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Abstract

The investor, who holds his portfolio and doesn't trade his shares, at current time can use the time series of the market trades that were made during the averaging interval with the securities of his portfolio to assess the current variance of the portfolio. We show how the time series of trades with the securities of the portfolio determine the time series of trades with the portfolio as a single market security. The time series of portfolio trades determine the return and variance of the portfolio in the same form as the time series of trades with securities determine their returns and variances. The description of any portfolio and any single market security is equal. The time series of portfolio trades define the decomposition of the portfolio variance by its securities. If the volumes of trades with all securities are assumed constant, the decomposition of the portfolio variance coincides with Markowitz's (1952) expression of variance. However, the real markets expose random volumes of trades. The portfolio variance that accounts for the randomness of trade volumes is a polynomial of the *4th degree* in the variables of relative amounts invested into securities and with the coefficients different from covariances of securities returns. We discuss the possible origin of the latent and unintended assumption that Markowitz (1952) made to derive his result. Our description of the portfolio variance that accounts for the randomness of real trade volumes could help the portfolio managers and the majors like BlackRock's Aladdin and Asimov, JP Morgan, and the U.S. Fed to adjust their models and forecasts to the reality of random markets.

Keywords : portfolio variance, portfolio theory, random trade volumes
JEL: C0, E4, F3, G1, G12

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1. Introduction

More than seventy years ago, Markowitz (1952) described the portfolio variance $\Theta(t,t_0)$ (1.2) as the quadratic form in variables $X_j(t_0)$ of relative amounts invested into the securities with the coefficients equal to the covariances $\theta_{jk}(t,t_0)$ (1.3) of the returns of the securities that compose the portfolio. This result allowed Markowitz to formulate the principles of optimal selection of the portfolio with higher returns under lower variance. Since then, portfolio theory has been further developed by many contributions (Pogue, 1970; Markowitz, 1991; Rubinstein, 2002; Cochrane, 2014; Elton et al., 2014; Boyd et al., 2024). However, Markowitz's expression of the portfolio variance $\Theta(t,t_0)$ (1.2) remains unchanged.

We believe that Markowitz's (1952) result is well known and needs no additional clarifications. We follow Markowitz and consider the portfolio that was collected of $j=1,.. J$ securities in the past at time t_0. The mean return $R(t,t_0)$ (1.1) of the portfolio at time t takes the form:

$$R(t,t_0) = \sum_{j=1}^{J} R_j(t,t_0)\, X_j(t_0) \tag{1.1}$$

We denote the mean return $R_j(t,t_0)$ of the security j at time t with respect to time t_0 in the past. The coefficients $X_j(t_0)$ in (1.1) denote the relative amounts invested into security j in the past at time t_0. All prices are adjusted to the current time t. Markowitz (1952) derived the portfolio variance $\Theta(t,t_0)$ (1.2) as a quadratic form in the variables of the relative amounts $X_j(t_0)$ invested into security j:

$$\Theta(t,t_0) = \sum_{j,k=1}^{J} \theta_{jk}(t,t_0) X_j(t_0)\, X_k(t_0) \tag{1.2}$$

The coefficients $\theta_{jk}(t,t_0)$ (1.3) in (1.2) denote the covariances of returns of securities j and k:

$$\theta_{jk}(t,t_0) = E\left[\left(R_j(t_i,t_0) - E[R_j(t_i,t_0)]\right)\left(R_k(t_i,t_0) - E[R_k(t_i,t_0)]\right)\right] \tag{1.3}$$

The expression of the portfolio variance $\Theta(t,t_0)$ (1.2) for decades served successfully as a basis for the optimal portfolio selection and of the portfolio theory as a whole.

The dependence of the portfolio variance on the variances of the securities that compose the portfolio determines the key issue for the methods of optimal portfolio selection. Actually, any valuable results in economics and finance are the consequences of particular approximations and assumptions. We restudy Markowitz's expression of the portfolio variance $\Theta(t,t_0)$ (1.2) and show that it gives the correct assessment of the portfolio variance only if the volumes of market trades with all securities that compose the portfolio are assumed to be constant during the averaging interval. Meanwhile, the time series of volumes of trades with the securities reveal their high irregularity or randomness during any reasonable interval. We derive a market-based expression of the portfolio variance $\Theta(t,t_0)$ that accounts for the impact of the randomness of the time series of volumes of market trades with the securities. Our expression of the portfolio variance $\Theta(t,t_0)$ differs from (1.2) and is a polynomial of the 4th degree in the variables $X_j(t_0)$ of the relative amounts invested into securities.

We highlight that our description of the portfolio variance that accounts for the randomness of trade volumes has nothing in common with numerous studies (Karpoff, 1986; Lo and Wang, 2001; Goyenko, 2024) that consider different issues of the trading volume.

It is evident that our market-based portfolio variance $\Theta(t,t_0)$, which takes into account the randomness of the volumes of market trades with the securities of the portfolio, changes the existing methods for optimal selection and the optimal portfolio compositions. In our paper we don't study this separate and complex problem. Investors and portfolio managers can use our results to adjust their procedures of optimal portfolio selections with the randomness of the volumes of market trades. However, we describe and explain the essence of two economic approximations that determine two expressions of the portfolio variance. The investors should take care of the consequences when and how they use one of these approximations.

In Section 2, we study the portfolio that was collected by the investor in the past at time t_0 of shares of $j=1,2,..J$ securities. We assume that the investor holds his portfolio unchanged and doesn't trade the shares of his portfolio. To assess at current time t the average return and variance of his portfolio, the investor observes the time series of all market trades that were performed with all securities of his portfolio during the averaging interval. We show how the investor should transform the time series of market trades with the securities of his portfolio to obtain the time series that describe the trades with his portfolio as a single market security. The portfolio's time series equally describe the return and variance of the portfolio that is composed of many securities $j=1,2,..J$, $J>>1$, and the portfolio that is composed of a single security $J=1$. In Section 3, we derive how the time series of values and volumes of trades with a single security j of the portfolio and the time series of values and volumes of trades with the portfolio as a single security determine their average prices, returns, and the variances of prices and returns. In Section 4, we describe the decomposition of average price, return, and their variances of the portfolio by the securities that compose the portfolio. The decomposition of the portfolio variance $\Theta(t,t_0)$ by its securities is a polynomial of the 4th degree in the variables of the relative amounts $X_j(t_0)$ (1.1; 1.2) invested into securities, and coefficients of this polynomial differ from the covariances $\theta_{jk}(t,t_0)$ (1.3). We show that when all volumes of trades that were performed with the securities during the averaging interval are assumed constant, the expression of the portfolio variance takes the form (1.2). In Section 5, we discuss the imaginary hypothesis that may explain the unintended latent approximation, which led Markowitz to his form of the portfolio variance (1.2). The conclusion is in Section 6.

We collect most calculations in Appendices A – D. In App. A, we derive the expressions of the market-based means and variances of prices and returns of a market security. In App. B, we derive the market-based covariances between prices and returns of two securities. In App. C, we derive the decompositions of the market-based means and variances of prices and returns of the portfolio by its securities. In App. D, we explain the economic sense of the distinctions between the market-based and the frequency-based assessments of the statistical moments of prices and returns. All prices are adjusted to the current time *t*.

2. Time series of trades with the portfolio as a single market security

We assume that at time t_0 in the past, the investor composed his portfolio of shares of $j=1,2,..J$ market securities. The investor holds his portfolio unchanged and doesn't trade the shares of his portfolio. Let us denote the investor's portfolio at time t_0 in the past by the number $U_j(t_0)$ and the values $C_j(t_0)$ of shares of marker securities $j=1,2,..J$. The prices $p_j(t_0)$ per share of each security j obey trivial equations:

$$C_j(t_0) = p_j(t_0)U_j(t_0) \quad ; \quad j = 1, \ldots J \tag{2.1}$$

The prices $p_j(t)$ and the values $C_j(t)$ of the shares $U_j(t_0)$ of security j can change in time t, but the number of shares $U_j(t_0)$ of each security j in the portfolio remains constant. We denote the value $Q_\Sigma(t_0)$ and the volume $W_\Sigma(t_0)$ or the number of shares of the portfolio at time t_0:

$$Q_\Sigma(t_0) = \sum_{j=1}^{J} C_j(t_0) \quad ; \quad W_\Sigma(t_0) = \sum_{j=1}^{J} U_j(t_0) \tag{2.2}$$

We define the price $s(t_0)$ (2.3) per one share of the portfolio similarly (2.1):

$$Q_\Sigma(t_0) = s(t_0)W_\Sigma(t_0) \quad ; \quad s(t_0) = \sum_{j=1}^{J} p_j(t_0)\, x_j(t_0) \quad ; \quad x_j(t_0) = \frac{U_j(t_0)}{W_\Sigma(t_0)} \tag{2.3}$$

We determine the portfolio at time t_0 in the past by its value $Q_\Sigma(t_0)$, volume $W_\Sigma(t_0)$, price $s(t_0)$, and by the set of corresponding values $C_j(t_0)$, volumes $U_j(t_0)$, and prices $p_j(t_0)$ of the securities $j=1,2,..J$ that compose the portfolio. Relations (2.3) decompose the price $s(t_0)$ per share of the portfolio by the prices $p_j(t_0)$ (2.1) of its securities. The coefficients $x_j(t_0)$ define the relative numbers of the shares $U_j(t_0)$ of security j in the total number of shares $W_\Sigma(t_0)$ of the portfolio.

The investor doesn't trade the shares of his portfolio but at the current time t is looking for the assessments of the means and variances of the prices and returns of his portfolio. To do that, the investor observes the time series of market trades that were performed with all securities of his portfolio during the averaging interval. For convenience, we assume that market trades with all securities $j=1,..J$ that compose the portfolio occur simultaneously at the same time t_i with a short time span $\varepsilon>0$ between the trades and assume that ε is constant and is the same for the trades with all securities $j=1,..J$. For each averaging time interval Δ, the number of market trades during the interval Δ is finite $i=1,..N$. For the current time t we denote the time averaging interval Δ (2.4) and consider the N terms of the time series of market trades at time t_i during Δ:

$$\Delta = \left[t - \frac{\Delta}{2};\ t + \frac{\Delta}{2}\right] \quad ; \quad t_{i+1} = t_i + \varepsilon \ \in \Delta \quad ; \quad i = 1, \dots N \quad ; \quad N \cdot \varepsilon = \Delta \quad ; \quad \varepsilon > 0 \tag{2.4}$$

We assume that during Δ (2.4), N trades were performed with each security $j=1,2,..J$ of the portfolio. During Δ (2.4), each trade with the value $C_j(t_i)$ and volume $U_j(t_i)$ at time t_i defines the price $p_j(t_i)$ (2.5) with security j:

$$C_j(t_i) = p_j(t_i)U_j(t_i) \quad ; \quad t_i \in \Delta \quad ; \ i = 1, \dots N \ ; \ \ j = 1,..J \tag{2.5}$$

The selection of the interval Δ (2.4) raises a question: what should be the duration of the averaging interval Δ (2.4) that can provide the investor with a reliable market-based assessment of the return and variance of his portfolio? What happens if the investor decides to sell his portfolio soon during Δ of the same duration? The investor hopes that the return of the portfolio after a possible sale would be close to his current assessment of the return and the variance during the period of the portfolio's sale would be close to the current assessment of the variance. To get this, the sale of the portfolio as an additional volume of trades should not disturb a lot the statistical properties of market trades with the securities during the averaging interval Δ (2.4). To obtain that, the volumes of trades with each security $j=1,2,..J$ in the market that are made during Δ (2.4) should be much more than the number of shares $U_j(t_0)$ (2.1) of security j in the portfolio of the investor. Simply speaking, to derive the reliable assessment of returns and variance of his portfolio, the investor should choose the averaging interval Δ that guarantees that the numbers of shares of each security j of his portfolio are less, for example, than 1-3 % of the total volumes of trades that were made with security j during Δ (2.4).

The investor can assess the total value $C_{\Sigma j}(t)$ and volume $U_{\Sigma j}(t)$ of trades that were made with security j during Δ as:

$$C_{\Sigma j}(t) = \sum_{i=1}^{N} C_j(t_i) \quad ; \quad U_{\Sigma j}(t) = \sum_{i=1}^{N} U_j(t_i) \quad ; \quad j = 1,..J \tag{2.6}$$

The investor selects averaging interval Δ (2.4) so that, due to (2.2), it guarantees:

$$U_j(t_0) \ll U_{\Sigma j}(t) \quad ; \quad W_\Sigma(t_0) = \sum_{j=1}^{J} U_j(t_0) \ll \sum_{j=1}^{J} U_{\Sigma j}(t) = \sum_{j=1}^{J}\sum_{i=1}^{N} U_j(t_i) \tag{2.7}$$

These simple considerations highlight a rather important issue: the assessments of returns and variance of the portfolio that is composed of shares of $j=1,..J$ securities depend on the duration of the averaging interval Δ (2.4). The more shares and securities in the portfolio, the longer the averaging interval Δ (2.4) should be.

