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3 July 2025

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July 4, 2025

Abstract

We characterize stable market structures under price-competition in differentiated markets when multiple cartels may form. Market structures without cartelisation are never stable and always involve multiple small cartels, and, but for one 'knife-edge' case, only involves multiple small cartels. Combined with the result that the unique stable market structure under quantity-competition is also characterised by multiple small cartels, this underscores the importance of considering the possibility of multiple cartels in competition policy. Comparing stable market structures under price and quantity competition, we find that prices and profits are higher under price-competition whenever the market is sufficiently differentiated or sufficiently concentrated.

JEL Classification: C70; D43; L13.

Keywords: multiple cartels; stable cartels; price competition; differentiated markets

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1 Introduction

A fundamental question of enquiry in economics pertains to how market structures influence market interactions and economic outcomes, and an essential component of such analyses relates to the strategies that firms may employ to alter the form or extent of market competition. Generally, in markets without any form of friction or externalities, perfect competition amongst the firms delivers market outcomes that generate the highest amount of economic surplus. The corollary is that any form of collusion amongst firms that impedes competition in the marketplace is undesirable. A prime example of this concern relates to cartel formation. Firms may have the incentive of forming a cartel and restricting economic activity in the market in question by colluding on higher prices, or restricting output, in order to obtain a market outcome that is more profitable than what would emerge as a result of competition amongst the firms. Furthermore, not only does cartelisation restrict economic activity in the concerned market, but it may potentially adversely affect economic activity in upstream and/or downstream markets as well.

As a testimony to the significance of this concern, there is a large body of literature that has explored various aspects of cartelisation. Since the working of a cartel requires firms in the cartel to either restrict output or increase prices, the early literature (see, for instance, Stigler 1950), by surmising that it is in the incentive of each firm to not be part of a cartel but instead free-ride on the cartel by further expanding output or further lowering price (instead of restricting output or increasing price, as the case may be), de-emphasised the concern that cartels may impede economic efficiency on the grounds that cartels are unstable arrangements. However, the subsequent game-theoretic literature has developed a more refined understanding of the strategic aspects of cartel formation: while a firm may indeed have such an incentive to free-ride on the cartel, the cartel may also respond in a manner that, once this response is taken into account, it is actually less profitable for a firm to not partake in the cartel. This strategic consideration may then lend stability to cartels. However, the existing literature has predominantly maintained the a priori assumption that at most one cartel may be formed. In contrast, in this paper, in the context of price competition in a differentiated market, we examine the firms' incentives to form cartels, and analyse stable market structures when multiple cartels may exist in the market.

The first step towards such an analysis involves spelling out the appropriate notion of cartel stability. It is of course possible that all, some, or none of the firms form a cartel. In the single cartel framework, where at most one cartel may be formed, we define a market structure as the set of firms in the cartel (if any) and the set of independent firms (if any), whereas, in the multiple cartels framework, it is the collection of cartels (if any) and the set of independent firms (if any). The firms in a cartel and each independent firm choose their prices simultaneously with the objective of maximising the aggregate cartel profit and the individual profit, respectively. An *equilibrium* of a market structure is a profile of prices such that neither any cartel nor any independent firm obtains a higher aggregate/individual profit by unilaterally deviating to a different (set of) price(s). Following d'Aspremont, Jacquemin, Gabszewicz and Weymark (1983), a market structure is stable in the single cartel framework if a firm in a cartel does not find it more profitable to exit the cartel and operate independently (i.e. it is internally stable), and if a firm outside the cartel does not find it more profitable to join the cartel (i.e. it is externally stable). And, following Khan and Peeters (2024), a market structure is stable in the multiple cartels framework if, in addition to the above two conditions, a firm in a cartel does not find it more profitable to leave its current cartel and either join another cartel or form a new cartel with an independent firm, and if an independent firm does not find it more profitable to form another cartel with another independent firm. That is, a market structure is stable if no firm can initiate a profitable unilateral deviation from that market structure.

Clearly, the multiple cartels framework subsumes the single cartel framework, and the two additional conditions for stability in the multiple cartels framework are necessitated purely due to the possibility that multiple cartels may form. It also follows from the above that, in both frameworks, an analysis of stability requires a comparison of *equilibrium profits* before and after a unilaterally initiated change by a firm in the market structure. So, the first result we establish is that any market structure in the multiple cartels framework – and by implication, in the single cartel framework as well – has a unique equilibrium.

The stable market structures in the single cartel framework are determined by searching for k-firm cartels such that: *(i)* the equilibrium profit of a firm in the cartel is at least as much as the equilibrium profit it obtains on leaving the cartel and operating independently in the market structure that now has a cartel with $k-1$ firms, and *(ii)* the equilibrium profit of a firm operating independently outside the cartel is at least as much as the equilibrium profit it receives on joining the cartel so that the market structure is described by a $(k+1)$-firm cartel. A direct computation and comparison of the equilibrium profits reveals that the market structure with a three-firm cartel is the unique stable market structure.

The intuition of this result is as follows. There are two countervailing forces that are relevant in the determination of stable market structures. Firstly, when *any* subset of firms forms a cartel, then coordinated price setting in the cartel results in firms in the cartel setting higher prices as the cartel internalises the externality of price competition; and, since prices are strategic complements, the other firms also set higher prices in response. Compared to the situation where *all* firms compete amongst themselves, this results not only in higher equilibrium prices but also higher equilibrium profits across the board. This describes the incentive of a firm to be in a cartel. Secondly, even though a market structure with a cartel of *any* size leads to a higher profit than the market structure where *all* firms compete independently, it is also simultaneously true that an independent firm's equilibrium profit exceeds that of a firm in the cartel. This is because the independent firm free-rides on the cartel due to which its equilibrium price is lower, and its equilibrium profit is higher, than a firm in the cartel. This describes the incentive of a firm to not be part of the cartel. Hence, a firm may prefer to operate independently outside the cartel rather than to be part of the cartel, but, at the same time, being a part of some cartel is more profitable than the situation where all firms compete amongst themselves. Stability is obtained when these two opposing forces are in a state of balance. Intuitively, this implies that a market structure without cartels cannot be stable but, at the same time, the cartel cannot be too large. This balance is attained in the single cartel framework when there are three firms in the cartel.

In the multiple cartels framework, the direct approach of computing and comparing the relevant equilibrium profits to determine the stable market structures is infeasible owing to the sheer number of comparisons that have be conducted – the number of possible market structures increases exponentially in the number of firms. This leads us to develop novel arguments involving dynamic adjustment processes that causes a transition of the market structure between the two equilibria where the profits need to be compared. This enables the relevant equilibrium profit comparison on the basis of which we find that: *(i)* any market structure which has a cartel with more than three firms is not stable, *(ii)* any market structure with more than two independent firms is not stable, and a market structure with an independent firm is stable only if each of the other firms is in some three-firm cartel, and *(iii)* any market structure that satisfies the above two conditions is stable. Hence, if there are ten firms, then the market structure where all firms are in some two-firm cartel is stable as is the market structure where there are three three-firm cartels and an independent firm as is the market structure where there are two three-firm cartels and two two-firm cartels.

There are three primary implications of these results. Firstly, in context of the once held view that cartels are inherently unstable, we find that a stable market structure necessarily involves cartelisation. Secondly, in the the context of the literature's predominant assumption that at most one cartel may exist in the market, we find that whenever multiple cartels are feasible (i.e. there are at least four firms), there always exists a stable market structure involving multiple cartels; furthermore, whenever there are at least five firms, any stable market structure necessarily involves multiple cartels. Hence, the literature's predominant assumption is not innocuous, and is, in fact, with substantial loss of generality. Lastly, even though the firms are ex-ante identical in all aspects, in a stable market structure, some asymmetry amongst the firms may emerge. For instance, in the same market, one firm may be in a three-firm cartel while another firm may either be in a two-firm cartel or operate independently. This naturally leads to ex-post asymmetry in the market outcomes in the form of equilibrium price dispersion and equilibrium profit dispersion in spite of ex-ante symmetry amongst the firms.

The market structures that emerge as being stable in the multiple cartels framework is intimately linked in a very intuitive manner to the stable market structure in the single cartel framework. In the single cartel framework, neither the market structure where all firms are independent nor the market structure where there is a single two-firm cartel is stable. In the former case, one independent firm obtains a higher profit by forming a two-firm cartel with another independent firm; in the latter case, an independent firm (as well as the two-firm cartel) becomes more profitable on joining the two-firm cartel thus forming a three-firm cartel. In the multiple cartels framework, this translates to the following: whenever there is an independent firm, and there is either another independent firm or another two-firm cartel, then it is more profitable for all concerned to form a two or three-firm cartel (as the case may be). Furthermore, in the single cartel framework, a market structure where there is a cartel with more than three firms is not stable as it is more profitable for a firm in the cartel to exit the cartel and operate independently; a similar statement holds in the multiple cartels framework for exactly the same reason. Thus, the stable market structure in the single cartel framework, by imposing bounds on the size of a cartel and the number of independent firms in the multiple cartels framework, has a direct bearing on the stable market structures in the multiple cartels framework.

Interestingly, the key takeaway of our paper – that stable market structures comprise of multiple small cartels – complements and resonates with the results on the market structures that are stable under quantity competition. In the context of the differentiated markets that we consider in this paper, Khan and Peeters (2024) find that when firms compete in quantities (instead of prices as in the current paper), then the unique stable market structure is characterised by each firm being in some two-firm cartel (subject to the integer constraint). Thus, once again, market structures described by small yet multiple cartels emerge as being stable. The intuition behind this (under both forms of competition) can be traced back to reasons provided earlier. On the one hand, there are clear benefits to cartelisation that stem from internalisation of the externality of market competition. But, on the other hand, if a firm is part of a "large" cartel, then it has to restrict output or increase price to a larger extent – this discourages cartel participation and encourages free-riding. Multiple small cartels emerge as being optimal in this regard. Firms are able to reap the benefits of cartelisation, and, at the same time, since the cartels are small, the extent to which firms have either to restrict output or to increase price (compared to the situation where they are not in any cartel) is not as severe. This lends stability to market structures with multiple small cartels.

Finally, even though we find that "similar" market structures are stable under both price competition and quantity competition, a question that is particularly germane concerns the market prices and profits that firms are able to earn under these two types of competition. The relevance of this issue arises from the fact that, while this distinction is irrelevant when the market is monopolistic, it makes a material difference in oligopolistic markets. For instance, when firms compete in homogeneous markets without cartelisation, (Bertrand) price competition (under the standard assumptions) leads to more competitive prices and lower firm profits than (Cournot) quantity competition; Singh and Vives (1985) show that the same holds in differentiated markets. However, as we have discussed, all firms competing amongst themselves is not a stable market structure – this motivates us to compare the market outcomes corresponding to the stable market structures under both forms of competition. While the conventional wisdom is that price competition leads to more competitive market outcomes, this may now be mitigated by two factors. Firstly, the cartels under price competition may be larger than the ones under quantity competition (because three-firm cartels are stable under price competition), and larger-sized cartels internalise the externality of market competition to a greater extent. Secondly, the larger-sized cartels under price competition also imply that there are fewer competing units. Both these factors promote softer market competition, and thereby less competitive market outcomes, under price competition. Indeed, in line with this intuition, we find that prices and profits are higher under price competition provided the market is sufficiently differentiated or sufficiently concentrated.

Our paper's focus on stable market structures under static price competition in differentiated markets when one allows for the formation of multiple cartels is a marked departure from the existing literature. As mentioned earlier, d'Aspremont et al. (1983) is the seminal work that considers the equilibrium effect of a firm joining/leaving a cartel on stability of the cartel when only one cartel can be formed, and frames cartel stability in homogeneous markets in terms of the internal stability and external stability criteria. In differentiated markets, cartel stability has primarily been studied in the context of the supergame framework (Deneckere, 1983; Majerus, 1988; Ross, 1992) rather than the standard static framework we adopt in this paper. The issue of multiple cartels in a static framework has also received scant attention: the existing literature has focussed on multiple cartels in either a supergame framework (Eaton and Eswaran, 1998; Bos and Marini, 2022), or via a cooperative approach (Espinosa and Inarra, 2000), or in situations where firms must be a part of one of exactly two possible coalitions without the possibility of operating independently outside the two coalitions (Palsule-Desai, 2015; Lambertini et al., 2022). Lastly, as discussed earlier and as is also discussed in substantial detail later in the paper, Khan and Peeters (2024), which examines stable market structures under static quantity competition in differentiated markets, is the most closely related paper which complements both the research question as well as the results of this paper.

Finally, it is pertinent to clarify that the questions of cartel stability in the single cartel framework and multiple cartels framework differs substantially from the apparently similar questions of profitability of mergers (Deneckere and Davidson, 1985) and the analysis of merger/takeover waves (Fauli-Oller, 2000; Qiu and Zhou, 2007), respectively. This is because, in contrast to a cartel where each firm in the cartel retains its existence, the firms that merge cease to exist individually; as a result, the internal stability criterion is not relevant for mergers while, in cartel stability, it plays a crucial role by determining the incentive of a firm to free-ride on a cartel. This distinction between analysis of profitability of mergers and cartel stability remains valid even when, in a homogeneous market, the merged entity chooses the number of subsumed units that continue to operate in the post-merger situation (Kamien and Zhang, 1990), or in a differentiated market where the merged entity chooses the number of product variants to be offered (Lommerud and Sorgard, 1997).

The structure of the paper is as follows. We introduce the model in Section 2. We analyse and discuss the results of the single cartel framework and the multiple cartels framework in Section 3 and Section 4. We present a comparative discussion of stable market structures under price competition and quantity competition in Section 5, and conclude in Section 6. Finally, the formal proofs are presented in the Appendix.

