The Ubiquitous Giffen

Miller, Anne

28 June 2025

This paper is a development of MPRA paper No. 121455

The Ubiquitous Giffen
by
Anne G Miller

Abstract

This paper shows that a demand equation derived by adding two bounded leaning-S-shaped utilities includes the inferior-Giffen response.

A leaning-S-shaped, bounded cardinal utility, ($0 \leq u \leq 1$), for a single commodity is identified as a representation of the individual's experience of fulfilment of a need – deprivation (increasing marginal utility (MU)), subsistence (a point of inflection), sufficiency (diminishing MU), and either satiation at finite consumption with the possibility of surfeit, or satiation at infinite consumption.

The separability rule states that utilities of commodities fulfilling the same need are weakly separable (multiplicative) and those of commodities fulfilling two different needs are strongly separable (additive).

Functional forms are derived from a utility function created by *adding* two *normal* distribution functions with satiation at infinity, the parameters of which have meaningful psychological interpretations. The indifference map, demand and Engels curve diagrams are explored.

Concave- and convex-to-the-origin indifference curves, (the former defining 'dysfunctional poverty', leading to disequilibrium in the derived functional forms), are separated by a straight-line indifference curve with slope defined by the relative-intensities-of-need.

Convex-to-the-origin indifference curves enable optimisation even for deprivation in one need. The boundaries between superior and inferior responses, and between inferior normal and inferior Giffen, are reflected in envelope curves in the derived functional form diagrams.

The inferior-Giffen experience occurs when an individual responds to a price increase for an abundant, cheaper good by consuming more of it, enabled by relinquishing some consumption of a more expensive commodity fulfilling a different need, of which s/he is already extremely deprived.

Keywords:

Bounded cardinal utility includes increasing marginal utility expressing deprivation; additive separability for different needs; dysfunctional poverty leads to involuntary unemployment and disequilibrium; envelope curves reflect inferior responses; the straight-line indifference curve determines the equilibrium price and survival endowments.

1 Introduction

This paper aims to introduce some psychology into mainstream cardinal utility theory in the form of separate fundamental human needs. There is an extensive literature on the ontology and epistemology of needs in philosophy (Lawson [1], Yamamori [2]), and there are many papers about systems of needs in psychology (Maslow [3], Doyal and Gough [4]). Ward and Lasen [5] provide ‘An Overview of Needs Theories behind Consumerism’, examining ‘the development of hierarchical needs theory from Maslow to Gough’. But, apart from Miller [6], there does not seem to have been an attempt to introduce the concept of separate needs into utility theory.

The method creates a cardinal utility function for two commodities fulfilling separate needs, based on two propositions. The first extends the seminal work of Van Praag [7] on bounded S-shaped cardinal utilities for single commodities, which can be seen to express the experiences of the individual at different stages of the fulfilment of a need – deprivation, subsistence, sufficiency, satiation and possible surfeit. Secondly, the separability rule is applied, such that weak separability (multiplicativity) is used for choices concerning commodities fulfilling the same need, and strong separability (additivity) for choices concerning commodities fulfilling two different needs.

A functional form for the demand equations is derived from the utility function. The equations are used to create the indifference curve map and diagrams for the demand and Engels curves for the two dependent variables. The diagrams are then examined, interpreted, and outcomes inferred.

In section 2, Van Praag’s seminal, bounded S-shaped utility for a single commodity, is shown to illustrate the different stages of fulfilment of a need.

A utility function for commodities fulfilling separate needs is created from the addition of two normal distribution functions in section 3. The indifference curve map displays both concave- and convex-to-the-origin indifference curves, separated by a straight-line indifference curve.

In section 4, the utility function is maximised subject to a budget constraint using the Lagrange multiplier method. The optimality condition (equation 4) is used to explore the boundaries between superior and inferior, and between inferior normal and inferior-Giffen, experiences on the indifference curve map.

Section 5 derives the functional form for the demand equation.

The focuses of section 6 are the demand and Engels diagrams of the two commodities, highlighting the envelope curves associated with the boundaries between superior and inferior responses on the demand diagrams and those between inferior normal and inferior-Giffen on the Engels curve diagrams. It also presents both demand diagrams and both Engels diagrams together within Figure 4, sharing the four axes.

The conclusion in section 7 summarises the shape and separability assumptions about utility, indicating how both the inferior and the inferior-Giffen experiences materialise from them.

2 Van Praag's leaning-S-shaped utility function for a commodity

The Law of Demand, based on diminishing marginal utility, remained unchallenged until van Praag [7] created an S-shaped utility function that is bounded below and above, ($0 \leq u \leq 1$). One of the aims of his ground-breaking work, which he calls 'a neo-cardinal theory of consumer behaviour', was to enable interpersonal welfare comparisons to be made, thus partially solving the non-measurability problem of utility. It has been developed and applied successfully by The Leyden School (for example, Van Herwaarden and Kapteyn [8], Hagenaars [9] and van Praag and Kapteyn [10]).

The introduction of the concept of separate needs into cardinal utility theory requires two further stages in addition to van Praag's bounded utility.

- The recognition that the bounded S-shaped utility for a commodity represents the various experiences of an individual through the fulfilment of a need.

- The application of the separability rule between commodities fulfilling the same need or fulfilling separate needs. This is introduced and developed in section 3.

The different experiences through the fulfilment of need are illustrated in Figure 1[1].

The bounded S-shaped utility for a commodity (good, service or event), q_i, is recognised as representing both the intensity-of-need of the individual and his/her experience through the various stages of fulfilment of a need – deprivation (increasing marginal utility (MU)), subsistence (a point of inflection), sufficiency (diminishing MU), and either satiation at finite consumption with the possibility of surfeit, or satiation at infinite consumption.

A bounded S-shaped utility has the following properties:

> An individual's experience of consumption, q_i, of the i'th commodity, (good, service or event), $-\infty \leq q_i \leq \infty$, for i = 1, 2, …, m, can be represented by a continuous, smooth, single-valued, 'leaning-S-shaped', utility function, $u_i = u(q_i)$, $0 \leq u_i \leq 1$, bounded below and above, but marginal utility, u_i', is always less than infinity.

The assumptions that utility, $u_i = u(q_i)$, and reaches a minimum, $u_i = 0$, at $q_i = \text{min}_i$ and a maximum, $u_i = 1$, at $q_i = \text{sat}_i$ where sat_i could be either finite or infinite, are necessary conditions for utility to be bounded below and above. $q_i \geq 0$ represents consumption of some personal resources, but a minimum could occur at $q_i \leq 0$, because individuals can receive free satisfiers provided by natural circumstances – such as the warmth of the sun could heat a home, before fuel is consumed.

A point of inflection occurs at $q_i = \mu_i$, representing a 'subsistence' threshold, (which is not the same as the committed consumption parameter, or survival level, in the Stone-Geary utility function from which the Linear Expenditure System is derived).

---

[1] Figures 1-4 were created using Seppo Mustonen's program SURVO [11].

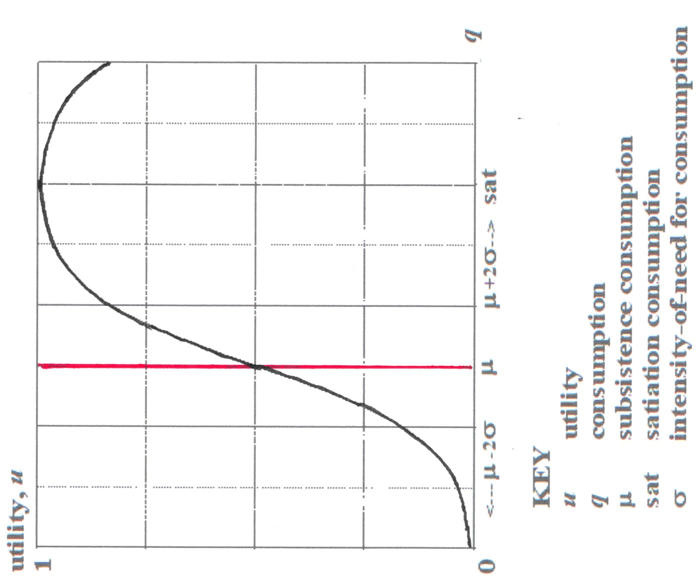


FIGURE 1 LEANING-S-SHAPED U-FN

Consuming less than μ_i, where MU is increasing, implies ‘deprivation’. Consumption greater than subsistence, where MU is positive but diminishing, may be labelled ‘sufficiency’. Maximum utility implies ‘satiation’ of that need, while satiation at finite consumption offers the possibility of a ‘surfeit’. Obviously, for satiation at infinite consumption there would be no experience of surfeit.

Parameter σ_i in Figure 1 is a measure of the intensity-of-need for the i’th commodity. The range of q_i, ($\mu_i \pm 1.96.\sigma_i$), indicates where MU is experienced most intensely. The smaller is σ_I, the steeper is the slope of the $u_i(q_i)$ function around the parameter μ_i, and the more intense is the need.