Now, let us notice that the changes of the scale λ of the values $C_j(t_i)$ and volumes $U_j(t_i)$ of trades at time t_i with security j during Δ (2.4) don't change the statistical properties of the price $p_j(t_i)$. Let us define the normalized values $c_j(t_i)$ and volumes $u_j(t_i)$ of trades (2.8)

$$c_j(t_i) = \lambda \cdot C_j(t_i) \qquad ; \qquad u_j(t_i) = \lambda \cdot U_j(t_i) \tag{2.8}$$

The change of scale (2.8) transforms the equations (2.5) into (2.9):

$$c_j(t_i) = p_j(t_i)\, u_j(t_i) \quad or \quad \lambda \cdot C_j(t_i) = p_j(t_i)\, \lambda \cdot U_j(t_i) \tag{2.9}$$

It is obvious that random values $C_j(t_i)$, $c_j(t_i)$, and volumes $U_j(t_i)$, $u_j(t_i)$ (2.8; 2.9) define the same statistical properties of price $p_j(t_i)$ of security j. Let us apply these useful relations, and for each security $j=1,2,..J$ of the portfolio, choose the scale λ_j

$$\lambda_j = \frac{U_j(t_0)}{U_{\Sigma j}(t)} \tag{2.10}$$

The changes of scale (2.10) for each security $j=1,2,..J$ of the portfolio define normalized values $c_j(t_i)$ and volumes $u_j(t_i)$ (2.11):

$$c_j(t_i) = \lambda_j \cdot C_j(t_i) \qquad ; \qquad u_j(t_i) = \lambda_j \cdot U_j(t_i) \tag{2.11}$$

The relations (2.10; 2.11) guarantee that the total normalized volume $u_{\Sigma j}(t)$ (2.12) of trades with each security $j=1,..J$ of the portfolio during Δ (2.4) equals the number of shares $U_j(t_0)$ of security j that compose the investor's portfolio:

$$u_{\Sigma j}(t) = \sum_{i=1}^{N} u_j(t_i) = \frac{U_j(t_0)}{U_{\Sigma j}(t)} \sum_{i=1}^{N} U(t_i) = U_j(t_0) \tag{2.12}$$

For each security j, the time series of normalized values $c_j(t_i)$ and volumes $u_j(t_i)$ (2.8) describe the trade of precisely $U_j(t_0)$ shares of the investor's portfolio during Δ (2.4).

Let us consider at time t_i the sums of the trades with all securities $j=1,2,..J$, which compose the portfolio, and introduce the volumes $W(t_i)$ and values $Q(t_i)$ (2.13) of the trades with the portfolio aa a single security:

$$Q(t_i) = \sum_{j=1}^{J} c_j(t_i) \qquad ; \qquad W(t_i) = \sum_{j=1}^{J} u_j(t_i) \tag{2.13}$$

The relations (2.13) replace the initial time series of the values $C_j(t_i)$ and volumes $U_j(t_i)$ (2.5) of market trades with securities $j=1,2,..J$ with the time series (2.13) that describe the values $Q(t_i)$ and volumes $W(t_i)$ of trades with the portfolio as a single market security. Similar to (2.5; 2.7), the equation (2.14) determines the portfolio price $s(t_i)$ at time t_i during Δ:

$$Q(t_i) = s(t_i)\, W(t_i) \qquad ; \qquad t_i \in \Delta \quad ; \quad i = 1, \dots N \tag{2.14}$$

From (2.12; 2.13), obtain that the total volume of trades $W_\Sigma(t)$ (2.15) at time t during Δ is a constant and is equal to the number of shares $W_\Sigma(t_0)$ (2.2) of the portfolio at time t_0:

$$W_\Sigma(t) = \sum_{i=1}^{N} W(t_i) = \sum_{j=1}^{J} \sum_{i=1}^{N} u_j(t_i) = \sum_{j=1}^{J} U_j(t_0) = W_\Sigma(t_0) \tag{2.15}$$

However, the total value $Q_\Sigma(t)$ (2.16) of trades with the portfolio at current time t during Δ is determined by current prices of the securities, and that results in:

$$Q_\Sigma(t) = \sum_{i=1}^{N} Q(t_i) = \sum_{j=1}^{J} \sum_{i=1}^{N} c_j(t_i) = \sum_{j=1}^{J} \sum_{i=1}^{N} p_j(t_i) u_j(t_i) \tag{2.16}$$

We remind the readers that the investor holds his portfolio unchanged and doesn't trade its shares. However, the observations at current time *t* of the trades that were performed with the securities of the portfolio during *Δ* (2.4) allow us to assess the average price of the portfolio *s(t)* (2.17). Similar to (2.3), obtain:

$$Q_{\Sigma}(t) = s(t)\, W_{\Sigma}(t) \tag{2.17}$$

The time series (2.13; 2.14) describe the values $Q(t_i)$, volumes $W(t_i)$, and prices $s(t_i)$ of the trades of the portfolio absolutely in the same way as the time series of the values $C_j(t_i)$, volumes $U_j(t_i)$, and prices $p_j(t_i)$ describe trades of each of the market securities $j=1,2,..J$. The trade volumes $W(t_i)$ (2.13) of the portfolio are formed by the normalized volumes $u_j(t_i)$ (2.11) of trades with each security $j=1,2,..J$ of the portfolio. The total normalized volume $u_{\Sigma j}(t)$ (2.12) of trades with each security *j* equals the number of shares $U_j(t_0)$ of that security in the portfolio at time t_0. Thus, the relations (2.12; 2.13; 2.15) prove that the time series of the volumes $W(t_i)$ (2.13) of trades of the portfolio as a single security during *Δ* (2.4) precisely conform to the number of shares $U_j(t_0)$ of each security *j* in the portfolio at time t_0.

3. Market-based return and variance of the portfolio as a single security

In this section we show that the time series of the values $Q(t_i)$ and volumes $W(t_i)$ (2.13-2.15) of trades with the portfolio as a single market security determine its return and variance in the same form as the time series of the values $C_j(t_i)$ and volumes $U_j(t_i)$ (2.5; 2.6) of trades with security *j* of the portfolio, or their normalized time series of values $c_j(t_i)$ and volumes $u_j(t_i)$ (2.11) determine the return and variance of market security *j*.

At first, let us consider the time series of normalized values $c_j(t_i)$ and volumes $u_j(t_i)$ (2.8) and determine the return and variance of security *j* that had $U_j(t_0)$ shares in the investor's portfolio at time t_0 in the past. The total value $c_{\Sigma j}(t)$ (3.1) of trades with security *j* during *Δ* (2.4) equals:

$$c_{\Sigma \mathrm{j}}(t) = \sum_{i=1}^{N} c_j(t_i) = \frac{U_j(t_0)}{U_{\Sigma \mathrm{j}}(t)} \sum_{i=1}^{N} C_j(t_i) = \frac{U_j(t_0)}{U_{\Sigma \mathrm{j}}(t)} C_{\Sigma \mathrm{j}}(t) \tag{3.1}$$

We use (2.9; 2.11; 2.12; 3.1) and define the average price $p_j(t)$ (3.2) of $U_j(t_0)$ shares of security *j* at current time *t* during *Δ* (2.4):

$$c_{\Sigma \mathrm{j}}(t) = p_j(t)\, u_{\Sigma \mathrm{j}}(t) \;\; ; \;\; p_j(t) = \frac{c_{\Sigma \mathrm{j}}(t)}{u_{\Sigma \mathrm{j}}(t)} = \frac{C_{\Sigma \mathrm{j}}(t)}{U_{\Sigma \mathrm{j}}(t)} = \frac{1}{u_{\Sigma \mathrm{j}}(t)} \sum_{i=1}^{N} p_j(t_i)\, u_j(t_i) = \frac{c_j(t;1)}{u_j(t;1)} \tag{3.2}$$

The same expression of the average price *p(t;1)* follows from (2.5; 2.6):

$$p_j(t) = \frac{C_{\Sigma \mathrm{j}}(t)}{U_{\Sigma \mathrm{j}}(t)} = \frac{1}{U_{\Sigma \mathrm{j}}(t)} \sum_{i=1}^{N} p_j(t_i)\, U_j(t_i) \tag{3.3}$$

One can easily find out that the average price $p_j(t)$ (3.2; 3.3) of $U_j(t_0)$ shares of security *j* during *Δ* (2.4) takes the form of volume weighted average price (VWAP) (Berkowitz et al., 1988;

Duffie and Dworczak, 2021). In (3.2) we define average normalized value $c_j(t;1)$ and volume $u_j(t;1)$ (3.4) of trades with $U_j(t_0)$ shares of security j during Δ (2.4).

$$c_j(t;1) = \frac{1}{N}\sum_{i=1}^{N} c_j(t_i) \quad ; \quad u_j(t;1) = \frac{1}{N}\sum_{i=1}^{N} u_j(t_i) \tag{3.4}$$

The normalized number $u_{\Sigma j}(t)$ (2.12) of shares of security j that were traded during Δ (2.4) equals the number of its shares $Uj(t_0)$ at time t_0. We define the instant return $R_j(t_i,t_0)$ (3.5) of the trade with security j at time t_i with respect to time t_0 in the past:

$$R_j(t_i,t_0) = \frac{p_j(t_i)}{p_j(t_0)} \tag{3.5}$$

We use the so-called gross return $R_j(t_i,t_0)$ (3.5) instead of the usual definition of return $r_j(t_i,t_0)$:

$$r_j(t_i,t_0) = R_j(t_i,t_0) - 1 = \frac{p_j(t_i)}{p_j(t_0)} - 1 \tag{3.6}$$

The variances of both definitions of return are the same. From (3.2; 3.3), obtain the average return $R_j(t,t_0)$ of $U_j(t_0)$ shares of security j at time t during Δ (2.4):

$$R_j(t,t_0) = \frac{p_j(t)}{p_j(t_0)} = \frac{p_j(t)u_{\Sigma j}(t)}{p_j(t_0)u_{\Sigma j}(t)} = \frac{c_{\Sigma j}(t)}{p_j(t_0)U_j(t_0)} = \frac{c_{\Sigma j}(t)}{C_j(t_0)} = \frac{U_j(t_0)}{U_{\Sigma j}(t)}\frac{C_{\Sigma j}(t)}{C_j(t_0)} \tag{3.7}$$

From (2.11; 3.2; 3.3; 3.7), obtain that the average return $R_j(t,t_0)$ of security j can be expressed as value weighted average return (3.8), which is weighted by the normalized volumes $u_j(t_i)$ (2.11) or equally by the volumes $U_j(t_i)$ (2.5; 2.6) of trades with security j:

$$R_j(t,t_0) = \frac{1}{u_{\Sigma j}(t)}\sum_{i=1}^{N} R_j(t_i,t_0)\, u_j(t_i) = \frac{1}{U_{\Sigma j}(t)}\sum_{i=1}^{N} R_j(t_i,t_0)\, U_j(t_i) \tag{3.8}$$

We present the derivation of the market-based variances of prices and returns of a single security in App. A. The market-based variance $\theta_j(t,t_0)$ (3.9) of return $R_j(t_i,t_0)$ (3.5) of security j that accounts for the impact of random volumes of market trades is determined by the market-based variance $\phi_j(t)$ (3.10) of its price $p_j(t_i)$ (2.9).

$$\theta_j(t,t_0) = E_m\left[\left(R_j(t_i,t_0) - R_j(t,t_0)\right)^2\right] = \frac{\phi_j(t)}{p_j^2(t_0)} \tag{3.9}$$

$$\phi_j(t) = E_m\left[(p_j(t_i) - p_j(t))^2\right] \tag{3.10}$$

In (3.9; 3.10), $E_m[..]$ denotes market-based averaging (A.3) that accounts for the randomness of volumes of trades. In App A we derive the market-based variance $\phi_j(t)$ (A.16; 3.11) of prices $p_j(t_i)$ (2.9) that depends on the time series of random values $C_j(t_i)$ and volumes $U_j(t_i)$ of trades with a security during Δ (2.4):

$$\phi_j(t) = \frac{\Psi_{Cj}(t) + p_j^2(t)\Psi_{Uj}(t) - 2p_j(t)\, cov\{C_j(t), U_j(t)\}}{U_j(t;2)} \tag{3.11}$$

The notions of functions in (3.11) are given in App. A. We denote the variance $\Psi_{Cj}(t)$ (3.12) of values of trades with security j during Δ (2.4):

$$\Psi_{Cj}(t) = E\left[\left(C_j(t_i) - C_j(t;1)\right)^2\right] = \frac{1}{N}\sum_{i=1}^{N}\left(C_j(t_i) - C_j(t;1)\right)^2 = C_j(t;2) - C_j^2(t;1) \quad (3.12)$$

We denote the variance $\Psi_{Uj}(t)$ (3.13) of volumes of trades with security *j* during Δ (2.4):

$$\Psi_{Uj}(t) = E\left[\left(U_j(t_i) - U_j(t;1)\right)^2\right] = \frac{1}{N}\sum_{i=1}^{N}\left(U_j(t_i) - U_j(t;1)\right)^2 = U_j(t;2) - U_j^2(t;1) \quad (3.13)$$

We denote covariance $cov\{C_j(t),U_j(t)\}$ (A.8) between the time series of values $C_j(t_i)$ and volumes $U_j(t_i)$ of trades with security *j* during Δ (2.4). The function $U_j(t;2)$ denotes the average square (A.2) of the volumes $U_j(t_i)$ of trades with security *j* during Δ (2.4).

