Sustainability of public debt, investment subsidies, and endogenous growth with heterogeneous firms and financial frictions

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Abstract

This study investigates the effect of public debt on growth, interest rates, and fiscal sustainability using a simple endogenous growth model with financial frictions and firm heterogeneity. Increases in public debt lead to higher real interest rates through financial markets, increase the cost of repaying public debt, and reduce private investment, resulting in lower long-run growth. Thus, large public debt is less sustainable. This study also examines the effect of investment subsidies financed by public debt and finds that they hinder economic growth in the long run unless the financial market is close to perfect. Therefore, increases in investment subsidies should be financed not only by issuing public bonds, but also through tax increases.

JEL classification: E62; H20; H60

Keywords: Sustainability of public debt, Financial frictions, Firm heterogeneity, Investment subsidies

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1 Introduction

Public debt has increased significantly in many developed countries, raising concerns among policymakers about how to ensure its sustainability.[1] Higher economic growth and lower interest rates are important factors in stabilizing the debt-to-GDP ratio. However, economic growth, interest rates, and public debt accumulation are determined by dynamic processes and are interrelated, making it difficult to assess the sustainability of public debt. Indeed, no consensus exists on what constitutes sustainable debt (e.g., D'Erasmo et al., 2016). Ramsey-type models with representative infinitely lived agents show that public debt cannot be sustainable if the government violates its transversality condition (e.g., Greiner, 2007, 2011, 2012, 2015; Kamiguchi and Tamai, 2012). Many empirical studies have tested whether the transversality condition holds (e.g., Afonso, 2005; Bohn, 1998; Hamilton and Flavin, 1986). Meanwhile, overlapping generations (OLG) models show that the government removes this constraint and can run a Ponzi game. Acknowledging the possibility of a public Ponzi game, many recent studies have analyzed fiscal sustainability in OLG models and defined fiscal sustainability as whether the ratio of public debt to GDP (or capital) converges to a stable level in the long run (Agénor and Yilmaz, 2017; Arai, 2011; Bräuninger, 2005; Chalk, 2000; Maebayashi and Konishi, 2021; Teles and Mussolini, 2014; Yakita, 2008, 2014).

Despite this large literature, to the best of my knowledge, few studies have examined long-run economic growth, interest rates, and the sustainability of public debt simultaneously. Moreover, although the abovementioned studies on the sustainability of public debt employ different types of models, they all assume that agents are homogeneous and financial markets are perfect.

This study first contributes to the literature by investigating the effect of public debt on growth, interest rates, and sustainability using a simple endogenous growth model with financial frictions and firm heterogeneity. Drawing on the recent literature, we also incorporate the Pareto distribution of firms' productivity into the impact of fiscal policy on growth when firms are heterogeneous (e.g., Arawatari et al., 2023; Jaimovich and Rebelo, 2017; Mino, 2015, 2016). Under financial frictions and firm heterogeneity, only high-productivity firms can borrow in the financial market. Low-productivity firms cannot borrow because of their credit constraints. Low-productivity agents then become lenders to high-productivity entrepreneurs (borrowers). A rise in government debt increases the issuance of government bonds and reduces the aggregate supply of credit in the financial market, thereby raising the interest rate. This increase in the interest rate reduces the number of firms by increasing the cost of borrowing for entrepreneurs. Public debt negatively affects economic growth by crowding out firms and their investments.

The second contribution of this study is to examine the growth effect of investment subsidies

[1] One way to address this problem would be fiscal consolidation efforts, especially in the European Union (EU).

financed by public debt. Governments offer various investment subsidies, including investment cost subsidies, research and development (R&D) subsidies, investment tax reductions, and direct investment grants (Kang, 2022). These investment subsidies are recognized as instruments for boosting economic growth. In 2022, the government budget allocations for R&D across the EU reached 0.74% of GDP, showing an increase of 5.4% compared with 2021 and an increase of 49.2% compared with 2012. According to the World Bank (2023), R&D expenditure in the United States increased from 2.67% to 3.46% of GDP between 2012 and 2021. Policy instruments to support young, growing, and innovative companies are also expected to boost investment and economic growth (European Commission, 2017). The public finances of these developed countries have relied heavily on public debt. However, to the best of my knowledge, the growth-enhancing effect of investment subsidy policy with heterogeneous firms has been studied only in the case of a balanced government budget. Therefore, it is important to examine the growth effect of investment subsidies financed by public debt. Furthermore, investigating whether the growth-enhancing effect of investment subsidies can improve the fiscal situation is worthwhile.

Herein, I develop an OLG model with endogenous growth, firm heterogeneity, and financial frictions. In the young period, agents earn wage income and draw productivity levels from the Pareto distribution. Agents who draw higher (lower) productivity than the cutoff level become entrepreneurs (lenders) and engage in (refrain from) production in old age. Additionally, the government issues public bonds to finance investment subsidies. The cutoff level of productivity is endogenously determined and it depends on fiscal policy.

The main findings of this study are as follows. First, an increase in public debt has two opposite effects on economic growth. One is the negative growth effect through the crowding-out effect of public debt on investment. The other is the positive effect stemming from an increase in the interest rate. The interest rate rises because an increase in government debt increases the issuance of government bonds and reduces the aggregate supply of credit in the financial market. A rise in the interest rate increases output and wage income, enhancing growth. This is because a rise in the interest rate heightens the cost of borrowing for entrepreneurs and reduces the number of firms, while increasing capital intensity and the marginal productivity of capital. The negative growth effect dominates the positive effect. Therefore, an increase in public debt hinders economic growth.

Second, a rise in the interest rate due to an increase in public debt raises interest payments and the growth in public debt. Meanwhile, the crowding-out effect of public debt on private investment reduces the burden of public expenditure on investment subsidies, thereby hampering the growth in public debt. The positive effect on the growth in public debt dominates the negative effect when the ratio of public debt to capital is large, making public debt unsustainable.