> Commodities with a large variance are commodities for which satisfaction comes rather slowly … Commodities with a small variance are commodities … of which one is quickly satisfied. For instance, life necessities have presumably a small variance ([1] van Praag, 1968, p.34).

The parameters can vary over time for an individual, and between different groups of people, according to demographic variables and other experiences.

Van Praag concentrated on the outcomes of an *n-variable*, *multiplicative*, *lognormal* distribution function, (n-Mult.LN-DF), with satiation at infinity for his utility function.

The utility function created here, (and on which Figures 2 – 4 are based), is the result of adding two bounded cardinal utilities based on the *normal* distribution function, (2-Add.N-DF), (but it does not have any probabilistic connotations in this context), again with satiation at infinity. The N-DF was chosen for pragmatic reasons, because it is quite tractable and is useful for illustrating many aspects of the theory, providing a reasonable approximation for that part of the leaning-S-shape around subsistence. Further, it has the added advantage that its two parameters, μ_i and σ_i, have important economic and psychological interpretations, and are potentially estimable. The parameter μ_i is the subsistence threshold between increasing and diminishing marginal utility, while σ_I is a measure of the intensity-of-need for the i’th commodity. Its symmetry implies that q_i = $\min_i$ could occur for $\min_i$ < 0, which could be explained by ‘free satisfiers’.

Thus, there are two ways in which the functional form derived here differs from that of van Praag.

- Van Praag makes a case for using the *log-normal* distribution function as his functional form ([1] 1968, pp.81, 86, 119), whereas for additive utilities, the *normal* distribution function is more tractable.

- Van Praag assumes that the relationship between the utilities from commodities is *multiplicative*, both without and with dependence (substitutes and complements), whereas the utility function created here is based on *added* utilities..

The analysis of the new functional form will concentrate on the responses around subsistence. Thus, the implications of higher consumption being close to satiation well before infinity will not be considered further here.

3 Separate needs and the indifference curve map

The separability rule states:

> a group of commodities that satisfy the same need are weakly separable, that is, based on multiplicative utilities (with or without dependence), and groups of satisfiers, each group satisfying a different need, are strongly separable, that is, based on additive utilities.

The discussion of ‘separability’ and ‘the grouping of commodities’ in the economics literature (Green [12] and Deaton and Muellbauer [13], for example) often comes across as though they are afterthoughts, and it tends to centre on whether the utilities gained from the consumption of different commodities are additive or multiplicative.

The separability rule gives rise to two very different indifference curve maps. The multiplicative one is reminiscent of the familiar convex-to-the-origin indifference maps found in textbooks, (some sample diagrams of which can be seen in van Praag [7] p.88), and will not be discussed further here.

Following Mallman and Nudlar [14], and to a lesser extent Maslow [4], it is assumed here that there are a few separable fundamental human needs and that these are universal and ahistoric. Needs are satisfied by an infinite diversity of culturally determined satisfiers. Needs cannot be observed directly, but only through the effects of their satisfiers, or lack thereof.

Max-Neef proposed a system,

> composed of nine fundamental human needs: permanence (or subsistence), protection, affection, understanding, participation, leisure, creation, identity (or meaning) and freedom … fundamental needs are finite, few and classifiable … fundamental needs are the same in all cultures and all historical periods. What changes, both over time and through cultures, is the form or the means by which these needs are satisfied ([15] pp. 49-50).

The 2-Add.N-DF utility function is the sum of two normal distribution functions representing consumption, q_i, $-\infty < q_i < +\infty$, i = 1, 2, where the i’th commodity fulfils the i’th need. The sum is scaled equally such that utility, u, lies between 0 and 1.

The ‘2-Add.N-DF’ **utility function** is:

$$u(q_1, q_2) = \tfrac{1}{2} F_1(q_1) + \tfrac{1}{2} F_2(q_2)$$

$$u(q_1, q_2) = \frac{1}{2}\int_{-\infty}^{q_1} \frac{\exp\left[-(R_1-\mu_1)^2/2\sigma_1^2\right]}{\sigma_1.\sqrt{2\pi}}\, dR_1 + \frac{1}{2}\int_{-\infty}^{q_2} \frac{\exp\left[-(R_2-\mu_2)^2/2\sigma_2^2\right]}{\sigma_2.\sqrt{2\pi}}\, dR_2 \qquad (1)$$

where u, $0 \le u \le 1$, is utility,
$\mu_1, \mu_2 \ge 0$ are subsistence parameters, and
$\sigma_1, \sigma_2 > 0$ are parameters representing intensity-of-need for commodities 1 and 2.

It would be almost impossible to use equation (1) to create an indifference curve map. Fortunately, as Johnson and Kotz ([16] p. 244) state ‘The shape of this [logistic] distribution is quite similar to that of the normal density function’.

$$P(t) = \frac{e^{t}}{[1 + e^{t}]^2} = \frac{e^{-t}}{[1 + e^{-t}]^2}.$$

This was used to create the indifference curve map in Figure 2, adjusted for location and scale.

$$q_2 = \mu_2 - \{\log [(0.5 * \text{bracket}) / (u * \text{bracket} - 0.5) - 1] \} / (1.82/\sigma_2), \qquad (2)$$

where u is utility, and bracket = $(1 + \exp (- (1.82/\sigma_1) * (q_1 - \mu_1)))$.

A straight-line indifference curve, BA, separates the concave-to-the-origin indifference curves in the triangular area B0A surrounding the origin, from the convex ones.

Individuals receive endowments of unearned consumption during their lifetimes, from their families, local communities, education, and via unearned income and state benefits. Let C_1 and C_2 be endowments of q_1 and q_2.

The concave-to-the-origin indifference curves represent dysfunctional poverty, because the individual is trapped on a corner solution on one or other axis, faced with choices that would not increase his/her utility, unless favourable relative prices enabled him/her to optimise in a convex-to-the-origin situation.

If area B0A represents dysfunctional poverty, and the convex-to-the-origin indifference curves present optimisation choices to the individual, then the straight-line indifference curve, BA, can be identified as a poverty line between those two experiences.

The indifference curve map is divided into **four quadrants** by the subsistence parameters, μ_1 and μ_2, with point (μ_1, μ_2) labelled as E, leaving a border of deprivation beside the two axes.

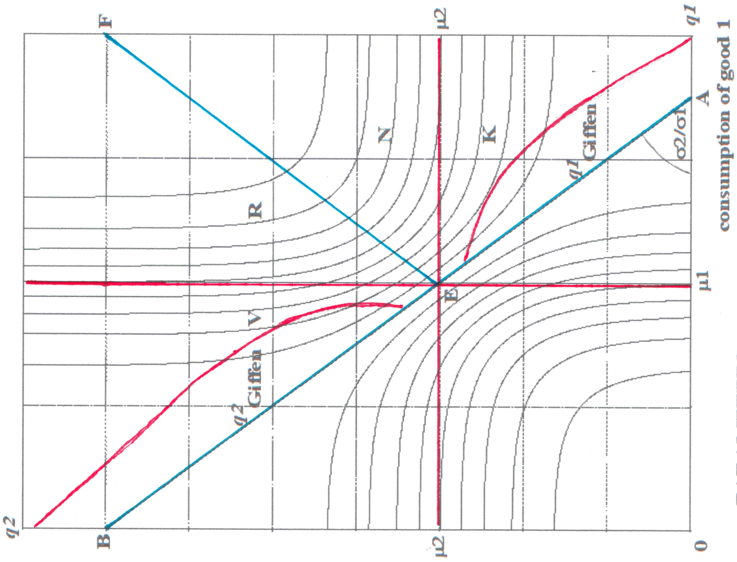
EF is the locus of points where the slopes of the convex-to-the-origin indifference curves are the same as that of BA. The equation for BA is $q_2 = \mu_2 - (\sigma_2/\sigma_1).(q_1 - \mu_1)$.

The **convex-to-the-origin indifference curves** can be divided into four areas:

- a rhomboid labelled V, bounded by the q_2-axis, BE and $q_1 = \mu_1$, in which the individual has sufficient of q_2, but is deprived of q_1
- a triangular area labelled R, bounded by $q_1 = \mu_1$ and the locus EF
- a triangular area labelled N, bounded by EF and $q_2 = \mu_2$, and
- a rhomboid labelled K, bounded by $q_2 = \mu_2$, EA and the q_1-axis, in which the individual has sufficient of q_1, but is deprived of q_2.

This leads to **three levels of fulfilment** for two different needs:

- The lowest level is dysfunctional poverty (concave-to-the-origin indifference curves) from which it is very difficult for the individual to extract him/herself without favourable relative prices or extra endowments.



PARAMETERS

$\mu 1, \mu 2$	*subsistence consumption for goods 1 and 2*
$\sigma 1, \sigma 2$	*intensity-of-need for goods 1 and 2*
BA	*a poverty line*
EF	*locus of tangents with same slope as BA*

FIGURE 2 INDIFFERENCE CURVES FOR SATIATION AT INFINITY

- Functional poverty is an intermediate level occurring where the individual is deprived in one need, but has sufficient in the other to enable him/her to optimise, improving his/her situation, as in areas K and V.

