Granularity Shock: A Small Perturbation Two-Factor Model

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This paper proposes a small perturbation two-factor model designed to capture granularity risk, extending the classical Vasicek Asymptotic Single Risk Factor (ASRF) portfolio loss model. By applying the Lyapunov Central Limit Theorem, we demonstrate that, for small Herfindahl-Hirschman Index (HHI) values, granularity risk – conditional on market risk – is approximately proportional to a standard normal random variable. Instead of analyzing heterogeneous portfolios directly, we focus on a homogeneous portfolio subject to a small perturbation induced by granularity risk. We propose the *Vasicek-Herfindahl portfolio loss distribution*, which extends the Vasicek portfolio loss distribution to account for portfolio concentration. Utilizing this distribution, we derive closed-form granularity adjustments for the probability density function (PDF) and cumulative distribution function (CDF) of portfolio loss, as well as for Value at Risk (VaR) and Expected Shortfall (ES). We compare our primary results with existing findings and validate them through Monte Carlo simulations.

Keywords
Credit portfolio model; Granularity adjustment; Value at Risk; Expected Shortfall

1 Introduction

The Vasicek (1987) model, founded on the Law of Large Numbers (LLN), assumes perfect granularity by considering a homogeneous portfolio with equal weights for all exposures. Under this assumption, the portfolio loss converges almost surely to its conditional expectation given the systematic risk factor. However, this framework is inadequate for heterogeneous portfolios with varying loan sizes, where a residual "granularity risk" persists due to deviations from perfect diversification.

Gordy (2003) demonstrated that, under mild regularity conditions, the portfolio loss in a large heterogeneous portfolio converges almost surely to its conditional expectation given the market factor. He also highlighted the importance of the Herfindahl-Hirschman Index (HHI) as a key measure of granularity adjustment (GA).

The foundation for further study of GA to VaR was provided by Gouriéroux *et al* (2000), who calculated the first and second derivatives of VaR.

Emmer & Tasche (2005) obtained GA to VaR for both the general case of loss distribution and for the case of the Vasicek model. The formula of GA to VaR for the case of the Vasicek model was refined by Gordy & Lutkebohmert (2013).

Voropaev (2011) then moved on to studying the behavior of the portfolio loss PDF and granularity adjustments to VaR and ES, using a moment-based method.

Since the Vasicek model, based on the LLN, is not suitable for capturing granularity effects, it is advisable to adapt it for such cases using Lyapunov's Central Limit Theorem (CLT). This

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approach allows to account for variations in loan seizes within the portfolio and provides a modeling of loss distribution under granular conditions.

Currently, there is a “granularity gap” in the regulation of credit risk. On one hand, a primitive archaic approach is used that considers the sizes of loans within the portfolio but ignores the correlations between these assets. On the other hand, the more advanced Internal Ratings-Based (IRB) approach accounts for correlations but neglects the varying sizes of loans in the portfolio. This paper aims to fill this gap concerning small values of HHI. The Vasicek-Herfindahl portfolio loss distribution introduced in our paper can be used to regulate a bank’s economic capital.

This paper is organized as follows:

• Section 2 introduces the main focus of our study and provides essential background on the Vasicek model.

• Section 3 examines the behavior of the portfolio loss random variable as the Herfindahl-Hirschman Index (HHI) approaches zero. We also derive the Vasicek-Herfindahl portfolio loss distribution and analyze its key properties. Furthermore, we compute Value at Risk (VaR) and the corresponding adjustment (GA to VaR) using our methodology, comparing these results with those presented by Emmer & Tasche (2005).

• Section 4 extends this analysis to calculate Expected Shortfall (ES) and its adjustment (GA to ES) within our framework.

• Section 5 explores the impact of adding a new loan to an existing heterogeneous portfolio, demonstrating how our model adapts to such changes.

• Section 6 provides a review of the approach proposed by Emmer & Tasche (2005).

• Section 7 discusses the moment-based approach of Voropaev (2011).

• Section 8 examines Vasicek’s (2002) attempt to incorporate granularity effects into credit risk modeling.

• Section 9 concludes the paper with a summary of key findings and implications.

2 Model Framework

Consider a portfolio with n loans, where the weight of the k-th loan is w_k, satisfying:

$$\sum_{k=1}^{n} w_k = 1 \tag{2.1}$$

with the constraint $w_k \geq 0$ for each k.

The degree of concentration within the portfolio is measured by the Herfindahl-Hirschman Index (HHI), defined as:

$$h_n = \sum_{k=1}^{n} {w_k}^2 \tag{2.2}$$

This metric plays a central role in analyzing how portfolio concentration influences risk exposure.

The portfolio loss is defined as:

$$Loss(\{X_k\}_{k=1}^n, Y) = \sum_{k=1}^{n} w_k \, l(X_k, Y) \tag{2.3}$$

where:

- Y is a standard normal random variable representing systematic (market) risk,
- Each X_k is a standard normal random variable capturing individual (idiosyncratic) risk,
- The set $\{X_k\}_{k=1}^n$ and Y are assumed to be independent and identically distributed (i.i.d.),
- The indicator variable $l(X_k, Y)$ equals 1 if loan k defaults and 0 otherwise.

The expected value of the default indicator for each loan is:

$$\mathbb{E}[l(X_k, Y)] = PD \tag{2.4}$$

and its variance is:

$$var[l(X_k, Y)] = PD(1 - PD) \tag{2.5}$$

where PD is the probability of default.

Assume that $l(X_k, y)$ is equal to 1 with probability $p(y)$ (the default) and 0 otherwise, where y is a realization of the market shock Y. The random variable $l(X_k, Y)$, conditional on the market shock Y, follows a Bernoulli distribution:

$$l|Y \sim Bernoulli(p(Y)) \tag{2.6}$$

The conditional mean of the default indicator is:

$$\mathbb{E}[l|Y] = p(Y) \tag{2.7}$$

and its conditional variance is:

$$\sigma^2(p(Y)) = p(Y)(1 - p(Y)) \tag{2.8}$$

Following Vasicek (2002), the conditional probability of default for each loan is specified as:

$$p(Y) = \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho}Y}{\sqrt{1-\rho}}\right) \tag{2.9}$$

where

• $\rho \in [0,1]$ is the asset correlation,

• Φ is the standard normal cumulative distribution function (CDF),

• Φ^{-1} is the inverse of Φ.

The conditional expected loss of the portfolio given Y is expressed as:

$$p(Y) = \mathbb{E}[Loss|Y] \tag{2.10}$$

The Vasicek CDF is given by:

$$F_V(x; PD, \rho) = \Phi\left(-p^{-1}(x)\right) = \Phi\left(\frac{\sqrt{1-\rho}\Phi^{-1}(x) - \Phi^{-1}(PD)}{\sqrt{\rho}}\right) \tag{2.11}$$

If all weights w_i are equal, then, by the Law of Large Numbers (LLN), the random value $Loss|Y$ converges in probability to its conditional mean $p(Y)$:

$$Loss|Y \xrightarrow{P} p(Y) \tag{2.12}$$

(as established in Vasicek (2002)).

3 Asymptotic Loss

Due to the Lyapunov CLT, the portfolio loss converges in distribution to an *asymptotic loss* L:

$$Loss(\{X_k\}_{k=1}^{n}, Y) \xrightarrow{d} L(Z, V) = V + \sqrt{h}\sigma(V)Z \tag{3.1}$$

where:

• $Z \sim \mathcal{N}(0,1)$: a standard normal random variable, independent of V. It generates *granularity risk*.

• $V = p(Y) \sim Vasicek(PD, \rho)$, where $Vasicek(PD, \rho)$ represents the Vasicek loan loss distribution.

• $\sigma(V) = \sqrt{V(1-V)}$.

• $h = \lim_{n\to\infty} h_n$.

Proof details are provided in the Appendix 1.

The heterogeneous portfolio loss risk encompasses not only market risk but also the granularity risk, which is represented by the term $\sqrt{h}\sigma(V)Z$. It is important to note that granularity risk is influenced by market risk.

If $h = 0$ (perfect granularity), then $L(Z, V) = V$. Conversely, the case where $h = 1$ indicates full concentration, occurring when the weight of one of the loans is 1 and the weights of all the others are 0.

The range of the function $Loss(\{X_k\}_{k=1}^{n}, Y)$ is the unit interval $[0,1]$, while the range of the function $L(Z, V)$ is $\mathbb{R}$. However, when $h = 0$, the range of the function $L(Z, V)$ is narrowed to the unit interval $[0,1]$.

The asymptotic portfolio loss risk L is the sum of the "classical" Vasicek portfolio loss risk V and the granularity risk $H = \sqrt{h}\sigma(V)Z$:

$$L = V + H \tag{3.2}$$

Conditional mean of the loss given V:

$$\mathbb{E}[L|V] = V \tag{3.3}$$

Conditional variance:

$$var[L|V] = h\sigma^2(V) \tag{3.4}$$

Equations (3.2) – (3.4) illustrates that interpreting $\mathbb{E}[L|V]$ as exclusively representing systematic risk and $L - \mathbb{E}[L|V]$ as purely capturing idiosyncratic risk can be misleading. This is because the residual term $Z\sigma(V)\sqrt{h} = L - \mathbb{E}[L|V]$ is influenced by the market risk Y.

When $h \ll 1$, then the granularity risk H is considered a small perturbation to the portfolio loss risk:

$$GA^L = \sqrt{h}\sigma(V)Z \tag{3.5}$$

The random variable $L \sim VH(PD, \rho, h)$, where $VH(PD, \rho, h)$ represents the Vasicek-Herfindahl loan loss distribution.

The Vasicek-Herfindahl CDF is given by:

$$F_{VH}(x; PD, \rho, h) = \mathbb{P}[L(h) < x] = \int_0^1 \Phi\left(\frac{x - v}{\sqrt{h}\sigma(v)}\right) dF_V(v; PD, \rho) \tag{3.6}$$

Proof details are provided in the Appendix 2.

The Value at Risk $VaR_\alpha(L(h)) = x(\alpha, h)$ is the root of the integral equation:

$$1-\alpha = F_{VH}(x(\alpha,h);PD,\rho,h) = \int_0^1 \Phi\left(\frac{x(\alpha,h)-v}{\sqrt{h}\sigma(v)}\right) dF_V(v;PD,\rho) \tag{3.7}$$

where α is the confidence level. For brevity, when writing $x(\alpha,h)$, we omit including the parameters PD and ρ. The numerical value of this root can be easily determined using known parameters PD, ρ, h, and α.