Third, investment subsidies financed by public debt hinder economic growth in the long run, although they have a positive impact on growth in the short run. Moreover, investment subsidies make public debt less sustainable or increase the public debt-to-GDP ratio in the long-run steady state. These results are robust unless the financial market is close to perfect. This is because, first, investment subsidies lower the barrier to entrepreneurship and increase the number of less productive firms, which decreases aggregate productivity. Second, the increase in these firms heightens aggregate demand for credit in the financial market and leads to upward pressure on the interest rate. A rise in the interest rate increases the cost of servicing public debt and crowds out private investment. These two effects have a significant and negative impact on economic growth in the short run, even though investment subsidies encourage investment. Public debt increases due to the financing costs of subsidies and owing to a rise in the interest rate in the financial market, thereby worsening the fiscal situation. In the long run, this rise in the interest rate increases output and wage income, enhancing growth (crowding-in effect of debt), while it increases public debt financing and decreases investment (crowding-out effect of debt), hindering growth. Since financial frictions (difficulties in borrowing) decrease the demand of credit and lowers the interest rate and economic growth, the positive long-run growth effect is weaker than the negative effect. This advocates that increases in investment subsidies should be financed not only by issuing public bonds but also through tax increases unless the financial market is close to perfect. This is an important policy implication for many developed countries that have increased investment subsidies while relying heavily on public debt for fiscal management.

Related Literature

Chalk (2000), de la Croix and Michel (2002), and Yakita (2014) examine the sustainability of public debt in OLG models and conclude that a Ponzi game by governments is possible.[2] Fiscal sustainability in OLG models is often defined as the convergence of public debt to a sustainable level in the long run. Chalk (2000) and Maebayashi (2023) examine this issue under specific fiscal policy rules in OLG models. Chalk (2000) employs a constant deficit rule, while Maebayashi (2023) implements the fiscal consolidation rule based on the Stability and Growth Pact in the EU. However, as these studies are based on exogenous growth models, they ignore the long-run endogenous growth effect from a non-decreasing return to capital (Romer, 1986).

This study is closely related to those of Bräuninger (2005), Yakita (2008), Arai and Kunieda (2010), Arai (2011), Teles and Mussolini (2014), Agénor and Yilmaz (2017), Maebayashi and Konishi (2021), and Futagami and Konishi (2023), which also investigate the sustainability of public debt when the government plays a Ponzi game in OLG models with an endogenous

[2]Tirole (1985) uses Diamond's (1965) OLG model and show that positive bubbles can exist when the economy is dynamically inefficient. This result can also imply that the government's Ponzi game is possible.

growth structure.[3] Bräuninger (2005), Arai (2011), Teles and Mussolini (2014), and Maebayashi and Konishi (2021) find that public debt has a negative effect on long-run growth (Saint-Paul, 1992); however, they all assume a constant interest rate over time because they use the standard AK model (Romer, 1986). Yakita (2008) and Agénor and Yilmaz (2017) include public capital in the final production function (Futagami et al., 1993) and capture its positive external effects on growth and interest rates. The present study differs from these previous studies because it considers the positive growth effect of investment subsidies directly on firms and endogenous movements of interest rates through the financial market structure. Minami and Horii (2025) examine the growth effects of debt-financed R&D subsidies in a continuous-time OLG model (Blanchard 1985). They show that debt-financed R&D subsidies do not enhance long-run growth unless R&D productivity is sufficiently high. Unlike Minami and Horii (2025), this study considers heterogeneous firms and financial frictions and finds that the long-run growth effect of debt-financed investment subsidies depends on the degree of financial frictions. Although Arai and Kunieda (2010) similarly consider financial market imperfection with heterogeneous agents, they assume a uniform distribution of individual productivity as well as risk neutrality, and ignore the investment subsidy policy. In contrast to Arai and Kunieda (2010), this study incorporates more realistic growth effects under investment subsidy policies through microfoundations with risk-averse utility and a Pareto distribution of firms' productivity, as mentioned in the following literature.

Recent trends in the growth theory literature incorporate the heterogeneity of individuals or firms into endogenous growth models (e.g., Arawatari et al., 2023; Jaimovich and Rebelo, 2017; Mino, 2015, 2016). Mino (2016), Jaimovich and Rebelo (2017), and Arawatari et al. (2023) consider the effect of tax and fiscal policies when firms differ in their productivity under a Pareto distribution. These studies show that the effects of such public policies on growth significantly differ from that in the (homogeneous) representative agent economy. To the best of my knowledge, studies on investment subsidy policies in this context are somewhat limited. Morimoto (2018) studies R&D subsidy policy under heterogeneity in individual productivity and shows that R&D subsidies increase economic growth when they are not very large. However, Morimoto (2018) considers a balanced budget to finance the subsidies. This study contributes to the literature by considering the effect of investment subsidies financed by public debt and showing that the growth effect of subsidies can be negative unless the financial market is close to perfect.

[3] Greiner (2007, 2011, 2012, 2015), Kamiguchi and Tamai (2012), and Miyazawa et al. (2019) investigate the sustainability of public debt in representative infinitely lived agent models with an endogenous growth structure in which a Ponzi game by the government is impossible (according to the transversality condition).

2 Model

2.1 Entrepreneurial households

Consider an economy comprising two types of households living in two periods. The number of households is normalized to one. Entrepreneurial households can invest in capital, hire young labor, and use capital to produce goods. The production technology follows the form used in Mino (2016):

$$y_{i,t} = \mathcal{F}(z_{i,t-1}k_{i,t}, n_{i,t}K_t) = A(z_{i,t-1}k_{i,t})^{\alpha}(n_{i,t}K_t)^{1-\alpha}, \quad A > 0 \quad i \in [0,1], \tag{1}$$

where $y_{i,t}$, $k_{i,t}$, $n_{i,t}$, and K_t denote output, capital, labor, and aggregate capital, respectively. Aggregate capital has a positive external effect on production (e.g., Romer, 1986). We assume that capital $k_{i,t}$ is broadly viewed to include both ICT capital (knowledge capital) related to innovation and development (R&D) and non-ICT capital.[4] Here, $z_{i,t}$ is the production efficiency of the firm owned by the type i entrepreneurial household.

In the young period, each entrepreneurial household draws $z_{i,t}$ from a Pareto distribution whose cumulative distribution is given by

$$F(z) = 1 - z^{-\varphi}, \quad 1 \leq z < \infty, \quad \varphi > 1. \tag{2}$$

Here, a lower (higher) value of φ means a higher (lower) degree of heterogeneity in the production technology. Following Itskhoki and Moll (2014), Liu and Wang (2014), and Mino (2015), we assume that $z_{i,t}$ is independent and identically distributed (iid) both over time and across agents.