The relations (3.9; 3.11) determine market-based variance $\theta_j(t,t_0)$ (A.29; 3.14) of return $R_j(t_i,t_0)$ (3.5) of security *j* during Δ (2.4):

$$\theta_j(t,t_0) = \frac{\Psi_{Cj}(t) + R_j^2(t,t_0)\Psi_{C_{0j}}(t,t_0) - 2R_j(t,t_0)\, cov\{C_j(t),C_{0j}(t,t_0)\}}{C_{0j}(t,t_0;2)} \quad (3.14)$$

The notations in (3.14) are given in App. A. (A.24; A.26; A.30-A.32) and index *j* denotes security *j=1,..J* of the portfolio. The relations (3.11; 3.14) describe the market-based variances of prices and returns that account for the randomness of the volumes $U_j(t_i)$ of trades with security *j* during Δ (2.4). If one assumes that the volumes $U_j(t_i)$ of trades with security *j* during Δ (2.4) are constant, then variances of price and return of security *j* take the usual simple expressions (App.A.).

If $U_j(t_i)=const$, then the VWAP $p_j(t)$ (3.2; 3.3) takes the conventional simple form:

$$p_j(t) = \frac{1}{U_{\Sigma j}(t)}\sum_{i=1}^{N} p_j(t_i)\, U_j(t_i) \;\rightarrow\; p_j(t) = \frac{1}{N}\sum_{i=1}^{N} p_j(t_i) \quad (3.15)$$

The average return $R_j(t,t_0)$ (3.8) of security *j* during Δ (2.4) takes the form (3.16):

$$R_j(t,t_0) = \frac{1}{U_{\Sigma j}(t)}\sum_{i=1}^{N} R_j(t_i,t_0)\, U_j(t_i) \;\rightarrow\; R_j(t,t_0) = \frac{1}{N}\sum_{i=1}^{N} R_j(t_i,t_0) \quad (3.16)$$

The variance $\phi_j(t)$ (3.11) of prices $p_j(t_i)$ of security *j* takes the form (3.17)

$$\phi_j(t) = \frac{1}{N}\sum_{i=1}^{N}\left(p_j(t_i) - p_j(t)\right)^2 \quad (3.17)$$

The variance $\theta_j(t,t_0)$ (3.14) of return $R_j(t_i,t_0)$ of security *j* takes the form (3.18):

$$\theta_j(t,t_0) = \frac{1}{N}\sum_{i=1}^{N}\left(R_j(t_i,t_0) - R_j(t,t_0)\right)^2 \quad (3.18)$$

We underline that the expressions of the variance $\phi_j(t)$ (3.17) of prices and the variance $\theta_j(t,t_0)$ (3.18) of return $R_j(t_i,t_0)$ (3.16) of security *j* result from the assumption that all volumes of trades $U_j(t_i)=const$ with security *j* during Δ (2.4) are constant. In App. D., we discuss in more detail the distinctions between two approximations of a random and a constant trade volumes $U_j(t_i)$.

The goal of the above considerations of rather common definitions of average price, return, and their variances of a particular security *j* was to demonstrate that they are expressed by the time series of normalized values $c_j(t_i)$ and volumes $u_j(t_i)$ or equally by the time series of values

$C_j(t_i)$ and volumes $U_j(t_i)$ of trades with security j during Δ (2.4). That presents the evidence that the time series of the value $Q_\Sigma(t_0)$, volume $W_\Sigma(t_0)$, and price $s(t_0)$ (2.2; 2.3) that describe the initial state of the portfolio at time t_0 in the past and the time series of the values $Q(t_i)$, volumes $W(t_i)$, and prices $s(t_i)$ (2.13; 2.14) at time t_i of market trades with the portfolio as a single market security determine the return and variance of the portfolio completely in the same form as (3.7; 3.8) and (3.11; 3.14). That is the result of the transformation of the time series of the values $C_j(t_i)$ and volumes $U_j(t_i)$ of trades with the securities $j=1,2,..J$ that compose the portfolio into the time series that describe the values $Q(t_i)$ and volumes $W(t_i)$ of trades with the portfolio as a single security.

The simple substitutions (3.18) of variables:

$$C_j(t_i) \rightarrow Q(t_i) \quad ; \quad U_j(t_i) \rightarrow W(t_i) \quad ; \quad p_j(t_i) \rightarrow s(t_i) \tag{3.18}$$

give the expressions of the portfolio price, return, and their variances. From (2.17), obtain:

$$s(t) = \frac{Q_\Sigma(t)}{W_\Sigma(t)} = \frac{1}{W_\Sigma(t)} \sum_{i=1}^{N} s(t_i)\, W(t_i) \tag{3.19}$$

$$R(t,t_0) = \frac{s(t)}{s(t_0)} = \frac{Q_\Sigma(t)}{Q_\Sigma(t_0)} = \frac{1}{W_\Sigma(t)} \sum_{i=1}^{N} R(t_i,t_0) W(t_i) \quad ; \quad R(t_i,t_0) = \frac{s(t_i)}{s(t_0)} \tag{3.20}$$

$$\Phi(t) = \frac{\Psi_Q(t) + s^2(t)\Psi_W(t) - 2s(t)\, cov\{Q(t),W(t)\}}{W(t;2)} \tag{3.21}$$

$$\Theta(t,t_0) = \frac{\Psi_Q(t) + R^2(t,t_0)\Psi_{Q_0}(t,t_0) - 2R(t,t_0)\, cov\{Q(t),Q_0(t,t_0)\}}{Q_0(t,t_0;2)} \tag{3.22}$$

The above expressions describe market-based mean price $s(t)$ (3.19), return $R(t,t_0)$ (3.20), price variance $\Phi(t)$ (3.21), and return variance $\Theta(t,t_0)$ (3.22) of the portfolio during Δ (2.4). These market-based expressions account for the impact of random volumes $U_j(t_i)$ of market trades with securities $j=1,2,..J$ that compose the portfolio. We repeat that the investor collected this portfolio at time t_0 in the past and holds his portfolio unchanged and doesn't trade the shares of his portfolio. The investor observes the time series of the values $C_j(t_i)$ and volumes $U_j(t_i)$ of market trades with securities of his portfolio that were performed during Δ. The expressions (3.19-3.22) are the result of these observations.

It is evident that if all volumes $U_j(t_i)$ of trades with all securities that compose the portfolio are assumed constant during Δ, the expressions (3.19-3.22) take simple forms:

$$s(t) = \frac{1}{W_\Sigma(t)} \sum_{i=1}^{N} s(t_i)\, W(t_i) \;\rightarrow\; s(t) = \frac{1}{N} \sum_{i=1}^{N} s(t_i) \tag{3.23}$$

$$R(t,t_0) = \frac{1}{N} \sum_{i=1}^{N} R(t_i,t_0) \tag{3.24}$$

$$\Phi(t) = \frac{1}{N} \sum_{i=1}^{N} \big(s(t_i) - s(t)\big)^2 \tag{3.25}$$

$$\Theta(t,t_0) = \frac{1}{N} \sum_{i=1}^{N} \big(R(t_i,t_0) - R(t,t_0)\big)^2 \tag{3.26}$$

The market-based expressions (3.19-3.22) of average price $s(t)$, return $R(t,t_0)$, price variance $\Phi(t)$, and return variance $\Theta(t,t_0)$ of the portfolio present them in forms that coincide with the expressions for a single security. However, the investors that are looking for the optimal compositions of their portfolio by the set of $j=1,..J$ securities need the expressions that describe the decompositions of the portfolio properties by its securities. Actually, the definitions of the time series of the values $Q(t_i)$, volumes $W(t_i)$, and prices $s(t_i)$ (2.13; 2.14) of trades with the portfolio as a single security allow us to present the decomposition of the average price $s(t)$ (3.19), return $R(t,t_0)$ (3.20), price variance $\Phi(t)$ (3.21), and return variance $\Theta(t,t_0)$ (3.22) of the portfolio by its securities.

4. Decomposition of the portfolio variance by its securities

In this section we present the decompositions of the average price $s(t)$ (3.19), return $R(t,t_0)$ (3.20), price variance $\Phi(t)$ (3.21), and return variance $\Theta(t,t_0)$ (3.22) of the portfolio by its securities. The derivation is based on the results of Apps. A-D, and we refer there for detail.

Let us consider the time series of the values $Q(t_i)$, volumes $W(t_i)$, and prices $s(t_i)$ (2.13; 2.14) of trades with the portfolio as a single market security.

4.1 Decomposition of the mean price s(t):

The decomposition of the mean price $s(t)$ (3.19) of the portfolio at time t by the mean prices $p_j(t)$ (A.3) of the securities $j=1,2,..J$ is given in (C.2):

$$s(t) = \sum_{j=1}^{J} p_j(t)\ x_j(t_0) \quad ; \quad x_j(t_0) = \frac{U_j(t_0)}{W_\Sigma(t_0)} \tag{4.1}$$

The coefficients $x_j(t_0)$ in (4.1) describe the relative numbers of shares of the security j in the portfolio (2.3). We use $E_m[..]$ to denote market-based mathematical expectation and highlight its difference from the frequency-based mathematical expectation $E[..]$ (see App. A; App. D).

4.2 Decomposition of the return R(t,t₀):

We use (3.20) and (2.13; 2.14) and obtain the decomposition of return $R(t,t_0)$ (C.12; C.13):

$$R(t,t_0) = \frac{s(t)}{s(t_0)} = \sum_{j=1}^{J} \frac{p_j(t)}{p_j(t_0)} \frac{p_j(t_0)U_j(t_0)}{s(t_0)W_\Sigma(t_0)} = \sum_{j=1}^{J} R_j(t,t_0)\ X_j(t_0) \tag{4.2}$$

The functions $X_j(t_0)$ define the relative amount invested into security j of the portfolio at time t_0 in the past and coincide with (1.1; 1.2):

$$X_j(t_0) = \frac{p_j(t_0)U_j(t_0)}{s(t_0)W_\Sigma(t_0)} = \frac{C_j(t_0)}{Q_\Sigma(t_0)} \tag{4.3}$$

4.3 Decomposition of the variance Φ(t) of prices:

The average and average square of the portfolio values $Q(t;1)$, $Q(t;2)$ and volumes $W(t;1)$, $W(t;2)$ are determined in (A.2). We introduce coefficients of variation of the values $\psi(t)$, volumes $\chi(t)$, and their normalized covariance $\varphi(t)$ (3.6) of portfolio trades:

$$\psi^2(t) = \frac{\Psi_Q(t)}{Q^2(t;1)} \quad ; \quad \chi^2(t) = \frac{\Psi_W(t)}{W^2(t;1)} \quad ; \quad \varphi(t) = \frac{cov\{Q(t),W(t)\}}{Q(t;1)W(t;1)} \tag{4.4}$$

$$W(t;2) = W^2(t;1) + \Psi_W(t) = W^2(t;1)[1 + \chi^2(t)] \tag{4.5}$$

The relations (4.4; 4.5) allow us to transform the market-based variance $\Phi(t)$ (3.31) into (4.6):

$$\Phi(t) = \frac{\psi^2(t) - 2\,\varphi(t) + \chi^2(t)}{1+\chi^2(t)}\, s^2(t) \tag{4.6}$$

The decomposition of the variance $\Phi(t)$ (3.31; 4.6) of prices $s(t_i)$ of the portfolio by the covariances of normalized values and volumes of the securities $j=1,..J$ that compose the portfolio takes the form (see B.8; B.9; C.6-C.9).

$$\Phi(t) = \frac{1}{1+\chi^2(t)}\Big[\sum_{j,k=1}^{J} \psi_{jk}(t)\, p_j(t)p_k(t)\, x_j(t_0)x_k(t_0) - $$

$$-2\sum_{j,k,l=1}^{J} \varphi_{jk}(t)\, p_j(t)p_l(t)\, x_j(t_0)x_k(t_0)x_l(t_0)$$

$$+\sum_{jklf=1}^{J} \chi_{jk}(t)\, p_l(t)p_f(t)\, x_j(t_0)x_k(t_0)x_l(t_0)x_f(t_0)\Big] \tag{4.7}$$

If all trade volumes $u_j(t_i)$ of all securities $j=1,2,..J$ of the portfolio during the interval Δ (2.4) are assumed constant, then (4.7) takes the form (4.8). The functions $\sigma_{jk}(t)$ (B.16) present the covariances of prices $p_j(t_i)$ and $p_k(t_i)$ of two securities j and k of the portfolio during Δ.

$$\Phi(t) = \psi^2(t)s^2(t) = \sum_{j,k=1}^{J} \sigma_{jk}(t)\, x_j(t_0)x_k(t_0) \tag{4.8}$$

The derivation of (4.6-4.8) is given in App. C, (C3-C.10).