Since a closed-form solution for $x(\alpha,h) = F_{VH}^{-1}(1-\alpha;h)$ to the integral equation (12.1) is generally unavailable, we approximate $x(\alpha,h)$ via a first order Taylor expansion around $h=0$:

$$\begin{aligned} VaR_\alpha(L(h)) = x(\alpha,h) &= x(\alpha) + h\left.\frac{\partial x(\alpha,\chi)}{\partial\chi}\right|_{\chi=0} + o(h) \\ &= x(\alpha) - \frac{h}{2}\left.\frac{\frac{\partial}{\partial v}\left(\sigma^2(v)f_V(v;PD,\rho)\right)}{f_V(v;PD,\rho)}\right|_{v=x(\alpha)} + o(h) \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} &\left.\frac{\frac{\partial}{\partial v}\left(\sigma^2(v)f_V(v;PD,\rho)\right)}{f_V(v;PD,\rho)}\right|_{v=x(\alpha)} \\ &\qquad = 1-2x(\alpha) + \frac{x(\alpha)\big(1-x(\alpha)\big)}{\varphi\left(\Phi^{-1}\big(x(\alpha)\big)\right)}\left(\Phi^{-1}\big(x(\alpha)\big) + \sqrt{\frac{1-\rho}{\rho}}\Phi^{-1}(\alpha)\right) \end{aligned} \tag{3.9}$$

and

$$x(\alpha) = x(\alpha,h=0) = \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right) \tag{3.10}$$

is the root of the equation

$$1-\alpha = F_V(x(\alpha);PD,\rho) \tag{3.11}$$

Proof details are provided in the Appendix 3.

Based on the approach outlined by Gouriéroux et al (2000), Emmer & Tasche (2005, Remark 2.3) with a correction from Gordy & Marrone (2012) (reversing the sign before $\Phi^{-1}\big(x(\alpha)\big)$ in (3.9)) derived the same formula (3.8) with precision up to notation ($q_{1-\alpha}(X) = \Phi^{-1}(\alpha)$, $\Phi\left(\frac{c-\sqrt{\rho}q_{1-\alpha}(X)}{\sqrt{1-\rho}}\right) = x(\alpha)$, $c = \Phi^{-1}(PD)$, $X = Y$, where the left indicates the notation from Emmer & Tasche (2005), and the right — ours).

Figure 1 ($h \in [0,0.01]$) and Figure 2 ($h \in [0,1]$) illustrate the accuracy of the VaR approximations across different levels of HHI. The exceptional accuracy of the formula $x(\alpha,h) = F_{VH}^{-1}(1-\alpha;h)$ in the case of small HHI values h is expected, as it is grounded in the Lyapunov Central Limit Theorem. However, what is particularly noteworthy is that the first-order Taylor series approximation of $VaR_\alpha(L(h))$ around $h = 0$ demonstrates a high level of accuracy for h values up to approximately 0.2 and continues to provide reasonable estimates even up to 0.3. This highlights the robustness of the approximation beyond its initial range of applicability.

The model's applicability is reinforced by empirical data such as Skridulytė & Freitakas (2012), who report maximum HHI levels around 0.24 in the Lithuanian banking sector.

The downward outliers observed in Figure 2 of the simulated data are a consequence of the decreasing accuracy of the Monte Carlo method as granularity increases. To mitigate this effect, it is necessary to increase the number of simulations.

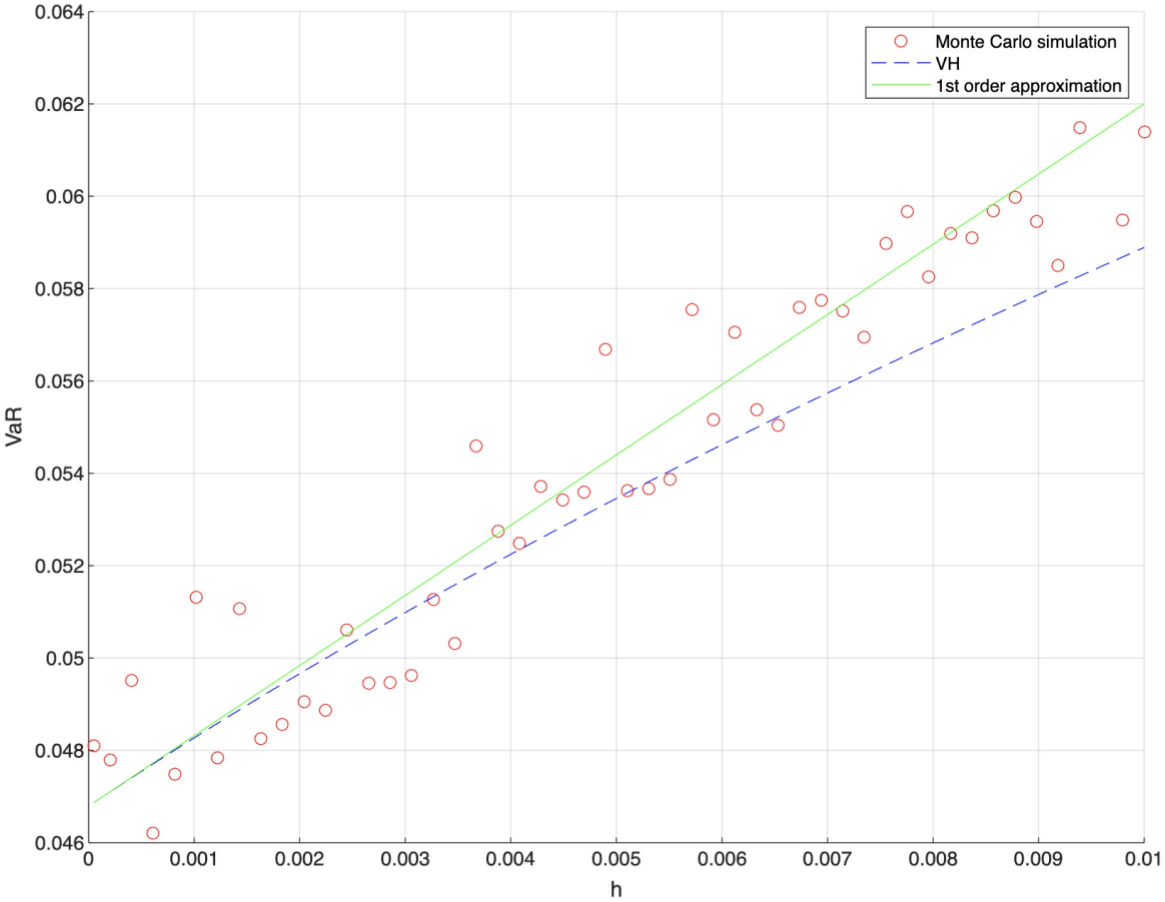

Figure 1. Dependence of Value at Risk (VaR) on HHI $h \in [0, 0.01]$. *Comparison of the simulated VaR at level* α, $VaR_\alpha(Loss(\{X_k\}_{k=1}^n, Y))$ *(red circles), with the theoretical* $VaR_\alpha(L(h)) = F_{VH}^{-1}(1-\alpha; h)$ *(blue dashed line), and the first-order Taylor series approximation of* $VaR_\alpha(L(h))$ *around* $h = 0$ *(green solid line). The Monte Carlo simulation involved 20 000 runs. Parameters used: default probability* $PD = 0.01$, *number of loans* $n = 20\,000$, *confidence level* $\alpha = 0.01$.

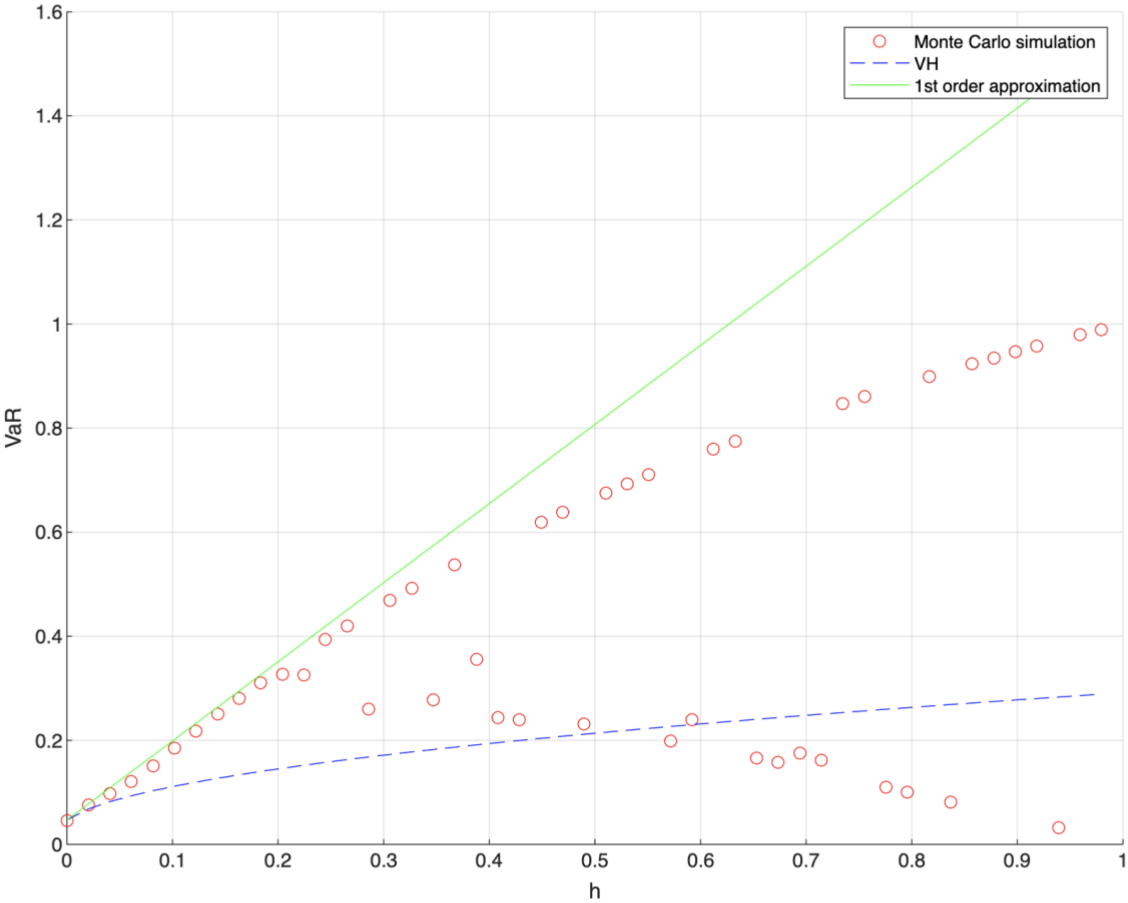

Figure 2. Dependence of Value at Risk (VaR) on HHI $h \in [0,1]$.