- The optimum level enables the individual to experience sufficiency in both needs, as in areas N and R.

4 The budget constraint and the properties of the convex-to-the-origin indifference curves

C_1 and C_2 are endowments of q_1 and q_2, valued at prices p_1 and p_2 respectively, where p_1 and $p_2 \geq 0$. $M \geq 0$, where M is full income.

Survival income $= \mu_1.p_1 + \mu_2.p_2$.
Full income: $M = C_1.p_1 + C_2.p_2$.
Supernumerary income: $Z = M$ – survival income
$= (C_1 - \mu_1).p_1 + (C_2 - \mu_2).p_2.$

A linear budget constraint is expressed as:

$$q_2 = (M - q_1.p_1)/\mathrm{p}_2. \tag{3}$$

Maximising $u(q_1,\ q_2)$ subject to the budget constraint M, and using the Lagrange Multiplier method leads to:

$$\frac{du(q1,q2)}{dq_1} = \frac{\frac{1}{2}.exp\left[-(q_1-\mu_1)^2/2\sigma_1^2\right]}{\sigma_1.\sqrt{2\pi}} - \lambda.p_1 = 0. \tag{1a}$$

$$\frac{du(q1,q2)}{dq_2} = \frac{\frac{1}{2}.exp\left[-(q_2-\mu_2)^2/2\sigma_2^2\right]}{\sigma_2.\sqrt{2\pi}} - \lambda.p_2 = 0. \tag{1b}$$

$$\frac{exp\left[-(q_1-\mu_1)^2/2\sigma_1^2\right]}{exp\left[-(q_2-\mu_2)^2/2\sigma_2^2\right]} = \frac{\sigma_1.p_1}{\sigma_2.p_2}. \tag{1c}$$

This yields the **optimality condition**:

$$\left(\frac{q_2-\mu_2}{\sigma_2}\right)^2 - \left(\frac{q_1-\mu_1}{\sigma_1}\right)^2 = ln\left(\frac{\sigma_1.p_1}{\sigma_2.p_2}\right)^2 \tag{4}$$

The optimality condition describes a family of hyperbolae with respect to own price, whose asymptotes are the straight-line indifference curve and its mirror image, EF. It also describes the *income-consumption locus* for a given price ratio, p_1/p_2, on the indifference curve map.

The locus of the **threshold between q_1 being superior and its being inferior** on the indifference curve map can be found by re-arranging equation (4) in terms of q_1 and differentiating with respect to q_2.

$$q_1 = \mu_1 + \sigma_1.\left[\left(\frac{q_2-\mu_2}{\sigma_2}\right)^2 - 2.ln\left(\frac{\sigma_1.p_1}{\sigma_2.p_2}\right)\right]^{\frac{1}{2}} \tag{4a}$$

$$\frac{dq_1}{dq_2} = \frac{1}{2}.\left[\left(\frac{q_2-\mu_2}{\sigma_2}\right)^2 - 2.ln\left(\frac{\sigma_1.p_1}{\sigma_2.p_2}\right)\right]^{\frac{-1}{2}}.\left(\frac{2q_2-2\mu_2}{\sigma_2}\right).\left(\frac{\sigma_1}{\rho_2}\right).$$

$$\frac{dq_1}{dq_2} = \frac{\left(\frac{q_2-\mu_2}{\sigma_2}\right).\left(\frac{\sigma_1}{\sigma_2}\right)}{\sqrt{\left[\left(\frac{q_2-\mu_2}{\sigma_2}\right)^2 - 2.ln\left(\frac{\sigma_1.p_1}{\sigma_2.p_1}\right)\right]}} = 0. \quad (5)$$

By setting dq_1/dq_2 = 0 in equation (5), the **locus for this threshold** is found to be coincidental with $q_2 = \mu_2$, for $q_1 > \mu_1$. This is the boundary between areas labelled K and N on the indifference curve map. Thus, in area K, the individual's reaction to a rise in price, p_1/p_2, will be to increase q_1. Commodity 1 reacts as an inferior good in area K.

Similarly, the boundary for q_2 is $q_1 = \mu_1$ for $q_2 > \mu_2$, which is the boundary between q_2 reacting to price changes as an inferior good in area labelled V and as superior in area R on the indifference curve map.

- In the upper right-hand quadrant of Figure 2, both commodities are experienced as superior goods, (additivity and positive diminishing marginal utilities always yield superior characteristics). With additive utilities, the two goods are net substitutes for each other.
- Inferior responses occur for a commodity that fulfils a need sufficiently but is combined with a commodity of which the individual is deprived, as anticipated by Berg [17]. Good 1 responds as inferior in the rhomboid area labelled K in Figure 2, bounded by the q_1-axis, EA and $q_2 = \mu_2$ for $q_1 > \mu_1$, (Dougan [18], Silberberg and Walker [19]). That the Giffen experience is associated with a straight-line indifference curve, adjacent to a triangular non-solution space, was anticipated by Davies [20].
- In area V, in that part of the left-hand border where the indifference curves are convex-to-the-origin, the consumer is deprived of good 1, (with increasing MU), and, following Hirschleifer's terminology ([21], chap.4), good 1 is here termed an ultra-superior good. Kohli [22] calls this experience an 'anti-Giffen good', but 'anti-inferior' would be more accurate. *Good 2* is experienced as an inferior good in area V.

Boundary between inferior normal and inferior-Giffen responses

The locus of points for the threshold between q_1 responding as inferior normal and its being inferior-Giffen is derived in the Appendix. It is obtained from the numerator of equation (A7), leading to equation (A7a). It is more complex than the superior-inferior boundary.

$$\sqrt{\left[exp\left(\left(\frac{q_2-\mu_2}{\sigma_2}\right)^2 - \left(\frac{q_1-\mu_1}{\sigma_1}\right)^2\right)\right]} .\left(\frac{q_1}{\sigma_1}\right).\left(\frac{q_2-\mu_2}{\sigma_2}\right) + 1 = 0. \quad \text{(A7a)}$$

This locus does not go through point E, (μ_1, μ_2), but meets the straight-line indifference curve BA at a point which can be found from the solution to the quadratic equation in q_1 within the denominator of equation (A7) yielding equation (A7b):

$$\sqrt{\left[exp\left(\left(\frac{q_2-\mu_2}{\sigma_2}\right)^2 - \left(\frac{q_1-\mu_1}{\sigma_1}\right)^2\right)\right]} .\left(\frac{\sigma_2}{\sigma_1}\right).[1 - q_1(q_1-\mu_1)/\sigma_1^2] = 0$$

$[1 - q_1(q_1 - \mu_1)/\sigma_1^2] = 0,$

$$q_1 = [\mu_1 \pm \sqrt{(\mu_1^2 + 4.\sigma_1^2)}]/2. \tag{A7b}$$

The boundary between q_1 being inferior normal and inferior-Giffen must be found by solving equation (A7a) numerically, to find solutions for q_2 for given values of $q_1 > [\mu_1 \pm \sqrt{(\mu_1^2 + 4.\sigma_1^2)}]/2$. This boundary has been drawn for q_1 in area K and for q_2 in area V in Figure 2.

Consider an individual whose endowment C_1 of good 1 lies half-way between μ_1 and A in Figure 2:

- With an endowment C_2 of good 2, lying between 0 and BA, the individual will experience dysfunctional poverty.
- With an endowment of C_2 lying between BA and the Giffen boundary, s/he is experiencing 'Giffen-deprivation' in area K. Good 1 responds as Giffen when more is consumed of this cheaper, abundant commodity, q_1, as its price, p_1/p_2, rises, enabled by consuming less of the more expensive good, q_2, in which the individual is already very deprived.
- When the endowment, C_2, is great enough, the individual responds as inferior normal in the 'normal-deprived' part of area K, to a rise in p_1/p_2.
- When $C_2 > \mu_2$, the individual has sufficient of both commodities and responds as superior to a rise in p_1/p_2.

Similarly, the experience of the individual from the perspective of increasing q_2, matches that of increasing q_1.

Rather than categorising the commodity, it is the consumer's *experience of, and response to, the fulfilment of a need* by a commodity, in combination with a good that fulfils another need, that should be categorised as ultra-superior, superior, inferior-normal or inferior-Giffen. This would appear to confirm Spiegel's belief 'that Giffen goods are far more pervasive than is generally believed' ([23], p.137 and [24], pp. 45-47; Weber [25]). That the challenge of formulating a utility function for the elusive 'Giffen *good*' (as opposed to the pervasive Giffen *experience*) continues to engage economists is evidenced by Sørensen [26], Jensen and Miller [27], Moffatt [28], Haagsma [29] and Biederman [30].