4.4 Decomposition of the variance $\Theta(t,t_0)$ of returns:

We present the derivation of decomposition of the variance $\Theta(t,t_0)$ (4.9) of returns in App. C (C.14-C.17). The decomposition of the variance $\Theta(t,t_0)$ (3.11) of returns of the portfolio by its securities results from the decomposition of the variance $\Phi(t)$ (4.7) of prices.

$$\Theta(t,t_0) = \frac{1}{1+\chi^2(t)}\Big[\sum_{j,k=1}^{J} \psi_{jk}(t)\, R_j(t,t_0)R_k(t,t_0)\, X_j(t_0)X_k(t_0) - $$

$$-2\sum_{j,k,l=1}^{J} \varphi_{jk}(t)\, R_j(t,t_0)R_l(t,t_0)\, X_j(t_0)X_k(t_0)X_l(t_0)$$

$$+\sum_{jklf=1}^{J} \chi_{jk}(t)\, R_l(t,t_0)R_f(t,t_0)\, X_j(t_0)X_k(t_0)X_l(t_0)X_f(t_0)\Big] \tag{4.9}$$

The decomposition (4.9) of the variance $\Theta(t,t_0)$ (3.14) of returns of the portfolio is a polynomial of the 4th degree in the variables of relative amounts $X_j(t_0)$ invested into security j of the portfolio. The expression (4.9) differs a lot from Markowitz's form of the portfolio variance $\Theta(t,t_0)$ (1.2) as a quadratic form in variables of $X_j(t_0)$. The only cause of these distinctions is the impact of random volumes of market trades with the securities of the portfolio. For the approximation when all volumes of trades with all securities $j=1,2,..J$ during Δ (2.4) are assumed constant, the variance $\Theta(t,t_0)$ (3.14; 4.9) takes the form (1.2) that was derived by Markowitz (1952):

$$\Theta(t,t_0) = \sum_{j,k}^{J} \theta_{jk}(t,t_0)\, X_j(t_0) X_k(t_0) \tag{4.10}$$

The covariances $\theta_{jk}(t,t_0)$ are determined in (B.17).

We underline that one should consider the variances of any portfolio in the same way as the variances of any tradable market security. The portfolio variance of prices $\Phi(t)$ (3.21) and the variance of returns $\Theta(t,t_0)$ (3.22) have the same expressions as the variances of prices $\phi_j(t)$ (3.11) and returns $\theta_j(t,t_0)$ (3.14) of any market security j. The decompositions of the portfolio variances of prices $\Phi(t)$ (4.7) and returns $\Theta(t,t_0)$ (4.9) by its securities are the result of the compositions of time series of the values $Q(t_i)$, volumes $W(t_i)$, and prices $s(t_i)$ (2.13; 2.14) of trades with the portfolio as a single market security by the corresponding time series that describe the values $c_j(t_i)$, volumes $u_j(t_i)$, and prices $p_j(t_i)$ (2.9; 2.11) of trades of the securities that compose the portfolio. The expressions of the portfolio variance $\Theta(t,t_0)$ (3.22; 4.9) highlight that the impact of risks of securities of the portfolio on the portfolio variance or portfolio risk has more complex dependence than was assumed by Markowitz (1.2).

5. A hypothesis that may explain the origin of Markowitz's variance

Finally, we consider a hypothesis that may explain the emergence of the unintended assumptions that result in Markowitz's decomposition of the portfolio variance $\Theta(t,t_0)$ (1.2). We propose that at first, Markowitz derived the decomposition of the portfolio return $R(t,t_0)$ (1.1; 4.2) by the mean returns $R_j(t,t_0)$ of its securities. The expression (1.1) defines the portfolio return $R(t,t_0)$ as a linear form of the mean returns $R_j(t,t_0)$ of its securities with coefficients that equal to the relative amounts $X_j(t_0)$ invested into securities at time t_0. Further, Markowitz made a latent assumption that at time t_i the instant random returns $R_j(t_i,t_0)$ of the securities define the random return $R(t_i,t_0)$ of the portfolio at time t_i in the same form as (1.1):

$$R(t_i,t_0) = \sum_{j=1}^{J} R_j(t_i,t_0)\; X_j(t_0) \tag{5.1}$$

This "almost obvious" assumption (5.1) immediately results in (1.1) and (1.2). However, it is evident that the transition from (1.1) to (5.1) hides an approximation that neglects all factors with zero means but non-zero average squares that could significantly disturb the variance (1.2) of the portfolio. Our market-based consideration of this problem confirms that.

The time series of the values $Q(t_i)$, volumes $W(t_i)$, and prices $s(t_i)$ (2.13; 2.14) of trades of the portfolio as a single security reveals a more complex dependence of the random returns $R(t_i,t_0)$ of the portfolio on random returns $R_j(t_i,t_0)$ of its securities. From (2.14; C.11), obtain the instant random return $R(t_i,t_0)$ as a result of trade with the portfolio at time t_i:

$$R(t_i,t_0)=\frac{s(t_i)}{s(t_0)}=\frac{Q(t_i)}{s(t_0)W(t_i)}=\sum_{j=1}^{J}\frac{c_j(t_i)}{p_j(t_0)u_j(t_i)}\frac{p_j(t_0)U_j(t_0)}{s(t_0)W_\Sigma(t_0)}\frac{W_\Sigma(t_0)}{W(t_i)}\frac{U_j(t_i)}{U_{\Sigma j}(t)} \quad (5.2)$$

The use (2.11; A.24) and (C.13), transforms (5.2) into (5.3):

$$R(t_i,t_0)=\sum_{j=1}^{J}R_j(t_i,t_0)\,X_j(t_0)\frac{W_\Sigma(t_0)}{W(t_i)}\frac{U_j(t_i)}{U_{\Sigma j}(t)} \quad (5.3)$$

The average of (5.2) with the help of (3.19) and (4.1) give the same expression of the average return $R(t,t_0)$ (4.2) of the portfolio as (1.1). If one assumes that all volumes $U_j(t_i)$ of trades with all securities $j=1,..J$ of the portfolio during Δ (2.4) are constant, obtain:

$$U_j(t_i)=\frac{U_{\Sigma j}(t)}{N}\ ;\ W(t_i)=\frac{W_\Sigma(t_0)}{N}\Rightarrow\frac{W_\Sigma(t_0)}{W(t_i)}\frac{U_j(t_i)}{U_{\Sigma j}(t)}=1 \quad (5.4)$$

In this case, the random returns $R(t_i,t_0)$ (5.3) of the portfolio take the form (5.1). That is a result of the assumption that all volumes of trades with all securities $j=1,..J$ of the portfolio during the averaging interval Δ (2.4) are assumed constant. Markowitz's expression of the portfolio variance $\Theta(t,t_0)$ (1.2) is a direct consequence of the (5.1). That clarifies the essence and limitations of Markowitz's expression of the portfolio variance $\Theta(t,t_0)$ (1.2), which is valid only if all trade volumes $U_j(t_i)$ during Δ (2.4) with all securities $j=1,..J$ of the portfolio are assumed constant.

If the investor whants assess the variance $\Theta(t,t_0)$ of his portfolio that accounts for the randomness of the volumes of trades with the securities of his portfolio during the averaging interval Δ (2.4), he should consider (5.3). Let us denote the fluctuations of returns of the portfolio $\delta R(t_i,t_0)$ and of the securities $\delta R_j(t_i,t_0)$:

$$\delta R(t_i,t_0)=R(t_i,t_0)-R(t,t_0)\ ;\ \ \delta R_j(t_i,t_0)=R_j(t_i,t_0)-R_j(t,t_0) \quad (5.5)$$

The term $\frac{U_j(t_i)}{U_{\Sigma j}(t)}$ in (5.3) has the meaning of the random share of the trade volume $U_j(t_i)$ at time t_i with security j in the total volume of trades $U_{\Sigma j}(t)$ during Δ (2.4). The term $\frac{W(t_i)}{W_\Sigma(t_0)}=\frac{W(t_i)}{W_\Sigma(t)}$ in (5.3) is the random share of the volume $W_j(t_i)$ of trade at time t_i with the portfolio during Δ (2.4) in the total volume $W_\Sigma(t)$ or total number of shares $W_\Sigma(t_0)$ of the portfolio $W_\Sigma(t)= W_\Sigma(t_0)$.

If one accounts for the randomness of trade volumes, these factors significantly disturb (5.1) and result in distinctions of the portfolio variance $\Theta(t,t_0)$ from the expression (1.2).

Let us denote $\delta W_j(t_i,t_0)$ (5.6) as a measure of randomness of the volumes $U_j(t_i)$ of trades at time t_i with security j with respect to the volumes $W(t_i)$ of trades with the portfolio as a single security

$$\delta W_j(t_i,t_0)=\frac{W_\Sigma(t_0)}{U_{\Sigma j}(t)}\frac{U_j(t_i)}{W(t_i)}-1 \quad (5.6)$$

Then, from (1.1; 5.3; 5.5; 5.6), obtain:

$$\delta R(t_i,t_0)=\sum_{j=1}^{J}[\delta R_j(t_i,t_0)+R_j(t_i,t_0)\,\delta W_j(t_i,t_0)]\,X_j(t_0) \quad (5.7)$$

It is obvious that market-based mathematical expectation $E_m[...]$ of $\delta R(t_i,t_0)$ (5.7), which accounts for the impact of random volumes of trades with the securities of the portfolio and depends on the average price of the portfolio $s(t)=E_m[s(t_i)]$ (4.1; 4.2), equals zero:

$$E_m[\ \delta R(t_i, t_0)] = 0$$

However, the square of (5.7) gives:

$$\delta^2 R(t_i, t_0) = \sum_{j,k=1}^{J} \delta R_j(t_i, t_0)\, \delta R_k(t_i, t_0) X_j(t_0) X_k(t_0)$$

$$+2 \sum_{j,k=1}^{J} \delta R_j(t_i, t_0) R_k(t_i, t_0) X_j(t_0) X_k(t_0) \delta W_k(t_i, t_0) +$$

$$\sum_{j,k=1}^{J} R_j(t_i, t_0) R_k(t_i, t_0) X_j(t_0) X_k(t_0) \delta W_j(t_i, t_0) \delta W_k(t_i, t_0) \qquad (5.8)$$

The variance $\Theta(t,t_0)$ of the portfolio is a market-based mathematical expectation of $\delta^2 R(t_i,t_0)$:

$$\Theta(t, t_0) = E_m[\ \delta^2 R(t_i, t_0)] \qquad (5.9)$$

The averaging of the first term of (5.8) give (5.10; 511) that coincide with Markowitz's expression of the variance $\Theta(t,t_0)$ (1.2):

$$E_m\left[\sum_{j,k=1}^{J} \delta R_j(t_i, t_0)\, \delta R_k(t_i, t_0) X_j(t_0) X_k(t_0)\right] = \sum_{j,k}^{J} \theta_{jk}(t, t_0)\, X_j(t_0) X_k(t_0) \qquad (5.10)$$

$$\theta_{jk}(t, t_0) = E_m[\delta R_j(t_i, t_0) \delta R_k(t_i, t_0)] \qquad (5.11)$$

However, the averaging of (5.8) reveals additional terms (5.12; 5.13) that significantly change the expression of the portfolio variance $\Theta(t,t_0)$

$$2\ \sum_{j,k=1}^{J} E_m\left[R_j(t_i, t_0) R_k(t_i, t_0) \delta W_k(t_i, t_0)\right] X_j(t_0) X_k(t_0) \qquad (5.12)$$

$$\sum_{j,k=1}^{J} E_m\, [R_j(t_i, t_0) R_k(t_i, t_0) \delta W_j(t_i, t_0) \delta W_k(t_i, t_0)] X_j(t_0) X_k(t_0) \qquad (5.13)$$

We derived the portfolio variance $\Theta(t,t_0)$ (5.9-5.13) that accounts for the randomness of the volumes of trades with securities of the portfolio in Section 4.

Actually, the decomposition (1.1) of the mean portfolio return $R(t,t_0)$ by the mean returns $R_j(t,t_0)$ of its securities doesn't cause the similar decomposition of the random returns (5.1). That was a latent assumption of Markowitz. The impacts of random volumes of market trades with the securities of the portfolio cause the random returns $R(t_i,t_0)$ of the portfolio to have a more complex form (5.2; 5.3), and the portfolio variance $\Theta(t,t_0)$ takes the form (3.11; 4.9). However, the simplicity of Markowitz's expression of the portfolio variance $\Theta(t,t_0)$ (1.2) resulted in its being in use as the basis for optimal portfolio selection for more than 70 years.

6. Conclusion

The investor who holds his portfolio and doesn't trade its shares can use the time series of market trades with the securities of the portfolio to assess portfolio return and variance in the same form as he assesses return and variance of any market security. The transformations of the time series of market trades with securities that compose the portfolio determine the time series of trades with the portfolio as a single market security. That establishes the equality between the description of any portfolio and any single market security.

The decomposition of the portfolio's variance by its securities results from the dependence of the portfolio trade time series on the time series of trades with the securities and is a polynomial of the 4th degree in the variables of the relative amounts $X_j(t_0)$ invested into securities. The only cause of the distinctions from Markowitz's expression of the portfolio variance $\Theta(t,t_0)$ (1.2), which has a quadratic form, is the impact of the random volumes of trades with the securities. Markowitz's decomposition of the portfolio variance $\Theta(t,t_0)$ (1.2) is valid when all volumes of trades with all securities of the portfolio are assumed constant during the averaging interval. The current methods for selecting the portfolio with higher returns under lower variance that are based on decomposition $\Theta(t,t_0)$ (1.2) are valid only for this approximation that neglects the impact of random trade volumes.

The market-based portfolio selection that accounts for the influence of random volumes of market trades is more difficult. The expression of market-based portfolio variance $\Theta(t,t_0)$ (3.11; 4.9) reveals that the dependence of the portfolio risk on the risks of the securities that compose the portfolio is a more complex problem than it was described by Markowitz (1.2).