3.1 Symmetry Properties

The Vasicek distribution $F_V(x; PD, \rho)$ exhibits notable symmetry characteristics:

$$F_V(1/2 - x; 1/2 - p, \rho) - 1/2 = 1/2 - F_V(1/2 + x; 1/2 + p, \rho)$$

for each ρ. This property follows directly from the equation presented by Vasicek (2002, p.4):

$$F_V(x; PD, \rho) = 1 - F_V(1 - x; 1 - PD, \rho)$$

From this property, it immediately follows that:

$$F_V(1/2; 1/2, \rho) = 1/2$$

$$VaR_{\alpha=1/2}(L(h = 0)) = x_{\alpha=1/2}(PD = 1/2, \rho) = 1/2$$

It is also straightforward to see that the density function satisfies:

$$f_V(1/2 - x; 1/2 - p, \rho) = f_V(1/2 + x; 1/2 + p, \rho)$$

and

$$f_V(1/2; 1/2, \rho) = 1$$

The Vasicek-Herfindahl distribution also exhibits a similar symmetry:

$$F_{VH}(x; PD, \rho, h) = 1 - F_{VH}(1 - x; 1 - PD, \rho, h) \tag{3.12}$$

for each ρ and h. See proof in Appendix 4. From this, it follows that:

$$F_{VH}(1/2; 1/2, \rho, h) = 1/2$$

and consequently:

$$VaR_{\alpha=1/2}(L(h)) = x_{\alpha=1/2}(PD = 1/2, \rho, h) = 1/2$$

for each ρ and h. These symmetry properties are useful for validating the accuracy of software used to compute VaR.

4 Expected Shortfall

The Expected Shortfall at level α is defined as:

$$ES_\alpha(L) = \mathbb{E}[L | L > VaR_\alpha(L)] = \frac{1}{\alpha}\int_0^\alpha VaR_\gamma\,(L) d\gamma \tag{4.1}$$

The Taylor expansion of ES around $h = 0$ becomes:

$$ES_\alpha\big(L(h)\big) = ES_\alpha\big(L(0)\big) + \frac{h}{2\alpha}\sigma^2\big(x(\alpha)\big) f_V(x(\alpha); PD, \rho) + o(h) \tag{4.2}$$

(see proof in Appendix 5), where

$$\begin{aligned} ES_\alpha\big(L(0)\big) &= \frac{1}{\alpha}\int_0^\alpha VaR_\gamma\,(L(h=0)) d\gamma = \frac{1}{\alpha}\int_0^\alpha \Phi\left(\frac{\sqrt{\rho}\Phi^{-1}(1-\gamma) + \Phi^{-1}(PD)}{\sqrt{1-\rho}}\right) d\gamma \\ &= \frac{1}{\alpha}\Phi_2\left(\Phi^{-1}(1-\alpha), \frac{\Phi^{-1}(PD)}{\sqrt{1-\rho}}; \sqrt{\rho}\right) \end{aligned} \tag{4.3}$$

where $\Phi_2(.,.;\rho)$ is the bivariate standard normal CDF with correlation ρ.

5 Incorporating an Additional Loan into a Portfolio

The approach presented in this paper enables the derivation of analytical formulas for risk metrics also in more complex scenarios.

Let us extend our model by adding a new loan to an existing portfolio with HHI h. This new loan has characteristics identical to the other credits and carries a weight $w \in [0,1]$. The HHI of a new portfolio is

$$\bar{h} = (1 - w)^2 h + w^2 = h + w^2 + o(\|h, w^2\|) \tag{5.1}$$

The Value-at-Risk (VaR) of asymptotical loss of the new portfolio at confidence level α is:

$$VaR_\alpha\big(L(\bar{h})\big) = x\big(\alpha, \bar{h}\big) = x(\alpha) - \frac{h + w^2}{2} \frac{\frac{\partial}{\partial v}\big(\sigma^2(v) f_V(v; PD, \rho)\big)}{f_V(v; PD, \rho)}\Bigg|_{v=x(\alpha)} + o(\|h, w^2\|)$$

(5.2)

The change in VaR resulting from adding this new loan is:

$$VaR_\alpha\big(L(\bar{h})\big) - VaR_\alpha(L(h)) = -\frac{w^2}{2} \frac{\frac{\partial}{\partial v}\big(\sigma^2(v) f_V(v; PD, \rho)\big)}{f_V(v; PD, \rho)}\Bigg|_{v=x(\alpha)} + o(\|h, w^2\|)$$

(5.3)

This result extends naturally to multiple added loans:

$$VaR_\alpha\big(L(\bar{h})\big) - VaR_\alpha(L(h)) = -\frac{\sum_{k=1}^{N} {w_k}^2}{2} \frac{\frac{\partial}{\partial v}\big(\sigma^2(v) f_V(v; PD, \rho)\big)}{f_V(v; PD, \rho)}\Bigg|_{v=x(\alpha)} + o(\|h, \{w_k^2\}_{k=1}^{N}\|)$$

(5.4)

6 Approach of Emmer & Tasche (2005)

The approach used by Emmer & Tasche (2005) to find GA to VaR is based on the transformation:

$$\Lambda(h, c) = cL(h) + (1 - c)\mathbb{E}(L(h)|V)$$

(6.1)

The key idea is to decompose this transformed loss variable into two parts:

- A component $\mathbb{E}(L(h)|V)$, which, by the LLN, does not depend on the heterogeneity parameter h,

- A small perturbation term $c\big(L(h) - \mathbb{E}(L(h)|V)\big)$, where the residual $L(h) - \mathbb{E}(L(h)|V)$ encapsulates the influence of h.

Thus,

$$\Lambda(h, c) = \mathbb{E}(L(h)|V) + c\big(L(h) - \mathbb{E}(L(h)|V)\big)$$

(6.2)

This approach utilizes the method developed by Gouriéroux *et al* (2000) to compute the derivatives of $VaR_\alpha(X + cY)$ with respect to the parameter c. This enables us to expand $VaR_\alpha(\Lambda(h, c)) = x(\alpha, h, c)$ into a Taylor series around $c = 0$. By evaluating this series at $c = 1$, we can approximate $VaR_\alpha(L(h)) = x(\alpha, h)$, since $\Lambda(h, c = 1) = L(h)$. However, a significant limitation of this method is the absence of a rigorous proof guaranteeing the convergence of the Taylor series at $c = 1$. This raises questions about the accuracy and reliability of the approximation in practice.

Emmer and Tasche (2005) derived the first and second derivatives of VaR with respect to c:

$$\left.\frac{\partial x(\alpha,h,c)}{\partial c}\right|_{c=0} = 0$$
(6.3)

$$\left.\frac{\partial^2 x(\alpha,h,c)}{\partial c^2}\right|_{c=0} = -\left.\frac{\frac{\partial}{\partial v}\left(var[L(h)|V=v]f_V(v;PD,\rho)\right)}{f_V(v;PD,\rho)}\right|_{v=x(\alpha)}$$
(6.4)

Using these derivatives, they obtained a second-order Taylor expansion of VaR:

$$x(\alpha,h) = x(\alpha,h,c=1) = x(\alpha) - \frac{1}{2}\left.\frac{\frac{\partial}{\partial v}\left(var[L(h)|V=v]f_V(v;PD,\rho)\right)}{f_V(v;PD,\rho)}\right|_{v=x(\alpha)} + o(c^2=1?)$$
(6.5)

Next, we demonstrate that our approach – combining the application of the Lyapunov Central Limit Theorem (CLT) to derive the asymptotic loss distribution of a heterogeneous portfolio with a first-order Taylor series expansion of VaR with respect to the Herfindahl-Hirschman Index (HHI) – yields consistent and equivalent results. Moreover, we establish that the convergence challenges typically associated with the series expansion are effectively mitigated within our framework, ensuring reliable and robust approximations.

Since asymptotic loss

$$L(h) = V + \sqrt{h}\sigma(V)Z$$

it follows that

$$\mathbb{E}(L(h)|V) = V$$

and

$$\Lambda(h,c) = V + c\sqrt{h}\sigma(V)Z$$
(6.6)

Thus, this transformation just replaces the Herfindahl-Hirschman Index h with c^2h:

$$\Lambda(h,c) = L(c^2h)$$
(6.7)

The right-hand side of the equation

$$\begin{aligned}\left.\frac{\partial x(\alpha,h,c=1)}{\partial h}\right|_{h=0} = \left.\frac{\partial x(\alpha,h)}{\partial h}\right|_{h=0} &= -\left.\frac{\frac{\partial}{\partial v}\left(\sigma^2(v)f_V(v;PD,\rho)\right)}{2f_V(v;PD,\rho)}\right|_{v=x(\alpha)} \\ &= -\left.\frac{\frac{\partial}{\partial v}\left(h\sigma^2(v)f_V(v;PD,\rho)\right)}{2hf_V(v;PD,\rho)}\right|_{v=x(\alpha)} \\ &= -\left.\frac{\frac{\partial}{\partial v}\left(var[L(h)|V=v]f_V(v;PD,\rho)\right)}{2hf_V(v;PD,\rho)}\right|_{v=x(\alpha)}\end{aligned}$$
(6.8)

coincides, up to a factor of $(2h)^{-1}$, with the right-hand side of equation (7) from Emmer & Tasche (2005):

$$\left.\frac{\partial^2 x(\alpha,h,c)}{\partial c^2}\right|_{c=0} = -\left.\frac{\frac{\partial}{\partial v}\left(var[L(h)|V=v]f_V(v;PD,\rho)\right)}{f_V(v;PD,\rho)}\right|_{v=x(\alpha)}$$

(6.9)

Note that Emmer & Tasche (2005) denote the density of random variable $V = \mathbb{E}(L(h)|V)$ as γ_L instead of our notation $f_V(v;PD,\rho)$.