5 The demand equation

The demand equation is derived in the Appendix.

$$q_1 = \mu_1 + \frac{\left(\frac{Z}{p_2}\right).x \; - b.\sqrt{\left[\left(\frac{Z}{p_2}\right)^2 + (x^2 - b^2).\left(\sigma_1^2.2.ln\left(\frac{x}{b}\right)\right)\right]}}{(x^2 - b^2)}. \tag{A9}$$

An alternative expanded version is obtained by dividing numerator and denominator by x^2, (where $x = p_1/p_2$, (relative prices), and b = σ_2/σ_1, (relative intensities-of-need)), and by expanding Z/p_2 yielding:

$$q_1 = \mu_1 + \frac{((C_1-\mu_1).+(C_2-\mu_2)/x) - \frac{b}{x}.\sqrt{\left[((C_1-\mu_1).+(C_2-\mu_2)/x)^2 - \left(1-\left(\frac{b}{x}\right)^2\right).\left(\sigma_1^2..2.ln\left(\frac{b}{x}\right)\right)\right]}}{\left(1-\left(\frac{b}{x}\right)^2\right)} \quad \text{(A10)}$$

Equation (10) demonstrates that the dependent variable, q_1, is a non-linear function of the independent variables, 'own' relative price, ($x = p_1/p_2$), and endowments, C_1 and C_2, with parameters, μ_1, μ_2, σ_1 and σ_2/σ_1. The demand equations for q_1 and q_2 are symmetric and homogeneous of degree zero in p_1, p_2 and Z.

The strong separability assumption allows the demand equation for any two commodities to be estimated independently of any other commodity fulfilling another need.

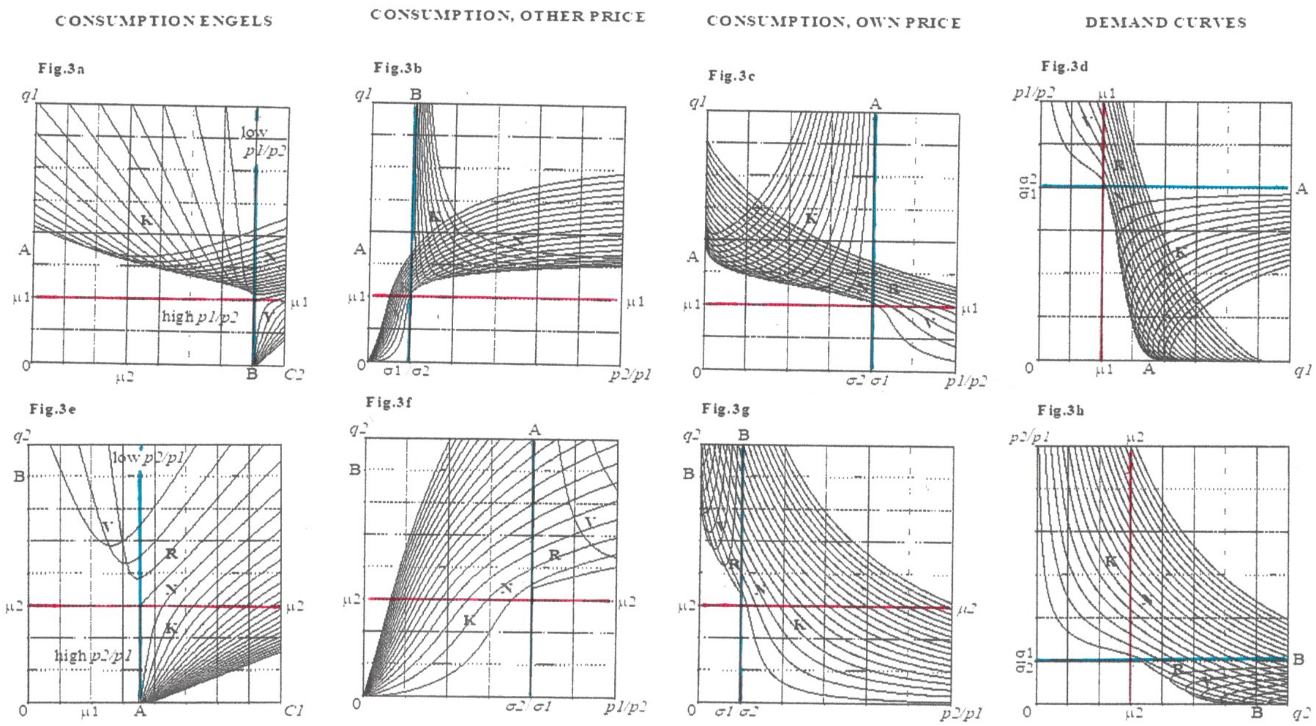
6 The derived functional form diagrams

The top row of Figure 3 comprises four diagrams, each with dependent variable, q_1. The first diagram is of Engels curves, where consumption, q_1, of good 1 is plotted against endowments of the other good, C_2. In the second and third diagrams, the dependent variable, q_1, is plotted against 'other price', p_2/p_1, and 'own price', p_1/p_2, respectively. The fourth diagram rotates the axes to present the demand diagram in its more familiar orientation. The lower row repeats the exercise for q_2. The areas K, N, R and V from the indifference curve map are identified on these diagrams.

The presentation of the eight diagrams together in Figure 3 enables patterns to be discerned. It also demonstrates that, although the demand curves have essentially the same pattern, (best illustrated in figure 3b), they can appear to be very different in figures 3c, 3f and 3g.

To accommodate the effect of constraining $q_2 \geq 0$ on the Map, when the individual is in dysfunctional poverty, leading to corner solutions on the axes bordering the non-solution space, equations (9) and (10) must be qualified such that $0 \leq q_1 \leq M/p_1$. Thus, if $q_1 < 0$, substitute $q_1 = 0$. If $q_1 > M/p_1$, substitute $q_1 = M/p_1$. Similarly, for $0 \leq q_2 \leq M/p_2$. These have been indicated for q_2, in the demand diagram, Figure 3h, and in the Engels diagram, Figure 3e, below.

The following paragraphs trace the shapes of the *demand curves for good 2* in Figure 3h. It is assumed that C_2 = zero and thus q_2 is a function of own price and endowments of the other commodity, $q_2(p_2/p_1, C_1)$. The curves take different shapes depending on the level of endowment, C_1. Each shape starts with $p_2/p_1 = 0$. At zero price, the individual can consume his/her fill, represented here by $q_2 >$ B on the q_2 axis in Fig.3h. At B, utility is near the satiation of need 2 by good 2.



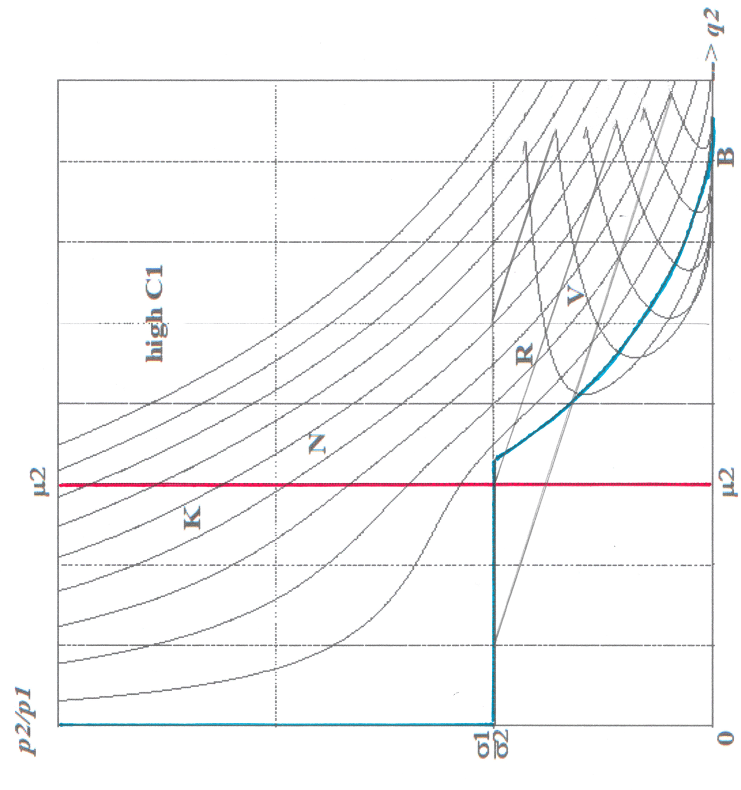
KEY dependent variables:
independent variables:

q1, q2 consumption
p1, p2 prices
C1, C2 endowments of q1, q2

$\mu 1$, $\mu 2$ subsistence consumption
$\sigma 1$, $\sigma 2$ intensity of need
B, A survival endowments

K N R V relate to indifference map

FIGURE 3 DERIVED FUNCTIONAL FORMS



VARIABLES
$q2$ consumption, good 2
$p2/p1$ relative prices
$C1$ endowment of good 1
K N R and V relate to FIGURE 2.