After realizing z_i, each entrepreneurial household maximizes its lifetime utility:

$$U_{i,t}^{j} = (1-\beta)\ln c_{i,t}^{y,j} + \beta\left[(1-\gamma)\ln c_{i,t+1}^{o,j} + \gamma \ln x_{i,t+1}^{j}\right], \quad j \in \{e, l\} \tag{3}$$

subject to the budget and credit constraints. Here, $c_{i,t}^{y,e}$, $c_{i,t+1}^{o,e}$, and $x_{i,t+1}^{e}$ represent consumption in the young period and old age and bequests by entrepreneurial households that produce goods (active entrepreneurs, hereafter), while $c_{i,t}^{y,l}$, $c_{i,t+1}^{o,l}$, and $x_{i,t+1}^{l}$ represent consumption by those who do not produce goods (non-active entrepreneurs hereafter), respectively.

In the young period, active entrepreneurs supply one unit of labor inelastically, earn wage income w_t, and inherit from parents $x_{i,t}$.[5] With this income, they can borrow from non-active

[4] ICT capital includes hardware, communication, and software, whereas non-ICT capital includes transport equipment and non-residential construction; agricultural products, metal products, and machinery other than hardware and communication equipment; and other products of non-residential gross fixed capital formation (see OECD, 2010).

[5] Even with endogenous labor supply, our main results are robust if we do not consider the bequest motive.

entrepreneurs and invest in capital $k_{i,t+1}$. Here, let us denote $d_{i,t}$ as private debt. Then, the net worth of active entrepreneurs is $a_{i,t+1} = k_{i,t+1} - d_{i,t}$. Investment $k_{i,t+1}$ is subsidized by the government at the rate of σ_k. Since total income is allocated to consumption and net worth, the budget constraint of active entrepreneurs in the young period is represented as

$$c_{i,t}^{y,e} = w_t + x_{i,t} + \sigma_k k_{i,t+1} - a_{i,t+1}, \qquad a_{i,t+1} = k_{i,t+1} - d_{i,t}. \tag{4}$$

Note that $x_{i,t}$ depends on whether parents are active or non-active entrepreneurs, as we see next. However, this is not critical to the macroeconomy when we aggregate all agents, as we see later.[6]

In old age, active entrepreneurs produce final goods with the production function (1). They hire young labor $n_{i,t+1}$ in generation $t+1$ and use capital $k_{i,t+1}$ installed in period t for production. Profit $\pi_{i,t+1}$ is sales $y_{i,t}$ minus both wage payments and private debt repayments. Active entrepreneurs in old age allocate it into consumption $c_{i,t+1}^{o,e}$ and bequest to their children $x_{i,t+1}^{e}$. Thus, the budget constraint of active entrepreneurs in old age is written as

$$c_{i,t+1}^{o,e} = \pi_{i,t+1} - x_{i,t+1}^{e}, \tag{5}$$

$$\pi_{i,t+1} = y_{i,t+1} - w_{t+1} n_{i,t+1} - R_{t+1} d_{i,t}, \tag{6}$$

where $R_t(= 1 + r_t)$ is the gross interest rate when r_t denotes the interest rate. We assume full capital depreciation because we consider a period to be about 30 years.[7] Furthermore, the financial market is assumed to be imperfect in the following sense. Entrepreneurs face a credit constraint such that

$$d_{i,t} \leq \lambda k_{i,t+1}. \tag{7}$$

If $\lambda = 1$, the financial market is perfect, while no borrowing is available if $\lambda = 0$, meaning that λ is the degree of (im)perfection of the financial market.

From (1), (3), (4), (5), (6), and (7), the first-order conditions (FOCs) with respect to $n_{i,t+1}$,

[6]Even without the bequest motive $x_{i,t}$, our main results remain robust. However, without $x_{i,t}$, the investment levels of all firms becomes the same, which is somewhat unrealistic. Furthermore, through bequests $x_{i,t}$, public debts held by lenders can have a crowding-in effect on investment. For these reasons, and for future reference, we allow for the presence of the bequest motive.

[7]Without full capital depreciation ($\delta \neq 1$), the first term on the right-hand side of (6) is replaced by $y_{i,t+1} + (1 - \delta)k_{i,t+1}$ if we denote $\delta \in [0, 1]$ as capital depreciation.

$d_{i,t}$, and $k_{i,t+1}$ are given by

$$n_{i,t+1}; \qquad w_{t+1} = (1-\alpha)\frac{y_{i,t+1}}{n_{i,t+1}}, \tag{8}$$

$$d_{i,t}; \qquad \frac{1-\beta}{c_{i,t}^{y,e}} = \frac{\beta(1-\gamma)R_{t+1}}{c_{i,t+1}^{o,e}} + \mu_{i,t}, \tag{9}$$

$$k_{i,t+1}; \qquad \frac{(1-\beta)(1-\sigma_k)}{c_{i,t}^{y,e}} = \frac{\beta(1-\gamma)}{c_{i,t+1}^{o,e}}\frac{\partial \pi_{i,t+1}}{\partial k_{i,t+1}} + \lambda\mu_{i,t}, \tag{10}$$

$$\mu_{i,t}(\lambda k_{i,t+1} - d_{i,t}) = 0, \ \ \mu_{i,t} \geq 0, \ \ \lambda k_{i,t+1} - d_{i,t} \geq 0, \tag{11}$$

$$\mu_{i,t} = \frac{\beta(1-\gamma)}{1-\sigma_k-\lambda}\frac{\alpha(y_{i,t+1}/k_{i,t+1}) - (1-\sigma_k)R_{t+1}}{c_{i,t+1}^{o,e}}, \tag{12}$$

where $\mu_{i,t}$ is the Lagrangian multiplier associated with the debt constraint and represents the investment wedge between the marginal product of capital $\alpha(y_{i,t+1}/k_{i,t+1})$ and marginal cost $(1-\sigma_k)R_{t+1}$ in (12). Note that (12) is derived from (9) and (10) with (1), and (6). If the credit constraint is not binding, $\mu_{i,t} = 0$ holds from (11). We assume that entrepreneurs produce goods as long as their profits are not negative. Thus, from (11) and (12), the credit constraint (7) binds when

$$\alpha(y_{i,t+1}/k_{i,t+1}) \geq (1-\sigma_k)R_{t+1}. \tag{13}$$

From (1) and (8), we obtain

$$y_{i,t} = Az_{i,t-1}k_{i,t}\left[\frac{(1-\alpha)AK_t}{w_t}\right]^{\frac{1-\alpha}{\alpha}}, \tag{14}$$

which together with (13) yields the cutoff value of z as

$$z_t^* = \frac{(1-\sigma_k)R_{t+1}}{\alpha A}\left[\frac{w_{t+1}}{(1-\alpha)AK_{t+1}}\right]^{\frac{1-\alpha}{\alpha}}. \tag{15}$$