Hence

$$\left.\frac{\partial x(\alpha,h,1)}{\partial h}\right|_{h=0} = \frac{1}{2h}\left.\frac{\partial^2 x(\alpha,h,c)}{\partial c^2}\right|_{c=0}$$

(6.10)

Let us prove that this equality holds true.

Based on the Taylor series expansion of the Value at Risk:

$$VaR_\alpha\big(\Lambda(h,c)\big) = x(\alpha,h,c) = VaR_\alpha\big(L(c^2h)\big) = x(\alpha,c^2h)$$
$$= x(\alpha) + c^2h\left.\frac{\partial x(\alpha,\chi)}{\partial \chi}\right|_{\chi=0} + o(c^2h)$$

(6.11)

we have:

$$\left.\frac{\partial x(\alpha,h,c)}{\partial c}\right|_{c=0} = 0$$

(6.12)

$$\left.\frac{\partial^2 x(\alpha,h,c)}{\partial c^2}\right|_{c=0} = 2h\left.\frac{\partial x(\alpha,\chi)}{\partial \chi}\right|_{\chi=0}$$

(6.13)

Q.E.D.

7 Approach of Voropaev (2011)

Based on the information obtained in our paper about the behavior of the portfolio loss distribution near zero of the HHI, let us consider the approach of Voropaev (2011).

Transformation of variables $v = x - u$ allows us to write the Vasicek-Herfindahl function PDF (11.10) in the form of formula (3.1) of Voropaev (2011):

$$f^*(x) = \int_{-\infty}^{\infty} g(u|x-u)f(x-u)\,du$$

(7.1)

where

$$g(u|x) = \frac{\varphi\left(\frac{u}{\sqrt{h}\sigma(x)}\right)}{\sqrt{h}\sigma(x)} \theta(x)\theta(1-x)$$

(7.2)

$$f^*(x) = f_{VH}(x; PD, \rho, h)$$

$$f(x) = f_V(x; PD, \rho)$$

(7.3)

By expanding the integrand into a Taylor series around $u = x$, we obtain

$$g(u|x-u)f(x-u) = \sum_{k=0}^{\infty} (-1)^k \frac{u^k}{k!} \frac{\partial^k}{\partial x^k} \left(g(u|x)f(x)\right)$$

(7.4)

and

$$f^*(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!} \frac{d^k}{dx^k} \left(f(x) m_k(x)\right)$$

(7.5)

where $m_k(x)$ is the k^{th} conditional moment of the distribution H given $V = x$:

$$m_k(x) = \int_{-\infty}^{\infty} u^k g(u|x)\, du$$

(7.6)

Note that the function $g(u|x)$ was not specified by Voropaev. We will proceed further to demonstrate that his approach ultimately leads to the same results as our method.

It follows from equations (7.2) and (7.6) that

$$m_k(x) = \theta(x)\theta(1-x) \int_{-\infty}^{\infty} u^k \frac{\varphi\left(\frac{u}{\sqrt{h}\sigma(x)}\right)}{\sqrt{h}\sigma(x)} du$$

(7.7)

Since $\frac{\varphi\left(\frac{u}{\sqrt{h}\sigma(x)}\right)}{\sqrt{h}\sigma(x)}$ is the even function of u, odd moments are equal to zero. Hence,

$$f^*(x) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} \frac{d^{2k}}{dx^{2k}} \left(f(x) m_{2k}(x)\right)$$

(7.8)

Since

$$\int_{-\infty}^{\infty} u^{2k} exp(-au^2)\, du = \sqrt{\frac{\pi}{a}} \frac{(2k-1)!!}{(2a)^k}$$

(7.9)

then

$$m_{2k}(x) = \theta(x)\theta(1-x) \int_{-\infty}^{\infty} u^{2k} \frac{\varphi\left(\frac{u}{\sqrt{h}\sigma(x)}\right)}{\sqrt{h}\sigma(x)} du = \theta(x)\theta(1-x)\left(h\sigma^2(x)\right)^k (2k-1)!! \tag{7.10}$$

Hence, by employing Voropaev's approach, we derive the equation:

$$f^*(x) = \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!} h^k \frac{\partial^{2k}}{\partial x^{2k}} \left(\sigma^{2k}(x) f_V(x; PD, \rho)\right) \tag{7.11}$$

In contrast, our approach leads to the formula (11.34). Since

$$\frac{(2k-1)!!}{(2k)!} = \frac{1}{k!}\left(\frac{1}{2}\right)^k \tag{7.12}$$

both equalities coincide, leading to the result:

$$f_{VH}(x; PD, \rho, h) = f^*(x) \tag{7.13}$$

8 Vasicek's Attempt

From formula (10) in Vasicek (2002, p. 8), after obvious transformations, the formula

$$var[L] = h\Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); 1) + (1-h)\Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); \rho) - PD^2 \tag{8.1}$$

follows for the unconditional variance, taking into account:

$$\begin{gathered} a = 1 \\ b = \Phi^{-1}(PD) \\ H = T \\ \Phi_2(x, x; 1) = \Phi(x) \end{gathered}$$

However, the following formula from Vasicek (2002) is erroneous:

$$var[L] \approx (\rho + (1-\rho)h)\varphi^2\left(\Phi^{-1}(PD)\right) \tag{8.2}$$

Let's demonstrate how this error occurred.

Vasicek used the tetrachoric expansion of the bivariate normal CDF:

$$\Phi_2(x, x; \rho) \approx \Phi^2(x) + \rho\varphi^2(x) \tag{8.3}$$

$$\Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); \rho) \approx PD^2 + \rho\varphi^2\left(\Phi^{-1}(PD)\right) \tag{8.4}$$

Applying this expansion to the case $\rho = 1$ yields the incorrect result:

$$\Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); 1) \approx PD^2 + \varphi^2\left(\Phi^{-1}(PD)\right)$$

(8.5)

whereas, in fact,

$$\Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); \rho) \xrightarrow[\rho \to 1^-]{} PD \tag{8.6}$$

As a result, Vasicek arrived at the incorrect approximation:

$$\begin{aligned} var[L] &\approx h\left(PD^2 + \varphi^2\big(\Phi^{-1}(PD)\big)\right) + (1-h)\left(PD^2 + \rho\varphi^2\big(\Phi^{-1}(PD)\big)\right) - PD^2 \\ &= (\rho + (1-\rho)h)\varphi^2\big(\Phi^{-1}(PD)\big) \end{aligned} \tag{8.7}$$

instead of the correct approximation:

$$\begin{aligned} var[L] &\approx hPD + (1-h)\left(PD^2 + \rho\varphi^2\big(\Phi^{-1}(PD)\big)\right) - PD^2 \\ &= hPD(1-PD) + (1-h)\rho\varphi^2\big(\Phi^{-1}(PD)\big) \end{aligned} \tag{8.8}$$

Furthermore, on page 8, Vasicek presented equation (12):

$$\mathbb{P}[L \le x] = F_V(x; p, \rho + h(1-\rho)) \tag{8.9}$$

without proper justification. The fallacy of this formula is demonstrated in Figure 3, where the function

$$dF_V(x) = \frac{F_V(x; p, \rho + h(1-\rho)) - F_V(x; p, \rho)}{h} \tag{8.10}$$

is compared to the corresponding simulated function

$$dF_s(x) = \frac{f_s(x; p, \rho, h) - f_s(x; p, \rho, 0)}{h} \tag{8.11}$$

as well as the function

$$dF_{VH}(x) = \frac{F_{VH}(x; p, \rho, h) - F_{VH}(x; p, \rho, 0)}{h} = \frac{F_{VH}(x; p, \rho, h) - F_V(x; p, \rho)}{h} \tag{8.12}$$

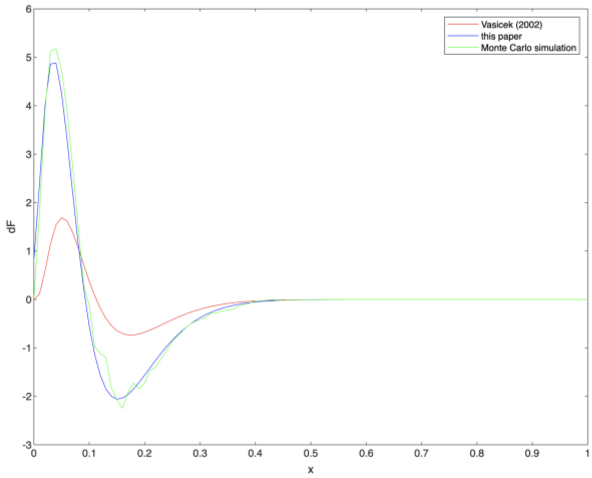

Figure 3. Comparison of $\mathrm{d}F_V(x)$ *(red line) with the corresponding simulated function (green line) and with* $\mathrm{d}F_{VH}(x)$ *(blue line). Number of Monte Carlo simulations: 20 000. The parameters used: PD* $= 0.1$, $\rho = 0.1$, $n = 20\,000$, $h = 0.01$.

The poor quality of Vasicek's attempt to assess the granularity effect is evident in Figure 3.

9 Conclusion

We propose a novel methodology for evaluating granularity risk, providing an alternative to:

- The approach of Emmer & Tasche (2005), which is grounded in the framework established by Gouriéroux et al. (2000),
- The moment-based methodology of Voropaev (2011).

Our approach employs the Lyapunov Central Limit Theorem (CLT) to derive the asymptotic loss distribution of a heterogeneous portfolio. Additionally, we utilize this framework to perform a Taylor series expansion of the Value at Risk (VaR) around the Herfindahl-Hirschman Index (HHI) at $h = 0$.

We demonstrate that, for small values of the Herfindahl-Hirschman Index (HHI), granularity risk, conditional on market risk, is proportional to a standard normal random variable. Instead of studying the behavior of a heterogeneous portfolio, we examine the behavior of a homogeneous portfolio subjected to a small perturbation induced by granularity risk.