PARAMETERS
$\mu 2$ subsistence good 2
$\sigma 1/\sigma 2$ relative intensity-of-need
B survival endowment, good 2

FIGURE 3h DEMAND CURVE, GOOD 2

For low endowments, C_1 < A and $p_2/p_1 = 0$, the demand curve starts at q_2 > B. As p_2/p_1 increases, q_2 decreases initially through section R before increasing steeply through section V in a U-shaped curve to a level of q_2 > B. The demand curve in Figure 3h has been modified, by the addition of a line from point q_2 > B to its value when $p_2/p_1 = \sigma_1/\sigma_2$, representing the movement of the budget line down the q_2 axis as price increases on the Map, but constrained by $q_1 \geq 0$. When $p_2/p_1 = \sigma_1/\sigma_2$, the individual faces disequilibrium. The disjointed demand curve jumps to $q_2 = 0$ as p_2/p_1 increases. As price increases further, the demand curve moves up the p_2/p_1 axis at $q_2 = 0$.

For C_1 = A: for $p_2/p_1 = 0$, the demand curve starts at q_2 > B. As p_2/p_1 increases, q_2 decreases in a negative-sloping curve through section R to $q_2 = [\mu_2 \pm \sqrt{(\mu_2^2 + 4.\sigma_2^2)}]/2$ (equation A7b) in section V, where $p_2/p_1 = \sigma_1/\sigma_2$ (and the budget line is co-incidental with BA). As p_2/p_1 increases, it jumps to $q_2 = 0$, creating a disjointed demand curve, and then moves up the p_2/p_1 axis.

For C_1 > A: for $p_2/p_1 = 0$, the demand curve again starts at q_2 > B. As p_2/p_1 increases, q_2 follows an optimisation path, decreasing first through section R, then briefly through section N and finally into section K, in a familiar negative-sloping curve. When $p_2/p_1 = \infty$, $q_2 = 0$.

An envelope curve around the series of U-shaped demand curves associated with low values of C_1 and low values of p_2/p_1 indicates the boundary between q_2 responding as superior in area R and inferior in area V. The equation for the envelope curve for a good (derived in the Appendix) is:

$$q_2 = \mu_2 + \sigma_2.\sqrt{\left[+2.\ln\left(\frac{p_1\sigma_1}{p_2\sigma_2}\right)\right]}\ ,\ for \frac{p_2}{p_1} < \frac{\sigma_1}{\sigma_2}. \qquad \text{(A12)}$$

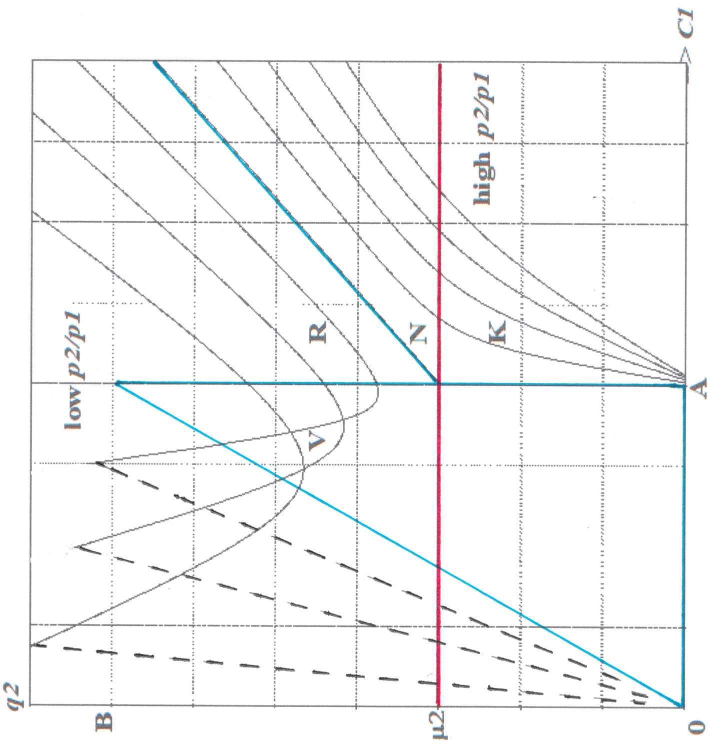
The increasing demand curves in section V represent the inferior-Giffen experience. Demand curves associated with deprivation tend to be elastic.

A few economists (Stonier and Hague [31] p.77; Hirschleifer [21] pp. 98 and 114) have tried to draw a series of demand curves for a commodity as it transforms from superior to inferior (or from an inferior-normal to an inferior-Giffen). With hindsight, it should have been intuitively obvious that there might be an envelope curve on demand curves, if one assumes that the demand curves usually have negative slopes. If they shift upwards as unearned income increases for superior goods, and they also shift upwards as unearned income decreases for inferior goods, then an envelope must occur on the boundary between a good being inferior and its being superior.

Starting with the Engels curve for the given price of $p_2/p_1 = \sigma_1/\sigma_2$, the budget line is parallel to the straight-line indifference curve BA on the Map. As C_1 increases from C_1 = 0 to C_1 = A, C_2 also increases from 0 to B. The individual is in dysfunctional poverty and in disequilibrium, because equal utility is offered as a corner solution at each end of the budget line.

This is represented on Figure 3e by both $q_2 = 0$ tracing the C_1 axis as C_1 increases from 0 to A, and at the same time, q_2 increases, moving along the q_2 axis, in a straight line, as C_2 increases from 0 to B. At $q_2 = 0$ at A and q_2 = B, the budget is co-incidental with BA and the individual faces an infinity of options between these two points. However, as C_1 continues to increase further, the Engels curves jump from B and A to point E,

(μ_1, μ_2), combining to exit via the locus EF on the Map, forming a positive-sloping straight line in Figure 3e.

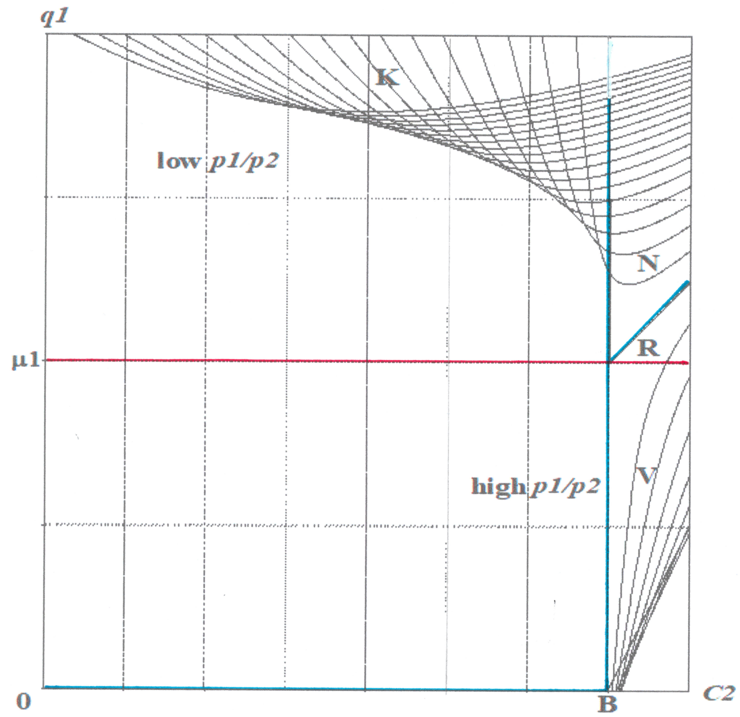


VARIABLES
q2 consumption
C1 endowments
p2/p1 price of consumption

PARAMETERS
μ2 subsistence consumption
A survival endowment

K N R and V relate to FIGURE 2.

FIGURE 3e CONSUMPTION ENGELS CURVES



VARIABLES		PARAMETERS	
$q1$	consumption of good 1	$\mu 1$	subsistence good 1
$C2$	endowments	B	survival endowment
$p1/p2$	relative price of good 1		

K, N, R and V relate to FIGURE 2.