2 Model

There are $n \geq 2$ firms who compete simultaneously in a differentiated market, a la Shubik (1980), with price as the strategic variable. The price of firm i, for any $i \in \{1, \ldots, n\}$, is denoted by p_i, and $(p_1, \ldots, p_n)$ denotes the profile of prices set by the n firms. We assume that there are no fixed costs of production, and that firms produce on demand at an identical marginal cost equal to c. The demand of firm $i \in \{1, \ldots, n\}$ at the profile of prices $(p_1, \ldots, p_n)$ is given by

$$q_i(p_1, \ldots, p_n) = v - p_i - \gamma(p_i - \tfrac{1}{n} \textstyle\sum_{j=1}^{n} p_j).$$

The demand function expresses that, in a differentiated market where firms compete in prices, the demand faced by a firm not only depends inversely on its own price but also on how its price compares to the price set by the other firms. Specifically, if one holds fixed the price set by a firm, the demand faced by it increases if its price is lower than the average market price, and decreases if its price is higher than the average market price. Hence, if a subset of the other firms increase their price, then the firm's demand increases. The responsiveness of a firm's demand to how its price compares with the average market price is captured by the parameter $\gamma \geq 0$. A smaller γ implies a greater degree of market differentiation for the reason that when γ is smaller, a firm's demand is relatively less responsive to the difference between a firm's price and the average market price – that is, with a smaller γ, a firm's demand is relatively more driven by its own price. We focus on the situation where the market is sufficiently differentiated in that $\gamma \in (0, \frac{2}{3}]$. This demand function of firm i can also be expressed in terms of what is perhaps a more familiar form of a firm's demand function in differentiated markets:

$$q_i(p_1, \ldots, p_n) = v - \beta\, p_i + \delta \textstyle\sum_{j \neq i} p_j$$

where $\beta = \frac{n+(n-1)\gamma}{n} > 0$ and $\delta = \frac{\gamma}{n} > 0$.

We define a *market structure* as a (possibly empty) set of $k^I \geq 0$ firms that operate independently, and a partitioning of the remaining firms, where each partition acts as a cartel.

Given that firms are symmetric with respect to the demand conditions and cost structures, we can restrict market structures to tuples $(k_1, \ldots, k_m, k^I)$ with $m \geq 0$, $k_\ell \geq 2$ for $\ell \in \{1, \ldots, m\}$, $k^I \geq 0$ and $\sum_{\ell=1}^{m} k_\ell + k^I = n$. Here m is an integer that refers to the number of cartels, k_ℓ is the size of (i.e. the number of firms in) cartel $\ell \in \{1, \ldots, m\}$, and, as mentioned above, k^I is the number of firms acting independently. In what follows, we will differentiate between the *single cartel framework*, where at most one cartel exists, and the *multiple cartels framework*, where more than one cartel may exist.

The profit of firm i at the profile of prices $(p_1, \ldots, p_n)$ is given by the profit function $\pi_i(p_1, \ldots, p_n) = (p_i - c)q_i(p_1, \ldots, p_n)$. An independent firm chooses its own price with the objective of maximising its own profit while a cartel chooses the prices of the firms in the cartel with the objective of maximising the aggregate profit of the firms in the cartel.

The *equilibrium* of a given market structure is a tuple of prices $(p_1^*, \ldots, p_n^*)$ such that neither an independent firm nor a cartel has a unilateral profitable deviation. Hence, in an equilibrium profile of prices, if one holds fixed the prices chosen by the other firms, an independent firm/cartel cannot obtain a higher profit by choosing a price that is different from its chosen price(s) in the tuple $(p_1^*, \ldots, p_n^*)$.

The objective of this paper is to analyse market structures that are *stable* and, as we will shortly see, this involves comparison of equilibrium outcomes across market structures. So, in the following theorem, we first establish that each and every market structure has a unique equilibrium.

Theorem 1. *Each market structure has a unique equilibrium. This equilibrium is symmetric in that firms in similar positions (including firms which belong to different cartels of equal size) set equal prices. Further, in this equilibrium, each firm enjoys positive profit (that is, each firm sets a price greater than the marginal cost, and receives positive demand).*

In addition to existence and uniqueness of equilibrium for each market structure, the theorem also states that the equilibrium is symmetric for firms in similar positions. That is, not only do all the firms within a particular cartel set the same price but, if there are multiple cartels of the same size, then the price set by each firm in each of these cartels of equal size is identical. It follows that all of these firms also obtain identical profits. Similarly, the independent firms share the same price and accrue the same level of profit. Furthermore, in this equilibrium, each firm obtains positive profit.

Finally, we remark that the fact that each market structure has a unique equilibrium does not automatically imply that each market structure is 'stable'. In the following sections, we will define stability of a market structure for two different frameworks. In the section that follows immediately after, we analyse stability under the assumption that has generally been maintained in the literature that at most one cartel may exist in the market, and we refer to this as the *single cartel framework*. However, since there is no a priori reason or justification as to why at most one cartel may exist in the market, this will be succeeded by an analysis of stability without such an artificial imposition, and we refer to this as the *multiple cartels framework*. Here, it is pertinent to mention that, in addition to the lack of an a priori reason for restricting attention to the single cartel framework, the relevance of the multiple cartels framework also comes from the fact that Khan and Peeters (2024) find that in differentiated markets where firms compete by choosing quantities, a market structure with cartelisation is stable only if there are multiple cartels. As we will see, the results we obtain here have a similar flavour, and after presenting the results, we will discuss and compare stable market structures in differentiated markets when firms compete in prices (as in the current paper) and when firms compete in quantities (as in Khan and Peeters 2024).

3 Stable market structures: Single cartel framework

In this section, we assume that at most one single cartel may form. Let this cartel, C, consist of k firms, with $2 \leq k \leq n$; the remaining $n-k$ firms act independently. Firms in the cartel C choose the prices $(p_i)_{i \in C}$ to maximise the cartel's aggregate profit $\sum_{i \in C} \pi_i = \sum_{i \in C}(p_i - c)q_i(p_1, \ldots, p_n)$ while each independent firm $j \notin C$, if it exists, chooses its price p_j to maximise its own profit $\pi_j = (p_j - c)q_j(p_1, \ldots, p_n)$. Solving the respective system of first-order conditions presented in the proof of Theorem 1, we find the equilibrium prices

$$p^C(k) = \frac{(2n+(2n-k)\gamma)nv+[(n+(n-k)\gamma)(2n+[n+k-1]\gamma)+(n+(n-1)\gamma)(n-k)\gamma]c}{(2n+[n+(n-k)-k]\gamma)(2n+[n+k-1]\gamma)}$$

and

$$p^I(k) = \frac{(2n+(2n-1)\gamma)nv+[(n+(n-1)\gamma)(2n+[n+(n-k)-k]\gamma)+(n+(n-k)\gamma)k\gamma]c}{(2n+[n+(n-k)-k]\gamma)(2n+[n+k-1]\gamma)}$$

for each firm in the k-firm cartel and each independent firm, respectively. These prices result in profits of

$$\pi^C(k) = \frac{(2n+(2n-1)\gamma)^2(n+(n-k)\gamma)n(v-c)^2}{[(2n+[n+(n-k)-k]\gamma)(2n+[n+k-1]\gamma)]^2}$$

and

$$\pi^I(k) = \frac{(2n+(2n-k)\gamma)^2(n+(n-1)\gamma)n(v-c)^2}{[(2n+[n+(n-k)-k]\gamma)(2n+[n+k-1]\gamma)]^2}$$

for each firm in the k-firm cartel and each independent firm, respectively. It is easily seen that $p^C(k) > p^I(k)$, which, due to the demand function being a decreasing function of price, leads to $q^C(k) < q^I(k)$. Moreover, we find $\pi^C(k) < \pi^I(k)$, i.e. in any single-cartel market structure, an independent firm enjoys a higher profit than a cartel firm. However, this by itself does not imply that a cartel cannot be *stable*.

In order to define stability succinctly, we use $k = 1$ to define the particular market structure where all firms act independently. We simply extend this to the expressions above to incorporate $k = 1$ so that $p^C(1) = p^I(1)$ and $\pi^C(1) = \pi^I(1)$ refer to the price and profit of an independent firm when all firms operate independently. With this in place, we use the stability notion proposed in d'Aspremont et al. (1983).

Definition 1. *A cartel with* $2 \leq k \leq n$ *is* internally stable *if* $\pi^C(k) \geq \pi^I(k-1)$. *A cartel with* $1 \leq k \leq n-1$ *is* externally stable *if* $\pi^I(k) \geq \pi^C(k+1)$. *A cartel* $1 \leq k \leq n$ *is stable if it is both internally and externally stable, where we assume* $k = 1$ *to be internally stable and* $k = n$ *to be externally stable.*

The definition states that, in order to be stable, a cartel must be both internally stable and externally stable. A cartel is internally stable if no firm belonging to the cartel finds it more profitable to leave the cartel and, instead, operate as an independent firm, while a cartel is externally stable if no independent firm finds it more profitable to join the cartel. We emphasise that when we examine profitability of a firm joining a cartel, or leaving a cartel that it is a part of, we compare the *equilibrium profit* before and after a firm joins/leaves a cartel. As mentioned above, the market structure where all firms are independent (i.e. $k = 1$) is assumed to satisfy internal stability trivially while the complete cartel (i.e. $k = n$) is assumed to satisfy external stability trivially.

We are now in a position to characterise stable market structures in the single cartel framework.

Theorem 2. *In any market with number of firms* n*, a stable market structure always exists, it is unique, and it is described by* $k = 2$ *when* $n = 2$*, and* $k = 3$ *when* $n \geq 3$*.*

According to this theorem, the only stable market structure is the complete two-firm cartel in case $n = 2$, the complete three-firm cartel in case $n = 3$, and the incomplete three-firm cartel in case $n \geq 4$. This implies three things in particular. Firstly, each firm operating independently can never be a stable arrangement – the stable market structure must involve cartelisation. Secondly, in a stable market structure, irrespective of the number of firms that are active on the market, no more than three firms can be part of a cartel. Thirdly, whenever the number of firms $n \geq 4$, we obtain equilibrium price dispersion and equilibrium profit dispersion in the stable market structure in spite of each firm's demand and cost conditions being identical – this is because independent firms set a lower price, but earn a higher profit, than the three firms in the cartel.

The intuition underlying this result is based on the fact that market structures that emerge as being stable strike a balance between two countervailing tendencies: the profitability of collusion on the one hand, and a firm's incentive to not participate in collusion itself but instead free-ride on the benefits of the collusive behaviour of the other firms on the other hand. We briefly elaborate on each of these in turn.

Firstly, if we begin from a situation where all firms operate independently, then there is an obvious benefit to collusive behaviour because price competition amongst the firms causes both the equilibrium price and the per-firm equilibrium profit to be lower than what they potentially can be if firms were to collude amongst themselves. In fact, this benefit accrues even in case of an incomplete cartel where a strict subset of the firms collude. That is, even though the independent firms are more profitable than a firm in the cartel, a firm in the cartel obtains a higher profit than what it would earn if all firms were independent; this holds irrespective of the size of the cartel. This betterment in profit is made possible by a cartel – be it a complete or an incomplete cartel – internalising the externalities of price competition amongst the firms in the cartel, and suitably increasing the price. Indeed, even in case of an incomplete cartel, the higher price set by the firms in the cartel is complemented by the firms that operate independently outside the cartel also increasing their price – though not to the extent that would be observed if these firms were also in the cartel – due to prices being strategic complements. This results in the external instability of both a market structure where all firms operate independently, and a market structure with an incomplete two-firm cartel. In the former case, two independent firms benefit by forming a two-firm cartel; in the latter case of an incomplete two-firm cartel, a third firm benefits by joining the cartel to create a three-firm cartel.

Secondly, and on the other hand, if we consider any complete or incomplete cartel, then a firm in the cartel may have the incentive of leaving the cartel because it can then free-ride on the firms which remain in the cartel. This generates the internal instability of any cartel with at least four firms. In any such cartel, if a firm exits the cartel and operates independently, then profit-maximising behaviour results in it reducing its price (compared to what its price would be when it is in the cartel) which then provides it with a higher profit.

The balance between these two opposing forces is struck when there are exactly three firms in the cartel. A firm in such a cartel does not find it more profitable to exit the cartel and, instead, operate independently while none of the independent firms find it more profitable to join the cartel. The firms that operate independently set a lower price than the firms in the cartel, and owing to the higher demand they face on account of the lower price, obtain a higher profit. This also results in equilibrium price dispersion in spite of them being identical.

In the next section, and in contrast to the current section, we analyse stable market structures without the artificial restriction that at most one cartel can exist in the market. We will see that, firstly, the assumption of a single cartel framework is not innocuous in that stable market structures involve multiple cartels, and, secondly, the stable market structures in the multiple cartels framework are related in an intuitive way to the market structure that is stable in the single cartel framework.

4 Multiple cartel framework

We begin by defining stable market structures in the multiple cartels framework in Subsection 4.1. Next, we analyse and present the results of the stability analysis in Subsection 4.2. Finally, we discuss the nature of the stable market structures by examining the prices set, and profits obtained, by the firms in the stable market structures in Subsection 4.3.

4.1 Stable market structures: Definition

The first step in the analysis of stable market structures, while allowing for the possibility of multiple cartels, involves appropriately defining when a market structure is stable. We use the definition of stability proposed in Khan and Peeters (2024), which is a natural extension of the stability notion of d'Aspremont et al. (1983) that was used in the previous section.