An intriguing observation has emerged: although the function $VaR_\alpha(L(h)) = F_{VH}^{-1}(1-\alpha; h)$ provides a good approximation of the simulated function $VaR_\alpha(Loss(\{X_k\}_{k=1}^n, Y))$ only for small HHI values, the first-order Taylor series expansion of $VaR_\alpha(L(h))$ around $h = 0$ exhibits high accuracy for h values up to approximately 0.2. Moreover, the approximation continues to provide reasonable estimates even for h approaching 0.3 (see model parameters in the caption of Figure 1).

It was also unexpected that the second-order Taylor expansion of $Loss$ around $c = 0$, as presented in Emmer and Tasche (2005) (see Chapter 6), coincided with the first-order Taylor expansion around $h = 0$. The reason for this coincidence was elucidated in our study.

10 Appendix 1

10.1 Lyapunov CLT

Suppose $\{\xi_i\}$ is a sequence of independent random variables, each with finite mean μ_i and variance ${\sigma_i}^2$. Define:

$$ {s_n}^2 = \sum_{i=1}^{n} {\sigma_i}^2 \tag{10.1} $$

Lyapunov's condition states that if, for some $\delta > 0$,

$$ \lim_{n\to\infty} \frac{1}{{s_n}^{2+\delta}} \sum_{i=1}^{n} \mathbb{E}\left(|\xi_i - \mu_i|^{2+\delta}\right) = 0 \tag{10.2} $$

then the normalized sum converges in distribution to a standard normal:

$$ \frac{1}{s_n} \sum_{i=1}^{n} (\xi_i - \mu_i) \xrightarrow{d} \mathcal{N}(0,1) \tag{10.3} $$

Let us apply the Lyapunov CLT to our problem. We have:

$$ \xi_i = w_i l(X_i, y) $$

$$ \mu_i = w_i p(y) $$

$$ s_n = \sqrt{h_n}\sigma(p(y)) $$

where

$$ h_n = \sum_{k=1}^{n} {w_k}^2 $$

is the Herfindahl-Hirschman Index. In the new variables, formula (10.3) is transformed into the following form:

$$ \frac{\sum_{i=1}^{n} w_i \left(l(X_i, y) - p(y)\right)}{\sqrt{h_n}\sigma(p(y))} \xrightarrow{d} \mathcal{N}(0,1) \tag{10.4} $$

Taking into account equation (2.3), this formula can be expressed as:

$$ \frac{Loss(\{X_k\}_{k=1}^{n}, y) - p(y)}{\sqrt{h_n}\sigma(p(y))} \xrightarrow{d} \mathcal{N}(0,1) \tag{10.5} $$

Thus, the portfolio loss converges in distribution to the asymptotic loss L:

$$ Loss(\{X_k\}_{k=1}^{n}, Y) \xrightarrow{d} L(Z, Y) = p(Y) + \sqrt{h}\sigma(p(Y))Z \tag{10.6} $$

where $Y, Z \sim \mathcal{N}(0,1)$ are independent, and

$$h = \lim_{n\to\infty} h_n$$

This heterogeneous portfolio loss risk encompasses not only market risk but also granularity risk, which is represented by the term $\sqrt{h}\sigma(p(Y))Z$. It is important to note that granularity risk is influenced by market risk.

If $h = 0$ (perfect granularity), then $L(Z,Y) = p(Y)$. Conversely, the case where $h = 1$ indicates full concentration, occurring when the weight of one of the loans is 1 and the weights of all the others are 0.

The range of the function $Loss(\{X_k\}_{k=1}^n, Y)$ is the unit interval $[0,1]$, while the range of the function $L(Z,Y)$ is $\mathbb{R}$. However, when $h = 0$, the range of the function $L(Z,Y)$ is narrowed to the unit interval $[0,1]$.

10.2 Applicability of the Lyapunov CLT

The applicability of our approach is constrained by the limits of the Lyapunov CLT. Let $\delta = 1$. We need to verify:

$$\lim_{n\to\infty} \frac{1}{s_n{}^3} \sum_{i=1}^{n} \mathbb{E}[|\xi_i - \mu_i|^3] = 0 \tag{10.7}$$

which is equivalent to:

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n w_i{}^3}{\left(\sum_{j=1}^n w_j{}^2\right)^{3/2}} = 0 \tag{10.8}$$

Proof.

We start with the expression:

$$\mathbb{E}[|\xi_i - \mu_i|^3] = w_i{}^3\mathbb{E}[|l(X_k, y) - p(y)|^3] = w_i{}^3 p(y)(1-p(y))\big((1-p(y))^2 + p^2(y)\big) \tag{10.9}$$

Thus, we have:

$$\lim_{n\to\infty} \frac{1}{s_n{}^3} \sum_{i=1}^{n} \mathbb{E}\left[|\xi_i - \mu_i|^3\right] = \frac{\mathbb{E}[|l(X_k, y) - p(y)|^3]}{\sigma^3(p(y))} \lim_{n\to\infty} \frac{\sum_{i=1}^n w_i{}^3}{(\sum_{i=1}^n w_i{}^2)^{3/2}}$$
$$= \frac{(1-p(y))^2 + p^2(y)}{\sqrt{p(y)\big(1-p(y)\big)}} \lim_{n\to\infty} \frac{\sum_{i=1}^n w_i{}^3}{\left(\sum_{j=1}^n w_j{}^2\right)^{3/2}} \tag{10.10}$$

Q.E.D.

To simulate the random variable $Loss$, it is necessary to use a set $w_k \geq 0$, $k = 1, \dots, n$, such that:

$$\sum_{k=1}^{n} w_k = 1$$
$$\sum_{k=1}^{n} {w_k}^2 \ll 1$$
$$\frac{\sum_{k=1}^{n} {w_k}^3}{\left(\sum_{j=1}^{n} {w_j}^2\right)^{3/2}} \ll 1$$

(10.11)

We use the geometric progression defined as follows:

$$w_k = (1-s)s^{k-1}$$

(10.12)

where $0 < s < 1$. Let

$$h = \sum_{k=1}^{\infty} {w_k}^2 = \frac{1-s}{1+s} \Rightarrow s = \frac{1-h}{1+h}$$

(10.13)

Now we can evaluate:

$$\frac{\sum_{k=1}^{\infty} {w_k}^3}{\left(\sum_{j=1}^{\infty} {w_j}^2\right)^{3/2}} = \frac{\sum_{k=1}^{\infty} s^{3k}}{\left(\sum_{j=1}^{\infty} s^{2j}\right)^{3/2}} = \frac{(1-s^2)^{3/2}}{1-s^3}$$

(10.14)

If s is chosen in the left neighborhood of 1, then this ratio is close to 0.

11 Appendix 2

11.1 Vasicek-Herfindahl Distribution

Let the random variable $V = p(Y)$ such that $V \sim Vasicek(PD, \rho)$, where $Vasicek(PD, \rho)$ represents the Vasicek loan loss distribution.

Define

$$H = Z\sigma(V)\sqrt{h}$$

(11.1)

where

$$\sigma(x) = \sqrt{x(1-x)}$$

(11.2)

The asymptotic portfolio loss risk L is the sum of the "classical" Vasicek portfolio loss risk V and the granularity risk H:

$$L = V + H = V + Z\sigma(V)\sqrt{h}$$

(11.3)

The conditional variance of the portfolio loss given V is

$$var[L|V] = h\sigma^2(V)$$

(11.4)

Equation (11.3) illustrates that interpreting $\mathbb{E}[L|V]$ as exclusively representing systematic risk and $L - \mathbb{E}[L|V]$ as purely capturing idiosyncratic risk can be misleading. This is because the residual term $Z\sigma(V)\sqrt{h} = L - \mathbb{E}[L|V]$ is influenced by the market risk Y.

When $h \ll 1$, then the granularity risk $H = Z\sigma(V)\sqrt{h}$ is considered a small perturbation to the portfolio loss risk:

$$GA^L = Z\sigma(V)\sqrt{h} \tag{11.5}$$

The PDF of the portfolio loss L is given by:

$$\begin{aligned} f_L(x) = f_{V+H}(x) &= \int_0^1 f_{H|V}(x - v|v) f_V(v) dv = \int_0^1 f_{H|V}(x - v|v) dF_V(v) \\ &= \int_0^1 \frac{\varphi\left(\frac{x - v}{\sqrt{h}\sigma(v)}\right)}{\sqrt{h}\sigma(v)} dF_V(v) \end{aligned} \tag{11.6}$$

where $\varphi(x)$ is the standard normal PDF. Similarly, the CDF of the portfolio loss L is:

$$\begin{aligned} F_L(x) = F_{V+H}(x) = \mathbb{P}[V + H < x] &= \int_0^1 \mathbb{P}[H < x - v|V = v] f_V(v) dv \\ &= \int_0^1 F_{H|V}(x - v|v) f_V(v) dv = \int_0^1 \Phi\left(\frac{x - v}{\sqrt{h}\sigma(v)}\right) dF_V(v) \end{aligned} \tag{11.7}$$

The PDF of the random variable H, conditional on the random variable V, is given by:

$$f_{H|V}(x|v) = \frac{\varphi\left(\frac{x}{\sqrt{h}\sigma(v)}\right)}{\sqrt{h}\sigma(v)} \tag{11.8}$$

The CDF of the random variable H, conditional on the random variable V, is given by:

$$F_{H|V}(x|v) = \Phi\left(\frac{x}{\sqrt{h}\sigma(v)}\right) \tag{11.9}$$

Now we introduce the Vasicek-Herfindahl PDF of the portfolio loss:

$$f_{VH}(x; PD, \rho, h) = \int_0^1 \frac{\varphi\left(\frac{x - v}{\sqrt{h}\sigma(v)}\right)}{\sqrt{h}\sigma(v)} dF_V(v; PD, \rho) \tag{11.10}$$

and the Vasicek-Herfindahl CDF:

$$F_{VH}(x; PD, \rho, h) = \int_0^1 \Phi\left(\frac{x - v}{\sqrt{h}\sigma(v)}\right) dF_V(v; PD, \rho)$$

(11.11)

Using integration by parts, the function can be transformed into the following form:

$$F_{VH}(x; PD, \rho, h) = \int_0^1 F_V(v; PD, \rho) d\Phi\left(\frac{v - x}{\sqrt{h}\sigma(v)}\right)$$

(11.12)

11.2 Model Validation Using Monte Carlo Simulation

The simulated CDF is given by:

$$F_s(x) = \sum_{k=1}^{N} I(Loss_k \leq x)$$

(11.13)

where $I(x)$ is the indicator function, $Loss_k$ is the k^{th} Monte Carlo simulation of the random variable $Loss$, and N is the total number of simulations.