FIGURE 3a CONSUMPTION ENGELS CURVES

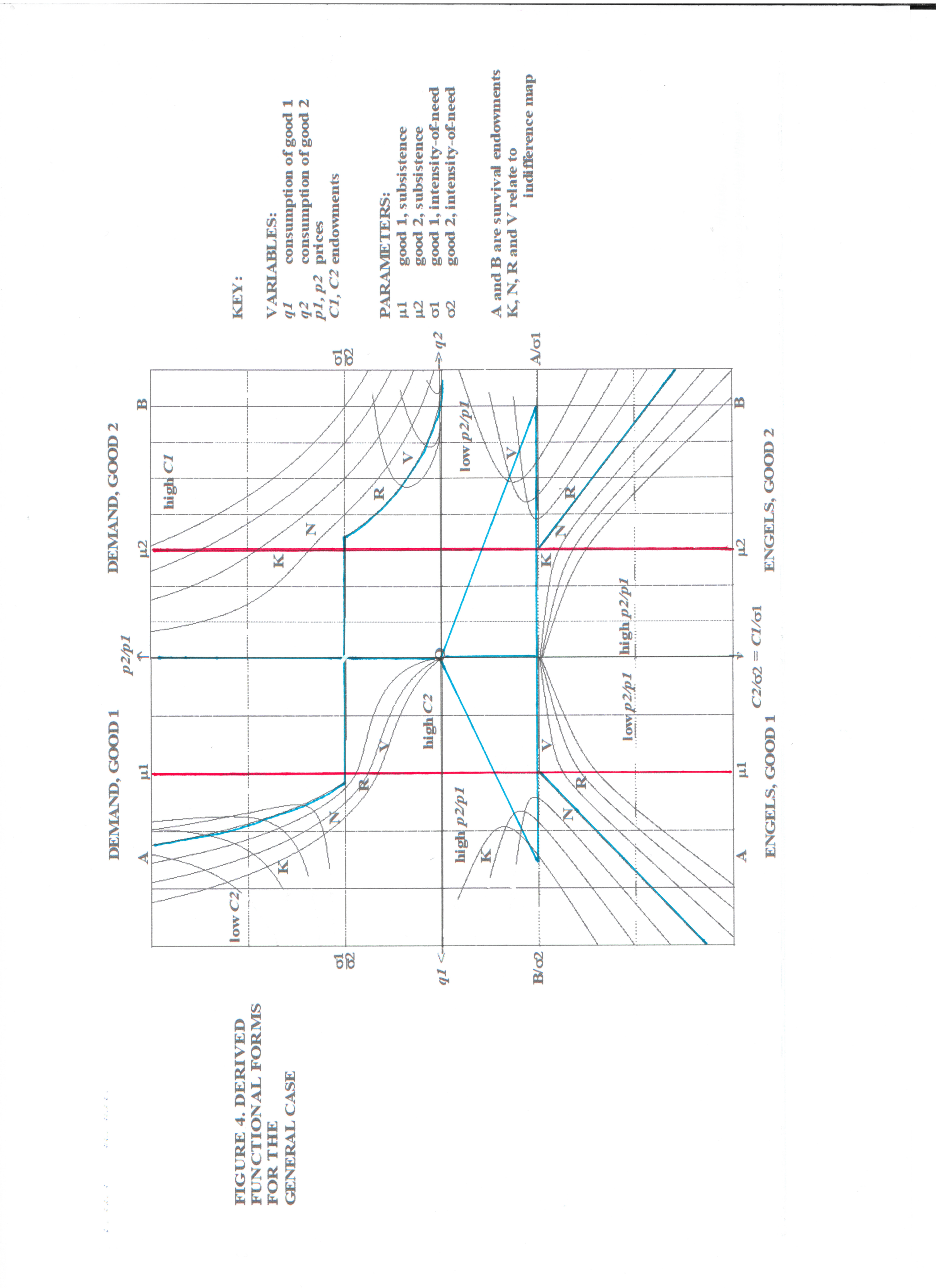


FIGURE 4. DERIVED FUNCTIONAL FORMS FOR THE GENERAL CASE

For a higher-level price, $p_2/p_1 > \sigma_1/\sigma_2$, and $C_1 = 0$, the individual is in dysfunctional poverty. As C_1 increases, $q_2 = 0$ offers higher utility than $q_2 = C_2$. This is illustrated on the Engels diagram by $q_2 = 0$ travelling up the C_1 axis until C_1 = A, where the individual can start to optimise. As C_1 increases further, q_2 increases, passing through section K, and on through section N.

For a lower-level price, $p_2/p_1 < \sigma_1/\sigma_2$, with endowment C_1 increasing from $C_1 = 0$ and C_2 also increasing from $C_2 = 0$, the individual is again in dysfunctional poverty, where $q_2 = C_2$ offers higher utility than $q_2 = 0$. This is represented by dashed Engels curves for good 2 in Figure 3e. This situation persists until $q_2 > B$, where the individual begins to optimise through section V. Initially q_2 decreases as the optimisation passes through the Giffen-deprived part of section V, but increases again through the 'normal-deprived' part of V before increasing through section R.

An envelope curve is perceptible around the series of three U-shaped Engels curves and ending at point (C_1 = A and $q_2 = \mu_2$), reflecting the boundary between inferior normal and inferior-Giffen experiences associated with low endowments. This is illustrated more clearly in Figure 3a for good 1.

Figure 4 has been created from Figs 3b, 3h, 3a and 3e of Figure 3, but with their axes rotated so that they can share the axes for q_1, p_2/p_1, q_2 and endowments. The straight-line indifference curve BA determines the survival endowments in Fig.3a and Fig.3e, and the scales have been adjusted to reflect this, with a common axis $C_2/\sigma_2 = C_1/\sigma_1$. An increase in C_1 for a fixed price inevitably brings about a corresponding increase in C_2. The purpose of Figure 4 is to give a visual impression of how the different diagrams fit together, and to observe the emerging patterns.

The two upper diagrams illustrate the range of the corner solutions clearly ($q_1 = 0$, and $q_2 = 0$). In the top-right demand diagram, if $p_2/p_1 = 0$ and thus good 2 is free, then the individual can consume as much as s/he wants, but the relative price of good 1 is infinite, ($p_1/p_2 = \infty$), and the individual cannot afford any of it. This a case of the poverty-stricken consumer choosing his/her own cheaper deprivation.

7 Conclusion

A leaning-S-shaped utility for a commodity expresses the experiences of an individual during the different stages of fulfilment of a need – deprivation, subsistence, sufficiency and in this model, satiation at infinity. The separability rule is applicable to leaning-S-shaped utilities and while multiplicativity can represent two commodities fulfilling the *same* need, additivity represents each fulfilling a *different need*. In either case, an individual can experience various combinations of deprivation and sufficiency. A commodity being experienced as inferior is always associated with deprivation in a 'different needs' model.

The inferior-Giffen experience occurs when an individual responds to a price increase for an abundant, cheaper good by consuming more of it, enabled by relinquishing some consumption of a more expensive commodity fulfilling a different need, of which s/he is already extremely deprived. Rather than categorising a commodity as a Giffen good, it is an individual's *experience* of the relevant circumstances that determines his/her response to the change in price as inferior-Giffen.

Acknowledgements

I am grateful to Peter Fisk for introducing me to van Praag's work, to David A Williams, David Sterratt and Geoffrey Wyatt for mathematical and computing advice, and for helpful comments on earlier versions of this paper from Paul Hare, Douglas Mair, Mike Danson, Prabir Bhattacharya and Otto Lehto.

Funding Sources

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

REFERENCES

[17] Berg, Morten, "Giffen's Paradox Revisited." *Bulletin of Economic Research*, 39 (January 1987) 79 – 89.

[30] Biederman, Daniel K. "A strictly-concave, non-spliced, Giffen-compatible utility function", *Economic Letters*, 131(C) (2015) 24 – 28. https://doi.org/10.1016/j.econlet.2015.03.024.

[20] Davies, John E. "Giffen Goods, the Survival Imperative, and the Irish Potato Culture", *Journal of Political Economy*, 102(3) (June 1994) 547 – 65.

[13] Deaton, A and Muellbauer, J. *Economics and Consumer Behaviour*, Cambridge University Press, Cambridge, 1980.

[18] Dougan, W.R. "Giffen Goods and the Law of Demand", *Journal of Political Economy*, 90(4) (August 1982) 809 – 15.

[4] Doyal, Len and Gough, Ian *A Theory of Human Need*, Macmillan, London, 1991.

[12] Green, H.A.J. *Consumer Theory*, Macmillan, London, 1976.

[29] Haagsma, Rein "A Convenient Utility Function with Giffen Behaviour", *International Scholarly Research Network*, (2012), Article ID 608645.

[9] Hagenaars, Aldi J.M. *The Perception of Poverty*, North Holland, Amsterdam, 1986.

[21] Hirschleifer, Jack *Price Theory and Applications*, Prentice-Hall, New Jersey, 1976.

[27] Jensen, Robert T. and Miller, Nolan H. "Giffen behaviour and subsistence consumption", *The American Economic Review*, 98(4) (2008) 1553 – 1577. https://doi.org/10.1257/aer.98.4.1553.

[16] Johnson, Norman L. and Kotz, Samuel *Continuous univariate distributions -1*, Houghton Mifflin (Wiley), Boston, 1970.

[22] Kohli, Ulrich "Inverse Demand and Anti-Giffen Goods", *European Economic Review*, 27(3) (April 1985) 397 – 404.

[1] Lawson, T. *Economics and Reality*, Routledge, London, 1997.

[14] Mallman, C.A., and Nudlar, Oscar, *Human Development in its Social Context*, Hodder and Stoughton, London, 1986.

[3] Maslow, Abraham H. "A Theory of Human Motivation", *Psychological Review*, 50 (1943) 370 – 396. https://dx.doi.org/10.1037/h0054346

[15] Max-Neef, Manfred “Human-scale economics: the challenges ahead”, in Paul Ekins (ed), *The Living Economy: a New Economics in the Making*, Routledge and Kegan Paul, London, 1986.

[6] Miller, A.G. “A Needs-Based Demand Theory.” In P Vanden Abeele (Ed), *Proceedings of the thirteenth colloquium of the International Association for Research in Economic Psychology*, Volume II, IAREP, Leuven, Autumn 1988.