To forecast the portfolio variance $\Theta(t,t_0)$ (3.11; 4.9) at horizon T, one should predict the time series of the values and volumes of market trades with all securities of the portfolio at the same horizon T during the averaging interval Δ (2.4). That significantly complicates the forecasts of the portfolio variance and the methods for selecting optimal portfolios with lower variance and higher return. In this paper we don't consider these problems.

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Appendix A. Market-Based Means and Variances of a Security

This Appendix gives brief derivations of the market-based means and variances of prices and returns of a market security that are based on the results (Olkhov, 2022-2025).

Let us consider the equation (2.5) on the values $C(t_i)$, volumes $U(t_i)$, and prices $p(t_i)$ at time t_i, $i=1,..N$, of market trades with a security during Δ (2.4):

$$C(t_i) = p(t_i)\, U(t_i) \tag{A.1}$$

We assess the *n-th* statistical moments of trade values $C(t;n)$ and volumes $U(t;n)$ by a finite number of N terms of time series during Δ in a generally accepted form:

$$C(t;n) = E[C^n(t_i)] = \frac{1}{N}\sum_{i=1}^{N} C^n(t_i) \quad ; \quad U(t;n) = E[U^n(t_i)] = \frac{1}{N}\sum_{i=1}^{N} U^n(t_i) \tag{A.2}$$

We denote mathematical expectation $E[..]$ of random trade values and volumes and recall that (A.2) gives the approximations of statistical moments by a finite number N of terms. The equation (A.1) prohibits independent definitions of statistical moments of values $C(t_i)$, volumes $U(t_i)$, and prices $p(t_i)$. We consider the trade values $C(t_i)$ and volumes $U(t_i)$ as the random variables that determine the market-based mean price $p(t)$ (A.3) as the ratio of the total value $C_\Sigma(t;1)$ to the total volume $U_\Sigma(t;1)$ (A.4) of market trades that equals volume weighted average price (VWAP) (Berkowitz et al., 1988; Duffie and Dworczak, 2021):

$$p(t) = E_m[p(t_i)] = \frac{C_\Sigma(t;1)}{U_\Sigma(t;1)} = \frac{1}{U_\Sigma(t;1)}\sum_{i=1}^{N} p(t_i)U(t_i) = \sum_{i=1}^{N} p(t_i)\mu(t_i;1) = \frac{C(t;1)}{U(t;1)} \tag{A.3}$$

We note $E_m[..]$ the market-based mathematical expectation to underline the distinctions with the generally accepted mathematical expectation $E[..]$ (A.2) (Shiryaev, 1999; Shreve, 2004), which we call the frequency-based. We clarify the relations between the market-based $E_m[..]$ and the frequency-based $E[..]$ mathematical expectations in App. D. The total values $C_\Sigma(t;1)$ to total volumes $U_\Sigma(t;1)$ (A.4) of market trades takes the form:

$$C_\Sigma(t;1) = \sum_{i=1}^{N} C(t_i) \quad ; \quad U_\Sigma(t;1) = \sum_{i=1}^{N} U(t_i) \tag{A.4}$$

The function $\mu(t_i,1)$ (A.5) in (A.3) has the meaning of the weight function.

$$\mu(t_i;1) = \frac{U(t_i)}{U_\Sigma(t;1)} \quad ; \quad \sum_{i=1}^{N} \mu(t_i;1) = 1 \tag{A.5}$$

To derive the variance of price $\phi(t)$ (A.6) of a market security

$$\phi(t) = E_m\left[\left(p(t_i) - p(t)\right)^2\right] = var\{p(t), p(t)\} \tag{A.6}$$

one should consider the squares (A.7) of the equation (A.1):

$$C^2(t_i) = p^2(t_i)U^2(t_i) \tag{A.7}$$

The equation (A.7) determines how the *2nd* statistical moments of trade values $C(t;2)$, volumes $U(t;2)$ (A.2), and their covariance $cov\{C(t),U(t)\}$ (A.8) determine the variance of price $\phi(t)$ (A.6).

$$cov\{C(t), U(t)\} = E\big[\big(C(t_i) - C(t;1)\big)\big(U(t_i) - U(t;1)\big)\big] =$$
$$= \frac{1}{N}\sum_{i=1}^{N}\big(C(t_i) - C(t;1)\big)\big(U(t_i) - U(t;1)\big) \quad \text{(A.8)}$$

The equation (A.7) determines the weight function $\mu(t_i,2)$ (A.9) that is similar to (A.3; A.5):

$$\mu(t_i;2) = \frac{U^2(t_i)}{\sum_{i=1}^{N} U^2(t_i)} \quad ; \quad \sum_{i=1}^{N}\mu(t_i;2) = 1 \quad \text{(A.9)}$$

The average $E_m[p^2(t_i)]$ must be consistent with the mean price $p(t)= E_m[p(t_i)]$ (A.3) that is determined by the weight functions $\mu(t_i,1)$ (A.5). To derive $E_m[p^2(t_i)]$ and the price variance $\phi(t)$ (A.6) that is consistent with the mean price $p(t)$ (A.3) we define:

$$\phi(t) = E_m[(p(t_i) - p(t))^2] = \sum_{i=1}^{N}\big(p(t_i) - p(t)\big)^2\mu(t_i;2) = E_m[p^2(t_i)] - p^2(t) \quad \text{(A.10)}$$

We highlight that the mean price $p(t)$ (A.3) in (A.10) is determined by the weight function $\mu(t_i,1)$ (A.5), but not by $\mu(t_i,2)$ (A.9). The definition of the price variance $\phi(t)$ (A.10) ties up the VWAP $p(t)$ (A.3; A.5) and the averaging by the weight function $\mu(t_i,2)$ (A.9). That defines the consistent values of the price variance $\phi(t)$ and $E_m[p^2(t_i)]$. We refer to Olkhov (2022-2023) for further clarifications. One can calculate (A.10) as follows:

$$\phi(t) = \sum_{i=1}^{N} p^2(t_i)\, w(t_i;2) - 2p(t)\sum_{i=1}^{N} p(t_i)\mu(t_i;2) + p^2(t) \quad \text{(A.11)}$$

From (A.2) and (A.7; A.9), obtain

$$\sum_{i=1}^{N} p^2(t_i)\,\mu(t_i;2) = \frac{1}{\frac{1}{N}\sum_{i=1}^{N} U^2(t_i)}\frac{1}{N}\sum_{i=1}^{N} C^2(t_i) = \frac{C(t;2)}{U(t;2)} \quad \text{(A.12)}$$

$$\sum_{i=1}^{N} p(t_i)\mu(t_i;2) = \frac{1}{\frac{1}{N}\sum_{i=1}^{N} U^2(t_i)}\frac{1}{N}\sum_{i=1}^{N} C(t_i)\,U(t_i) = \frac{E[C(t)U(t)]}{U(t;2)} \quad \text{(A.13)}$$

We denote the joint mathematical expectation $E[C(t)U(t)]$ of the values and volumes:

$$E[C(t)U(t)] = \frac{1}{N}\sum_{i=1}^{N} C(t_i)\,U(t_i) = C(t;1)U(t;1) + cov\{C(t), U(t)\} \quad \text{(A.14)}$$

From (A.12-A.14), obtain

$$\phi(t) = \frac{C(t;2) - 2p(t)C(t;1)U(t;1) - 2p(t)cov\{C(t),U(t)\} + p^2(t)U(t;2)}{U(t;2)} =$$
$$\frac{C(t;2) - C^2(t;1) + C^2(t;1) - 2p(t)C(t;1)U(t;1) + p^2(t)U^2(t;1) + p^2(t)\big[U(t;2) - U^2(t;1)\big] - 2p(t)cov\{C(t),U(t)\}}{U(t;2)} \quad \text{(A.15)}$$

Finally, from (A.3; A.15), obtain the market-based variance $\phi(t)$ (A.16) of price of the security:

$$\phi(t) = \frac{\Psi_C(t) + p^2(t)\Psi_U(t) - 2p(t)\, cov\{C(t),U(t)\}}{U(t;2)} \quad \text{(A.16)}$$

In (A.16) we denote the variance $\Psi_C(t)$ (A.17) of trade values and the variance $\Psi_U(t)$ (A.18) of trade volumes during Δ:

$$\Psi_C(t) = E\left[\big(C(t_i) - C(t;1)\big)^2\right] = \frac{1}{N}\sum_{i=1}^{N}\big(C(t_i) - C(t;1)\big)^2 = C(t;2) - C^2(t;1) \quad \text{(A.17)}$$

$$\Psi_U(t) = E\left[\big(U(t_i) - U(t;1)\big)^2\right] = \frac{1}{N}\sum_{i=1}^{N}\big(U(t_i) - U(t;1)\big)^2 = U(t;2) - U^2(t;1) \quad \text{(A.18)}$$

The mean price $p(t)$ (A.3) and the variance $\phi(t)$ (A.16) of the price of a market security account for the impact of random volumes $U(t_i)$ of market trades during Δ (1.4).

If one considers the approximation for which all trade volumes $U(t_i)=U$ are constant during Δ (2.4), then from (A.3) and (A.9; A.10), obtain the frequency-based approximations of the mean price $p(t)$ (A.19) and variance $\phi(t)$ (A.20) of prices of a market security:

$$if \;\; U(t_i) = U - const \;\; \Rightarrow \; \mu(t_i;1) = \frac{U(t_i)}{\sum_{i=1}^{N} U(t_i)} = \frac{1}{N} \quad ; \quad \mu(t_i;2) = \frac{U^2(t_i)}{\sum_{i=1}^{N} U^2(t_i)} = \frac{1}{N}$$

$$p(t) = E_m[p(t_i)] = \sum_{i=1}^{N} p(t_i)\mu(t_i;1) = \frac{1}{N}\sum_{i=1}^{N} p(t_i) \tag{A.19}$$

$$\phi(t) = E_m\left[\big(p(t_i) - p(t)\big)^2\right] = \sum_{i=1}^{N}\big(p(t_i) - p(t)\big)^2 \mu(t_i;2) = \frac{1}{N}\sum_{i=1}^{N}\big(p(t_i) - p(t)\big)^2 \tag{A.20}$$

The usual frequency-based assessments (A.19; A.20) neglect the impact of random trade volumes on the mean and variance of the price of a security. The expressions (A.19; A.20) use only random time series of prices $p(t_i)$, $i=1,..N$ (Shiryaev, 1999; Shreve, 2004; Elton et al., 2014). The neglecting of the impact of random trade volumes could result in significant errors for the assessments of the means and variances of big stakes of market securities and multi-billion portfolios. The use of market-based means and variances of prices (A.3; A.16) that account for the impact of random volumes of market trades is mandatory for those who design reliable large market and macroeconomic models and forecasts. In particular, it is important for the developers of market and macroeconomic models like BlackRock's Aladdin and Asimov, JP Morgan, and the U.S. Fed.

The derivation of higher market-based *n-th* statistical moments that determine market-based price probability with higher accuracy is given in Olkhov (2022).

The derivations of the market-based mean and variance of returns are given in Olkhov (2023). However, the description of the mean and variance of returns with respect to the price of the market security at a specific time t_0 in the past when the investor has collected his portfolio is a much simpler problem. We consider the gross return $R(t_i,t_0)$ of price $p(t_i)$ of a market security at time t_i with respect to its price $p(t_0)$ in the past at time t_0 as:

$$R(t_i, t_0) = \frac{p(t_i)}{p(t_0)} \tag{A.21}$$

The variance (A.23) of gross return $R(t_i,t_0)$ (A.21) and net return $r(t_i,t_0)$ (A.22) is the same:

$$r(t_i, t_0) = \frac{p(t_i) - p(t_0)}{p(t_0)} = R(t_i, t_0) - 1 \tag{A.22}$$

$$var\{r(t, t_0)\} = E[(r(t_i, t_0) - E[r(t_i, t_0)])^2] = E[(R(t_i, t_0) - E[R(t_i, t_0)])^2] = var\{R(t, t_0)\} \tag{A.23}$$

The derivation of the mean and variance of returns (A.21) is much more convenient than for (A.22). To describe return $R(t_i,t_0)$ (A.21), we introduce the equation (A.24), alike to (A.1):

$$C(t_i) = p(t_i) \cdot U(t_i) = \frac{p(t_i)}{p(t_0)} \cdot p(t_0)U(t_i) = R(t_i,t_0)C_0(t_i,t_0)$$

$$C(t_i) = R(t_i,t_0)C_0(t_i,t_0) \quad ; \quad C_0(t_i,t_0) = p(t_0)U(t_i) \qquad \text{(A.24)}$$

The function $C_0(t_i,t_0)$ in (A.24) describes the value of the current trade volume $U(t_i)$ at the price $p(t_0)$ in the past at time t_0. The return $R(t_i,t_0)$ (A.21) at time t_i is the ratio of the current trade value $C(t_i)$ of the trade volume $U(t_i)$ to its past value $C_0(t_i,t_0)$. The use of (A.24) results in the derivation of the market-based mean return $R(t,t_0)$ that is averaged during Δ (2.4) in the form that coincides with VWAP $p(t)$ (A.3; A.19):

$$R(t,t_0) = E_m[R(t_i,t_0)] = \frac{1}{\sum_{i=1}^{N} C_0(t_i,t_0)} \sum_{i=1}^{N} R(t_i,t_0)C_0(t_i,t_0) = \frac{C(t;1)}{C_0(t,t_0;1)} \qquad \text{(A.25)}$$