Substituting (14) and (15) into the left-hand side (LHS) and the right-hand side (RHS) of (13) respectively, we can rewrite (13) as

$$z_{i,t} \geq z_t^*.$$

This indicates that entrepreneurs who draw $z_{i,t} \geq z_t^*$ produce and become active entrepreneurs. They also become borrowers because the debt constraint (7) is binding. Conversely, credit constraints are ineffective for entrepreneurs who draw $1 \leq z_{i,t} < z_t^*$.[8] In the competitive final good

[8]From (2), the minimal value of z is 1.

market, firms with $z_{i,t} < z_t^*$ cannot compete with those with z^*. Thus, entrepreneurs who own firms with $z_{i,t} < z_t^*$ give up production (i.e., become non-active entrepreneurs) and become lenders. We show in Lemma 1 that the indifference between becoming borrowers and lenders holds for agents with $z_{i,t} = z_t^*$.

Moreover, (15) indicates that investment subsidies reduce the barriers to becoming an entrepreneur z^* and increase the number of borrowers for given interest rates R_{t+1}.

Utility maximization with respect to $x_{i,t+1}^e$ yields

$$x_{i,t+1}^e = \gamma \pi_{i,t+1}. \tag{16}$$

Substituting (8) and $d_{i,t} = \lambda k_{i,t+1}$ into (6), we obtain

$$\pi_{i,t+1} = \alpha y_{i,t+1} - R_{t+1} \lambda k_{i,t+1}. \tag{17}$$

From (4), (5), (9), (12), (16), (17), and $d_{i,t} = \lambda k_{i,t+1}$, we obtain

$$k_{i,t+1} = \frac{\beta}{1 - \sigma_k - \lambda}(w_t + x_{i,t}). \tag{18}$$

(18) indicates that a higher degree of financial market imperfection (a lower value of λ) reduces the investment of each firm. Conversely, a larger investment subsidy to firms increases the investment of each firm.

We move onto the case of lenders (non-active entrepreneurs). Lenders (entrepreneurs who draw $z_{i,t} < z_t^*$ and do not engage in production) maximize their utility (3) subject to the following budget constraints:

$$c_{i,t}^{y,l} = w_t + x_{i,t} - l_{i,t} - q_t b_{i,t+1}, \tag{19}$$

$$c_{i,t+1}^{o,l} = R_{t+1} l_{i,t} + b_{i,t+1} - x_{i,t+1}^l, \tag{20}$$

where $l_{i,t}$ is the loan to active entrepreneurs, while $b_{i,t+1}$ is the quantity of government bonds purchased and q_t is the price of government bonds. We assume that government bonds are discount bonds with a maturity of 1 period and a face value of 1 (Maebayashi and Tanaka, 2022). The no-arbitrage condition between lending to active entrepreneurs and buying treasuries equates these rates of return as

$$R_{t+1} = 1/q_t. \tag{21}$$

(19) indicates that in the young period, lenders supply one unit of labor inelastically, earn wage

income w_t, and inherit from parents $x_{i,t}$, as active entrepreneurs do. They allocate this income toward loans to active entrepreneurs $l_{i,t}$, purchases of government bonds $q_t b_{i,t+1}$, and consumption $c_{i,t}^{y,l}$. (20) implies that the total return from lending to the private sector $R_{t+1} l_{i,t}$ and public sector $(1/q_t) q_t b_{i,t+1}$ in old age is divided into consumption $c_{i,t+1}^{o,l}$ and bequests $x_{i,t+1}^{l}$.

Lenders' FOCs with respect to $l_{i,t} + q_t b_{i,t+1}$ and $x_{i,t+1}^{l}$ result in

$$l_{i,t} + q_t b_{i,t+1} = \beta \left(w_t + x_{i,t} \right), \tag{22}$$

$$x_{i,t+1}^{l} = \gamma (R_{t+1} l_{i,t} + b_{i,t+1}). \tag{23}$$

We summarize the discussion so far in the following lemma.

Lemma 1. *The indifference between being an active entrepreneur (borrower) and an non-active entrepreneur (lender) holds for households whose productivity is z_t^* (see Appendix A for this proof). Households with $z_{i,t} \geq z_t^*$ become active entrepreneurs (borrowers) while those with $z_{i,t} < z_t^*$ become non-active entrepreneurs (lenders) in period t.*

Hereafter, we omit the index i for simplicity.

2.2 Government

The government owes a given amount of debt (denoted by B_t) at the beginning of period t. It repays this debt and finances investment subsidies by issuing new bonds, indicating that it is playing a Ponzi game, as in Yakita (2014) and Maebayashi and Tanaka (2022).[9] This simple approach to the government budget aims to clearly study the growth effect of investment subsidies financed by public debt. A more comprehensive approach, including income tax revenue, is provided in Section 7. Here, recall that investment subsidies are provided only to active entrepreneurs ($z_t \geq z_t^*$) and government bonds are purchased by lenders ($1 \leq z_t < z_t^*$). Thus, the government's budget constraint in period t is given by

$$q_t B_{t+1} = B_t + \sigma_k \int_{z_t^*}^{\infty} k_{t+1} dF(z_t), \tag{24}$$

$$B_{t+1} = \int_{1}^{z_t^*} b_{t+1} dF(z_t) \quad \left(B_t = \int_{1}^{z_{t-1}^*} b_t dF(z_{t-1}) \right). \tag{25}$$

[9]Recall that a Ponzi game by the government is possible in OLG models as noted in the Introduction. Previous studies (e.g., Maebayashi and Tanaka, 2022; Yakita, 2014) investigate fiscal sustainability in a Ponzi game played by the government. Maebayashi and Tanaka (2022) consider public debt finance only for the repayment of debt ($q_t B_t = B_t$), while Yakita (2014) considers both the repayment of debt and net income transfers to young labor ($q_t B_t = B_t +$ net transfers).