[28] Moffatt, Peter G. “A Class of Indirect Utility Functions Predicting Giffen Behaviour.” In Heijman, Wim and von Mouche, Pierre (Eds), *New Insights into the Theory of Giffen Goods*, Springer-Verlag, Heidelberg, 2012, pp. 127 – 41.

[11] Mustonen, Seppo, *SURVO: An Integrated Environment for Statistical Computing and Related Areas*, Survo Systems Ltd, Helsinki, 1992.

[19] Silberberg, Eugene, and Walker, Donald A. “A Modern Analysis of Giffen’s Paradox”, *International Economic Review,* 25 (October 1984) 687 – 94.

[26] Sørensen, Peter N. “Simple Utility Functions with Giffen Demand”, *Economic Theory*, 31(2) (2007) 367 – 370. https://doi.org/10.1007/s00199-006-0086-6

[23] Spiegel, Uriel “The Case of a Giffen Good”, *Journal of Economic Education*, 25 (Spring 1994) 137 – 47.

[24] Spiegel, Uriel “The Case of a Giffen Good: Reply”, *Journal of Economic Education*, 28(1) (Winter 1997) 45 – 47.

[31] Stonier, A.W., and Hague, D.C. *A Textbook in Economic Theory*. Longman, London, 1980.

[8] Van Herwaarden, Floor G., and Kapteyn, Arie “Empirical Comparison of the Shape of Welfare Functions”, *European Economic Review*, 15(3) (March 1981) 261 – 86. https://doi.org/10.1016/S0014-2921(81)80002-5.

[7] Van Praag, B.M.S. *Individual Welfare Functions and Consumer Behaviour*, North Holland, Amsterdam, 1968.

[10] Van Praag, B.M.S. and Kapteyn, A.J. “How sensible is the Leyden individual welfare function of income? A reply”, *European Economic Review*, 38(9) (December 1994) 1817 – 25. https://doi.org/10.1016/0014-2921(94)90052-3

[5] Ward, David and Lasen Marta “An Overview of Needs Theories behind Consumerism.” *Journal of Applied Economic Sciences*, 2009.

[25] Weber, Christian E. “The Case of a Giffen Good: Comment”, *Journal of Economic Education*, 28(1) (Winter 1997) 36 – 44.

[2] Yamamori, Toru “The concept of need in Adam Smith”, *Cambridge Journal of Economics*, 41(2) (March 2017) 327 – 347. https://doi.org/10.1093/cje/bew049.

MATHEMATICAL APPENDIX

The Appendix includes the following:

- The derivation for an equation for the boundary between inferior normal and inferior-Giffen responses (to be solved numerically).
- The derivation of the functional form for the demand equation.
- The derivation of an equation for the envelope curve on the demand curves associated with the boundary between superior and inferior responses.

Boundary between *q1* responding as inferior normal and its being inferior-Giffen
The equation for the budget passing through two endowments, (C_1, C_2), is given by

$$\frac{p_1}{p_2} = \frac{(C_2 - q_2)}{(q_1 - C_1)}$$

The optimality condition in equation (4) gives the locus of points describing the *price ratio-consumption locus* for a given income, *M*, on the indifference curve map.

The price ratio, p_1/p_2, is substituted into the optimality condition, equation (4), eliminating prices from equation (6),

$$\left(\frac{q_2 - \mu_2}{\sigma_2}\right)^2 - \left(\frac{q_1 - \mu_1}{\sigma_1}\right)^2 = ln\left(\frac{\sigma_1 . p_1}{\sigma_2 . p_2}\right)^2 \quad (4)$$

$$exp\left[\left(\frac{(q_2 - \mu_2)}{\sigma_2}\right)^2 - \left(\frac{(q_1 - \mu_1)}{\sigma_1}\right)^2\right] = \left[\left(\frac{(C_2 - q_2)}{(q_1 - C_1)}\right) . \left(\frac{\sigma_1}{\sigma_2 .}\right)\right]^2 . \quad (6)$$

To simplify the notation, let $\left(\frac{q_2 - \mu_2}{\sigma_2}\right)^2 - \left(\frac{q_1 - \mu_1}{\sigma_1}\right)^2$ = S.

Re-arranging equation (6) in terms of C_2 gives:

$$C_2 = q_2 + (q_1 - C_1) . \left(\frac{\sigma_2}{\sigma_1}\right) . \sqrt{[exp(S)]}.$$

Differentiating C_2 with respect to q_1 and q_2 yields the following:

$$\frac{dC_2}{dq_1} = \left(\frac{\sigma_2}{\sigma_1}\right) . \sqrt{[exp(S)]} + (q_1 - C_1) . \left(\frac{\sigma_2}{\sigma_1}\right) . \frac{1}{2} . [exp(S)]^{-\frac{1}{2}} . \exp(S) . (-2)(q_1 - \mu_1)/\sigma_1^2 .$$

$$\frac{dC_2}{dq_1} = [1 - (q_1 - C_1) . (q_1 - \mu_1)/\sigma_1^2] . \left(\frac{\sigma_2}{\sigma_1}\right) . \sqrt{[exp(S)]} .$$

$$\frac{dC_2}{dq_2} = 1 + (q_1 - C_1) . \left(\frac{\sigma_2}{\sigma_1}\right) . \frac{1}{2} . [exp(S)]^{-1/2} exp(S) . (+2)(q_2 - \mu_2)/\sigma_2^2 .$$

$$\frac{dC_2}{dq_2} = \left[1 + (q_1 - C_1).(q_2 - \mu_2)/(\sigma_2^2).\left(\frac{\sigma_2}{\sigma_1}\right).\sqrt{[exp(S)]}\right].$$

Using implicit differentiation, dq_1/dq_2 is obtained and set equal to zero, eliminating C_2. C_1 is set equal to zero and the equation multiplied through by –1, resulting in equation (7).

$$\frac{dq_1}{dq_2} = -\frac{dC_2}{dq_2} \Big/ \frac{dC_2}{dq_1}$$

$$\frac{dq_1}{dq_2} = -\frac{1+(q_1-C_1).(q_2-\mu_2)/(\sigma_2^2).\left(\frac{\sigma_2}{\sigma_1}\right).\sqrt{[exp(S)]}}{\left[[1-(q_1-C_1).(q_1-\mu_1)/\sigma_1^2].\left(\frac{\sigma_2}{\sigma_1}\right).\sqrt{[exp(S)]}\right]} = 0.$$

$$\frac{dq_1}{dq_2} = +\frac{1+q_1.(q_2-\mu_2)/(\sigma_2^2).\left(\frac{\sigma_2}{\sigma_1}\right).\sqrt{[exp(S)]}}{\left[[1-q_1.(q_1-\mu_1)/\sigma_1^2].\left(\frac{\sigma_2}{\sigma_1}\right).\sqrt{[exp(S)]}\right]} = 0. \qquad (7)$$

Thus, the locus for the threshold between q_1 being inferior normal and inferior-Giffen is given by the numerator of equation (7):

$$\sqrt{\left[exp\left(\left(\frac{q_2-\mu_2}{\sigma_2}\right)^2 - \left(\frac{q_1-\mu_1}{\sigma_1}\right)^2\right)\right]}.\left(\frac{q_1}{\sigma_1}\right).\left(\frac{q_2-\mu_2}{\sigma_2}\right) + 1 = 0. \qquad (7a)$$

This locus does not go through the point E, (μ_1, μ~~2~~), but cuts the straight-line indifference curve BA at a point which can be found from the solution to the quadratic equation in q_1 within the denominator of equation (7) as follows:

$$\sqrt{\left[exp\left(\left(\frac{q_2-\mu_2}{\sigma_2}\right)^2 - \left(\frac{q_1-\mu_1}{\sigma_1}\right)^2\right)\right]}.\left(\frac{\sigma_2}{\sigma_1}\right).[1 - q_1(q_1-\mu_1)/\sigma_1^2] = 0$$

$[1 - q_1(q_1 - \mu_1)/\sigma_1^2] = 0,$

$q_1^2 - \mu_1.q_1 - \sigma_1^2 = 0.$

$q_1 = [\mu_1 \pm \sqrt{(\mu_1^2 + 4.\sigma_1^2)}]/2.$ (7b)

Equation (7a) must be solved numerically to find the solutions for q_2 for given values of $q_1 > [\mu_1 \pm \sqrt{(\mu_1^2 + 4.\sigma_1^2)}]/2$.