The average $C_0(t,t_0;1)$ (A.26) is determined similar to (A.2):

$$C_0(t,t_0;1) = \frac{1}{N}\sum_{i=1}^{N} C_0(t_i,t_0) = p(t_0)U(t;1) \qquad \text{(A.26)}$$

From (A.25), obtain:

$$\frac{C_0(t_i,t_0)}{\sum_{i=1}^{N} C_0(t_i,t_0)} = \frac{U(t_i)}{\sum_{i=1}^{N} U(t_i)} = \mu(t_i;1)$$

The market-based mean return $R(t,t_0)$ (A.25) takes the form (A.3; A.27):

$$R(t,t_0) = \sum_{i=1}^{N} R(t_i,t_0)\mu(t_i;1) = \frac{E_m[p(t_i)]}{p(t_0)} = \frac{p(t)}{p(t_0)} \qquad \text{(A.27)}$$

From (A.16) obtain the variance $\theta(t,t_0)$ (A.28; A.29) of return of a market security:

$$\theta(t,t_0) = E_m\left[\left(R(t_i,t_0) - R(t,t_0)\right)^2\right] = \frac{E_m\left[\left(p(t_i)-p(t)\right)^2\right]}{p^2(t_0)} = \sum_{i=1}^{N}\left(R(t_i,t_0) - R(t,t_0)\right)^2 \mu(t_i;2) \qquad \text{(A.28)}$$

$$\theta(t,t_0) = \frac{\phi(t)}{p^2(t_0)} = \frac{\Psi_C(t) + R^2(t,t_0)\Psi_{C_0}(t,t_0) - 2R(t,t_0)\, cov\{C(t),C_0(t,t_0)\}}{C_0(t,t_0;2)} \qquad \text{(A.29)}$$

Function $\Psi_{C0}(t,t_0)$ (A.30) determines the variance of the past value $C_0(t_i,t_0)$ and $cov\{C(t),C_0(t,t_0)\}$ (A.31) determines the covariance of the current $C(t_i)$ and past $C_0(t_i,t_0)$ trade values. The mean squares of the past values $C_0(t,t_0;2)$ (A.32) are determined alike to (A.2):

$$\Psi_{C_0}(t) = \frac{1}{N}\sum_{i=1}^{N}\left(C_0(t_i,t_0) - C_0(t,t_0;1)\right)^2 = C_0(t,t_0;2) - C_0^2(t,t_0;1) \qquad \text{(A.30)}$$

$$cov\{C(t),C_0(t,t_0)\} = \frac{1}{N}\sum_{i=1}^{N}\left(C(t_i) - C(t;1)\right)\left(C_0(t_i,t_0) - C_0(t,t_0;1)\right) \qquad \text{(A.31)}$$

$$C_0(t,t_0;2) = \frac{1}{N}\sum_{i=1}^{N} C_0^2(t_i,t_0) \qquad \text{(A.32)}$$

The relations (A.25-A.32) determine the mean and variance of returns of a market security with respect to its price $p(t_0)$ in the past at time t_0.

If one considers the approximation for which all trade volumes $U(t_i)$ are assumed constant, then, similar to (A.19; A.20), from (A.25-A.32), obtain the frequency-based approximations of the mean $R(t,t_0)$ (A.33) and variance $\theta(t,t_0)$ (A.34) of returns of a market security:

$$R(t,t_0) = E_m[R(t_i,t_0)] = \frac{1}{N}\sum_{i=1}^{N} R(t_i,t_0) \qquad \text{(A.33)}$$

$$\theta(t,t_0) = E_m\left[\left(R(t_i,t_0) - R(t,t_0)\right)^2\right] = \frac{1}{N}\sum_{i=1}^{N}\left(R(t_i,t_0) - R(t,t_0)\right)^2 \quad \text{(A.34)}$$

The generally accepted frequency-based expressions of the mean $R(t,t_0)$ (A.33) and variance $\theta(t,t_0)$ (A.34) of return describe the approximation for which all trade volumes are assumed constant. The frequency-based mean and variance (A.33; A.34) neglect the influence of the random volumes of market trades. Those who manage large stakes of securities and multi-billion portfolios should keep that in mind.

We highlight that Markowitz (1952) used the expression of the return $R(t,t_0)$ (1.1) of the portfolio that has absolutely the same form as VWAP $p(t)$ (A.3) and market-based average return $R(t,t_0)$ (A.25; A.27). From (1.1), obtain:

$$R(t,t_0) = \sum_{j=1}^{J} R_j(t,t_0)X_j(t_0) = \frac{1}{Q_\Sigma(t_0)}\sum_{J=1}^{J} R_j(t,t_0)C_j(t_0) \quad ; \quad X_j(t_0) = \frac{C_j(t_0)}{Q_\Sigma(t_0)} \quad \text{(A.35)}$$

It is obvious that the return $R(t,t_0)$ (1.1; A.35) of the portfolio matches the form and the meaning of VWAP $p_j(t)$ (A.3) and the mean return $R(t,t_0)$ (A.25). We call Markowitz's definition of the return $R(t,t_0)$ (1.1; A.35) of the portfolio Value Weighted Average Return, or VaWAR. We underline that there is no difference between determining the return of the portfolio $R(t,t_0)$ (A.35) via returns $R_j(t,t_0)$ of its numerous securities $j=1,2,\ldots J$, and determining the mean price (A.3) or mean return $R(t,t_0)$ (A.25) of a market security via its N trade values at time t_i during Δ (2.4). We consider that Markowitz (1952) has introduced the market-based averaging procedure as Value Weighted Averaging and Volume Weighted Averaging almost 35 years prior to Berkowitz et al. (1988).

Appendix B. Covariances of Prices and Returns of Securities *j* and *k*

The description of the market-based covariance $\sigma_{jk}(t)$ (B.1) of prices $p_j(t_i)$ and $p_k(t_i)$ (2.5) of securities j and k at time t during Δ (2.4) follows (Olkhov, 2025).

$$\sigma_{jk}(t) = cov\{p_j(t), p_k(t)\} = E_m[\left(p_j(t_i) - p_j(t)\right)\left(p_k(t_i) - p_k(t)\right)] \quad \text{(B.1)}$$

To define the market-based mathematical expectation $E_m[..]$ in (B.1), we consider the product (B.2) of two equations (2.5) that describe the securities j and k:

$$C_j(t_i)C_k(t_i) = p_j(t_i)p_k(t_i)\, U_j(t_i)U_k(t_i) \quad \text{(B.2)}$$

The same reasons that approve the derivation of the variance $\phi(t)$ (A.10) of prices allow determine the covariance $\sigma_{jk}(t)$ (B.3) of prices $p_j(t_i)$ and $p_k(t_i)$ in a similar form:

$$\sigma_{jk}(t) = \frac{1}{U_{jk}(t)}\frac{1}{N}\sum_{i=1}^{N}\left(p_j(t_i) - p_j(t)\right)\left(p_k(t_i) - p_k(t)\right)U_j(t_i)U_k(t_i) \quad \text{(B.3)}$$

$$U_{jk}(t) = E\left[U_j(t_i)U_k(t_i)\right] = \frac{1}{N}\sum_{i=1}^{N} U_j(t_i)U_k(t_i) \quad \text{(B.4)}$$

Simple transformations of (B.3) give:

$$\sigma_{jk}(t) = \frac{1}{U_{jk}(t)}\left[\frac{1}{N}\sum_{i=1}^{N} p_j(t_i)p_k(t_i)\,U_j(t_i)U_k(t_i) - p_k(t)\frac{1}{N}\sum_{i=1}^{N} p_j(t_i)U_j(t_i)U_k(t_i) - p_j(t)\frac{1}{N}\sum_{i=1}^{N} p_k(t_i)U_j(t_i)U_k(t_i)\right] + p_j(t)p_k(t)$$

$$\frac{1}{N}\sum_{i=1}^{N} p_j(t_i)p_k(t_i)\,U_j(t_i)U_k(t_i) = \frac{1}{N}\sum_{i=1}^{N} C_j(t_i)C_k(t_i) = E[C_j(t_i)C_k(t_i)]$$

$$\frac{1}{N}\sum_{i=1}^{N} p_j(t_i)U_j(t_i)U_k(t_i) = \frac{1}{N}\sum_{i=1}^{N} C_j(t_i)U_k(t_i) = E[C_j(t_i)C_k(t_i)]$$

From the above, obtain the expression for the covariance $\sigma_{jk}(t)$:

$$\sigma_{jk}(t) = \frac{E[C_j(t_i)C_k(t_i)] - p_k(t)E[C_j(t_i)U_k(t_i)] - p_j(t)E[U_j(t_i)C_k(t_i)]}{E[U_j(t_i)U_k(t_i)]} + p_j(t)p_k(t) \quad \text{(B.5)}$$

One can present the joint mathematical expectations of values and volumes as:

$$E[C_j(t_i)C_k(t_i)] = C_j(t;1)C_k(t;1) + cov\{C_j(t), C_k(t)\}$$

$$E[C_j(t_i)U_k(t_i)] = C_j(t;1)U_k(t;1) + cov\{C_j(t), U_k(t)\}$$

$$E[U_j(t_i)U_k(t_i)] = U_j(t;1)U_k(t;1) + cov\{U_j(t), U_k(t)\}$$

$$cov\{C_j(t), U_k(t)\} = \frac{1}{N}\sum_{i=1}^{N}[C_j(t_i) - C_j(t;1)][U_k(t_i) - U_k(t;1)] \quad \text{(B.6)}$$

Simple calculations give that the sum of terms with mean values and volumes equal zero:

$$C_j(t;1)C_k(t;1) - p_k(t)C_j(t;1)U_k(t;1) = C_j(t;1)[C_k(t;1) - p_k(t)U_k(t;1)] = 0$$

$$p_j(t)U_j(t;1)C_k(t;1) - p_j(t)p_k(t)U_j(t;1)U_k(t;1) = p_j(t)U_j(t;1)[C_k(t;1) - p_k(t)U_k(t;1)] = 0$$

Finally, obtain the covariance $\sigma_{jk}(t)$ (B.7) of prices $p_j(t_i)$ and $p_k(t_i)$ of the securities j and k:

$$\sigma_{jk}(t) = \frac{cov\{C_j(t),C_k(t)\} - p_k(t)cov\{C_j(t),U_k(t)\} - p_j(t)cov\{U_j(t),C_k(t)\} + p_j(t)p_k(t)cov\{U_j(t),U_k(t)\}}{U_{jk}(t)} \quad \text{(B.7)}$$

We underline that the market-based covariance $\sigma_{jk}(t)$ (B.7) of prices of securities j and k is determined by the covariances (B.6) of trade volumes and values of these securities.

The symmetry of terms $p_k(t)cov\{C_j(t),U_k(t)\}$ and $p_j(t)cov\{U_j(t),C_k(t)\}$ allows express them:

$$-p_k(t)cov\{C_j(t), U_k(t)\} - p_j(t)cov\{U_j(t), C_k(t)\} = -2p_k(t)cov\{C_j(t), U_k(t)\}$$

We define $\psi_{jk}(t)$, $\chi_{jk}(t)$, and $\varphi_{jk}(t)$ (B.8; B.9) alike to the coefficients of variations (4.4):

$$\psi_{jk}(t) = \frac{cov\{c_j(t),c_k(t)\}}{c_j(t;1)c_k(t;1)} = \frac{cov\{C_j(t),C_k(t)\}}{C_j(t;1)C_k(t;1)} \quad ; \quad \varphi_{jk}(t) = \frac{cov\{c_j(t),u_k(t)\}}{c_j(t;1)u_k(t;1)} = \frac{cov\{C_j(t),U_k(t)\}}{C_j(t;1)U_k(t;1)} \quad \text{(B.8)}$$

$$\chi_{jk}(t) = \frac{cov\{u_j(t),u_k(t)\}}{u_j(t;1)u_k(t;1)} = \frac{cov\{U_j(t),U_k(t)\}}{U_j(t;1)U_k(t;1)} \quad ; \quad U_{jk}(t) = U_j(t;1)U_k(t;1)[1 + \chi_{jk}(t)] \quad \text{(B.9)}$$

One can present (B.7) as:

$$\sigma_{jk}(t) = \frac{\frac{cov\{C_j(t),C_k(t)\}}{C_j(t;1)C_k(t;1)}C_j(t;1)C_k(t;1) - 2p_k(t)\frac{cov\{C_j(t),U_k(t)\}}{C_j(t;1)U_k(t;1)}C_j(t;1)U_k(t;1) + p_j(t)p_k(t)\frac{cov\{U_j(t),U_k(t)\}}{U_j(t;1)U_k(t;1)}U_j(t;1)U_k(t;1)}{U_j(t;1)U_k(t;1)[1+\chi_{jk}(t)]}$$

Functions $\psi_{jk}(t)$, $\chi_{jk}(t)$, and $\varphi_{jk}(t)$ (B.8; B.9) describe the covariances of trade values and volumes of securities j and k that are normalized to unit means. The expression for $U_{jk}(t)$

follows from (B.4). The use of (B.8; B.9) and relations between mean trade values $C_j(t;1)$, volumes $U_j(t;1)$, and prices $p_j(t)$ (A.3) gives the covariance $\sigma_{jk}(t)$ of prices:

$$\sigma_{jk}(t) = \frac{\psi_{jk}(t)-2\varphi_{jk}(t)+\chi_{jk}(t)}{1+\chi_{jk}(t)}\, p_j(t)p_k(t) \tag{B.10}$$

The expression (B.10) presents the covariance $\sigma_{jk}(t)$ of prices of securities j and k as covariances of normalized to unit means trade values and volumes of securities j and k.