The simulated PDF is:

$$f_s(x_j) = \frac{1}{N} \sum_{k=1}^{N} I(x_j \leq Loss_k < x_{j+1})$$

(11.14)

where

$$x_j = x_0 + j\Delta x$$

(11.15)

and Δx is the bin width.

The theoretical Vasicek-Herfindahl ΔCDF is given by:

$$\Delta F_{VH}(x; PD, \rho, h) = F_{VH}(x; PD, \rho, h) - F_{VH}(x; PD, \rho, 0)$$

(11.16)

The theoretical Vasicek-Herfindahl ΔPDF is defined as:

$$\Delta f_{VH}(x; PD, \rho, h) = f_{VH}(x; PD, \rho, h) - f_{VH}(x; PD, \rho, 0)$$

(11.17)

The simulated ΔCDF is expressed as:

$$\Delta F_s(x; h) = F_s(x; h) - F_s(x; 0)$$

(11.18)

The simulated ΔPDF is represented by:

$$\Delta f_s(x; h) = f_s(x; h) - f_s(x; 0)$$

(11.19)

The difference between the theoretical Vasicek-Herfindahl ΔCDF with the simulated ΔCDF is given by the equation:

$$\varDelta\varDelta F(x) = \varDelta F_{VH}(x) - \varDelta F_s(x) \tag{11.20}$$

Similarly, the difference between the theoretical Vasicek-Herfindahl ΔPDF and the simulated ΔPDF is represented as:

$$\varDelta\varDelta f(x) = \varDelta f_{VH}(x) - \varDelta f_s(x) \tag{11.21}$$

Figure 4 (with $h = 0.01$) and Figure 5 (with $h = 0.1$) illustrate the differences between the theoretical Vasicek-Herfindahl ΔCDF and ΔPDF and their corresponding simulated functions. Both figures demonstrate a decline in model quality as HHI values increase. Nevertheless, even with the relatively large value of $h = 0.1$, the model still accurately represents the shapes of both the PDF and CDF.

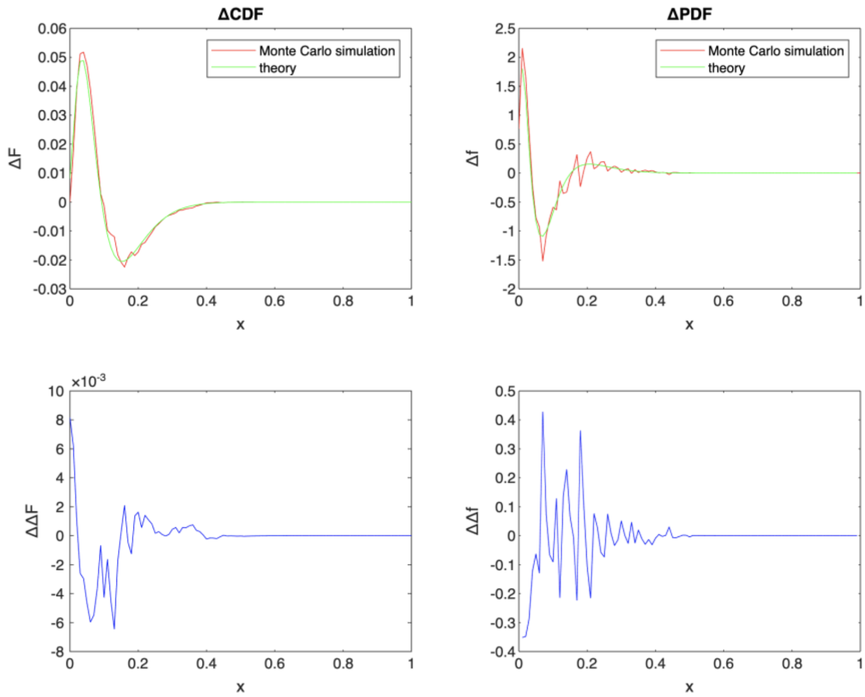

Figure 4. Theoretical Vasicek-Herfindahl ΔCDF and ΔPDF vs simulated functions (top row). Below the plots of the functions are the corresponding plots of the differences between theoretical and simulated functions. Number of Monte Carlo simulations: 20 000. Parameters used: PD=0.1, ρ=0.1, h=0.01, n=20 000.

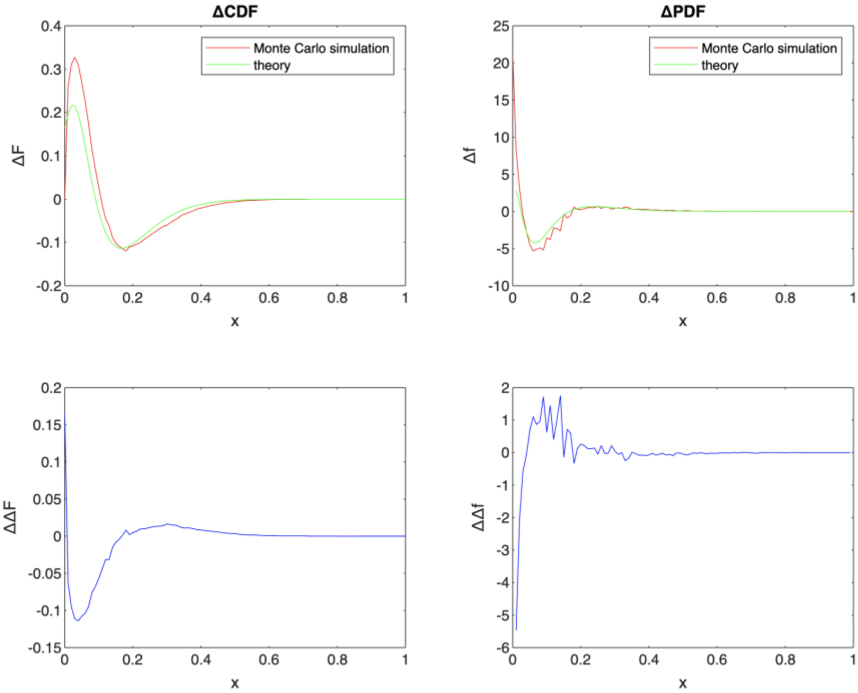

Figure 5. Theoretical Vasicek-Herfindahl ΔCDF and ΔPDF vs simulated functions (top row). Below the plots of the functions are the corresponding plots of the differences between theoretical and simulated functions. Number of Monte Carlo simulations: 20 000. Parameters used: PD=0.1, ρ=0.1, h=0.1, n=20 000.

11.3 Properties of the Vasicek-Herfindahl Distribution

11.3.1 Normalization Property of the PDF

The total area under the PDF curve is equal to 1:

$$\int_{-\infty}^{\infty} f_{VH}(x;PD,\rho,h)dx = 1 \tag{11.22}$$

Proof.

$$\int_{-\infty}^{\infty} f_{VH}(x;PD,\rho,h)dx = \int_0^1 \frac{dF_V(v;PD,\rho)}{\sqrt{h}\sigma(v)} \int_{-\infty}^{\infty} \varphi\left(\frac{x-v}{\sqrt{h}\sigma(v)}\right)dv = \int_0^1 dF_V(v;PD,\rho)$$
$$= F_V(v;PD,\rho)|_0^1 = 1 \tag{11.23}$$

Q.E.D.

11.3.2 Expected Loss

The unconditional mean of the asymptotic loss L is equal to the unconditional probability of default:

$$\mathbb{E}[L] = PD \tag{11.24}$$

Proof.

$$\mathbb{E}[L] = \mathbb{E}\left[V + Z\sqrt{V(1-V)h}\right] = \mathbb{E}[V] + \sqrt{h}\mathbb{E}[Z]\mathbb{E}\left[\sqrt{V(1-V)}\right] = \mathbb{E}[V] = PD \tag{11.25}$$

Q.E.D.

11.3.3 Variance

The unconditional variance of the asymptotic loss L is a linear function of h:

$$var[L] = PD(1-PD)h + (1-h)var[V] \tag{11.26}$$

where

$$var[V] = var\big[\mathbb{E}[L|V]\big] = \Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); \rho) - PD^2 \tag{11.27}$$

Proof.

$$\begin{aligned}\mathbb{E}[L^2] = \mathbb{E}&\left[V^2 + 2VZ\sqrt{V(1-V)h} + Z^2V(1-V)h\right] \\ &= \mathbb{E}[V^2] + 2\sqrt{h}\mathbb{E}[Z]\mathbb{E}\left[V\sqrt{V(1-V)}\right] + h\mathbb{E}[Z^2]\mathbb{E}[V(1-V)] \\ &= \mathbb{E}[V^2] + h\mathbb{E}[V(1-V)] = (1-h)\mathbb{E}[V^2] + h\mathbb{E}[V] \\ &= (1-h)\Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); \rho) + hPD\end{aligned} \tag{11.28}$$

$$\begin{aligned}var[L] = \mathbb{E}[L^2] - \mathbb{E}^2[L] &= (1-h)\Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); \rho) + hPD - PD^2 = \\ &= (1-h)(\Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); \rho) - PD^2) + (1-h)PD^2 + hPD - PD^2 \\ &= (1-h)var[V] + PD(1-PD)h\end{aligned} \tag{11.29}$$

Q.E.D.