However, prior to stating the definition, we present the perhaps obvious clarification that a firm may join/form a cartel with other firms if and only if none of the involved firms face a reduction in its profit. This is because the formation of a cartel requires all the cartel members to cooperate while setting their price, and neither a firm nor an existing cartel can be coerced into cooperation. In the single cartel framework, there was no need to explicitly mention this because, firstly, if an independent firm finds it more profitable to join a cartel, then each cartel firm must find this more profitable as well. This is because the independent firm obtains a higher profit than a cartel firm (recall that when $k \geq 2$, $\pi^I(k) > \pi^C(k)$), and if it joins the cartel, then, due to symmetry, all the firms accrue an identical profit of $\pi^C(k+1)$. Hence, if the independent firm experiences an improvement in profit (i.e. $\pi^C(k+1) > \pi^I(k)$), then so do the other firms in the cartel (i.e. $\pi^C(k+1) > \pi^C(k)$ again). Secondly, beginning from the situation where all firms are independent, if an independent firm finds it more profitable to form a cartel with another independent firm, then, due to symmetry, so does the latter.

While these two properties are still relevant and hold true in the multiple cartels framework as well, there are other possibilities where the interest of all involved firms may not coincide. For instance, it may be more profitable for a firm in a cartel to exit the cartel and form another cartel with an independent firm, but the independent firm may experience a profit deterioration on doing so. Hence, it would decline to be part of the cartel, and the cartel will not be formed.

With this clarification, we now present the definition of a stable market structure in the multiple cartels framework.

Definition 2. *The market structure $(k_1, \ldots, k_m, k^I)$, that comprises of $m \geq 1$ cartels, with $k_\ell \geq 2$ firms in the ℓth cartel, and $k^I \geq 0$ independent firms, is* stable *if and only if:*

1. *A firm belonging to a cartel does not find it more profitable to leave the cartel to become an independent firm.*

2. *An independent firm does not find it more profitable to join an existing cartel.*

3. *A firm belonging to a cartel does not find it more profitable to leave the cartel to join another cartel, or form a new cartel with an independent firm.*

4. *An independent firm does not find it more profitable to form a new cartel with another independent firm.*

The market structure where all firms are independent is stable *if two independent firms do not find it more profitable to form a two-firm cartel.*

It is evident from a comparison of the definitions of stability in the two frameworks that the conditions for *stable cartelisation* in the single cartel framework are necessary for stability in the multiple cartels framework. This is because the first/second condition above is a generalisation of the internal/external stability condition of a cartel in the single cartel framework. The two additional conditions now become necessary because of the possibility of the formation of multiple cartels. In the particular case of the market structure where all firms are *independent*, the stability conditions are identical in the single cartel framework and the multiple cartels framework. We reiterate that the evaluation of profitability of joining/leaving/forming a cartel is based on a comparison of the *equilibrium profits* before and after the firm in question joins/leaves/forms a cartel.

4.2 Stable market structures: Characterisation

In the analysis of stability, in order for market structures consisting of more than one cartel to even be feasible, we assume $n \geq 4$. We proceed by a series of lemmas which culminate in the characterisation of the stable market structures.

Lemma 1. *In any stable market structure, there is at most one independent firm.*

The lemma, the proof of which is presented in the appendix, is intimately connected to Theorem 2, which establishes that, in the single cartel framework, the market structure where all firms operate independently cannot be stable – whenever such a market structure exists, two (independent) firms gain in profit by forming a cartel. In the multiple cartels framework, this translates to the following: whenever two independent firms exist, with other firms possibly conducting their market activity in multiple cartels, the two firms gain in profit by forming a two-firm cartel. Here, we underscore that, in the single cartel framework, such a statement is true only when all firms are independent but not more generally whenever two independent firms exists – this is because of the a priori assumption that formation of more than one cartel is not permitted in the single cartel framework.

In order to establish the lemma, the following needs to be proved. Consider any market structure where there exists at least two independent firms, and, since we are in the multiple cartels framework, let other firms be organised in any arbitrary manner (in the context of cartelisation). A comparison of the equilibrium profit of these two firms in the equilibrium of two different market structures – one where these two firms are independent, and the other where these two independent firms form a two-firm cartel – should reveal that the equilibrium profit of each independent firm is greater in the latter situation. Since the other firms can be organised in any arbitrary manner, obtaining such a result would show that a market structure with at least two independent firms can never be stable. The challenge lies in the fact that it is necessary to compare the profit in two different equilibria, after all firms and cartels have adjusted their price in response to the two independent firms forming a cartel. Thus, the nature of this exercise involves a 'non-local' comparison of profit. And, this comparison needs to be conducted for each and every possible market structure, the number of which increases exponentially in the number of firms.

The manner in which we exploit Theorem 2 to accomplish this is as follows. We use the original price competition game to define two modified price competition games – one which is played by any two independent firms while holding fixed the prices of the other firms, and the other which is played by the other firms while holding fixed the prices of the two afore-mentioned firms. (Formally, the prices of the firms that are held fixed are lumped together with the intercept v of the demand function, and then the concerned firms play the price-competition game only amongst themselves.) We start with the equilibrium profile of the original game when the two firms are independent. Now, from this point on, these games are played alternately beginning with the two independent firms. Clearly, by the definition of the equilibrium, the price of the firms in the equilibrium profile of prices of the original game is also the equilibrium profile of prices in each modified game.

But now, let the independent firms collude in their modified price-competition game. Since the prices of the other firms are fixed, this results in the two firms increasing their price, and by Theorem 2, the two firms must gain in profit. Next, the other firms respond to this cartelisation-induced price increase by playing their modified price-competition game. Since prices are strategic complements, the other firms also increase their price. This causes the two firms in question to face a higher demand at the same price thus improving their profit even more. Moreover, in the next round, when these two firms, who are now in a cartel, again play their modified price-competition game, their demand expands owing to the increase in prices set by the other firms. As a result, the two-firm cartel further increases its price, and experiences a further improvement in profit. This process continues, and in each step of this process, due to the reasons outlined above, the two firms in question observe an increment in their profit.

We show that the alternating play of the two modified price-competition games converges to the equilibrium of the original price-competition game when the two independent firms form a cartel, and the other firms maintain the arbitrary manner in which they organised their market activity. Thus, we have an adjustment process that results in a transition from the equilibrium where the two firms are independent to the equilibrium where these two firms collude. Along each step of this adjustment process, the two firms experience a gain in profit.

Hence, if one compares the equilibrium profits, it clearly must be that the two firms obtain a higher equilibrium profit when they collude.[1]

Summarily, this lemma reveals a clear connection between stability in the single cartel framework and the multiple cartels framework – it is in fact the incentive of two firms to form a two-firm cartel in the single cartel framework in the market structure where all firms are independent that directly leads to two independent firms also having the incentive to form a two-firm cartel in the multiple cartels framework.

Corollary 1. *A single three-firm cartel, which is the unique stable market structure in the single cartel framework when $n \geq 3$, is not stable in the multiple cartels framework when $n \geq 5$.*

In the context of Corollary 1, which rules out the unique stable market structure in the single cartel framework (i.e. a single three-firm cartel) from being stable in the multiple cartels framework whenever the number of firms $n \geq 5$, we now show that a single three-firm cartel is in fact stable when $n = 4$. Since this cartel satisfies the first two conditions of Definition 2 of a stable market structure by virtue of being stable in the single cartel framework, and because the last condition is trivially satisfied (on account of the existence of only one independent firm), we only need to verify whether condition 3 is satisfied, i.e. whether $\pi^I(3,1) \geq \pi^C(2,2)$ holds. The usual computations show that $\pi^I(3,1) = \frac{4(3\gamma+4)(5\gamma+8)^2(v-c)^2}{[9\gamma^2+64\gamma+64]^2}$ and $\pi^C(2,2) = \frac{2(\gamma+2)(v-c)^2}{[\gamma+4]^2}$, and that $\pi^I(3,1) \geq \pi^C(2,2)$ is indeed satisfied. So, a single three-firm cartel is stable when $n = 4$. Alternatively stated, a market structure with an independent firm is stable when $n = 4$, and we will steadily generalise this for $n \geq 4$ firms in Theorem 3 where we show that a market structure with an independent firm is stable when $n \geq 4$ but if and only if all the other firms are organised in three-firm cartels.

While Lemma 1 above restricts the number of independent firms to at most one in a stable market structure, Lemma 2 below restricts the size of the cartels in a stable market structure.

Lemma 2. *A market structure that includes a cartel with four or more firms is not stable.*

The intuition and the arguments behind Lemma 2 above, much like the previous Lemma 1, has its roots in the characterisation of the stable market structures in the single cartel framework (Theorem 2). In the single cartel framework, any cartel which has more than three firms is internally unstable, and this is also true when the market may contain multiple cartels. While

[1]We highlight that the strategic complementarity of the game is the key property which underlies this result. Specifically, given Theorem 2, the result of Lemma 1 holds for any game of strategic complements.

the internal stability of such cartels simply carries over, the arguments needed to prove this in the multiple cartels framework are more involved. In the single cartel framework, in order to establish internal instability, it was possible to obtain the result by comparing the equilibrium profit of a firm in a cartel with more than three firms with its equilibrium profit after it exited the cartel to operate independently. Here, the complexity stems not only from the fact that one has to compare the profit in two different equilibria, which makes the comparison a 'non-local' exercise, but the number of feasible market structures is exponentially increasing in the number of firms thus rendering a direct comparison of equilibrium profits infeasible.

As a result, we take recourse to a more sophisticated version of the type of the arguments used to prove Lemma 1. In order to appreciate the necessity of a more complex argumentation, it may be recalled that in the previous lemma, we were able to define an adjustment process that led from one equilibrium where two independent firms exist to another equilibrium where the two independent firms form a cartel, and that along each step of the adjustment process, the two firms gained in profit. The gain in profit was initially triggered by the two firms increasing their price due to collusion, and the gain in profit was sustained thereafter by each set of firms increasing their price alternately (due to prices being strategic complements), leading to a demand expansion effect on the complementary set of firms, thereby increasing their profit. In this case, this argument fails because we now wish to reason that when a firm leaves a cartel, it obtains a higher profit in the new equilibrium. The very act of leaving a cartel implies a reduction in prices. As we will argue, while the initial act of exiting the cartel and reducing the price is obviously profit improving for the exiting firm, price being strategic complements, the reduction in price leads to a sequence of prices reductions; this has a demand contracting effect which, in fact, reduces profit. So, while the exiting firm experiences an initial bump in profit, it is followed by a sequence of profit decrements, and it is nigh impossible to reason at this level of generality that the initial boost in profit dominates the sum of the subsequent reductions in profit.

We circumvent this complication by taking any cartel with at least four firms, and, after a firm in the cartel exits the cartel and starts operating independently, we compare the profitability of leaving the cartel in the single cartel framework with the profitability of leaving the cartel in the multiple cartels framework (where other cartels may also exist in the market). Using the type of adjustment process outlined in the discussion following Lemma 1, we prove that the changes in profit of the firm along the adjustment process – that leads from the equilibrium when the firm is in the cartel to the equilibrium when the firm operates independently – is more favourable in the multiple cartels framework than in the single cartel framework. Since exiting the cartel is more profitable in the single cartel framework, it follows that exiting the cartel is at least as profitable in the multiple cartels framework as well.[2]

Now, Lemma 1 and Lemma 2 taken together imply that, in a stable market structure, only two-firm cartels and three-firm cartels, and at most one independent firm, may exist. In the next lemma, we refine the set of stable market structures further, and state that, in a stable market structure, an independent firm and a two-firm cartel cannot exist simultaneously.

Lemma 3. *In any stable market structure that includes an independent firm, there is no two-firm cartel.*

The intuition behind this lemma can again be traced to the single cartel framework where we saw that a two-firm cartel is externally unstable – it is more profitable for an independent firm (and also the firms in the two-firm cartel) to join the cartel. Any market structure with at least one two-firm cartel and at least one independent firm is not stable for exactly the same reason, and to establish this, we rely on arguments similar to the one used to prove Lemma 1.[3]

These three lemmas imply that there are only four possible types of market structures that may potentially be stable in the multiple cartels framework: $(3, \ldots, 3)$, $(3, \ldots, 3, 1)$, $(2, \ldots, 2)$ and $(3, \ldots, 3, 2, \ldots, 2)$. Lemma 4 states that each of these market structures is indeed stable, provided, of course, that the number of firms $n \geq 4$ makes its formation feasible.

Lemma 4. *A market structure is stable if and only if (i) it only contains three-firm cartels and possibly one independent firm, (ii) it only contains two-firm cartels, or (iii) it only contains two-firm and three-firm cartels.*

In the proof of the lemma, we show that each and every feasible deviation from any one of the above market structures is not profitable, and a substantial part of the proof is accomplished by using the arguments and the results of the preceding lemmas.

For instance, take the $(3, \ldots, 3)$ market structure. We know from Lemma 3 that an independent firm and a two-firm cartel gain in profit by forming a three-firm cartel. For this reason, it is not a profitable deviation for a firm to leave the three-firm cartel and operate independently. The only other feasible deviation involves a firm joining another cartel to form

[2]At a more general level, the properties of the strategic situation which lie behind Lemma 2 is that the game is one of both gross substitutes (i.e. one's payoff is increasing in others' strategies) and strategic substitutes (i.e. one's marginal payoff is increasing in others' strategies), and satisfies the aggregative property as well (i.e. one's payoff can be expressed as a function of one's own strategy and the sum of others' strategies).

[3]We point out that, just as in the case of Lemma 1, this result holds for any game of strategic complements.

a four-firm cartel (leaving behind a two-firm cartel). If this is a profitable deviation for the firm, then, by Lemma 2, the firm can obtain a further improvement in profit by leaving the four-firm cartel and operating independently. But then, by Lemma 3, the firm can make yet another profit improvement by again joining the two-firm cartel (that it left behind on its initial exit from its three-firm cartel) thus causing a reversion to the initial $(3, \dots, 3)$ market structure. It is clearly not possible to experience a higher profit on reverting to the initial market structure – this impossibility contradicts the initial move being profitable. Thus, the market structure with all firms positioned in three-firm cartels is stable.