To derive the covariance $\theta_{jk}(t,t_0)$ of returns of the securities j and k with respect to their prices $p_j(t_0)$ and $p_k(t_0)$ in the past at time t_0 when the investor composed his portfolio, we introduce the equation (B.11) that has a form similar to (A.24) and (B.2) and obtain:

$$C_j(t_i)C_k(t_i) = R_j(t_i,t_0)R_k(t_i,t_0)C_{0j}(t_i,t_0)C_{0k}(t_i,t_0) \tag{B.11}$$

From (B.11), obtain the covariance $\theta_{jk}(t,t_0)$ of returns of securities j and k:

$$\theta_{jk}(t,t_0) = var\{R_j(t,t_0), R_k(t,t_0)\} = E_m\left[\left(R_j(t_i,t_0) - R_j(t,t_0)\right)\left(R_k(t_i,t_0) - R_k(t,t_0)\right)\right] =$$

$$E_m\left[\left(\frac{p_j(t_i)-p_j(t)}{p_j(t_0)}\right)\left(\frac{p_k(t_i)-p_k(t)}{p_k(t_0)}\right)\right] = \frac{\sigma_{jk}(t)}{p_j(t_0)p_k(t_0)} \tag{B.12}$$

From (B.7; B.12), obtain the covariance $\theta_{jk}(t,t_0)$ of returns:

$$\theta_{jk}(t,t_0) = \frac{cov\{C_j(t),C_k(t)\} - R_k(t,t_0)cov\{C_j(t),C_{0k}(t,t_0)\}}{C_{0jk}(t,t_0)} -$$

$$-\frac{R_j(t,t_0)cov\{C_{0j}(t,t_0),C_k(t)\} - R_j(t,t_0)R_k(t,t_0)cov\{C_{0j}(t,t_0),C_{0k}(t,t_0)\}}{C_{0jk}(t,t_0)} \tag{B.13}$$

The functions $C_{0j}(t_i,t_0)$ in (B.12) defines the past values of the current trade volume $U_j(t_i)$ at price $p_j(t_0)$ at time t_0. The function $C_{0jk}(t,t_0)$ in (B.13) describes the joint mathematical expectation (B.14) of the product of past values of securities j and k at time t_0

$$C_{0jk}(t,t_0) = E[C_{0j}(t_i,t_0)C_{0k}(t_i,t_0)] = \frac{1}{N}\sum_{i=1}^{N} C_{0j}(t_i,t_0)\, C_{0k}(t_i,t_0) \tag{B.14}$$

One can present the covariance $\theta_{jk}(t,t_0)$ (B.13) in the form similar to (B.10) and (A.29):

$$\theta_{jk}(t,t_0) = \frac{\sigma_{jk}(t)}{p_j(t_0)p_k(t_0)} = \frac{\psi_{jk}(t)-2\varphi_{jk}(t)+\chi_{jk}(t)}{1+\chi_{jk}(t)}\, R_j(t,t_0)R_k(t,t_0) \tag{B.15}$$

The market-based covariance $\theta_{jk}(t,t_0)$ (B.15) of returns of the securities j and k is determined by the coefficients of covariances $\psi_{jk}(t)$, $\varphi_{jk}(t)$ (B.8), and $\chi_{jk}(t)$ (B.9).

If one considers the approximation for which all trade volumes $U_j(t_i)$ with all securities that compose the portfolio are assumed constant during Δ, then the covariance $\sigma_{jk}(t)$ (B.10) and the covariance $\theta_{jk}(t,t_0)$ (B.15) take the frequency-based forms. If $U_j(t_i)=U_j$ constant, then:

$$cov\{C_j(t), U_k(t)\} = cov\{U_j(t), C_k(t)\} = cov\{U_j(t), U_k(t)\} = 0$$

$$cov\{C_j(t), C_k(t)\} = \frac{1}{N}\sum_{i=1}^{N}(C_j(t_i) - C_j(t;1))(C_k(t_i) - C_k(t;1)) =$$

$$= \frac{U_j U_k}{N}\sum_{i=1}^{N}(p_j(t_i) - p_j(t))(p_k(t_i) - p_k(t))$$

$$U_{jk}(t) = \frac{1}{N}\sum_{i=1}^{N} U_j U_k = U_j U_k$$

For that case, the covariance $\sigma_{jk}(t)$ (B.10) takes the frequency-based approximation (B.16):

$$\sigma_{jk}(t) = \frac{1}{N}\sum_{i=1}^{N}(p_j(t_i) - p_j(t))(p_k(t_i) - p_k(t)) \quad \text{(B.16)}$$

The covariance $\theta_{jk}(t,t_0)$ (B.13; B.15) takes the frequency-based approximation (B.17):

$$\theta_{jk}(t,t_0) = \frac{1}{N}\sum_{i=1}^{N}\left(R_j(t_i,t_0) - R_j(t,t_0)\right)\left(R_k(t_i,t_0) - R_k(t,t_0)\right) \quad \text{(B.17)}$$

Appendix C. The Decompositions of Means and Variances

The decompositions of the portfolio's mean price $s(t)$ (2.17; 3.19) and the variance $\Phi(t)$ (3.21) of prices and variance $\Theta(t,t_0)$ (3.22) of returns are determined by the time series of trade values $Q(t_i)$ and volumes $W(t_i)$ (2.13-2.17) that depend on the sums of the normalized values $c_j(t_i)$ and volumes $u_j(t_i)$ (2.11) of market trades of the securities $j=1,2,..J$, which compose the portfolio. The change of the order of sums defines the expressions of the decompositions.

C.1 The decomposition of the mean price s(t) of the portfolio.

We use (2.13; 2.14; 3.19), and obtain:

$$s(t) = \frac{1}{W_\Sigma(t_0)}\sum_{i=1}^{N} s(t_i)W(t_i) = \frac{1}{W_\Sigma(t_0)}\sum_{i=1}^{N} Q(t_i) = \frac{1}{W_\Sigma(t_0)}\sum_{i=1}^{N}\sum_{j=1}^{J} c_j(t_i) \quad \text{(C.1)}$$

We express $c_j(t_i)$ due to (2.10; 2.11), and change the order of sums:

$$s(t) = \frac{1}{W_\Sigma(t_0)}\sum_{i=1}^{N}\sum_{j=1}^{J} p_j(t_i)\, u_j(t_i) = \sum_{j=1}^{J}\frac{U_j(t_0)}{W_\Sigma(t_0)}\frac{1}{U_j(t_0)}\sum_{i=1}^{N} p_j(t_i)\, u_j(t_i)$$

From (A.3) and (2.10), obtain:

$$s(t) = \sum_{j=1}^{J} p_j(t)\; x_j(t_0) \quad ; \quad x_j(t_0) = \frac{U_j(t_0)}{W_\Sigma(t_0)} \quad \text{(C.2)}$$

We remind that $U_j(t_0)$ is a number of shares of the security j in the portfolio at time t_0. The investor holds his portfolio and the number of shares of each security remain unchanged. Relations (C.2) give the decomposition of the mean price $s(t)$ (C.1) of the portfolio during the averaging interval Δ (2.4) by the mean prices $p_j(t)$ (A.3) of the securities that compose the portfolio. Coefficients $x_j(t_0)$ in (C.2) describe the relative numbers of shares of the security j in the portfolio.

C.2 The decomposition of the variance Φ(t) of prices of the portfolio

To derive the decomposition of the variance $\Phi(t)$ (3.21) of prices of the portfolio by the securities $j=1,2,..J$ of the portfolio during Δ (2.4) we use (4.5; A.10; A.11) and substitut variables (3.18) to obtain the variance $\Phi(t)$ (C.3) of prices of the portfolio:

$$\Phi(t) = \frac{1}{W(t;2)}\frac{1}{N}\sum_{i=1}^{N}\left(s(t_i) - s(t)\right)^2 W^2(t_i) \quad \text{(C.3)}$$

We replace the notions (A.16-A.18) of securities by the similar notions of the portfolio:

$$\Psi_C(t) \to \Psi_Q(t) \quad ; \quad \Psi_U(t) \to \Psi_W(t) \quad ; \quad cov\{C(t), U(t)\} \to cov\{Q(t), W(t)\} \tag{C.4}$$

$$p(t) \to s(t) \quad ; \quad U(t;2) \to W(t;2) \tag{C.5}$$

Similar to (A.16), obtain the expression of the variance $\Phi(t)$ of prices of the portfolio as a function of the variances of the portfolio's values $\Psi_Q(t)$, volumes $\Psi_W(t)$ and their covariance $cov\{Q(t), W(t)\}$ and as a function of the coefficients of variation of the portfolio trade values $\psi(t)$, volumes $\chi(t)$, and their normalized covariance $\varphi(t)$ (4.4):

$$\Phi(t) = \frac{\Psi_Q(t) + s^2(t)\Psi_W(t) - 2s(t)\, cov\{Q(t), W(t)\}}{W(t;2)} = \frac{\psi^2(t) - 2\,\varphi(t) + \chi^2(t)}{1 + \chi^2(t)}\, s^2(t) \tag{C.6}$$

The definition of the values $Q(t_i)$ and volumes $W(t_i)$ (2.13) by sums of normalized values $c_j(t_i)$ and volumes $u_j(t_i)$ (2.11) help change the orders of sums and transform the variances of the portfolio's values $\Psi_Q(t)$, volumes $\Psi_W(t)$ and their covariance $cov\{Q(t), W(t)\}$:

$$\Psi_Q(t) = \frac{1}{N}\sum_{i=1}^{N}\big(Q(t_i) - Q(t;1)\big)^2 = Q(t;2) - Q^2(t;1) \ ; \ \Psi_W(t) = W(t;2) - W^2(t;1)$$

$$cov\{Q(t), W(t)\} = \frac{1}{N}\sum_{i=1}^{N}\big(Q(t_i) - Q(t;1)\big)\big(W(t_i) - W(t;1)\big) = E[Q(t_i)W(t_i)] - Q(t;1)W(t;1)$$

$$Q(t;2) = \frac{1}{N}\sum_{i=1}^{N} Q^2(t_i) = \frac{1}{N}\sum_{i=1}^{N}\sum_{j,k=1}^{J} c_j(t_i)c_k(t_i) = \sum_{j,k=1}^{J} E[c_j(t_i)c_k(t_i)]$$

$$E[Q(t_i)W(t_i)] = \frac{1}{N}\sum_{i=1}^{N} Q(t_i)W(t_i) = \sum_{j,k=1}^{J} E[c_j(t_i)u_k(t_i)]$$

$$W(t;2) = \frac{1}{N}\sum_{i=1}^{N} W^2(t_i) = \sum_{j,k=1}^{J} E[u_j(t_i)u_k(t_i)]$$

The use of (C.4; C.5) and (B.5-B.7) give the decomposition of the variance $\Phi(t)$ (C.6) of prices of the portfolio:

$$\Phi(t) = \frac{1}{W(t;2)}\sum_{j,k=1}^{J}\left[cov\{c_j(t), c_k(t)\} - 2s(t)cov\{c_j(t), u_k(t)\} + s^2(t)cov\{u_j(t), u_k(t)\}\right] \tag{C.7}$$

The use of functions $\psi_{jk}(t)$, $\chi_{jk}(t)$, and $\varphi_{jk}(t)$ (B.8; B.9) and (4.4; 4.5) transforms the decomposition of the variance $\Phi(t)$ (C.7) as:

$$\Phi(t) = \sum_{j,k=1}^{J}\frac{p_j(t)p_k(t)\psi_{jk}(t) - 2s(t)p_j(t)\varphi_{jk}(t) + s^2(t)\chi_{jk}(t)}{1 + \chi^2(t)}\, x_j(t_0)x_k(t_0) \tag{C.8}$$

The coefficients $x_j(t_0)$ in (C.8) define the relative numbers (2.3) of the shares $U_j(t_0)$ of securities j in the total number of shares $W_\Sigma(t_0)$ of the portfolio. However, the decomposition (C.8) hides the dependence of the decomposition of the mean price $s(t)$ (C.2) of the portfolio. Let us substitute (C.2) into (C.8) and obtain the final decomposition of the variance $\Phi(t)$ (C.9) of prices of the portfolio:

$$\Phi(t) = \frac{1}{1 + \chi^2(t)}\Big[\sum_{j,k=1}^{J}\psi_{jk}(t)\, p_j(t)p_k(t)\, x_j(t_0)x_k(t_0) -$$

$$-2\sum_{j,k,l}^{J}\varphi_{jk}(t)\, p_j(t)p_l(t)\, x_j(t_0)x_k(t_0)x_l(t_0)$$

$$+\sum_{jklf}^{J}\chi_{jk}(t)\, p_l(t)p_f(t)\, x_j(t_0)x_k(t_0)x_l(t_0)x_f(t_0)\Big] \tag{C.9}$$