11.4 Taylor Series of CDF and PDF

Let $z = h\sigma^2(v)$. Given that $0 < z \ll 1$, we expand $\Phi\big(u/\sqrt{z}\big)$ into a Taylor series around $z = 0$:

$$\Phi\left(\frac{u}{\sqrt{z}}\right) = \theta(u) + \sum_{k=1}^{\infty} (z/2)^k \frac{\delta^{(2k-1)}(u)}{k!} \tag{11.30}$$

(see proof in Appendix 6). Hence, the Vasicek-Herfindahl CDF can be written as:

$$
\begin{aligned}
F_{VH}(x;PD,\rho,h) &= \int_0^1 \Phi\left(\frac{x-v}{\sqrt{h}\sigma(v)}\right) dF_V(v;PD,\rho) \\
&= F_V(x;PD,\rho) \\
&+ \sum_{k=1}^{\infty} \frac{(h/2)^k}{k!} \int_0^1 \delta^{(2k-1)}(x-v)\sigma^{2k}(v) f_V(v;PD,\rho)dv \\
&= F_V(x;PD,\rho) \\
&- \sum_{k=1}^{\infty} \frac{(h/2)^k}{k!} \int_0^1 \delta^{(2k-1)}(v-x)\sigma^{2k}(v) f_V(v;PD,\rho)dv = F_V(x;PD,\rho) \\
&+ \sum_{k=1}^{\infty} \frac{(h/2)^k}{k!} \frac{\partial^{2k-1}}{\partial x^{2k-1}}\left(\sigma^{2k}(x) f_V(x;PD,\rho)\right) \\
&= F_V(x;PD,\rho) + \frac{h}{2}\frac{\partial}{\partial x}\left(\sigma^2(x) f_V(x;PD,\rho)\right) + o(h)
\end{aligned}
\tag{11.31}
$$

We used the Dirac delta function properties:

$$
\delta^{(k)}(-x) = (-1)^k \delta^{(k)}(x) \tag{11.32}
$$

$$
\int_{-\infty}^{+\infty} \delta^{(k)}(x) f(x)dx = (-1)^k f^{(k)}(0) \tag{11.33}
$$

The Vasicek-Herfindahl PDF is given by:

$$
\begin{aligned}
f_{VH}(x;PD,\rho,h) = \frac{\partial}{\partial x} F_{VH}(x;PD,\rho,h) &= \sum_{k=0}^{\infty} \frac{(h/2)^k}{k!} \frac{\partial^{2k}}{\partial x^{2k}}\left(\sigma^{2k}(x) f_V(x;PD,\rho)\right) \\
&= f_V(x;PD,\rho) + \frac{h}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2(x) f_V(x;PD,\rho)\right) + o(h)
\end{aligned}
\tag{11.34}
$$

Hence, the GA to CDF is given by:

$$
GA^{CDF} = \frac{h}{2}\frac{\partial}{\partial x}\left(\sigma^2(x) f_V(x;PD,\rho)\right) \tag{11.35}
$$

and the GA to PDF is expressed as:

$$
GA^{PDF} = \frac{h}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2(x) f_V(x;PD,\rho)\right) \tag{11.36}
$$

12 Appendix 3

12.1 Value at Risk (VaR)

The Value at Risk $VaR_\alpha(L(h)) = x(\alpha,h)$ is the root of the integral equation:

$$1-\alpha=\int_0^1 \Phi\left(\frac{x(\alpha,h)-v}{\sqrt{h}\sigma(v)}\right)dF_V(v;PD,\rho)$$

(12.1)

where α is the confidence level. The numerical value of this root can be easily determined using known parameters PD, ρ, h, and α. For brevity, when writing $x(\alpha,h)$, we omit including the parameters PD and ρ.

$x(\alpha,h)$ is the α-quantile of $L(h)$:

$$x(\alpha,h)=q_\alpha(L(h))$$

where, for any random variable X,

$$q_\alpha(X)=inf\{x\in\mathbb{R}:\mathbb{P}[X\le x]\ge\alpha\}$$

Since a closed-form solution for $x(\alpha,h)$ to the integral equation (12.1) is generally unavailable, we consider expanding $x(\alpha,h)$ into a Taylor series around $h=0$.

Differentiating the equality

$$1-\alpha=F_{VH}(x(\alpha,h),h)$$

(12.2)

with respect to h, we obtain:

$$\frac{\partial}{\partial v}F_{VH}(v,h)\bigg|_{v=x(\alpha,h)}\frac{\partial x(\alpha,h)}{\partial h}+\frac{\partial}{\partial h}F_{VH}(v,h)\bigg|_{v=x(\alpha,h)}=0$$

(12.3)

Note: parameters PD and ρ were omitted for brevity. Using the relation

$$\frac{\partial}{\partial v}F_{VH}(v,h)=f_{VH}(v,h)$$

(12.4)

we derive:

$$\frac{\partial x(\alpha,h)}{\partial h}\bigg|_{h=0}=-\frac{\frac{\partial}{\partial h}F_{VH}(v,h)}{f_{VH}(v,h)}\Bigg|_{v=x(\alpha),h=0}$$

(12.5)

where the α-quantile of $L(h=0)$

$$x(\alpha)=q_\alpha(L(h=0))=VaR_\alpha\big(L(h=0)\big)=\Phi\left(\frac{\Phi^{-1}(PD)-\sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right)$$

(12.6)

is the root of the equation

$$1-\alpha=F_V(x(\alpha);PD,\rho)$$

(12.7)

Since the following equalities hold:

$$\left.\frac{\partial F_{VH}(x;PD,\rho,h)}{\partial h}\right|_{h=0} = \frac{1}{2}\frac{\partial}{\partial x}\left(\sigma^2(x) f_V(x;PD,\rho)\right) \tag{12.8}$$

(see (11.31)) and

$$f_{VH}(x,PD,\rho,h=0) = f_V(x;PD,\rho) \tag{12.9}$$

then

$$\left.\frac{\partial x(\alpha,h)}{\partial h}\right|_{h=0} = -\left.\frac{\frac{\partial}{\partial v}\left(\sigma^2(v) f_V(v;PD,\rho)\right)}{2 f_V(v;PD,\rho)}\right|_{v=x(\alpha)} \tag{12.10}$$

Using the same methods applied in deriving formula (11.32), we can obtain the following expression for the partial derivative:

$$\frac{\partial x(\alpha,h)}{\partial h} = -\left.\frac{\sum_{k=0}^{\infty}\frac{(h/2)^k}{k!}\frac{\partial^{2k+1}}{\partial v^{2k+1}}\left(\sigma^{2(k+1)}(v) f_V(v;PD,\rho)\right)}{2\sum_{k=0}^{\infty}\frac{(h/2)^k}{k!}\frac{\partial^{2k}}{\partial v^{2k}}\left(\sigma^{2k}(v) f_V(v;PD,\rho)\right)}\right|_{v=x(\alpha,h)} \tag{12.11}$$

Taylor series expansion at $h = 0$:

$$\begin{aligned} VaR_\alpha(L(h)) = x(\alpha,h) &= x(\alpha) + h\left.\frac{\partial x(\alpha,\chi)}{\partial \chi}\right|_{\chi=0} + o(h) \\ &= x(\alpha) - h\left.\frac{\frac{\partial}{\partial v}\left(\sigma^2(v) f_V(v;PD,\rho)\right)}{2 f_V(v;PD,\rho)}\right|_{v=x(\alpha)} + o(h) \end{aligned} \tag{12.12}$$

It follows from equations (12.10) and $\sigma^2(x) = x(1-x)$ that:

$$\left.\frac{\partial x(\alpha,h)}{\partial h}\right|_{h=0} = -\frac{1}{2}(1-2v+v(1-v)\left.\frac{\partial}{\partial v} ln\left(f_V(v;PD,\rho)\right)\right|_{v=x(\alpha)} \tag{12.13}$$

Since the Vasicek PDF is given by:

$$f_V(x;PD,\rho) = \sqrt{\frac{1-\rho}{\rho}} exp\left\{-\frac{1}{2\rho}\left(\sqrt{1-\rho}\Phi^{-1}(x) - \Phi^{-1}(PD)\right)^2 + \frac{1}{2}\left(\Phi^{-1}(x)\right)^2\right\} \tag{12.14}$$

it follows that:

$$\frac{\partial ln f_V(x;PD,\rho)}{\partial x} = \frac{(2\rho-1)\Phi^{-1}(x) + \sqrt{1-\rho}\Phi^{-1}(PD)}{\rho\varphi\left(\Phi^{-1}(x)\right)} \tag{12.15}$$

From equations (12.6) and (12.15), we have:

$$\left.\frac{\partial}{\partial v} ln\big(f_V(v; PD, \rho)\big)\right|_{v=x(\alpha)} = \frac{\Phi^{-1}\big(x(\alpha)\big) + \sqrt{\frac{1-\rho}{\rho}}\Phi^{-1}(\alpha)}{\varphi\left(\Phi^{-1}\big(x(\alpha)\big)\right)}$$

(12.16)

It follows from equations (12.13) and (12.16) that:

$$\left.\frac{\partial x(\alpha, h)}{\partial h}\right|_{h=0} = -\frac{1}{2}\left(1 - 2x(\alpha) + \frac{x(\alpha)\big(1 - x(\alpha)\big)}{\varphi\left(\Phi^{-1}\big(x(\alpha)\big)\right)}\left(\Phi^{-1}\big(x(\alpha)\big) + \sqrt{\frac{1-\rho}{\rho}}\Phi^{-1}(\alpha)\right)\right)$$

(12.17)

Hence, the GA to VaR is:

$$GA^{VaR} = -\frac{h}{2}\left(1 - 2x(\alpha) + \frac{x(\alpha)\big(1 - x(\alpha)\big)}{\varphi\left(\Phi^{-1}\big(x(\alpha)\big)\right)}\left(\Phi^{-1}\big(x(\alpha)\big) + \sqrt{\frac{1-\rho}{\rho}}\Phi^{-1}(\alpha)\right)\right)$$

(12.18)

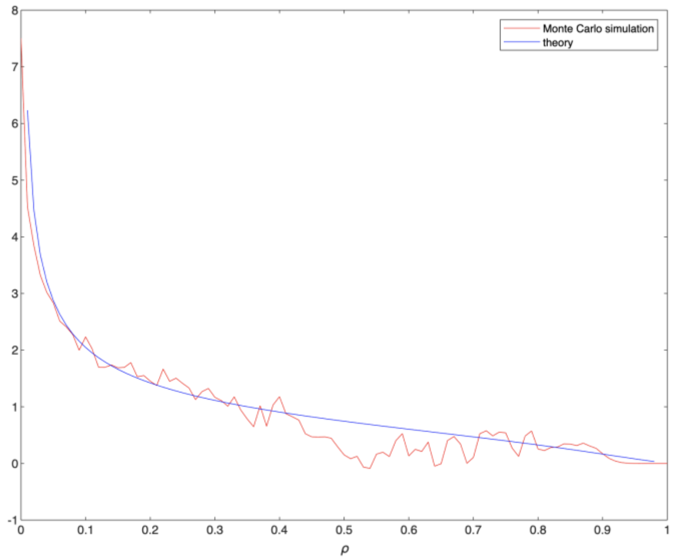

Figure 6. Comparison of the dependence on ρ of the simulated function $\frac{x(\alpha,h)-x(\alpha)}{h}$ (red line) and of the theoretical function $\left.\frac{\partial x(\alpha,h)}{\partial h}\right|_{h=0}$ (blue line). Number of Monte Carlo simulations: 20 000. The parameters used: $PD = 0.1$, $n = 15\ 000$, $h = 0.01$, $\alpha = 0.01$.