In the appendix, we show the unprofitability of each and every possible deviation in the other market structures. Here, we only highlight the particular deviation, where a firm in a three-firm cartel in a $(3, \dots, 3, 1)$ market structure exits its cartel with the intention of forming a two-firm cartel with the independent firm, to show the role of the assumption that a cartel can be formed only if none of the involved firms experience a profit deterioration. We show in the proof of the appendix that while this would be a profitable deviation for the firm in the cartel, it is not in the interest of the independent firm to form this cartel. This results in this two-firm cartel not being formed, and hence lends stability to the $(3, \dots, 3, 1)$ market structure against this particular deviation.

It is easily seen that for any $n \geq 4$, at least one of the above market structures is always possible, and this trivially implies existence of at least one stable market structure. We collect all of these results in the following theorem.

Theorem 3. *A stable market structure always exists. Any market structure which only contains (i) three-firm cartels and possibly one independent firm, (ii) two-firm cartels, or (iii) two-firm cartels and three-firm cartels, is stable. No other market structure is stable.*

Summarily, we find that stable market structures are characterised by the presence of multiple small cartels. The reason for this is that multiple small cartels permit the firms to obtain the benefit of being part of a cartel while making complete free-riding on the cartel unprofitable owing to the limited size of the cartels. Interestingly, in the context of the configuration of the multiple small cartels that emerge as being stable, not only is it the case that for a given n, more than one type of market structure can be stable (for instance, with $n = 6$, both market structures $(2, 2, 2)$ and $(3, 3)$ are stable) but, for a given n, there could be multiple ways in which a market structure with a combination of both two-firm cartels and three-firm cartels exists, and hence, is stable. For instance, when $n = 11$, both market structures $(3, 3, 3, 2)$ and $(3, 2, 2, 2, 2)$ are stable. We also underline that, as in the single cartel framework, equilibrium price dispersion may be observed in the stable market structures even though all the firms are identical. In the next subsection, we further discuss the nature of the stable market structures in terms of the prices set, and profits earned, by the firms.

4.3 Stable market structures: Market prices and firms' profit

Theorem 3 implies that for a given number of firms n, multiple stable market structures may exist, and we display this in Table 1. Since the market structure of the type $(3,\ldots,3,2,\ldots,2)$ can exist in multiple configurations whenever $n = 11$ and $n \geq 13$, it follows that there is a unique stable market structure only when $n = 5$. In this subsection, we discuss how the firms' prices and profits vary across the stable market structures. There are two comparisons that are relevant here: an intra-market comparison that relates to how prices and profits of different firms compare in the same market structure, and an inter-market comparison that relates to how prices and profits of firms compare across market structures.

n	stable market structures
4	$\{(3,\ldots,3,1),(2,\ldots,2)\}$
5 (mod 6)	$\{(3,\ldots,3,2,\ldots,2)\}$
6	$\{(3,\ldots,3),(2,\ldots,2)\}$
7 (mod 6)	$\{(3,\ldots,3,1),(3,\ldots,3,2,\ldots,2)\}$
8 (mod 6)	$\{(3,\ldots,3,2,\ldots,2),(2,\ldots,2)\}$
9 (mod 6)	$\{(3,\ldots,3),(3,\ldots,3,2,\ldots,2)\}$
10 (mod 6)	$\{(3,\ldots,3,1),(3,\ldots,3,2,\ldots,2),(2,\ldots,2)\}$
12 (mod 6)	$\{(3,\ldots,3),(3,\ldots,3,2,\ldots,2),(2,\ldots,2)\}$

Table 1: The stable market structures for different number of firms in the market. Note that the structure $(3,\ldots,3,2,\ldots,2)$ exists in multiple configurations if $n = 11$ or $n \geq 13$.

An intra-market comparison is pertinent in market structures of the type $(3,\ldots,3,2,\ldots,2)$ or $(3,\ldots,3,1)$ where cartels of unequal sizes exist. It is only in this case that we have equilibrium price dispersion and unequal distribution of profits amongst the firms. Clearly, firms in a three-firm cartel internalise the externality of price competition to a larger extent (compared to firms in a two-firm cartel or an independent firm), and this leads them to set higher prices. The firms in a two-firm cartel, or an independent firm, do not internalise the externality of price competition to the same degree, and free-ride to some extent on the three-firm cartels, and this results in these firms setting a lower price and obtaining a higher profit. So, for instance, in a $(3,2,2,2,2)$ market structure, a firm in the three-firm cartel sets a higher price but obtains a lower profit than a firm in a two-firm cartel.

An inter-market comparison is germane when multiple stable market structures exist (for a given number firms in the market). In this case, the differences in prices and profits across the different stable market structures is driven by differences in the size distribution of cartels, which, in turn, leads to a difference in the number of competing cartels. The intuition behind this is laid bare when one compares a market structure with at least one three-firm cartel with another market structure which comprises only of two-firm cartels. A three-firm cartel internalises the externality of price competition to a greater extent than a two-firm cartel, and this guides the former to set a relatively higher price. This relatively higher price set by a three-firm cartel also results in the other firms in that market structure to set a relatively higher price (compared to the other cartels/firms in the market structure with only two-firm cartels) due to prices being strategic complements. This intuition extends to the case where one has a market structure with a higher number of three-firm cartels, and is further accentuated by the fact that a market structure with greater number of three-firm cartels has fewer competing cartels/firms – this softens competition even more and leads to higher prices and higher profits across the board.

The implication of this is that the equilibrium price and equilibrium profit of a firm in a market structure which has only three-firm cartels is higher than that of a firm in a three-firm cartel in a market structure which has fewer (or zero) three-firm cartels. Similarly, for a firm that is not in a three-firm cartel, its equilibrium price and profit are higher when it is in a market structure with a higher number of three-firm cartels. So, if we compare, for $n = 11$, the $(3, 2, 2, 2, 2)$ market structure with the $(3, 3, 3, 2)$ market structure, then the equilibrium price and profit of a firm in a three-firm cartel is higher in the $(3, 3, 3, 2)$ market structure. Similarly, for a firm in a two-firm cartel, the equilibrium price and profit is higher in the $(3, 3, 3, 2)$ market structure. And, if we compare a $(3, \ldots, 3)$ market structure with a $(2, \ldots, 2)$ market structure, then each firm in the former market structure sets a higher price and earns a higher profit.

Finally, a direct comparison of prices and profits of firms in a $(3, \ldots, 3, 2, \ldots, 2)$ market structure and a $(3, \ldots, 3, 1)$ market structure reveals that a firm in a three-firm cartel sets a higher price and accrues a higher profit in the latter market structure. The intuition is that the free-riding by the solitary independent firm is comparatively less than the free-riding by the two-firm cartels (of which there are at least two if both these market structures are to be feasible simultaneously). We can combine the results of the inter-market comparisons above with this particular inter-market comparison to conclude that whenever the $(3, \ldots, 3, 1)$ and $(2, \ldots, 2)$ market structures are both feasible – this also implies that some $(3, \ldots, 3, 2, \ldots, 2)$

market structure is also feasible (except when $n = 4$) – then the price set, and profit received, by the firms in a three-firm cartel in the $(3, \ldots, 3, 1)$ market structure is higher than that of any firm in the $(2, \ldots, 2)$ market structure. This is because one can go from the $(3, \ldots, 3, 1)$ market structure to the $(2, \ldots, 2)$ market structure via a series of market structures of the type $(3, \ldots, 3, 2, \ldots, 2)$, with the number of three-firm cartels decreasing and the number of two-firm cartels increasing along this sequence; we have already reasoned that the price and profit of *any* firm is lower when there are fewer three-firm cartels in the market structure – so, the conclusion follows.[4]

5 Stable market structures: Price vs. quantity competition

One issue in oligopolistic markets that has historically attracted substantial attention concerns the impact of the firms' strategic variable on the market outcomes. For instance, in homogeneous oligopolistic markets, one has the well-known dichotomy between the situation where firms set prices and produce on-demand (i.e. Bertrand competition), which results in firms, under some conditions, obtaining zero profit even in an oligopolistic market on the one hand, and the situation where firms choose quantities and accept the market-clearing price (i.e. Cournot competition), which results in firms obtaining positive profits under the same market conditions on the other hand. Furthermore, Singh and Vives (1984) show that the feature that price competition leads to more competitive outcomes (i.e. lower market prices and lower profits) than quantity competition holds not only in the case of homogeneous markets but also in differentiated duopolies where the firms' products are substitutes.

In a similar vein, we will compare and discuss the market outcomes in the stable market structures when firms simultaneously set prices and produce on-demand (as in the current paper), and when firms simultaneously set quantities and accept the market-clearing price (as in Khan and Peeters 2024). This exercise is also motivated by the fact that the comparison mentioned in the preceding paragraph pertains to the case where all firms compete with each other – but, as we have already seen in the case of price competition, and as we will soon discuss in the case of quantity competition, all firms competing with each other is not a stable market structure. Hence, it is arguably more instructive to conduct this comparison at the stable market structures under each form of competition.

[4]For instance, for $n = 16$ we can go from the $(3, 3, 3, 3, 3, 1)$ market structure to the $(2, 2, 2, 2, 2, 2, 2, 2)$ market structure via the following series of transitions: $(3, 3, 3, 3, 3, 1) \to (3, 3, 3, 3, 2, 2) \to (3, 3, 2, 2, 2, 2, 2) \to (2, 2, 2, 2, 2, 2, 2, 2)$. As mentioned, along this sequence, the number of three-firm cartels decreases while the number of two-firm cartels increases.

In this context, we show in the appendix that the demand function $q_i = v - \frac{n+(n-1)\gamma}{n} p_i + \frac{\gamma}{n} \sum_{j \neq i} p_j$ of firm i that is relevant for price competition can be inverted to obtain the inverse demand function $p_i = v - \frac{n+\gamma}{(1+\gamma)n} q_i - \frac{\gamma}{(1+\gamma)n} \sum_{j \neq i} q_j$ of firm i that is relevant for quantity competition. In the context of this inverse demand function, where we continue to assume that $\gamma \leq \frac{2}{3}$, Khan and Peeters (2024) show that the unique stable market structure involves multiple small cartels as well, and is characterised by at most one firm operating independently with each of the other firms belonging to some two-firm cartel. So, if n is even, then each firm belongs to some two-firm cartel – in this case, because of symmetry, the firms produce identical quantities and earn identical profits. And, if n is odd, then one firm operates independently, and each of the other firms are in some two-firm cartel; the independent firm free-rides on the two-firm cartels, and produces a higher quantity and receives a higher profit. Thus, multiple small cartels emerge as being stable under quantity competition for exactly the same reason as under price competition – this market structure encourages participation in a cartel while discouraging complete free-riding. The quantities produced by a firm and its market-clearing price in the stable market structure are provided in the appendix.

We have already presented the stable market structures under price competition in Table 1. It follows that when firms compete in quantities, the cartels are never larger than the cartels formed under price competition; as a consequence, the number of competing cartels are never greater. One may now ask how the market prices and firms' profits in the stable market structures compare between price competition and quantity competition. On the one hand, the conventional wisdom is that price competition tends to lead to more competitive market prices and lower profits for the firms. But, on the other hand, this may now be counteracted by the fact that when firms compete in prices, cartel sizes may be larger implying that there is a comparatively greater internalisation of the externalities of competition, and that there are fewer number of competing cartels.

The manner in which these two competing considerations play out can be seen most clearly if we compare the equilibrium prices and profits in the $(2, \ldots, 2)$ market structure under both forms of competition on the one hand, and the equilibrium prices and profits in the $(2, \ldots, 2)$ market structure under quantity competition with the $(3, \ldots, 3)$ market structure under price competition (which requires n to be a multiple of 6) on the other hand. In the former comparison, both the market-clearing price as well as a firm's profit under quantity competition is higher than the price set, and profit obtained, by a firm under price competition. The reason is that, since the market structures are identical under both price competition and quantity competition, the nature of price competition leads to more competitive prices and lower profits. In the latter comparison, the market price as well as a firm's profit is higher under price competition whenever either there are few firms in the market (i.e. $n < 12$) or the market is sufficiently differentiated (i.e. $n \geq 12$ and $\gamma < \frac{n}{2n-6}$). The intuition is that when there are few firms, then, in case of price competition, the relatively greater internalisation of the externality of market competition by the larger-sized cartels, as well as the smaller number of competing cartels, elevates both market prices and profits, and this dominates the tendency of price competition to lead to more competitive market prices and lower profits. And, when there are a larger number of firms, then, in case of price competition, a sufficiently differentiated market acts as a bulwark against the competition intensifying effect of a larger number of competing cartels, the reason being that market competition is relatively softer when the market is more differentiated.