Here, let us define $\tilde{B}_{t+1} \equiv q_t B_{t+1}$. Then, we obtain $\tilde{B}_{t+1} = B_{t+1}/R_{t+1}$ from (21). Accordingly, (24) is transformed into

$$\tilde{B}_{t+1} = R_t \tilde{B}_t + \sigma_k \int_{z_t^*}^{\infty} k_{t+1} dF(z_t). \tag{26}$$

3 Equilibrium

Aggregate capital K_t is held by active entrepreneurs (borrowers) who draw $z_{t-1} \geq z_{t-1}^*$ in period $t-1$; therefore, it is represented by $\int_{z_{t-1}^*}^{\infty} k_{i,t} dF(z_{t-1}) = K_t$ (Mino, 2015, 2016). Using (2) and keeping in mind that z is iid, we can rewrite aggregate capital as[10]

$$K_t \left(= \int_{z_{t-1}^*}^{\infty} k_t dF(z_{t-1}) \right) = (z_{t-1}^*)^{-\varphi} \bar{k}_t. \tag{27}$$

where, $\bar{k}_t$ is the average level of k_t. Aggregating the credit constraint $d_t = \lambda k_{t+1}$, the equilibrium condition of the financial market is represented as

$$\int_1^{z_t^*} l_t dF(z_t) = \int_{z_t^*}^{\infty} d_t dF(z_t) = \lambda K_{t+1}. \tag{28}$$

The labor market clears as

$$N_t = \int_{z_{t-1}^*}^{\infty} n_{i,t} dF(z_{t-1}) = 1, \tag{29}$$

which indicates that total labor demand is equal to total labor supply, whose aggregate level is unity ($\int_1^{\infty} 1 dF(z_{t-1}) = 1$).

Using (2) and (27), we can also aggregate the production function (14) as

$$\begin{aligned} Y_t \left(= \int_{z_{t-1}^*}^{\infty} y_t dF(z_{t-1}) \right) &= \int_{z_{t-1}^*}^{\infty} A z_{t-1} k_t \left[\frac{(1-\alpha)AK_t}{w_t} \right]^{\frac{1-\alpha}{\alpha}} dF(z_{t-1}) \\ &= \frac{A\varphi}{\varphi - 1} z_{t-1}^* \left[\frac{(1-\alpha)AK_t}{w_t} \right]^{\frac{1-\alpha}{\alpha}} K_t. \end{aligned} \tag{30}$$

[10] Since z_t is iid across agents and over time,

$$K_t = \int_{z_{t-1}^*}^{\infty} k_t dF(z_{t-1}) di = \int_{z_{t-1}^*}^{\infty} dF(z_{t-1}) \bar{k}_t,$$

where, $\bar{k}_t$ is the average level of k_t. From (2), this is reduced to (27).

Substituting (15) into (30), we obtain

$$Y_t = \frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)} R_t K_t. \tag{31}$$

From (8), (29), and (31), we obtain $w_t N_t = w_t \cdot 1 = (1-\alpha)\int_{z_{t-1}^*}^{\infty} y_t dF(z_{t-1}) = (1-\alpha)Y_t$, leading to

$$w_t = \frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha) R_t K_t. \tag{32}$$

Substituting (32) into (15), we obtain

$$z_t^* = \left(\frac{\varphi}{\varphi-1}\right)^{\frac{1-\alpha}{\alpha}} \left(\frac{(1-\sigma_k)R_{t+1}}{\alpha A}\right)^{\frac{1}{\alpha}}. \tag{33}$$

Here, a larger z_t^* means a higher barrier to becoming active entrepreneurs (see below (15)), indicating a decrease in the number of firms. Then, (33) shows that an increase in R_{t+1} reduces the number of firms (i.e., an increase in z_t^*) because it increases the cost of borrowing for entrepreneurs. In addition, investment subsidies σ_k lower the barrier to becoming an entrepreneur z_t^* and increase the number of less productive firms. This lowers aggregate productivity and reduces both output, (31), and the wage rate, (32).

Let us continue with the aggregation of the other elements. Aggregating (18), using (27) and (32), and keeping in mind that x_t is independent of z_t, we obtain

$$K_{t+1} = \frac{\beta}{1-\sigma_k-\lambda}(z_t^*)^{-\varphi}\left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha)R_t K_t + X_t\right], \tag{34}$$

where $X_t \equiv \int_1^{\infty} x_t dF(z_{t-1}) = \int_1^{z_{t-1}^*} x_t^l dF(z_{t-1}) + \int_{z_{t-1}^*}^{\infty} x_t^e dF(z_{t-1})$. Here, X_t is derived using (16) and (23) associating with (2), (17), (25), (27), (28), (31), and (33) as

$$X_t = \gamma\left[\frac{\varphi(1-\sigma_k)}{\varphi-1} R_t K_t + B_t\right]. \tag{35}$$

See Appendix B for the derivations of (34) and (35). From (34) and (35), we obtain

$$K_{t+1} = \frac{\beta}{1-\sigma_k-\lambda}(z_t^*)^{-\varphi}\left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha+\alpha\gamma)R_t K_t + \gamma B_t\right]. \tag{36}$$

Through bequests X_t, public debts held by lenders (γB_t) have a crowding-in effect on investment.

Aggregating (22) and using (25) and (32), we obtain

$$\int_1^{z_t^*} l_t dF(z_t) = \beta \left[1 - (z_t^*)^{-\varphi}\right] \left[\frac{(1-\sigma_k)\varphi}{\alpha(\varphi - 1)}(1-\alpha)R_t K_t + X_t\right] - q_t B_{t+1}. \tag{37}$$

See Appendix B for the derivation of (37). Substituting (28) and (35) into (37), we obtain

$$\lambda K_{t+1} = \beta \left[1 - (z_t^*)^{-\varphi}\right] \left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi - 1)}(1-\alpha+\alpha\gamma)R_t K_t + \gamma B_t\right] - q_t B_{t+1}. \tag{38}$$

The RHS of (38) represents the aggregate supply of credit in the financial market, whereas the LHS shows aggregate demand for it. Increases in public borrowing (public debt issuance), $q_t B_{t+1}$, crowd out the aggregate supply of credit. This leads to upward pressure on the interest rate R_{t+1} in the financial market, as we see later.[11]

Associating (36) and (38) with (21) and $\tilde{B}_{t+1} \equiv q_t B_{t+1}$, we obtain the following asset market-clearing condition:

$$K_{t+1} + \tilde{B}_{t+1} = \beta R_t \left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi - 1)}(1-\alpha+\alpha\gamma)K_t + \gamma \tilde{B}_t\right] + \sigma_k K_{t+1}. \tag{39}$$

(39) indicates that an increase in public debt $\tilde{B}_{t+1}$ decreases private investment K_{t+1}. This is the familiar crowding-out effect of public debt $\tilde{B}_{t+1}$ on private investment K_{t+1}.

