)}$$

Assume $D = 0$ (i.e., $s_t = s_{t+1} = 0$). Then from (A2.2-a):

$$h_{t+1} = \frac{1 - vn_{t+1} - \left[\pi_1 n_t \left\{\frac{\lambda a(n_t)}{\delta}\right\}^{1/\sigma} + \pi_2\right]\omega_{t+1}}{\varsigma n_{t+1}} \tag{A2.6}$$

From (A2.3-a):

$$h_t \le \frac{\left[(1 - vn_t - \pi_2\omega_t)(H_t + \overline{H}) - \omega_{t+1}\pi_1 n_t \overline{H}\left\{\frac{A\delta\pi_1(1 - vn_{t+1})}{\varsigma}\right\}^{-1/\sigma}\right]}{(H_t + \overline{H})n_t\left[\varsigma + \omega_{t+1}\pi_1 A\left\{\frac{A\delta\pi_1(1 - vn_{t+1})}{\varsigma}\right\}^{-1/\sigma}\right]} \quad \text{and}$$

$$H_{t+1} \le \frac{A\left[(1 - vn_t - \pi_2\omega_t)(H_t + \overline{H}) - \omega_{t+1}\pi_1 n_t \overline{H}\left\{\frac{A\delta\pi_1(1 - vn_{t+1})}{\varsigma}\right\}^{-1/\sigma}\right]}{n_t\left[\varsigma + \omega_{t+1}\pi_1 A\left\{\frac{A\delta\pi_1(1 - vn_{t+1})}{\varsigma}\right\}^{-1/\sigma}\right]} \tag{A2.7}$$

Two steady states

Stagnant equilibrium

In a stagnant equilibrium, $c_{1,t} = c_{1,t+1} \equiv c_1$, $c_{2,t} = c_{2,t+1} \equiv c_2$ and $H_t = H_{t+1} = Ah\overline{H}/(1 - Ah) \equiv H$.[52] Then (A2.2), (A2.3), (A2.4) and (A2.5) are equivalent with:

$$\omega: \qquad \frac{u'(c_2)}{u'(c_1)} = \frac{\lambda a(n)}{\delta} \quad \text{or}$$

$$[\pi_1\omega n + Ds^m] = \left\{\frac{\lambda a(n)}{\delta}\right\}^{-1/\sigma}[1 - vn - \varsigma hn - \rho s - \pi_2\omega] \tag{A2.2-b}$$

$$H: \qquad \lambda\pi_1 a(n)[(1 - vn - \rho s) + \lambda a(n)\pi_2 Ds^m]\frac{A}{\varsigma} \le 1 \tag{A2.3-b}$$

[52] Therefore, $H + \overline{H} = \overline{H}/(1 - Ah)$.

$$N: \quad \begin{aligned} & -\{1-\lambda\pi_1 a(n)n\}\left[(v-\lambda a(n)\omega\pi_1\pi_2)(H+\overline{H})+\frac{\varsigma}{A}H\right] \\ & +\lambda\pi_1\{a(n)+a'(n)n\}\left[\frac{u(c_1)}{u'(c_1)}+\lambda\pi_2 a(n)\frac{u(c_2)}{u'(c_2)}\right]\le 0 \end{aligned} \qquad \text{(A2.4-b)}$$

$$s: \quad \frac{\lambda a(n)\pi_2 Dms^{m-1}}{\rho}\le 1 \qquad \text{(A2.5-b)}$$

Now assume, for simplicity, condition (y): $\pi_1=\pi_2=1$, $a(n)=1/n$ $(\varepsilon=1.0)$ and $\rho=\varsigma=\gamma=1$.[53] Then (A2.2,3,4,5-b) can be rewritten as:

$$\omega: \quad [\omega n+Ds^m]=\left(\frac{\delta n}{\lambda}\right)[1-vn-hn-s-\omega] \qquad \text{(A2.2-c)}$$

$$H: \quad \frac{\lambda}{n}[(1-vn-s)+\frac{\lambda}{n}Ds^m]A\le 1 \qquad \text{(A2.3-c)}$$

$$N: \quad -(1-\lambda)\left\{v+h-\frac{\lambda\omega}{n}\right\}\le 0 \qquad \text{(A2.4-c)}$$

$$s: \quad \frac{\lambda Dms^{m-1}}{n}\le 1 \qquad \text{(A2.5-c)}$$

Growth equilibrium

On the other hand, in a *growth* equilibrium, $H_{t+1}=AhH_t$ ($H_t>>\overline{H}$), $c_{1,t+1}=Ahc_{1,t}=Ahc_1$, $c_{2,t+1}=Ahc_{2,t}=Ahc_2$. Then (A2.2,3,4,5-b) are reduced to:

$$\omega: \quad \frac{u'(c_2)}{u'(c_1)}=\frac{\lambda a(n)}{\delta}(Ah)^{-\sigma} \quad \text{or}$$

$$[\pi_1\omega nAh+Ds^m]=\left\{\frac{\lambda a(n)}{\delta}\right\}^{-1/\sigma}Ah[1-vn-\varsigma hn-\rho s-\pi_2\omega] \qquad \text{(A2.2-d)}$$

$$H: \quad \lambda\pi_1 a(n)(Ah)^{-\sigma}[(1-vn-\rho s)+(Ah)^{-\sigma}\lambda a(n)\pi_2 Ds^m]\frac{A}{\varsigma}\le 1 \qquad \text{(A2.3-d)}$$