This intuition extends to cases where we compare other stable market structures under price competition with the corresponding stable market structure under quantity competition. Firstly, if we compare a similarly placed firm in the two different forms of market competition (i.e. if we compare an independent firm under price competition to an independent firm under quantity competition, or a firm in a two-firm cartel under price competition to a firm in a two-firm cartel under quantity competition), then its price is always higher under quantity competition owing to the first factor mentioned above. Interestingly, the firm's profit under price competition exceeds its counterpart's profit under quantity competition *only if* there is a larger cartel in the market structure under price competition, and, in addition, the market is sufficiently differentiated.[5] This is because the presence of the larger cartel under price competition allows the concerned cartel (which is smaller in size) to free-ride on the former, and this element of free-riding is *necessary* for the concerned cartel to obtain a higher profit than its counterpart under quantity competition. Secondly, if we compare a firm in a larger cartel (i.e. a three-firm cartel) under price competition to a firm in a smaller cartel (i.e. a two-firm cartel) under quantity competition, then *both* the price as well as the profit of a firm in the larger cartel under price competition is higher than the price and the profit of a firm in the smaller cartel under quantity competition whenever either there are few firms in the market or the market is sufficiently differentiated. Finally, due to the free-riding behaviour

[5] Hence, if we compare a firm's profit in the $(2, \ldots, 2)$ market structure in both forms of market competition, then the profit can never be higher under price competition. However, if we compare a firm in a two-firm cartel in a $(3, \ldots, 3, 2, \ldots, 2)$ market structure under price competition to a firm in a two-firm cartel in a $(2, \ldots, 2)$ market structure under quantity competition, then profit is higher under price competition only if the market is sufficiently differentiated – this is made possible by the presence of the three-firm cartels.

of the independent firm in either form of market competition, the (market-clearing) price of an independent firm in any form of market competition is lower than the (market-clearing) price of any firm in any cartel in any form of market competition.

These three principles can now be used to compare the price and the profit of any firm in any stable market structure under price competition with the price and the profit of any firm in the corresponding stable market structure under quantity competition. Hence, for example, the market price set and the profit received by a firm in a three-firm cartel in a $(3, \ldots, 3, 2, \ldots, 2)$ market structure under price competition is higher than the market-clearing price and profit of a firm in a two-firm cartel under quantity competition when either there are few firms in the market, or the market is sufficiently differentiated. And, the market-clearing price of any firm in a two-firm cartel under quantity competition is higher than the price set by any firm in a two-firm cartel under price competition, but the latter's profit is higher when the market is sufficiently differentiated.

Finally, it is also easy to see another general principle (that is relevant to the above example). The market differentiation threshold that is required for a firm in a larger cartel under price competition to obtain a higher profit than *any* firm in the corresponding market structure under quantity competition is stricter than the market differentiation threshold that is required for a firm in a smaller cartel (including the case of an independent firm) to obtain a higher profit than the same firm in the corresponding market structure under quantity competition. This is simply because, in the stable market structures under price competition, a firm in a smaller cartel obtains a higher profit than a firm in a larger cartel (recall Subsection 4.3). So, in the above example, whenever a firm in a three-firm cartel obtains a higher profit than a firm under quantity competition, then so does a firm in a two-firm cartel; however, the converse does not hold. In Figure 1, for $n = 30$ and $n = 60$, we illustrate the threshold for the market differentiation parameter γ, as a function of the fraction of firms that are in a three-firm cartel, below which the firms in a two-firm cartel (dashed line) and the firms in a three-firm cartel (dotted line) in the $(3, \ldots, 3, 2, \ldots, 2)$ stable market structure under price competition obtain a larger profit compared to the firms in the two-firm cartels in the $(2, \ldots, 2)$ stable market structure under quantity competition. The dark curves show these for $n = 30$ and the light curves for $n = 60$. In order to correctly interpret the figure, recall that the degree of market differentiation is decreasing in γ. We see that: *(i)* the dotted line is always below the dashed line implying that the constraint for firms in three-firm cartels to earn higher profits than the firms under quantity competition is stricter, *(ii)* as the fraction of firms in three-firm cartels increases, there is a relaxation in the constraint on how differentiated a market has to be in order for the firms to earn higher profits under price competition than under quantity competition, and *(iii)* for a given degree of differentiation, more three-firm cartels are needed for the firms in the three-firm cartel to earn larger profit than under quantity competition compared to the number needed for this to be the case for firms in a two-firm cartel.

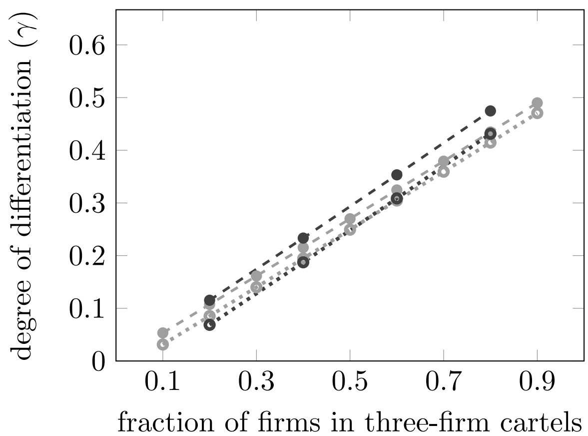

Figure 1: Firms in two-firm cartels in the $(3, \ldots, 3, 2, \ldots, 2)$ market structure under price competition earn a larger profit compared to the firms in the stable market structure under quantity competition for values of γ below the dashed curve; the dotted curve shows this for firms in three-firm cartels. The dark curves show this for $n = 30$ and the light curves for $n = 60$.

6 Conclusion

In this paper, in the context of cartel formation by firms, we analyse the stable market structures that emerge in differentiated markets when firms simultaneously set prices and produce on-demand. In contrast to the existing literature on cartelisation, which by and large assumes a priori that at most one cartel may form, we take the view that it is quite reasonable that more than one cartel may exist in the market. In fact, we find that stable market structures necessarily involve multiple small cartels, and apart from one knife-edge case, only involve these multiple small cartels. Interestingly, Khan and Peeters (2014) also find that market structures with multiple small cartels are stable under quantity competition. The intuition for stability of the market structures with multiple small cartels is that it provides the firms with profit-improving benefit that stems from being part of a cartel and, at the same time, owing to the small size of the cartels, deters firms from free-riding on other cartels by not participating in a cartel itself. The implication is that competition policy in general and cartel detection in particular should take cognizance of the possibility of the market being fragmented into multiple small cartels. If active efforts are not directed toward this possibility as well, then this form of cartelisation may not only remain undetected but may persist owing to this being a stable arrangement.

A Proofs

Proof of Theorem 1

Suppose there are $m \geq 0$ cartels and $k^I \geq 0$ independent firms. Define C_ℓ as the set of firms in cartel $\ell \in \{1, \ldots, m\}$, with k_ℓ being the number of firms in this set, and C^I as the set of independent firms, with k^I being the number of firms in this set. We will re-write the demand $q_i(p_1, \ldots, p_n) = v - p_i - \gamma(p_i - \frac{1}{n}\sum_{j=1}^n p_j)$ of firm i in its more computation friendly form $q_i(p_1, \ldots, p_n) = v - \frac{n+(n-1)\gamma}{n}p_i + \frac{\gamma}{n}\sum_{j=1, j\neq i}^n p_j$.

The cartel C_ℓ chooses the price of each firm in the cartel in order to maximise its aggregate profit $\sum_{j \in C_\ell} \pi_j = \sum_{j \in C_\ell}(p_j - c)[v - \frac{n+(n-1)\gamma}{n}p_j + \frac{\gamma}{n}\sum_{h \in C_\ell \setminus \{j\}} p_h + \frac{\gamma}{n}\sum_{h \notin C_\ell} p_h]$. The first-order condition with respect to the price p_i of a firm $i \in C_\ell$ is $v - 2\frac{n+(n-1)\gamma}{n}p_i + 2\frac{\gamma}{n}\sum_{j \in C_\ell \setminus \{i\}} p_j + \frac{\gamma}{n}\sum_{h \notin C_\ell} p_h + (\frac{n+(n-1)\gamma}{n})c - (k_\ell - 1)\frac{\gamma}{n}c = 0$. Similarly, the first-order condition with respect to the price $p_{i'}$ of firm $i' \in C_\ell$, with $i' \neq i$, is $v - 2\frac{n+(n-1)\gamma}{n}p_{i'} + 2\frac{\gamma}{n}\sum_{j \in C_\ell \setminus \{i'\}} p_j + \frac{\gamma}{n}\sum_{h \notin C_\ell} p_h + (\frac{n+(n-1)\gamma}{n})c - (k_\ell - 1)\frac{\gamma}{n}c = 0$. Subtracting the two first-order conditions, we obtain that $2(1+\gamma)(p_i - p_{i'}) = 0$, and this implies $p_i = p_{i'}$. Hence, firms in the same cartel set identical prices – so, we denote the price of each firm in cartel ℓ by p_ℓ^C.

Firm $i \in C^I$, in case it exists, chooses its own price p_i in order to maximise its own profit $\pi_i = (p_i - c)[v - \frac{n+(n-1)\gamma}{n}p_i + \frac{\gamma}{n}\sum_{h \notin C^I} p_h + \frac{\gamma}{n}\sum_{j \in C^I \setminus \{i\}} p_j]$. The first-order condition with respect to p_i is $v - 2\frac{n+(n-1)\gamma}{n}p_i + \frac{\gamma}{n}\sum_{h \notin C^I} p_h + \frac{\gamma}{n}\sum_{j \in C^I \setminus \{i\}} p_j + (\frac{n+(n-1)\gamma}{n})c = 0$. Similarly, for firm $i' \in C^I$, with $i' \neq i$, if it exists, we have that the first-order condition with respect to $p_{i'}$ is $v - 2\frac{n+(n-1)\gamma}{n}p_{i'} + \frac{\gamma}{n}\sum_{h \notin C^I} p_h + \frac{\gamma}{n}\sum_{j \in C^I \setminus \{i'\}} p_j + (\frac{n+(n-1)\gamma}{n})c = 0$. Subtracting the two first-order conditions, we find $[2(1+\gamma) - \frac{\gamma}{n}](p_i - p_{i'}) = 0$, which implies that $p_i = p_{i'}$. Hence, all independent firms (in case there are multiple of them) set identical prices, and we denote this price by p^I.

Using the symmetry obtained, the first-order condition for firms in cartel $\ell \in \{1, \ldots, m\}$ is $v - 2\frac{n+(n-1)\gamma}{n}p_\ell^C + 2\frac{\gamma}{n}(k_\ell - 1)p_\ell^C + \frac{\gamma}{n}\sum_{\ell' \neq \ell} k_{\ell'} p_{\ell'}^C + \frac{\gamma}{n}k^I p^I + (\frac{n+(n-1)\gamma}{n})c - (k_\ell - 1)\frac{\gamma}{n}c = 0$, and that of independent firms (in case they exist) is $v - 2\frac{n+(n-1)\gamma}{n}p^I + \frac{\gamma}{n}\sum_\ell k_\ell p_\ell^C + \frac{\gamma}{n}(k^I - 1)p^I + (\frac{n+(n-1)\gamma}{n})c = 0$. Defining $x_\ell^C \equiv \frac{\gamma}{n}k_\ell p_\ell^C$ for $\ell = 1, \ldots, m$, and $x^I \equiv \frac{\gamma}{n}k^I p^I$, we can write these first-order conditions in the form $\left[\frac{2(1+\gamma)n}{\gamma k_\ell} - 2\right]x_\ell^C - \sum_{\ell' \neq \ell} x_{\ell'}^C - x^I = v + \frac{n+(n-k_\ell)\gamma}{n}c$ for $\ell = 1, \ldots, m$ and $\left[\frac{2(1+\gamma)n}{\gamma k^I} - \frac{k^I+1}{k^I}\right]x^I - \sum_\ell x_\ell^C = v + \frac{n+(n-1)\gamma}{n}c$. This system of equations can be written in matrix-form as

$$\begin{pmatrix} \frac{2(1+\gamma)n}{\gamma k_1}-2 & -1 & \cdots & \cdots & -1 \\ -1 & \frac{2(1+\gamma)n}{\gamma k_2}-2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \frac{2(1+\gamma)n}{\gamma k_m}-2 & -1 \\ -1 & \cdots & \cdots & -1 & \frac{2(1+\gamma)n}{\gamma k^I}-\frac{k^I+1}{k^I} \end{pmatrix} \begin{pmatrix} x_1^C \\ x_2^C \\ \vdots \\ x_m^C \\ x^I \end{pmatrix} = \begin{pmatrix} v+\frac{n+(n-k_1)\gamma}{n}c \\ v+\frac{n+(n-k_2)\gamma}{n}c \\ \vdots \\ v+\frac{n+(n-k_m)\gamma}{n}c \\ v+\frac{n+(n-1)\gamma}{n}c \end{pmatrix}$$

In case there is no independent firm ($k^I = 0$), the last row and column in the matrix and the last rows of the two vectors are absent. We proceed by assuming $k^I > 0$ – the proof for the case $k^I = 0$ runs analogously.

Let us denote the non-homogeneous system of $m+1$ equations with $m+1$ unknowns as $Ax = v$. In order to prove that the system admits a unique solution, we will simply show that $\det(A) \neq 0$. Since $\det(A) = (-1)^{m+1}\det(-A)$, it suffices to show that $\det(-A) \neq 0$, the advantage of $-A$ being that all non-diagonal elements of $-A$ equal $+1$.

The matrix $-A$ can be written as $-A = D + ee^\top$, where e is the $(m+1)$-dimensional column vector with ones and D is a diagonal matrix with elements $d_{j,j} = -a_{j,j} - 1$ for $j = 1, \ldots, m+1$. We have that $d_{j,j} = 1 - \frac{2(1+\gamma)n}{\gamma k_j} < -1$ for $j = 1, \ldots, m$ and $d_{j,j} = \frac{1}{k^I} - \frac{2(1+\gamma)n}{\gamma k^I} < 0$, so that $\det(D) = \prod_{j=1}^{m+1} d_{j,j} \neq 0$, thus implying that D is invertible. By the matrix determinant lemma, it follows that $\det(-A) = \det(D + ee^\top) = (1 + e^\top D^{-1} e)\det(D)$. So, to prove that $\det(-A) \neq 0$, we need to show that $e^\top D^{-1} e = \sum_{j=1}^{m+1} \frac{1}{d_{j,j}} \neq -1$ or, equivalently, that $\sum_{j=1}^{m+1} -\frac{1}{d_{j,j}} \neq 1$.