From (26) and (27), we obtain

$$\tilde{B}_{t+1} = R_t \tilde{B}_t + \sigma_k K_{t+1}. \tag{40}$$

An increase in outstanding public debt increases the issuance of public bonds and worsens the fiscal condition. This is also the case for investment subsidies σ_k for a given investment K_{t+1}.

[11] There are two types of asset holdings: lending to firms and purchasing public bonds. Following Walras' law, we only have to focus on one of them (financial market or public bond market). This study focuses on the financial market and explains the findings based on it. However, we also briefly mention the public bond market. In equilibrium, the supply of bonds $q_t B_{t+1}$ by the government equals its demand by households for a given value of λK_{t+1}. An increase in the issuance of bonds by the government leads to an excess supply of bonds, which lowers the price of bonds q_t and leads to upward pressure on the interest rate R_{t+1} from (21).

4 Dynamic system and (un)sustainable paths of the economy

In this section, we derive the dynamic system of the economy and check whether public debt is sustainable.

Let us define $\theta \equiv \tilde{B}_t/K_t$ as the ratio of public debt to capital. From (39) and (40), we obtain

$$\frac{K_{t+1}}{K_t} = R_t \left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)} \beta(1-\alpha+\alpha\gamma) - (1-\beta\gamma)\theta_t \right], \tag{41}$$

$$\frac{\tilde{B}_{t+1}}{\tilde{B}_t} = R_t \left[\frac{\varphi\sigma_k(1-\sigma_k)}{\alpha(\varphi-1)} \beta(1-\alpha+\alpha\gamma)\theta_t^{-1} + [1-\sigma_k(1-\beta\gamma)] \right]. \tag{42}$$

Here, to ensure $K_{t+1}/K_t > 0$, we assume the following condition:

$$\theta_t < \bar{\theta} \equiv \frac{\varphi(1-\sigma_k)\beta(1-\alpha+\alpha\gamma)}{\alpha(\varphi-1)(1-\beta\gamma)}. \tag{43}$$

The last term of the RHS of (41) ($-(1-\beta\gamma)\theta_t$) indicates that the crowding-out effect of public debts on investment (see (39)) is stronger than the crowding-in effect public debts throuth bequests (see (41)).

From (41) and (42), we obtain

$$\theta_{t+1} = \frac{\frac{\varphi}{\alpha(\varphi-1)}\beta(1-\alpha+\alpha\gamma)\sigma_k(1-\sigma_k) + [1-\sigma_k(1-\beta\gamma)]\theta_t}{\frac{\varphi}{\alpha(\varphi-1)}\beta(1-\alpha+\alpha\gamma)(1-\sigma_k) - (1-\beta\gamma)\theta_t} \equiv \Lambda(\theta_t;\sigma_k). \tag{44}$$

(44) with (43) characterizes the dynamic system of the economy. The LHS of (44) represents the 45 degree line, while the right-hand side satisfies $\Lambda(0;\sigma_k) = \sigma_k > 0$, $\Lambda'(\theta_t;\sigma_k) > 0$, and $\Lambda''(\theta_t;\sigma_k) > 0$ ($\Lambda(0;0) = 0$, $\Lambda'(\theta_t;0) > 0$, and $\Lambda''(\theta_t;0) > 0$) obviously. The RHS of (44) intersects the LHS twice at the steady states S and U, as in Figure 1, if and only if

$$\left[1-\sigma_k(1-\beta\gamma) - \tfrac{\varphi}{\alpha(\varphi-1)}\beta(1-\alpha+\alpha\gamma)(1-\sigma_k)\right]^2 > 4(1-\beta\gamma)\tfrac{\varphi}{\alpha(\varphi-1)}\beta(1-\alpha+\alpha\gamma)\sigma_k(1-\sigma_k), \tag{45}$$

and

$$\frac{\varphi}{\alpha(\varphi-1)}\beta(1-\alpha+\alpha\gamma)(1-\sigma_k) - [1-\sigma_k(1-\beta\gamma)] > 0. \tag{46}$$

Let us denote the two stationary values of θ_t as θ_S^* at S and θ_U^* at U. From (43) and (44), $\theta_S^* < \theta_U^* < \bar{\theta}$ is satisfied if and only if

$$\Lambda(\bar{\theta};\sigma_k) > \bar{\theta}. \tag{47}$$

Thus, we arrive at the following lemma.

Lemma 2. *Two steady states, as represented by S and U in Figure 1, exist under* (45), (46)*, and* (47)*. The steady state S is stable, while U is unstable.*

[Figure 1]

Lemma 2 indicates that the ratio of public debt to capital θ_t converges to the stable steady-state value θ_S^* as long as the initial value is $\theta_0 < \theta_U^*$. Otherwise (the case of $\theta_0 > \theta_U^*$), θ_t continues to grow and eventually violates (43), making fiscal policy with debt financing unsustainable. Therefore, θ_U^* represents the maximum ratio of public debt to capital to ensure the sustainability of public debt. Thus, we obtain the following proposition.

Proposition 1. *Public debt is (not) sustainable if the initial debt to capital ratio θ_0 is smaller (larger) than θ_U^*. On sustainable transition paths, θ_t converges to the stable steady-state value θ_S^*.*

The definition of $\theta_t = \tilde{B}_t/K_t$ indicates that the growth in public debt ($\tilde{B}_{t+1}/\tilde{B}_t$) is larger (smaller) than that in private capital (K_{t+1}/K_t) for the fiscally unsustainable (sustainable) region $\theta_t > \theta_U^*$ ($\theta_S^* < \theta_t < \theta_U^*$). To see the intuition behind this, in the next section we examine how an increase in θ_t affects the interest rate as well as the growth in capital, GDP, and public debt.