The decomposition of the variance $\Phi(t)$ (C.9) of prices of the portfolio is a polynomial of the 4th degree by the relative numbers $x_j(t_0)$ (2.3) of the shares $U_j(t_0)$ of security j. The variance $\Phi(t)$ of prices of the portfolio (C.6; C.8; C.9) accounts for the impact of random trade volumes. For the approximation when all volumes $u_j(t_i)$ of all market trades with securities $j=1,2,\ldots J$, that compose the portfolio are assumed constant during Δ (2.4), then the variance $\Phi(t)$ (C.8; C.9) of prices takes the quadratic form (C.10) for $\sigma_{jk}(t)$ (B.16):

$$\Phi(t) = \sum_{j,k=1}^{J} \sigma_{jk}(t)\, x_j(t_0) x_k(t_0) \tag{C.10}$$

C.3 The decomposition of the mean return R(t,t₀) of the portfolio

The return $R(t_i,t_0)$ of the portfolio with price $s(t_i)$ (2.14) at time t_i during Δ (2.4) with respect to price $s(t_0)$ (2.3) of the portfolio at time t_0 follows (A.21):

$$R(t_i, t_0) = \frac{s(t_i)}{s(t_0)} = \frac{Q(t_i)}{s(t_0)W(t_i)} \tag{C.11}$$

The mean return $R(t,t_0)$ and its decomposition (C.12) follow the mean price $s(t)$ (C.1) of the portfolio and its decomposition (C.2):

$$R(t, t_0) = \frac{s(t)}{s(t_0)} = \sum_{j=1}^{J} \frac{p_j(t)}{p_j(t_0)} \frac{p_j(t_0)U_j(t_0)}{s(t_0)W_\Sigma(t_0)} = \sum_{j=1}^{J} R_j(t, t_0)\; X_j(t_0) \tag{C.12}$$

$$X_j(t_0) = \frac{p_j(t_0)U_j(t_0)}{s(t_0)W_\Sigma(t_0)} = \frac{C_j(t_0)}{Q_\Sigma(t_0)} \tag{C.13}$$

We remind that $p_j(t_0)$ (2.1) is the price of the security j of the portfolio at time t_0. The decomposition (C.12) coincides with (1.1) and the coefficients $X_j(t_0)$ (C.13) describe the relative amounts invested into security $j=1,2,..J$ at time t_0.

C.4 The decomposition of the variance Θ(t,t₀) of returns of the portfolio

The substitutions (C.4; C.5) define the variance $\Theta(t,t_0)$ (C.14) of returns of the portfolio, similar to the variance $\theta(t,t_0)$ (A.29) of returns of a security:

$$\Theta(t, t_0) = \frac{\Phi(t)}{s^2(t_0)} = \frac{\psi^2(t) - 2\,\varphi(t) + \chi^2(t)}{1 + \chi^2(t)}\, R^2(t, t_0) = \frac{\Psi_Q(t) + R^2(t,t_0)\Psi_{Q_0}(t,t_0) - 2R(t,t_0)\, cov\{Q(t), Q_0(t,t_0)\}}{Q_0(t,t_0;2)} \tag{C.14}$$

$$Q_0(t_i, t_0) = s(t_0)W(t_i) \tag{C.15}$$

$Q_0(t_i,t_0)$ (C.15) denotes the value of the current trade volume $W(t_i)$ of the portfolio in the past at price $s(t_0)$ at time t_0. The decomposition of the variance $\Theta(t,t_0)$ (C.16) of returns of the portfolio by the securities that compose the portfolio is completely the same as the decomposition of the variance $\Phi(t)$ (C.7) of prices of the portfolio.

$$\Theta(t, t_0) = \frac{1}{Q_0(t,t_0;2)} \sum_{j,k=1}^{J} [cov\{c_j(t), c_k(t)\} - 2R(t, t_0)cov\{c_j(t), c_{0k}(t, t_0)\} +$$
$$+ R^2(t, t_0)cov\{c_{0j}(t, t_0), c_{0k}(t, t_0)\}] \tag{C.16}$$

The function $Q_0(t,t_0;2)$ in (C.16) is determined similar to (A.32):

$$Q_0(t, t_0; 2) = \frac{1}{N} \sum_{i=1}^{N} Q_0^2(t_i, t_0)$$

The use of (4.4; 4.5) gives the decomposition of the variance $\Theta(t,t_0)$ (C.17) similar to (C.9):

$$\Theta(t,t_0) = \frac{1}{1+\chi^2(t)} \left[\sum_{j,k=1}^{J} \psi_{jk}(t)\, R_j(t,t_0) R_k(t,t_0)\, X_j(t_0) X_k(t_0) - \right.$$

$$-2 \sum_{j,k,l}^{J} \varphi_{jk}(t)\, R_j(t,t_0) R_l(t,t_0)\, X_j(t_0) X_k(t_0) X_l(t_0)$$

$$\left. + \sum_{jklf}^{J} \chi_{jk}(t)\, R_l(t,t_0) R_f(t,t_0)\, X_j(t_0) X_k(t_0) X_l(t_0) X_f(t_0) \right] \qquad \text{(C.17)}$$

The decomposition of the variance $\Theta(t,t_0)$ (C.17) of returns of the portfolio is a polynomial of the 4th degree by the relative amounts $X_j(t_0)$ (C.13) invested into the security j at time t_0. That is rather different from the quadratic form (1.2) derived by Markowitz (1952). Such distinctions highlight the influence of the random volumes $U_j(t_i)$ of trades with securities $j=1,2,..J$ that compose the portfolio. The market-based decomposition of the variance $\Theta(t,t_0)$ (C.17) makes the search for higher returns under lower variance a much more complex problem.

For the approximation when all trade volumes $u_j(t_i)$ with securities of the portfolio are assumed constant during Δ:

$$If\ u_k(t_i) - const, then: \ \ cov\{c_j(t), c_{0k}(t,t_0)\} = cov\{c_{0j}(t,t_0), c_{0k}(t,t_0)\} = 0$$

the portfolio variance $\Theta(t,t_0)$ (C.17) takes the form (1.2; C.18) that was derived by Markowitz.

$$\Theta(t,t_0) = \sum_{j,k}^{J} \theta_{jk}(t,t_0)\, X_j(t_0) X_k(t_0) \qquad \text{(C.18)}$$

We repeat that the variance $\Phi(t)$ (C.10) of prices of the portfolio and the variance $\Theta(t,t_0)$ (1.2; C.18) of the returns of the portfolio describe the approximation for which all volumes $U_j(t_i)$ of trades with all securities $j=1,2,..J$ of the portfolio are assumed constant during Δ (2.4). The expressions (C.10; 1.2) neglect the impact of random volumes of trades with securities.

Appendix D. Market-Based and Frequency-Based Statistical Moments

In this Appendix, we briefly discuss the economic meaning of the distinctions between the market-based and the frequency-based valuations of the statistical moments of prices and returns of market securities and of the portfolio. One can find more details in Olkhov (2022-2025). We use $E_m[..]$ to distinguish the market-based mathematical expectation from the frequency-based $E[..]$ that is generally accepted (Shiryaev, 1999; Shreve, 2004) and denote the market-based $p(t;n)$ and the frequency-based $\pi(t;n)$ (D.1) statistical moments of prices:

$$p(t;n) = E_m[p^n(t_i)] \quad ; \quad \pi(t;n) = E[p^n(t_i)] = \frac{1}{N} \sum_{i=1}^{N} p^n(t_i) \qquad \text{(D.1)}$$

In the main text and in the App.A-C we denoted the average price $p(t)$, but in this App.D we denote average price as $p(t;1)$ (D.1). We use a frequency-based definition to assess the *n-th* statistical moments of the values $C(t;n)$ and volumes $U(t;n)$ (A.2; D.2) of market trades:

$$C(t;n) = E[C^n(t_i)] = \frac{1}{N} \sum_{i=1}^{N} C^n(t_i) \quad ; \quad U(t;n) = E[U^n(t_i)] = \frac{1}{N} \sum_{i=1}^{N} U^n(t_i) \qquad \text{(D.2)}$$

Any averaging interval Δ (2.4) contains only a finite number N of terms of time series of trades, and (D.1; D.2) assess the frequency-based statistical moments by N terms. The equation (A.1; D.3) define relations between the values $C(t_i)$, volumes $U(t_i)$, and prices $p(t_i)$ of trade at time t_i:

$$C(t_i) = p(t_i)\, U(t_i) \tag{D.3}$$

The equation (D.3) prohibits the independent definitions of the average values, volumes, and prices. In App. A, we derived how the mean values $C(t;1)$ and volumes $U(t;1)$ define the VWAP $p(t;1)=p(t)$ (A.3), which differs from the definition of the frequency-based average price $\pi(t;1)$. However, in the approximation that all trade volumes $U(t_i)=U$ are assumed constant during Δ (2.4), from (D.2; D.3), obtain:

$$C(t;1) = E[C(t_i)] = \frac{1}{N}\sum_{i=1}^{N} C(t_i) = \frac{1}{N}\sum_{i=1}^{N} p(t_i)\, U(t_i) = U\frac{1}{N}\sum_{i=1}^{N} p(t_i) = U\pi(t;1) \tag{D.4}$$

Another representation ties up the frequency-based mean price $\pi(t;1)$ and the equation (D.3):

$$\pi(t;1) = \frac{1}{NU}\sum_{i=1}^{N} C(t_i) = \frac{1}{N}\sum_{i=1}^{N} \frac{C(t_i)}{U} = \frac{1}{N}\sum_{i=1}^{N} p(t_i) \tag{D.5}$$

The assumption that all volumes of trades are constant results in the frequency-based definition of the average price $\pi(t;1)$ (D.1; D.4) through $C(t;1)$ (D.2). To derive the frequency-based *n-th* statistical moment of price $\pi(t;n)$, one should take the *n-th* degree (D.6) of (D.3) and again assume $U(t_i)=U$ – const.

$$C^n(t_i) = p^n(t_i)U^n(t_i) \quad ; \quad n = 1,2,\ldots \tag{D.6}$$

From (D.2; D.6), follows:

$$C(t;n) = E[C^n(t_i)] = \frac{1}{N}\sum_{i=1}^{N} C^n(t_i) = \frac{1}{N}\sum_{i=1}^{N} p^n(t_i)U^n(t_i) = U^n\frac{1}{N}\sum_{i=1}^{N} p^n(t_i) = U^n\pi(t;n) \tag{D.7}$$

The representation (D.8) highlights the dependence of $p^n(t_i)$ on (D.6) and the ratio of the *n-th* degree of trade value $C^n(t_i)$ to the *n-th* degree of trade volume U^n that is determined by (D.6):

$$\pi(t;n) = \frac{1}{NU^n}\sum_{i=1}^{N} C^n(t_i) = \frac{1}{N}\sum_{i=1}^{N} \frac{C^n(t_i)}{U^n} = \frac{1}{N}\sum_{i=1}^{N} p^n(t_i) = \frac{C(t;n)}{U^n} \tag{D.8}$$

To define how the *n-th* statistical moments of trade values $C(t;n)$ (D.2) determine the *n-th* statistical moments of price $\pi(t;n)$, one should use the set of equations (D.6) for $n=1,2,..$. The more statistical moment of price $\pi(t;n)$ would be assessed, the higher the accuracy of the approximation of price probability could be obtained (Shiryaev, 1999; Shreve, 2004). The number m of equations (D.6) for $n=1,2,..m$ determines the approximation of price probability by the first m statistical moments of market trade values $C(t;n)$ (D.2).

The frequency-based statistical moments of price $\pi(t;n)$ (D1; D.6) are generally accepted (Shiryaev, 1999; Elton et al., 2014), but the limitations of such approximations are omitted. We show that the *n-th* statistical moments of trade values $C(t;n)$ (D.2) and equations (D.6) for $n=1,2,\ldots$ determine the frequency-based *n-th* statistical moments of price $\pi(t;n)$ (D.1; D.6) only for the approximation in which all trade volumes $U(t_i)=U$ are assumed constant during Δ (2.4). Otherwise, one should account for the impact of random trade volumes, consider the set of equations (D.6) for $n=1,2,..$, and derive the market-based statistical moments of price $p(t;1)$ (D.1; A.3), $\phi(t)$, $p(t;2)$ (D.1; A.10) (Olkhov, 2022).

The frequency-based assessments of the statistical moments of prices and returns neglect the randomness of market trade volumes. Market-based mean (A.3) and variance (A.16) of prices and mean (A.25) and variance (A.29) of returns of market securities account for the impact of random volumes of market trades.

That determines the economic essence of the distinctions between the market-based and the frequency-based descriptions of statistical moments of prices and returns. Our market-based description takes into account the impact of random time series of volumes of trade with the securities during the averaging interval Δ. The conventional frequency-based description neglects this effect and assumes that all trade volumes are constant during Δ (2.4). The investors, who manage large stakes of securities and multi-billion portfolios, and the developers of large market and macroeconomic models like BlackRock's Aladdin and Asimov, JP Morgan, and the U.S. Fed should keep that in mind.