For a cartel $\ell \in \{1, \ldots, m\}$, we have $-\frac{1}{d_{\ell,\ell}} = \frac{\gamma k_\ell}{2(1+\gamma)n - \gamma k_\ell}$. It is easily verified that, for any $1 < K \leq n$ and $1 \leq k \leq K-1$, the inequality $\frac{\gamma k}{2(1+\gamma)n-\gamma k} + \frac{\gamma(K-k)}{2(1+\gamma)n-\gamma(K-k)} \leq \frac{\gamma K}{2(1+\gamma)n-\gamma K}$ holds. This implies that $\sum_{\ell=1}^{m} -\frac{1}{d_{\ell,\ell}} \leq \frac{\gamma(n-k^I)}{2(1+\gamma)n-\gamma(n-k^I)}$, so that $\sum_{j=1}^{m+1} -\frac{1}{d_{j,j}} \leq \frac{\gamma(n-k^I)}{2(1+\gamma)n-\gamma(n-k^I)} + \frac{\gamma k^I}{2(1+\gamma)n-\gamma}$. The right-hand side of the expression can be shown to be strictly less than 1 (and this also applies in case $k^I = 0$). This leads to the overall conclusion that the matrix A is invertible, and the system of equations admits a unique solution $(x_1^C, \ldots, x_m^C, x^I)$ from which we obtain the unique equilibrium profile of prices $(p_1^C, \ldots, p_m^C, p^I)$. We also note that since each first order condition is linear and decreasing in price, the second-order conditions are obviously satisfied.

Finally, to complete the argument that $(p_1^C, \ldots, p_m^C, p^I)$ truly admits an equilibrium of the type described in the statement of the theorem, we need to prove that each of these prices, and the resulting profit of each firm, is positive. To this end, for each cartel $j = 1, \ldots, m$, we define $p_j = p_j^C$, and, in context of the set of independent firms and the price set by them, we define $p_{m+1} = p^I$, $C_{m+1} = C^I$, and $k_{m+1} = k^I$. Take $i \in \{1, \ldots, m+1\}$ for which $p_i = \min_{j=1,\ldots,m+1} p_j$. This implies that $p_i \leq \frac{1}{n} \sum_{j=1}^{m+1} k_j p_j$. Firstly, if $p_i < c$, then, using $v > c$ and $p_i \leq \frac{1}{n} \sum_{j=1}^{m+1} k_j p_j$, we obtain $q_i = v - p_i - \gamma(p_i - \frac{1}{n} \sum_{j=1}^{m+1} k_j p_j) > 0$, thereby implying $\pi_i < 0$. If p_i represents the price of a cartel (similarly, the independent firms), then the cartel (similarly, an independent firm) can do better by unilaterally deviating to $p_i = c$, because then $\pi_i = 0$. Therefore, in equilibrium, it must be $p_i \geq c$. Secondly, if $p_i = c$, then $\pi_i = 0$. But, since $q_i \geq v - c > 0$ (because $p_i \leq \frac{1}{n} \sum_{j=1}^{m+1} k_j p_j$), the cartel or an independent firm – depending on whose price p_i represents – can make a positive profit by a unilateral marginal increase in its price. Therefore, in equilibrium, $p_i > c$. Since $p_i = \min_{j=1,\ldots,m+1} p_j$, we obtain $p_j > c$ for all $j = 1, \ldots, m+1$.

Next, for the equilibrium profit of each firm to be positive, we need the demand of each firm to be positive at these (positive) prices. We highlight that this needs to proved explicitly as the non-negativity constraints that the demands cannot be negative were not included in the profit-maximisation problem of the cartels/independent firms. Since the demand is lowest for the firms with the highest price, take $i \in \{1, \ldots, m+1\}$ for which $p_i = \max_{j=1,\ldots,m+1} p_j$ thus implying that $q_i = \min_{j=1,\ldots,m+1} q_j$. First, if $q_i < 0$, so that $\pi_i < 0$, then the cartel or an independent firm – depending on whose price p_i represents – can do better by unilaterally deviating to $p_i = c$, because then $\pi = 0$. Therefore, in equilibrium, it must be that $q_i \geq 0$. Second, if $q_i = 0$, then $\pi_i = 0$ as well. In this case, either the concerned cartel or an independent firm can increase demand, and hence profit, by a unilateral marginal decrease in its price so that now $q_i > 0$, and this leads to its profit increasing to $\pi_i > 0$. Hence, in equilibrium, it must be that $q_i > 0$. As $q_i = \min_{j=1,\ldots,m+1} q_j$, it follows that $q_i > 0$ for all $j = 1, \ldots, m+1$. Since the price and demand faced by each firm is positive, we conclude that each firm enjoys a positive profit at the equilibrium price profile $(p_1^C, \ldots, p_m^C, p^I)$.

Proof of Theorem 2

Firstly, using the profit expressions presented in Section 2, it can be verified that $\pi^C(2) > \pi^I(1)$. This implies that the situation where each firm operates independently (i.e. the $k = 1$ situation) is not externally stable, and that a two-firm cartel is internally stable. Since any complete cartel satisfies external stability, it follows that a two-firm cartel is stable if $n = 2$. Secondly, it can also be verified that, when $n \geq 3$, then $\pi^C(3) > \pi^I(2)$. This implies that a two-firm cartel is not externally stable if $n \geq 3$, and that a three-firm cartel is internally stable; using the same reasoning, a three-firm cartel is stable if $n = 3$. Finally, it can be verified that, in the differentiated markets we consider, $\pi^C(k) < \pi^I(k-1)$ holds for $k \geq 4$. This implies that any cartel with $4 \leq k \leq n$ is not internally stable, and any cartel with $3 \leq k \leq n$ is externally stable. For this reason, a three-firm cartel is stable for all $n \geq 4$ as well. Existence and uniqueness follow trivially.

Proof of Lemma 1

Assume a market structure $(k_1, \ldots, k_m, k^I)$ with $k^I \geq 2$. We denote the price competition game in this (original) market structure as Γ. Let J be the set of any two firms in C^I, and let $-J$ be the collection of all other firms. Let (p^*_J, p^*_{-J}) denote the equilibrium in the game Γ. Due to symmetry, the two firms in J experience the same profit, and let this profit at these prices (p^*_J, p^*_{-J}) be denoted by π^*_J.

We consider the alternative price game, $\widehat{\Gamma}$, where the two firms in J form a cartel. Let $(\widehat{p}^*_J, \widehat{p}^*_{-J})$ denote the equilibrium in the game $\widehat{\Gamma}$, and $\widehat{\pi}^*_J$ denote the profit that each of the two firms in J experience at these prices. Since the price chosen by a firm is larger when it is in a cartel, we have that $\widehat{p}^*_J > p^*_J$. Next, since prices are strategic complements, we also have that $\widehat{p}^*_{-J} > p^*_{-J}$. That is, all firms will set higher prices in the game $\widehat{\Gamma}$ than in the game Γ.

In what follows, we will show that the equilibrium profit of the two firms in J is larger in the game $\widehat{\Gamma}$ than in the game Γ. To this end, we introduce two modified games, $\Gamma_J(p_{-J})$ and $\Gamma_{-J}(p_J)$. In the game $\Gamma_J(p_{-J})$, the firms in J take the prices of the firms in $-J$ as fixed, and choose price simultaneously to maximise profit. In the game $\Gamma_{-J}(p_J)$, all the firms in $-J$ take the prices of the firms in J as fixed and simultaneously choose prices; the firms in $-J$ which are in a cartel choose their prices to maximise the aggregate cartel profit while each independent firm in $-J$ chooses its own price to maximise its own profit.

Since the prices of the firms in $-J$ are held fixed in the game $\Gamma_J(p_{-J})$, the demand of a firm $i \in J$ in the game $\Gamma_J(p_{-J})$ can be written as $q_i = v_J(p_{-J}) - \frac{n+(n-1)\gamma}{n} p_i + \frac{\gamma}{n} p_{i'}$, where $v_J(p_{-J}) \equiv v + \frac{\gamma}{n} \sum_{j \in -J} p_j$, and i' is the other firm in J. Likewise, the demand of a firm $i \in -J$ in the game $\Gamma_J(p_{-J})$ is $q_i = v_{-J}(p_J) - \frac{n+(n-1)\gamma}{n} p_i + \frac{\gamma}{n} \sum_{j \in -J \setminus \{i\}} p_j$, where $v_{-J}(p_J) \equiv v + \frac{\gamma}{n} \sum_{j \in J} p_j$. That is, the modified games have a similar structure as the original game, with the set of firms restricted to the respective sets, and the parameter v is adjusted using the prices of the other firms that are taken to be fixed in the respective modified game.

We consider the dynamic adjustment process, where, at period $t = 0$, firms start with prices $p^0_J = p^*_J$ and $p^0_{-J} = p^*_{-J}$. Since (p^0_J, p^0_{-J}) is an equilibrium of the original game Γ, we have that p^0_J is an equilibrium of the game $\Gamma_J(p^0_{-J})$, and p^0_{-J} is an equilibrium of the game $\Gamma_{-J}(p^0_J)$ when the two firms in J operate independently. Now, let the cartel J be formed, with prices starting at (p^0_J, p^0_{-J}). In each odd period t, we set $p^t_{-J} = p^{t-1}_{-J}$, and the firms in J play the game $\Gamma_J(p^t_{-J})$, which results in equilibrium prices p^t_J. In each even period t, we set $p^t_J = p^{t-1}_J$, and the firms in $-J$ play the game $\Gamma_{-J}(p^t_J)$, which results in equilibrium prices p^t_{-J}. That is, firms in J adjust their prices in odd periods while firms in $-J$ adjust their prices in even periods. This process yield a sequence of prices $\{(p^t_J, p^t_{-J})\}^{\infty}_{t=0}$. Let π^t_J denote the profits that each of the two firms in J experience at prices (p^t_J, p^t_{-J}). We will show that in each odd step of the adjustment process, the prices increase and converge to equilibrium prices $(\widehat{p}^*_J, \widehat{p}^*_{-J})$ of the original game $\widehat{\Gamma}$, and that in each step of the adjustment process, the profits of the firms in J increase.

In period $t = 1$, noting that $n = 2$ in $\Gamma_J(p^t_{-J})$, external instability of market structure where all firms are independent in the single cartel framework (recall Theorem 2) implies that the when two firms in J collude and increase their price (i.e. $p^1_J > p^0_J$), they experience an increase in their profits. Due to the increase in price by the two firms in J, we have that $v_{-J}(p^2_J) = v_{-J}(p^1_J) > v_{-J}(p^0_J)$, which means that all firms in $-J$ experience an increase in demand in their period $t = 2$ game. In response, they increase their prices, i.e. $p^2_{-J} > p^1_{-J} = p^0_{-J}$, and this increase in prices leads to $v_J(p^2_{-J}) > v_J(p^1_{-J})$, due to which the profits of the firms in J increase further (without them making a change in price themselves in this period). Continuing this logic, we find that the dynamic adjustment process results two increasing sequences of prices, $\{(p^t_J)\}^{\infty}_{t=0}$ for all odd t for firms in J, and $\{(p^t_{-J})\}^{\infty}_{t=0}$ for all even t for firms in J, and an increasing sequence of associated profit levels, $\{\pi^t_J\}^{\infty}_{t=0}$ for all t for the firms in J.

Because $p^0_{-J} = p^*_{-J} < \widehat{p}^*_{-J}$, we have $v_J(p^0_{-J}) = v_J(p^*_{-J}) < v_J(\widehat{p}^*_{-J})$, and hence $p^1_J < \widehat{p}^*_J$. This results in $v_{-J}(p^2_J) = v_{-J}(p^1_J) < v_{-J}(\widehat{p}^*_J)$, and hence $p^2_{-J} < \widehat{p}^*_{-J}$. Iterating this argument, we find that $(p^t_J, p^t_{-J}) < (\widehat{p}^*_J, \widehat{p}^*_{-J})$ for all $t = 0, 1, \ldots$. This implies that the sequences $\{(p^t_J)\}^{\infty}_{t=0}$ for all odd t, and $\{(p^t_{-J})\}^{\infty}_{t=0}$ for all even t, are increasing sequences that are bounded from above, and hence converges to some p^{∞}_J and p^{∞}_{-J}. Since the equilibrium $(\widehat{p}^*_J, \widehat{p}^*_{-J})$ is unique (per Theorem 1), if this limit point $(p^{\infty}_J, p^{\infty}_{-J})$ is not equal to $(\widehat{p}^*_J, \widehat{p}^*_{-J})$, at least one of two sets of firms, J or $-J$, has an incentive to increase prices further. Hence, the increasing sequence $\{(p^t_J)\}^{\infty}_{t=0}$ must converge to $\widehat{p}^*_J$, and the increasing sequence $\{(p^t_{-J})\}^{\infty}_{t=0}$ must converge to $\widehat{p}^*_{-J}$. Since the profits of the two firms in J is increasing along each step of the process, we have that $\pi^t_J > \pi^0_J = \pi^*_J$ for all $t = 0, 1, \ldots$. From continuity of the profit functions it follows that $\widehat{\pi}^*_J = \widehat{\pi}^{\infty}_J > \pi^*_J$. This means that, irrespective of the structure of firms in $-J$, the firms in J enjoy higher equilibrium profits when forming a two-firm cartel than when they operate independently. Thus, a market structure with two or more than two independent firms cannot be stable in the multiple cartels framework.