13 Appendix 4

13.1 Symmetry Property

The distribution exhibits a symmetry property:

$$F_{VH}(x; PD, \rho, h) = 1 - F_{VH}(1 - x; 1 - PD, \rho, h)$$

(13.1)

This is similar to the symmetry property presented by Vasicek (2002, p.4):

$$F_V(x; PD, \rho) = 1 - F_V(1 - x; 1 - PD, \rho) \tag{13.2}$$

Proof.

$$\begin{aligned}
F_{VH}(x; PD, \rho, h) &= \int_0^1 \Phi\left(\frac{x - v}{\sqrt{h}\sigma(v)}\right) dF_V(v; PD, \rho) \\
&= -\int_0^1 \Phi\left(\frac{x - v}{\sqrt{v(1 - v)h}}\right) dF_V(1 - v; 1 - PD, \rho) \\
&= \int_0^1 \Phi\left(\frac{x - (1 - u)}{\sqrt{(1 - u)uh}}\right) dF_V(u; 1 - PD, \rho) = \\
&= 1 - \int_0^1 \Phi\left(\frac{(1 - x) - u}{\sqrt{h}\sigma(u)}\right) dF_V(u; 1 - PD, \rho) = 1 - F_{VH}(1 - x; 1 - PD, \rho)
\end{aligned} \tag{13.3}$$

Q.E.D.

14 Appendix 5

14.1 Expected Shortfall

The Expected Shortfall at level α is defined as:

$$ES_\alpha(L) = \mathbb{E}[L | L > VaR_\alpha(L)] = \frac{1}{\alpha}\int_0^\alpha VaR_\gamma\,(L) d\gamma \tag{14.1}$$

We expand $VaR_\gamma(L(h))$ into a Taylor series around $h = 0$:

$$VaR_\gamma(L(h)) = VaR_\gamma(L(h = 0)) + h\left.\frac{\partial VaR_\gamma(L(h))}{\partial h}\right|_{h=0} + o(h) \tag{14.2}$$

where $VaR_\gamma(L(h = 0)) = x(\gamma)$ is the root of the equation:

$$1 - \gamma = F_V(x(\gamma); PD, \rho) \tag{14.3}$$

Differentiating equation (14.3) with respect to γ, we obtain:

$$-1 = \frac{dx(\gamma)}{d\gamma}\left.\frac{\partial F_V(v; PD, \rho)}{\partial v}\right|_{v=x(\gamma)} = \frac{dx(\gamma)}{d\gamma} f_V(x(\gamma); PD, \rho) \tag{14.4}$$

which leads to:

$$\frac{dx(\gamma)}{d\gamma} = -\frac{1}{f_V(x(\gamma); PD, \rho)} \tag{14.5}$$

By the chain rule, the formula (12.10)

$$\left.\frac{\partial x(\gamma,h)}{\partial h}\right|_{h=0} = -\left.\frac{\frac{\partial}{\partial v}\left(\sigma^2(v)f_V(v;PD,\rho)\right)}{2f_V(v;PD,\rho)}\right|_{v=x(\gamma)} \tag{14.6}$$

is transformed into the following form:

$$\left.\frac{\partial x(\gamma,h)}{\partial h}\right|_{h=0} = \frac{1}{2}\frac{dx(\gamma)}{d\gamma}\left.\left(\frac{\partial}{\partial v}\left(\sigma^2(v)f_V(v;PD,\rho)\right)\right)\right|_{v=x(\gamma)} = \frac{1}{2}\frac{\partial}{\partial\gamma}\left(\sigma^2\big(x(\gamma)\big)f_V(x(\gamma);PD,\rho)\right) \tag{14.7}$$

Hence,

$$\int_0^\alpha \left.\frac{\partial x(\gamma,h)}{\partial h}\right|_{h=0} d\gamma = \left.\frac{1}{2}\sigma^2\big(x(\gamma)\big)f_V(x(\gamma);PD,\rho)\right|_{\gamma=0}^{\gamma=\alpha} = \frac{1}{2}\sigma^2\big(x(\alpha)\big)f_V(x(\alpha);PD,\rho) \tag{14.8}$$

It follows from equations (14.1) and (14.8) that

$$\left.\frac{\partial ES_\alpha(L(h))}{\partial h}\right|_{h=0} = \frac{1}{2\alpha}\sigma^2\big(x(\alpha)\big)f_V(x(\alpha);PD,\rho) \tag{14.9}$$

Therefore, the Taylor expansion of ES around $h = 0$ becomes:

$$ES_\alpha\big(L(h)\big) = ES_\alpha\big(L(0)\big) + \frac{h}{2\alpha}\sigma^2\big(x(\alpha)\big)f_V(x(\alpha);PD,\rho) + o(h) \tag{14.10}$$

$$\frac{ES_\alpha\big(L(h)\big) - ES_\alpha\big(L(0)\big)}{h} \approx \frac{1}{2\alpha}\sigma^2\big(x(\alpha)\big)f_V(x(\alpha);PD,\rho) \tag{14.11}$$

The GA to ES is

$$GA^{ES} = \frac{h}{2\alpha}\sigma^2\big(x(\alpha)\big)f_V(x(\alpha);PD,\rho) \tag{14.12}$$

For Monte Carlo simulations of conditional expectation, we use the formula

$$ES_\alpha(L) = \mathbb{E}[L|L > VaR_\alpha(L)] = \frac{\sum_{k=1}^N L_k I(L_k > VaR_\alpha(L))}{\sum_{k=1}^N I(L_k > VaR_\alpha(L))} \tag{14.13}$$

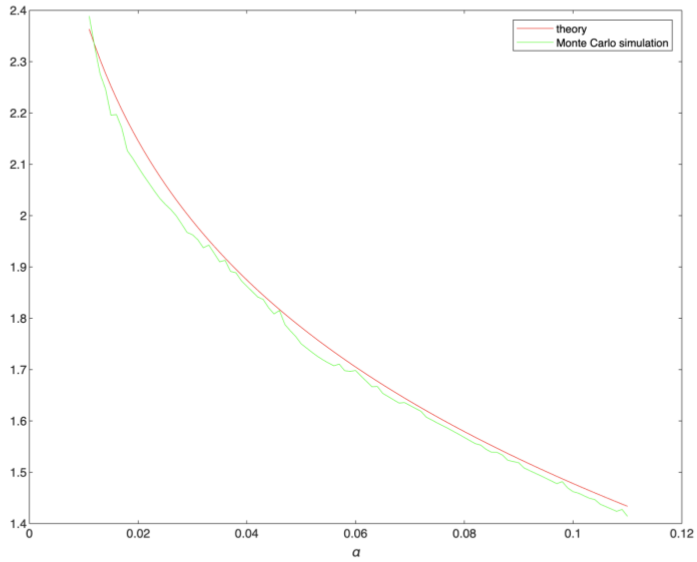

Figure 7. Comparison of the dependence on α of the simulated function $\frac{ES_\alpha(L(h))-ES_\alpha(L(0))}{h}$ *(green line) and of the theoretical function* $\left.\frac{\partial ES_\alpha(L(h))}{\partial h}\right|_{h=0}$ *(red line). Number of Monte Carlo simulations: 25 000. The parameters used: PD=0.1,* $\rho = 0.1$*, n=25 000, h=0.01.*

15 Appendix 6

The function $\Phi(u/\sqrt{z})$ satisfies the classical heat conduction (diffusion) equation:

$$\Phi_z(u/\sqrt{z}) = \frac{1}{2}\Phi_{uu}(u/\sqrt{z}) \tag{15.1}$$

The generalized heat equation

$$\frac{\partial^k}{\partial z^k}\Phi(u/\sqrt{z}) = 2^{-k}\frac{\partial^{2k}}{\partial u^{2k}}\Phi(u/\sqrt{z}) \tag{15.2}$$

can be proved by induction:

$$\begin{aligned}\frac{\partial^{k+1}}{\partial z^{k+1}}\Phi(u/\sqrt{z}) &= 2^{-k}\frac{\partial}{\partial z}\frac{\partial^{2k}}{\partial u^{2k}}\Phi(u/\sqrt{z}) = 2^{-k}\frac{\partial^{2k}}{\partial u^{2k}}\frac{\partial}{\partial z}\Phi(u/\sqrt{z}) \\ &= 2^{-(k+1)}\frac{\partial^{2(k+1)}}{\partial u^{2(k+1)}}\Phi(u/\sqrt{z})\end{aligned} \tag{15.3}$$

Given the limit representation of the Heaviside step function:

$$\Phi(x/\varepsilon) \xrightarrow[\varepsilon\to 0^+]{} \theta(x) \tag{15.4}$$

and relation between the Dirac delta function and the Heaviside step function:

$$\delta(x) = \theta'(x)$$

we have:

$$\lim_{z\to 0^+} \frac{\partial^k}{\partial z^k}\Phi\left(u/\sqrt{z}\right) = 2^{-k} \lim_{z\to 0^+} \frac{\partial^{2k}}{\partial u^{2k}}\Phi\left(u/\sqrt{z}\right) = 2^{-k} \frac{\partial^{2k}}{\partial u^{2k}}\theta(u) = 2^{-k}\delta^{(2k-1)}(u) \tag{15.5}$$

Using the above, the function $\Phi\left(u/\sqrt{z}\right)$ can be expanded into a Taylor series around $z = 0$:

$$\Phi\left(u/\sqrt{z}\right) = \theta(u) + \sum_{k=1}^{\infty} \frac{(z/2)^k}{k!}\delta^{(2k-1)}(u) \tag{15.6}$$

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