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[53] $\varepsilon=1.0$ denotes perfect inelasticity of altruism per child.

$$N: \quad \begin{aligned} & -\{1-\lambda\pi_1 a(n)n(Ah)^{1-\sigma}\}\left[(v+\varsigma h)-\lambda a(n)\omega\pi_1\pi_2(Ah)^{1-\sigma}\right]H \\ & +\lambda\pi_1\{a(n)+a'(n)n\}(Ah)^{1-\sigma}\left[\frac{u(c_1)}{u'(c_1)}+\lambda a(n)\pi_2(Ah)^{-\sigma}\frac{u(c_2)}{u'(c_2)}\right]\leq 0 \end{aligned} \qquad \text{(A2.4-d)}$$

$$s: \quad \frac{(Ah)^{-\sigma}\lambda a(n)\pi_2 Dms^{m-1}}{\rho}\leq 1 \qquad \text{(A2.5-d)}$$

Under condition (y),

$$\omega: \quad [\omega nAh+Ds^m]=\left(\frac{\delta n}{\lambda}\right)Ah[1-vn-hn-s-\omega] \qquad \text{(A2.2-e)}$$

$$H: \quad \frac{\lambda}{nh}[(1-vn-s)+\frac{\lambda}{Ahn}Ds^m]\leq 1 \qquad \text{(A2.3-e)}$$

$$N: \quad -(1-\lambda)\left\{v+h-\frac{\lambda\omega}{n}\right\}\leq 0 \qquad \text{(A2.4-e)[54]}$$

$$s: \quad \frac{\lambda Dms^{m-1}}{Ahn}\leq 1 \qquad \text{(A2.5-e)}$$

For example, it is easily seen, from (A2.6), that, in case of $D=0$, a steady state could be in a *growth* equilibrium, if:

$$1\leq\frac{A(1-vn-\pi_2\omega)}{n\left[\varsigma+\omega\pi_1 A\left\{\frac{A\delta\pi_1(1-vn)}{\varsigma}\right\}^{-1/\sigma}\right]} \qquad \text{(A2.8)}$$

Proposition A2-1: Assume that condition (y) is satisfied and that fertility decision is fixed at n. If the economy is in a *stagnant* equilibrium, then $h=0$ and the following condition is satisfied:

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[54] This is the same as (A2.4-c). For $\sigma\neq 1$, (7-3-e) is replaced with:

$$N: -\left[1-\lambda(Ah)^{1-\sigma}\right]\left\{v+h-\frac{\lambda\omega}{n}(Ah)^{1-\sigma}\right\}\leq 0$$

$$A\left[(1-nv)+(1-\lambda m)\left\{\frac{D(\lambda m)^m}{n}\right\}^{1/(1-m)}\right]<n$$

Proposition A2-2: Assume that condition (y) is satisfied and that fertility decision is fixed at n. If the economy is in a *growth* equilibrium, then the solution of the following equation,

$$x\left[(1-nv)+(1-m)\left(\frac{Dm^m}{A}\right)^{1/(1-m)} x^{1/(1-m)}\right]=1 \quad \text{where} \quad x\equiv\frac{\lambda}{nh} \quad (h=\frac{\lambda}{nx}),$$

satisfies $(A\lambda)/(nx)>1 \quad (Ah>1)$.

Proposition A2-3: (Rate of return in saving and bearing children)

In a stagnant steady state: $\dfrac{Ds^{m-1}}{\rho}\geq\dfrac{\omega\pi_1 n}{vn+\varsigma hn}$

In a growth steady state: $\dfrac{Ds^{m-1}}{\rho}\geq\dfrac{\omega\pi_1 nAh}{vn+\varsigma hn}$

In both cases: $1\geq\dfrac{\omega\pi_1\rho\lambda m}{vn+\varsigma hn}$

With equality in both states, if the economy is infinitely *egoistic,* $\lambda\to 0$.

Proposition A2-4: Assume $\varepsilon=1.0$, in which the generation holds perfectly inelastic altruism with number of child. Then the fertility n is at the corner solution if $\lambda>1$, but at the interior solution if $\lambda<1$.

Appendix 3- Supplementary explanation regarding short-run aspects of intra-family & inter-generational transfer and unfunded social security

In this appendix, I explain, using Figure 3 and on the basis of results obtained in Aoki (2007), some short-run aspects of intra-family & inter-generational transfer and unfunded social security, all of which support proposition 9-2 and corollary 9-1 to 4.

In Figure 3, $S(R)$ and $B(P)$ axes, beginning from original point O, denote support (receipt) and bequest (payment) between old parents and young adult children under no certainty premium transferred between the generations. a, c, d, z are indifference curves of old parents' indirect utility regarding insurance contract under their mortality risk, and b, e, f are those of young adult children. The contract curve, which satisfy both Pareto optimality and participation constraints with no actuarially fair insurance available, is GF, and the self-insuring contract curve with no intra-family risk sharing available is DJ. Assume that a fixed level of mandatory certainty premium transfer from young children to old parents, OO', takes place. Then corresponding to new original point and axes O', $S'(R')$ and $B'(P')$, DJ move to IK on line x, and GF does to LH on y, respectively. From lemmas 1 and 2 of Aoki (2007), the slopes of line x and y prove to be less than -1. This aspect also proves corollary 9-1 to 5, as well as proposition 2. For example, corollary 9-3 is derived by observing $O'I' > OD$.