Proof of Lemma 2

Assume a market structure $(k_1, \ldots, k_m, k^I)$ such that $\max_{\ell=1,\ldots,m} k_\ell \geq 4$. In case $m = 1$, the lack of stability follows from Theorem 2. Therefore, we focus on the situation $m \geq 2$. Let the set $J = C_\ell$ for an $\ell \in \{1, \ldots, m\}$ such that $k_\ell = \max_{j=1,\ldots,m} k_j \geq 4$ (i.e. J is the largest cartel), and let $-J$ denote the set of other firms, which, due to our assumption of $m \geq 2$, comprises of at least one other cartel.

Let Γ^m be the original price setting game with equilibrium prices $(p_J^{m,*}, p_{-J}^{m,*})$. Furthermore, let $\widehat{\Gamma}^m$ be the same game where one firm i in J leaves the cartel and acts independently, and let $(\widehat{p}_J^{m,*}, \widehat{p}_{-J}^{m,*})$ be the equilibrium prices in this game. The use of the superscript m is to signify that these games relate to the multiple cartels framework. We will show that firm i experiences a larger profit at prices $(\widehat{p}_J^{m,*}, \widehat{p}_{-J}^{m,*})$ than at prices $(p_J^{m,*}, p_{-J}^{m,*})$, as a result of which the first condition of the stability definition is not satisfied. We denote the intercept of the demand function in these two games Γ^m and $\widehat{\Gamma}^m$ by $v^m \equiv v$.

In order to make use of what we know from the single cartel framework (Theorem 2), we construct the game Γ^s where firms in J form a cartel, but where all firms in $-J$ operate independently. The use of the superscript s signifies that this is a game in the single cartel framework. Crucially, in the game Γ^s, we change the intercept parameter from v^m to v^s, where v^s is chosen so that the resulting equilibrium prices $(p_J^{s,*}, p_{-J}^{s,*})$ of the game Γ^s are such that $p_J^{s,*} = p_J^{m,*}$. Since the equilibrium prices are linearly increasing in the intercept (for any market structure), we can always find such a v^s, and it will be unique. Moreover, it must be $v^s > v^m$. For if not, i.e. whenever $v^s \leq v^m$, we would have $p_J^{s,*} < p_J^{m,*}$ and $p_{-J}^{s,*} < p_{-J}^{m,*}$, the reason for this being that $-J$ contains at least one cartel in the game Γ^m but none in Γ^s, and the cartel causes not only the price of the firms in the cartel to be higher but, because prices are strategic complements, also the other firms' prices to be higher in the game Γ^m. Furthermore, let $\widehat{\Gamma}^s$ be the same game where the same firm i in J leaves the cartel and acts independently, and let $(\widehat{p}_J^{s,*}, \widehat{p}_{-J}^{s,*})$ be the equilibrium prices in this game. We know from the cartel J not being internally stable in the single cartel framework that firm i experiences larger profit at prices $(\widehat{p}_J^{s,*}, \widehat{p}_{-J}^{s,*})$ than at prices $(p_J^{s,*}, p_{-J}^{s,*})$.

As in the proof of Lemma 1, we develop modified games for J and $-J$, but this time we will have two such modified games for each group of firms: $\Gamma_J^m(p_{-J})$, $\Gamma_{-J}^m(p_J)$, $\Gamma_J^s(p_{-J})$ and $\Gamma_{-J}^s(p_J)$. Note that, by construction, $p_J^{m,*} = p_J^{s,*}$ is the equilibrium in both the games

$\Gamma^m_J(p^{m,*}_{-J})$ and $\Gamma^s_J(p^{s,*}_{-J})$. Furthermore, $p^{m,*}_{-J}$ is the equilibrium of $\Gamma^m_{-J}(p^{m,*}_J)$, and, because $p^{m,*}_J = p^{s,*}_J$, we have that $p^{s,*}_{-J}$ is the equilibrium in $\Gamma^s_{-J}(p^{s,*}_J) = \Gamma^s_{-J}(p^{m,*}_J)$.

We consider two synchronously progressing dynamic adjustment processes which we label process m (which is relevant for the modified games in multiple cartels framework) and process s (which is relevant for the modified games in the single cartel framework). At period $t = 0$, process m starts at prices $p^{m,0}_J = p^{m,*}_J$ and $p^{m,0}_{-J} = p^{m,*}_{-J}$ while process s starts at prices $p^{s,0}_J = p^{s,*}_J$ and $p^{s,0}_{-J} = p^{s,*}_{-J}$. Now, let firm i leave the cartel J, with prices starting at $(p^{m,0}_J, p^{m,0}_{-J})$ and $(p^{s,0}_J, p^{s,0}_{-J})$ in the two processes. In each odd period t, we set $p^{m,t}_{-J} = p^{m,t-1}_{-J}$ and $p^{s,t}_{-J} = p^{s,t-1}_{-J}$, and firms in J, with firm i having left the cartel J, play the games $\Gamma^m_J(p^{m,t}_{-J})$ and $\Gamma^s_J(p^{s,t}_{-J})$, which results in equilibrium prices $p^{m,t}_J$ and $p^{m,s}_J$. In each even period t, we set $p^{m,t}_J = p^{m,t-1}_J$ and $p^{s,t}_J = p^{s,t-1}_J$, and firms in $-J$ play the games $\Gamma^m_{-J}(p^{m,t}_J)$ and $\Gamma^s_{-J}(p^{s,t}_J)$, which results in equilibrium prices $p^{m,t}_{-J}$ and $p^{s,t}_{-J}$. That is, firms in J adjust their prices in odd periods and firms in $-J$ adjust their prices in even periods. We reiterate that there is at least one cartel in $-J$ in process m, but none in process s, while from period 1 onwards, the set J comprises of the cartel $J \setminus \{i\}$ and the independent firm i in both processes. These processes yield the sequences of prices $\{(p^{m,t}_J, p^{m,t}_{-J})\}^{\infty}_{t=0}$ and $\{(p^{s,t}_J, p^{s,t}_{-J})\}^{\infty}_{t=0}$, and let $\pi^{m,t}_i$ and $\pi^{s,t}_i$ denote the profit that firm i experiences at these prices at the respective time period t in the respective process. Note that since $v^m_J(p^{m,0}_{-J}) = v^s_J(p^{s,0}_{-J})$ and $p^{m,0}_J = p^{s,0}_J$, we have that $\pi^{m,0}_i = \pi^{s,0}_i$.

We will show that $(p^{m,t}_J, p^{m,t}_{-J})$ converges to $(\widehat{p}^{m,*}_J, \widehat{p}^{m,*}_{-J})$, which will then imply by continuity that $\pi^{m,t}_i$ converges to $\widehat{\pi}^{m,*}_i$. And, making use of the knowledge that $\widehat{\pi}^{s,*}_i > \pi^{s,*}_i$, we will show that $\widehat{\pi}^{m,*}_i > \pi^{m,*}_i$.

In period $t = 1$, when firm i has left the cartel J, since the prices of the firms in $-J$ is held fixed, we still have $v^m_J(p^{m,1}_{-J}) = v^s_J(p^{s,1}_{-J})$. The fragmentation of the cartel caused by the exit of firm i results in a reduction in prices of each firm in J so that $p^{m,1}_J = p^{s,1}_J < p^{m,0}_J = p^{s,0}_J$, where the two equalities come from $v^m_J(p^{m,1}_{-J}) = v^s_J(p^{s,1}_{-J})$. The first equality also implies $\pi^{m,1}_i = \pi^{s,1}_i$. Now, from the single cartel framework, which is relevant for process s, we know that firm i's profit increases upon leaving cartel J. Therefore, we have that $\pi^{m,1}_i = \pi^{s,1}_i > \pi^{s,0}_i = \pi^{m,0}_i$, and firm i experiences an improvement in profit in both processes because of the price reduction.

Next, in period $t = 2$, because prices of firms in J are held fixed in every even period, we have $p^{m,2}_J = p^{m,1}_J$ and $p^{s,2}_J = p^{s,2}_J$. In addition, $p^{m,1}_J = p^{s,1}_J < p^{m,0}_J = p^{s,0}_J$ from above implies that firms in J reduce their prices by the same amount in both processes. So $v^m_{-J}(p^{m,2}_J)$ and $v^s_{-J}(p^{s,2}_J)$ fall by the same amount from $v^m_{-J}(p^{m,0}_J)$ and $v^s_{-J}(p^{s,0}_J)$. This identical fall in demand of each firm in $-J$ results in $p_{-J}^{m,2} < p_{-J}^{m,0}$ and $p_{-J}^{s,2} < p_{-J}^{s,0}$. The reason is that, if the firms in $-J$ would have been acting independently in process m, the drop in prices in process m would have been the same as in process s. Since there is at least one cartel in $-J$, the prices of firms in $-J$ drop less in process m than in process s, thus implying that the demand of each firm in J falls less in process m. That is, $v_J^m(p_{-J}^{m,2}) > v_J^m(p_{-J}^{s,2})$, and hence $\pi_i^{m,2} > \pi_i^{s,2}$.

In period $t = 3$, because the prices of firms in $-J$ are held fixed, we have that $p_{-J}^{m,3} = p_{-J}^{m,2}$ in process m, and $p_{-J}^{s,3} = p_{-J}^{s,2}$ in process s. Since the prices of all firms in $-J$ are lower at the end of period $t = 2$, we obtain that the demand of each firm in J falls in both processes, i.e. $v_J^m(p_{-J}^{m,3}) < v_J^m(p_{-J}^{m,1})$ and $v_J^s(p_{-J}^{s,3}) < v_J^s(p_{-J}^{s,1})$. So, in period $t = 3$, the price of each firm in J falls in both processes. However, since $v_J^m(p_{-J}^{m,3}) = v_J^m(p_{-J}^{m,2}) > v_J^s(p_{-J}^{s,2}) = v_J^s(p_{-J}^{s,3})$ (the inequality comes from the previous paragraph) while $v_J^m(p_{-J}^{m,1}) = v_J^s(p_{-J}^{s,1})$, prices fall less in process m than in process s. This results in $\pi_i^{m,3} > \pi_i^{s,3}$.

Continuing in this fashion, we find that in each even period, because of the existence of at least one cartel in $-J$ in process m but none in process s, the prices of the firms in $-J$ fall more in process s than in process m, and consequently the profit of firm i falls more in process s than in process m. In each odd period, due to the larger fall in prices of firms in $-J$ in process s than in process m in the previous even period, the demand of firms in J fall more in process s than in process m, and consequently the increase in profit of firm i is higher in process m than in process s. As a result, we find that $\pi_i^{m,t} > \pi_i^{s,t}$ for all $t \geq 2$.

Using a similar argument as in the proof of Lemma 1, we obtain that $(p_J^{m,t}, p_{-J}^{m,t})$ converges to $(\widehat{p}_J^{m,*}, \widehat{p}_{-J}^{m,*})$, and $(p_J^{s,t}, p_{-J}^{s,t})$ converges to $(\widehat{p}_J^{s,*}, \widehat{p}_{-J}^{s,*})$. By continuity, $\pi_i^{m,t}$ converges to $\widehat{\pi}^{m,*}$, and $\pi_i^{s,t}$ converges to $\widehat{\pi}^{s,*}$. Since $\pi_i^{m,t} > \pi_i^{s,t}$ for all $t \geq 2$, we have that $\widehat{\pi}^{m,*} \geq \widehat{\pi}^{s,*}$ (with the latter inequality actually being strict). Using the knowledge from the single cartel framework that $\widehat{\pi}_i^{s,*} > \pi_i^{s,*}$, we conclude $\pi_i^{m,*} = \pi_i^{s,*} < \widehat{\pi}_i^{s,*} \leq \widehat{\pi}^{m,*}$. Hence, irrespective of the structure of firms in $-J$, if firm i leaves the cartel J, which comprises of at least four firms, it experiences an increase in its profit in the multiple cartel framework.

Proof of Lemma 3

This proof is analogous to the proof of Lemma 1. Now, let J comprise of an independent firm and a two-firm cartel, and let $-J$ consist of the remaining firms. We know from the previous two lemmas that no other independent firms or cartels with four or more firms can exist in a stable market structure – so each firm in $-J$ must either be in some two-firm cartel or some three-firm cartel. The proof now proceeds exactly in the same manner as Lemma 1 by relating the external instability of the two-firm cartel in the single cartel framework to the profitability of collusion between the independent firm in J and the two-firm cartel in J. When the independent firm joins the two-firm cartel, the prices and profits of all firms in J rise, and this leads to similar sequences of increasing prices and profits which can then be used in exactly the same manner as in Lemma 1 to establish the external instability of the two-firm cartel in the multiple cartels framework.

Proof of Lemma 4

1. Stability of $(3, \ldots, 3)$.

Presented after the statement of Lemma 4.

2. Stability of $(2, \ldots, 2)$.

We know from Lemma 1 that two independent firms gain in profit by forming a cartel. For this reason, a firm never gains in profit on leaving the cartel and operating independently. The only other feasible deviation involves a firm joining another cartel to form a three-firm cartel (leaving behind an independent firm). In the initial $(2, \ldots, 2)$ market structure, each firm experiences a profit of $\pi_i(2, \ldots, 2) = \frac{(n+(n-2)\gamma)n(v-c)^2}{[2n+(n-2)\gamma]^2}$. If a firm, instead, joins another cartel to form a three-firm cartel, it experiences a profit of $\pi_i(3, 2, \ldots, 2, 1) = \frac{4(n+(n-3)\gamma)(n+(n-1)\gamma)^2(2n+(2n-1)\gamma)^2 n(v-c)^2}{[8n^3+4n^2(5n-6)\gamma+2n(8n^2-20n+9)\gamma^2+(4n^3-16n^2+15n-6)\gamma^3]^2}$. Since it is easily verified that the inequality $\pi_i(2, \ldots, 2) > \pi_i(3, 2, \ldots, 2, 1)$ holds, this is not a profitable deviation. We conclude that the market structure with all firms positioned in two-firm cartels is stable.