5 Interest rates and growth in capital, GDP, and public debt

We first derive the relationships among the behavior of entrepreneurs z_t^*, the (gross) interest rate R_{t+1}, and θ_t. Using (36) and (41) with $\tilde{B}_t \equiv B_t/R_t$ and $\theta_t \equiv \tilde{B}_t/K_t$ yields

$$z_t^* = \left\{ \left(\frac{\beta}{1-\sigma_k-\lambda} \right) \frac{\frac{\varphi}{\alpha(\varphi-1)}(1-\sigma_k)(1-\alpha+\alpha\gamma)+\gamma\theta_t}{\frac{\varphi}{\alpha(\varphi-1)}\beta(1-\sigma_k)(1-\alpha+\alpha\gamma)-(1-\beta\gamma)\theta_t} \right\}^{1/\varphi} \equiv z^*(\theta_t;\sigma_k). \quad (48)$$

Substituting (48) into (33), we obtain

$$R_{t+1} = \frac{\alpha A}{1-\sigma_k}\left(\frac{\varphi-1}{\varphi}\right)^{1-\alpha}\left(\frac{\beta}{1-\sigma_k-\lambda}\right)^{\frac{\alpha}{\varphi}}\left[\frac{\frac{\varphi}{\alpha(\varphi-1)}(1-\sigma_k)(1-\alpha+\alpha\gamma)+\gamma\theta_t}{\frac{\varphi}{\alpha(\varphi-1)}\beta(1-\sigma_k)(1-\alpha+\alpha\gamma)-(1-\beta\gamma)\theta_t}\right]^{\frac{\alpha}{\varphi}}$$
$$\equiv \mathcal{R}(\theta_t;\sigma_k). \quad (49)$$

Before examining the effects of θ_t on $z^*(\theta_t;\sigma_k)$ and $\mathcal{R}(\theta_t;\sigma_k)$, we consider the case without public debt finance, for which $\theta_t = 0$ in (48) and (49). We obtain $z^*(0;\sigma_k) = (1-\sigma_k-\lambda)^{-\frac{1}{\varphi}}$ and $\mathcal{R}(0;\sigma_k) = \frac{\alpha A}{1-\sigma_k}\left(\frac{\varphi-1}{\varphi}\right)^{1-\alpha}(1-\sigma_k-\lambda)^{-\frac{\alpha}{\varphi}}$, both of which are constant. Furthermore, a decrease (increase) in λ (i.e., larger (smaller) financial friction) lowers (raises) the interest rate

R_{t+1} and increases (reduces) the number of firms, i,e., a decrease (an increase) in z_t^*.[12] This is because a smaller (larger) λ leads to a tighter (looser) credit constraint (see (7)) and makes borrowing more difficult (easier), i.e., a decrease (increase) in the aggregate demand of credit, in the financial market. This decreases (increases) the interest rate R_{t+1} and reduces (raises) the hurdles to becoming an entrepreneur z_t^*.

Next, we examine the effect of θ_t on $z^*(\theta_t; \sigma_k)$ and $\mathcal{R}(\theta_t; \sigma_k)$. From (48) and (49), we derive the following lemma.

Lemma 3. $z^{*\prime}(\theta_t; \sigma_k) > 0$ *and* $\mathcal{R}'(\theta_t; \sigma_k) > 0$.

Increases in the public debt to capital ratio θ_t raise the interest rate, $\mathcal{R}'(\theta_t; \sigma_k) > 0$.[13] This is because an increase in government debt increases the issuance of government bonds (see (40)) and decreases the aggregate supply of credit in the financial market (see (38)). The rise in $\mathcal{R}(\theta_t; \sigma_k)$ (through an increase in θ_t) reduces the number of firms (i.e., an increase in z_t^*) because it raises the cost of borrowing for entrepreneurs (see (33)). Thus, $z^{*\prime}(\theta_t; \sigma_k) > 0$. Figure 2(b) shows this in a numerical example with the reasonable set of parameter values in Table 1 (see Appendix C for the choice of parameter values), while Figure 2(a) reproduces the dynamic system of θ_t presented in Section 4.

Furthermore, from (44) and (49), we obtain the following relationship between θ_t and the current interest rate R_t:

$$R_t = \Psi(\theta_t; \sigma_k) \;\text{ and }\; \Psi'(\theta_t; \sigma_k) = \frac{\mathcal{R}'(\theta_{t-1}; \sigma_k)}{\Lambda'(\theta_{t-1}; \sigma_k)} > 0, \tag{50}$$

indicating that the current interest rate R_t is increasing in θ_t. Figure 2(c) shows this relationship (50) numerically. This result is different from that in previous studies on the growth effect of public debt (e.g., Bräuninger, 2005; Futagami and Konishi, 2023; Saint-Paul, 1992) because these assume that the interest rate is constant over time because they adopt the standard AK model without firm heterogeneity and financial frictions.

[Table 1 and Figure 2]

Next, we derive the relationship between θ_t and the growth in capital, GDP, and public debt, shown as K_{t+1}/K_t, Y_{t+1}/Y_t, and B_{t+1}/B_t, respectively. Applying (50) to (41) and (42) and using

[12] This result is in line with Farhi and Tirole (2012), who show that a low pledgeability leads to a low interest rate.

[13] This result is in line with Bernanke and Gertler (1989), who show that bubbles increase the rate of return on savings and improve borrowers' net worth, which crowds in their future investments. Here, bubbles are similar to an increase in government debt in the sense that they crowd out private investments.

$\frac{Y_{t+1}}{Y_t} = \frac{R_{t+1}}{R_t}\frac{K_{t+1}}{K_t}$ (from (31)) and (49), we obtain

$$\frac{K_{t+1}}{K_t} \equiv g^K(\theta_t;\sigma_k) = \Psi(\theta_t;\sigma_k)\left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\beta(1-\alpha+\alpha\gamma)-(1-\beta\gamma)\theta_t\right], \tag{51}$$

$$\frac{\tilde{B}_{t+1}}{\tilde{B}_t} \equiv g^B(\theta_t;\sigma_k) = \Psi(\theta_t;\sigma_k)\left[\frac{\varphi\sigma_k(1-\sigma_k)}{\alpha(\varphi-1)}\beta(1-\alpha+\alpha\gamma)\theta_t^{-1}+1-\sigma_k(1-\beta\gamma)\right], \tag{52}$$

$$\frac{Y_{t+1}}{Y_t} \equiv g^Y(\theta_t;\sigma_k) = \mathcal{R}(\theta_t;\sigma_k)\left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\beta(1-\alpha+\alpha\gamma)-(1-\beta\gamma)\theta_t\right]. \tag{53}$$