Derivation of the demand equation

Survival income. $= \mu_1.p_1 + \mu_2.p_2$
full income: $M = C_1.p_1 + C_2.p_2.$
supernumerary income: $Z = (C_1 - \mu_1).p_1 + (C_2 - \mu_2).p_2$
$= M - \mu_1.p_1 - \mu_2.p_2,$
budget equation: $M = q_1.p_1 + q_2.p_2,$

Substituting for $q_2 = (M - q_1.p_1)/p_2$, from the budget constraint, and for $M = Z + \mu_1.p_1 + \mu_2.p_2$ from the supernumerary expenditure equation, into equation (4), yields an '**implicit demand equation**' (8):

$$q_2 = (Z + \mu_1.p_1 + \mu_2.p_2 - q_1.p_1)/p_2$$

$$\left(\frac{q_2 - \mu_2}{\sigma_2}\right)^2 - \left(\frac{q_1 - \mu_1}{\sigma_1}\right)^2 = ln\left(\frac{\sigma_1.p_1}{\sigma_2.p_2}\right)^2 \qquad (4)$$

$$\left[\frac{\frac{Z}{p_2} - (q_1 - \mu_1).x}{\sigma_2}\right]^2 = \left[\frac{(q_1 - \mu_1)}{\sigma_1}\right]^2 + ln\left[\frac{x}{b}\right]^2 \qquad (8)$$

where $x = p_1/p_2$ (relative prices); $b = \sigma_2/\sigma_1$ (relative intensities-of-need);

$$\frac{\sigma_1^2.\left(\frac{Z}{p_2}\right)^2 - 2.\sigma_1^2(q_1-\mu_1).x.\left(\frac{Z}{p_2}\right) + \sigma_1^2.(q_1-\mu_1)^2.x^2 - \sigma_2^2.(q_1-\mu_1)^2 - 2.\sigma_1^2.\sigma_2^2.ln\left(\frac{x}{b}\right)}{\sigma_1^2.\sigma_2^2} = 0.$$

$$\frac{(\sigma_1^2.x^2 - \sigma_2^2).(q_1-\mu_1)^2 - 2.\sigma_1^2.\left(\frac{Z}{p_2}\right).x.(q_1-\mu_1) + \sigma_1^2.\left(\frac{Z}{p_2}\right)^2 - 2.\sigma_1^2.\sigma_2^2.ln\left(\frac{x}{b}\right)}{\sigma_1^2.\sigma_2^2} = 0.$$

which is a quadratic equation in $(q_1 - \mu_1)$, which is solved using the negative square root, yielding **demand equation** (9) for the first commodity:

Let the root be $(q_1 - \mu_1) = \frac{-b \pm \sqrt{b^2 - 4.a.c}}{2.a}$;

$$a = (\sigma_1^2.x^2 - \sigma_2^2)\,; \quad b = -2.\sigma_1^2.x.Z/p_2\,; \quad c = \sigma_1^2.\left(\left(\frac{Z}{p_2}\right)^2 - \sigma_2^2.2.ln(x/b)\right).$$

$$(q_1 - \mu_1) = \frac{2.\sigma_1^2.\left(\frac{Z}{p_2}\right).x \pm \sqrt{\left[4.\sigma_1^4\left(\frac{Z}{p_2}\right)^2.x^2 - 4.(\sigma_1^2.x^2 - \sigma_2^2).\sigma_1^2.\left(\left(\frac{Z}{p_2}\right)^2 - .\sigma_2^2.2.ln(x/b)\right)\right]}}{2.(\sigma_1^2.x^2 - \sigma_2^2)}$$

$$(q_1 - \mu_1) = \frac{\sigma_1^2.\left(\frac{Z}{p_2}\right).x \pm \sqrt{\left[.\sigma_1^2.\sigma_2^2.\left(\frac{Z}{p_2}\right)^2 + (\sigma_1^2.x^2 - \sigma_2^2).(\sigma_1^2.\sigma_2^2.2.ln(x/b))\right]}}{(\sigma_1^2.x^2 - \sigma_2^2)}$$

Using the negative square root and dividing numerator and denominator by σ_1^2.

$$(q_1 - \mu_1) = \frac{\left(\frac{Z}{p_2}\right).x - \sqrt{\left[b^2.\left(\frac{Z}{p_2}\right)^2 + (x^2 - b^2).\left(\sigma_2^2.\left(\frac{\sigma_1^2}{\sigma_1^2}\right)2.ln(x/b)\right)\right]}}{(x^2 - b^2)}$$

$$q_1 = \mu_1 + \frac{\left(\frac{Z}{p_2}\right).x \; - \; b.\sqrt{\left[\left(\frac{Z}{p_2}\right)^2 + (x^2-b^2).\left(\sigma_1^2.2.ln\left(\frac{x}{b}\right)\right)\right]}}{(x^2-\,b^2)} \tag{9}$$

$Z/p_2 = (C_1 - \mu_1).p_1/p_2 + (C_2 - \mu_2).p_2/p_2$

An alternative expanded version is obtained by dividing numerator and denominator by x^2 and expanding Z/p_2 yielding:

$$q_1 = \mu_1 + \frac{((C_1-\mu_1).+(C_2-\mu_2)/x) \; - \frac{b}{x}.\sqrt{\left[\left((C_1-\mu_1).+(C_2-\mu_2)/x\right)^2 + \left(1-\left(\frac{b}{x}\right)^2\right).\left(\sigma_1^2..2.ln\left(\frac{x}{b}\right)\right)\right]}}{\left(1-\left(\frac{b}{x}\right)^2\right)} \tag{10}$$

Equation for the envelope curve on the demand equations representing the boundary between superior and inferior responses.

By differentiating q_1 in the demand function, equation (9), with respect to Z, and setting the partial derivative equal to zero, one obtains:

$$\frac{dq_1}{dZ} = \frac{\frac{x}{p_2} - \frac{2.b.Z}{p_2^2}.\frac{1}{2}.\left[\left(\frac{Z}{p_2}\right)^2 + (x^2-b^2).\left(\sigma_1^2.2.ln\left(\frac{x}{b}\right)\right)\right]^{-1/2}}{(x^2-\,b^2)} = 0.$$

Re-arranging this and squaring both sides, to express it in terms of Z, gives:

$$\left(\frac{x}{p_2}\right)^2.\left[\left(\frac{Z}{p_2}\right)^2 + \; (x^2 - \; b^2).\left(\sigma_1^2 .2. ln\left(\frac{x}{b}\right)\right)\right] = \; \left(\frac{b.Z}{p_2^2}\right)^2$$

$$\left(\frac{x}{p_2}\right)^2.\left(\frac{Z}{p_2}\right)^2 - \left(\frac{b.Z}{p_2^2}\right)^2 = -\left[\left(\frac{x}{p_2}\right)^2.(x^2 - \; b^2).\left(\sigma_1^2 .2. ln\left(\frac{x}{b}\right)\right)\right]$$

$$\left(\frac{Z}{p_2^2}\right)^2 (x^2 - b^2) = -\left[\left(\frac{x}{p_2}\right)^2.(x^2 - \; b^2).\left(\sigma_1^2 .2. ln\left(\frac{x}{b}\right)\right)\right]$$

$$\left(\frac{Z}{p_2^2}\right)^2 = -\left[\left(\frac{x}{p_2}\right)^2.\left(\sigma_1^2 .2. ln\left(\frac{x}{b}\right)\right)\right]$$

$$\left(\frac{Z}{p_2}\right)^2 = -\left[x^2.\left(\sigma_1^2 .2. ln\left(\frac{x}{b}\right)\right)\right]$$

$$Z^2 = -\left[p_2^2.x^2.\left(\sigma_1^2 .2. ln\left(\frac{x}{b}\right)\right)\right]$$

$Z = \sigma_{1.}p_1\sqrt{\left[-2.\,ln\left(\frac{x}{b}\right)\right]}$, if $(x/b) < 1$; that is, $x < b$, or $p_1/p_2 < \sigma_2/\sigma_1$.

$$Z = \sigma_{1.}p_1\sqrt{\left[+2.\,ln\left(\frac{b}{x}\right)\right]}\text{, if (b/}x\text{) > 1.} \tag{11}$$

Substituting for *Z* from equation (11) into equation (9) gives the envelope curve on the demand equations, for $p_1/p_2 \leq \sigma_2/\sigma_1$.

$$q_1 = \mu_1 + \frac{\left(\frac{Z}{p_2}\right).x \; - \; b.\sqrt{\left[\left(\frac{Z}{p_2}\right)^2 + (x^2 - b^2).\left(\sigma_1^2.2.ln(x/b)\right)\right]}}{(x^2 - b^2)} \tag{9}$$

$$q_1 = \mu_1 + \frac{\sigma_1\left(\frac{p_1}{p_2}\right).\sqrt{\left[+2.ln\left(\frac{b}{x}\right)\right]}x \; - \; b.\sqrt{\left[-\sigma_1^2.x^2.2ln\left(\frac{x}{b}\right) + (x^2 - b^2).\left(\sigma_1^2.2.ln(x/b)\right)\right]}}{(x^2 - b^2)}$$

$$q_1 = \mu_1 + \frac{\sigma_1.x^2.\sqrt{\left[+2.ln\left(\frac{b}{x}\right)\right]} \; - \; \sigma_1.b^2.\sqrt{\left[+\left(.2.ln(b/x)\right)\right]}}{(x^2 - b^2)}$$

$$q_1 = \mu_1 + \sigma_1.\sqrt{\left[+2.\,ln\left(\frac{b}{x}\right)\right]}\;,\; for\,\frac{p_1}{p_2} < \frac{\sigma_2}{\sigma_1}. \tag{12}$$