3. Stability of $(3, \ldots, 3, 1)$.

We know from Lemma 2 that it is not a profitable deviation for the independent firm to join one of the three-firm cartels and form a four-firm cartel. A cartel firm, on the other hand, can consider three types of deviations: (1) join another cartel, (2) become independent, and (3) form a new two-firm cartel with the independent firm.

In case it is more profitable for a cartel firm to join another cartel, thus making it more profitable to be in a four-firm cartel, then it is (by Lemma 2) even more profitable for this firm to leave the four-firm cartel and instead be independent. But then, Lemma 3 implies that it is even more profitable for this firm, which now operates independently, to form a three-firm cartel by joining the two-firm cartel it left behind on its initial exit from its three-firm cartel. However, this leads exactly to the initial market structure, and therefore, it is not possible for the firm to experience a profit increase. This shows that none of the first two feasible deviations are profitable.

Finally, we will show that in a $(3, \ldots, 3, 1)$ market structure, it is not profitable for the independent firm to form a two-firm cartel with a breakaway firm from a three-firm cartel;

that is, the independent firm does not consent to this deviation thus making it an infeasible deviation. Solving the system of first-order conditions for the situation with $m \geq 1$ three-firm cartels and 1 independent firm, and for the situation with $m-1$ three-firm cartels and 2 two-firm cartels, we find that the equilibrium profit of the independent firm i in the first situation is $\pi_i(3,\ldots,3,1) = \frac{(n+(n-1)\gamma)(2n+(2n-3)\gamma)^2 n(v-c)^2}{[4n^2+2n(3n-4)\gamma+(2n^2-7n+5)\gamma^2]^2}$ and that the equilibrium profit of the same firm in a two-firm cartel in the second situation is $\pi_i(3,\ldots,3,2,2) = \frac{(n+(n-2)\gamma)(2n+(2n-3)\gamma)^2 n(v-c)^2}{[4n^2+2n(3n-5)\gamma+2(n^2-4n+5)\gamma^2]^2}$. Since the former profit is larger than the second, we find that the independent firm would not consent to form a two-firm cartel with a breakaway firm from a three-firm cartel. Hence, this is not a feasible deviation. We thus conclude that the $(3,\ldots,3,1)$ market structure is stable.

4. Stability of $(3,\ldots,3,2,\ldots,2)$.

Firstly, if a firm in a two-firm cartel or a three-firm cartel exits its cartel and operates as an independent firm, then by the arguments in case 1 and case 2, this is not a profitable deviation.

Secondly, it is not more profitable for a firm in a three-firm cartel to exit the cartel and join a two-firm cartel to form a three-firm cartel. The reason is that this would result in exactly the same market structure with the firm simply moving from one three-firm cartel to another three-firm cartel – clearly, this cannot lead to a gain in profit.

Thirdly, it is not more profitable for a firm in a three-firm cartel to exit the cartel and join another three-firm cartel to form a four-firm cartel (thus leaving behind another two-firm cartel due to its exit from its three-firm cartel). For, if this deviation is profitable, then, by Lemma 2, it is even more profitable to exit the four-firm cartel and operate independently. But then, by Lemma 3, it is yet more profitable to join the two-firm cartel it left behind on its initial exit from its three-firm cartel. However, this causes a reversion to the initial market structure, and hence it is not possible for the firm to obtain a higher profit. As a result, this is not a profitable deviation either.

Fourthly, it is also not a profitable deviation for a firm in a two-firm cartel to exit the cartel and join a three-firm cartel to form a four-firm cartel (thus leaving behind an independent firm). For if this is more profitable, then the firm can make further profit gains by exiting the four-firm cartel and operating independently (recall Lemma 2), and, following this, gain even more in profit by forming a two-firm cartel with the independent firm it left behind on its initial exit from its two-firm cartel (recall Lemma 1). But this brings us back to the initial market structure implying that it is not possible for the firm to improve its profit; this contradiction shows the deviation is not profitable.

Finally, it is not a profitable deviation for a firm in a two-firm cartel to exit the cartel and join another two-firm cartel to form a three-firm cartel. This type of deviation causes the number of three-firm cartels to increase by one, the number of two-firm cartels to decrease by two, and the number of independent firms to increase by one. Let the number of three-firm cartels and two-firm cartels in the initial market structure be m_3 and m_2, respectively, so that, in the final market structure, their number is $m_3 + 1$ and $m_2 - 2$, respectively, in addition to which there is one independent firm. Solving the first-order conditions, the profit of the firm in the initial $(3, \ldots, 3, 2, \ldots, 2)$ market structure is $\pi_i(3, \ldots, 3, 2, \ldots, 2) = \frac{(n+(n-2)\gamma)(2n+(2n-3)\gamma)^2 n(v-c)^2}{4[2n^2+(3n-5)n\gamma+((n-4)n+m_2+3)\gamma^2]^2}$ while the profit that this firm receives in the final market structure $(3, \ldots, 3, 2, \ldots, 2, 1)$ is $\pi_i(3, \ldots, 3, 2, \ldots, 2, 1) = \frac{(n+(n-3)\gamma)(n+(n-1)\gamma)^2(2n+(2n-1)\gamma)^2 n(v-c)^2}{[4n^3+2(5n-6)n^2\gamma+((8n-21)n+2m_2+9)n\gamma^2+(2n^3-9n^2+8n-3+(2n-1)m_2)\gamma^3]^2}$. It can be easily verified that $\pi_i(3, \ldots, 3, 2, \ldots, 2) > \pi_i(3, \ldots, 3, 2, \ldots, 2, 1)$, and so this is not a profitable deviation.

We conclude from the absence of any feasible profitable deviation that the market structure with some firms positioned in some two-firm cartel with the other firms positioned in some three firm-cartel is stable.

Proof of Khan and Peeters (2024) equivalence

The system of demand functions of the firms

$$q_i = v - \tfrac{n+(n-1)\gamma}{n} p_i + \tfrac{\gamma}{n} \textstyle\sum_{j \neq i} p_j \qquad (i = 1, \ldots, n)$$

can be inverted to obtain the system of inverse demand functions

$$p_i = v - \tfrac{n+\gamma}{(1+\gamma)n} q_i - \tfrac{\gamma}{(1+\gamma)n} \textstyle\sum_{j \neq i} q_j \qquad (i = 1, \ldots, n).$$

In the inverse demand function, that is relevant in case of quantity competition, we denote $\beta \equiv \frac{n+\gamma}{(1+\gamma)n}$ and $\delta \equiv \frac{\gamma}{(1+\gamma)n}$. Clearly, $\delta < \beta$. Khan and Peeters (2024) show that in the case of quantity competition, when $\delta < \beta$, a market structure involving cartelisation is stable in the multiple-cartel framework if and only if

$$\begin{cases} \frac{\delta}{\beta}(3 - \frac{\delta}{\beta})n \leq 4\sqrt{1 + \frac{\delta}{\beta}} - 2(1 - \frac{\delta}{\beta})^2 & \text{when } n \text{ is even} \\ \frac{\delta}{\beta}(3 - \frac{\delta}{\beta})n \leq 4\sqrt{1 + \frac{\delta}{\beta}} - \frac{4 - 10\frac{\delta}{\beta} + 11(\frac{\delta}{\beta})^2 - 3(\frac{\delta}{\beta})^3}{2 - \frac{\delta}{\beta}} & \text{when } n \text{ is odd} \end{cases}$$

Moreover, it is uniquely stable if

$$\tfrac{\delta}{\beta}(3-\tfrac{\delta}{\beta})n < 2\tfrac{\delta}{\beta} + (2-\tfrac{\delta}{\beta})\big[2\sqrt{1+\tfrac{\delta}{\beta}} - (1-\tfrac{\delta}{\beta})\big].$$

Written in terms of γ, these conditions are identical to

$$\begin{cases} \frac{\gamma}{n+\gamma}(3-\frac{\gamma}{n+\gamma})n \leq 4\sqrt{1+\frac{\gamma}{n+\gamma}} - 2(1-\frac{\gamma}{n+\gamma})^2 & \text{if } n \text{ is even} \\ \frac{\gamma}{n+\gamma}(3-\frac{\gamma}{n+\gamma})n \leq 4\sqrt{1+\frac{\gamma}{n+\gamma}} - \frac{4-10\frac{\gamma}{n+\gamma}+11(\frac{\gamma}{n+\gamma})^2-3(\frac{\gamma}{n+\gamma})^3}{2-\frac{\gamma}{n+\gamma}} & \text{if } n \text{ is odd} \end{cases}$$

and

$$\tfrac{\gamma}{n+\gamma}(3-\tfrac{\gamma}{n+\gamma})n < 2\tfrac{\gamma}{n+\gamma} + (2-\tfrac{\gamma}{n+\gamma})\big[2\sqrt{1+\tfrac{\gamma}{n+\gamma}} - (1-\tfrac{\gamma}{n+\gamma})\big].$$

For the first condition, firstly, it can be shown that, in either case, the right-hand side minus the left-hand side is decreasing in γ. Secondly, it can be verified that the inequalities are satisfied at $\gamma = \frac{2}{3}$. So, from this, we conclude that the inequalities are satisfied for all $\gamma \in (0, \frac{2}{3}]$, and for all $n \geq 2$. For the second condition, after multiplying both sides by $(n+\gamma)^2$, and by using the fact that $\sqrt{(n+2\gamma)(n+\gamma)} > (n+\gamma)$, we find the inequality to be satisfied if $n^2(3-2\gamma) + n(7-2\gamma) + 4\gamma^2 > 0$, which is true for $\gamma \in (0, \frac{2}{3}]$.

We have characterised the stable market structures under price competition when $\gamma \leq \frac{2}{3}$, and, by the results contained in Khan and Peeters (2024), the above implies that, under the same demand conditions, when firms compete in quantities, the stable market structure involves multiple cartels, and is characterised by at most one independent firm with every other firm being in some two-firm cartel. So, if n is even, then each firm belongs to some two-firm cartel and sets its quantity equal to $\frac{n(1+\gamma)(v-c)}{2n+(n+2)\gamma}$ which leads to the each firm's market-clearing price being $\frac{(n+2\gamma)v+n(1+\gamma)c}{2n+(n+2)\gamma}$. And, if n is odd, then one firm operates independently and sets its quantity equal to $\frac{2n(n+\gamma)(1+\gamma)(v-c)}{4n^2+2n(n+3)\gamma+(n+3)\gamma^2}$ while each of the other firms are in some two-firm cartel and set quantity equal to $\frac{n(2n+\gamma)(1+\gamma)(v-c)}{4n^2+2n(n+3)\gamma+(n+3)\gamma^2}$; this leads to a market-clearing price of $\frac{2(n+\gamma)^2v+(2n^2+2n(n+1)\gamma+(n+1)\gamma^2)c}{4n^2+2n(n+3)\gamma+(n+3)\gamma^2}$ for the independent firm and $\frac{(2(n+\gamma)^2+n\gamma)v+(2n^2+n(2n+1)\gamma+(n+1)\gamma^2)c}{4n^2+2n(n+3)\gamma+(n+3)\gamma^2}$ for each of the other firms in the two-firm cartels.

In contrast, when firms compete in prices, and the number of firms n is even, then the market structures of the type $(3,\ldots,3)$, $(3,\ldots,3,2,\ldots,2)$, $(3,\ldots,3,1)$ and $(2,\ldots,2)$ are stable (whenever feasible); and, when the number of firms n is odd, then the market structures of the type $(3,\ldots,3)$, $(3,\ldots,3,2,\ldots,2)$ and $(3,\ldots,3,1)$ are stable (whenever feasible). Firms set their prices at $\frac{nv+(n+(n-3)\gamma)c}{2n+(n-3)\gamma}$ in the $(3,\ldots,3)$ market structure while they set their prices at $\frac{nv+(n+(n-2)\gamma)c}{2n+(n-2)\gamma}$ in the $(2,\ldots,2)$ market structure. In the $(3,\ldots,3,2,\ldots,2)$ market structure with m_3 three-firm cartels and m_2 two-firm cartels, such that $n = 3m_3 + 2m_2$, the price set by firms in a three-firm cartel is $\frac{2(n+(n-1)\gamma)nv+(2n^2+n(4n-8)\gamma+[(n-1)(2n-6)+2m_2]\gamma^2)c}{2(2n^2+n(3n-5)\gamma+[(n-2)^2+(m_2-1)]\gamma^2)}$ while the price set by firms in a two-firm cartel is $\frac{(2(n+(n-1)\gamma)-\gamma)nv+(2n^2+n(4n-7)\gamma+[(n-1)(2n-6)+2m_2]\gamma^2)c}{2(2n^2+n(3n-5)\gamma+[(n-2)^2+(m_2-1)]\gamma^2)}$. Finally, in a $(3,\ldots,3,1)$ market structure, firms in three-firm cartels set their prices at $\frac{(2(n+(n-1)\gamma)+\gamma)nv+(2n^2+n(4n-7)\gamma+(n-1)(2n-5)\gamma^2)c}{4n^2+2n(3n-4)\gamma+(n-1)(2n-5)\gamma^2}$ while the one independent firm sets its price at $\frac{(2(n+(n-1)\gamma)-\gamma)nv+(2n^2+n(4n-5)\gamma+(n-1)(2n-5)\gamma^2)c}{4n^2+2n(3n-4)\gamma+(n-1)(2n-5)\gamma^2}$.

We refer to Section 5 for a comparative discussion on the price/market-clearing price and profit of the firms under price competition and quantity competition.

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