An increase in the public debt to capital ratio θ_t has two opposite effects on $g^K(\theta_t;\sigma_k)$ and $g^Y(\theta_t;\sigma_k)$: (i) the negative growth effect resulting from the crowding-out effect of public debt on investment (see (39)) and (ii) the positive growth effect owing to a rise in the interest rate. The positive effect (ii) is called "the balance sheet effect" by Bernanke and Gertler (1989). The interest rate rises because increases in public debt lead to the greater issuance of public bonds (see (40)) and decrease the aggregate supply of credit in the financial market (see (38)). A rise in the interest rate increases output and wages (see (31) and (32)). This is because such an interest rate rise increases the cost of borrowing for entrepreneurs and reduces the number of firms, which increases capital intensity and the marginal productivity of capital. The negative effect dominates the positive one in the numerical example in Figure 2(d), indicating that both $g^K(\theta_t;\sigma_k)$ and $g^Y(\theta_t;\sigma_k)$ are decreasing in θ_t.[14]

The non-monotonic relationship between the growth in the interest rate R_{t+1}/R_t and θ_t creates the disparity between $g^K(\theta_t;\sigma_k)$ and $g^Y(\theta_t;\sigma_k)$. R_{t+1} is convex in θ_t (because so is z_t from Figure 2(b)), while R_t is concave in θ_t from Figure 2(c), resulting in $g^K(\theta_t;\sigma_k) > (<) g^Y(\theta_t;\sigma_k)$ for $\theta_S^* \leq \theta_t \leq \theta_U^*(\theta_t < \theta_S^*$ and $\theta_t > \theta_U^*)$.

We next examine the relationship between $g^B(\theta_t;\sigma_k)$ and θ_t. An increase in the public debt to capital ratio θ_t also has two opposite effects on $g^B(\theta_t;\sigma_k)$. First, an increase in this ratio θ_t raises the interest rate by decreasing the aggregate supply of credit in the financial market (see (38)). This rise in the interest rate boosts interest payments and $g^B(\theta_t;\sigma_k)$. Second, an increase in public debt crowds out private investment. Therefore, the burden of public spending on investment subsidies $\sigma_k K_{t+1}$ shrinks as the ratio of public debt to capital θ_t increases, leading to a reduction in $g^B(\theta_t;\sigma_k)$. The negative (positive) effects on $g^B(\theta_t;\sigma_k)$ dominate the positive (negative) effects on $g^B(\theta_t;\sigma_k)$ when θ_t is small (large), as Figure 2(d) shows.

We summarize these results in the following proposition.

[14]This result is in line with Hirano and Yanagawa (2017). They show that when the degree of pledgeability λ is relatively high, bubbles lower growth. In contrast to Hirano and Yanagawa (2017), the positive growth effect cannot dominate the negative one in this study even when λ is relatively low. This is because the crowding-out effect of debt is stronger in OLG models with finite-lived households than in Ramsey-type models with infinitely lived Ricardian households. Hirano and Yanagawa (2017) uses a Ramsey-type model.

Proposition 2. *An increase in θ_t increases the interest rate and decreases the growth rate of K_t and Y_t. The non-monotonic relationship between the growth in the interest rate R_{t+1}/R_t and θ_t creates the disparity between $g^K(\theta_t;\sigma_k)$ and $g^Y(\theta_t;\sigma_k)$. An increase in θ_t decreases the growth rate of B_t when θ_t is small, whereas it increases the growth rate of B_t when θ_t is large.*

Finally, we address the values of $\mathcal{R}(\theta_t;\sigma_k)$, $z^*(\theta_t;\sigma_k)$, $\Psi(\theta_t;\sigma_k)$, $g^K(\theta_t;\sigma_k)$, $g^Y(\theta_t;\sigma_k)$, and $g^B(\theta_t;\sigma_k)$ in the stable steady state S. From (44), (48) (with Lemma 2), (49), (50), (51), (52), and (53), we obtain the following proposition.

Proposition 3. *In the steady state S, the interest rate and cutoff value of z take the constant values of $\mathcal{R}(\theta_S^*;\sigma_k)(=\Psi(\theta_S^*;\sigma_k))$ and $z^*(\theta_S^*;\sigma_k)$ over time. Furthermore, K_t, Y_t, and $\tilde{B}_t$ grow at the same rate of $g(\theta_S^*;\sigma_k)\equiv g^K(\theta_S^*;\sigma_k)=g^Y(\theta_S^*;\sigma_k)=g^B(\theta_S^*;\sigma_k)$.*

Propositions 2 and 3 together with Figure 2 show the reason why fiscal policy with debt financing is unsustainable (sustainable) for $\theta_t>\theta_U^*$ ($0<\theta_t<\theta_U^*$).

6 Investment subsidies to firms

In this section, we examine the effects of introducing investment subsidies to firms on growth and fiscal sustainability. In this study, as noted in the Introduction, investment subsidies are financed through the issuance of public bonds rather than through tax finance as in previous studies (e.g., Morimoto, 2018).

From (44), we obtain

$$\left.\frac{\partial\Lambda(\theta_t;\sigma_k)}{\partial\sigma_k}\right|_{\sigma_k=0}=1+\frac{\beta\varphi(1-\alpha+\alpha\gamma)}{[\beta\varphi(1-\alpha+\alpha\gamma)+\alpha(\varphi-1)(1-\beta\gamma)\theta_t]^2}>0, \tag{54}$$

indicating that $\Lambda(\theta_t;\sigma_k)$ shifts upward in response to a marginal increase in σ_k evaluated at $\sigma_k=0$, as Figure 3 shows. Thus, we arrive at the following proposition.

Proposition 4. *Providing an investment subsidy for entrepreneurs (a) makes public debt less sustainable and (b) increases the ratio of public debt to capital θ_S^* in the steady state S.*

[Figure 3]

Figure 4(a) illustrates the case of an increase in σ_k from 0.01 to 0.04, justifying that increases in σ_k (i) make public debt less sustainable and (ii) increase the ratio of public debt to capital θ_S^* in the steady state S.

Investment subsidies to firms encourage investment by each firm ((18)) and promote economic growth $g^Y(\theta_t;\sigma_k)$ for θ_t as given, which we call the short-run effect of σ_k on $g^Y(\theta_t;\sigma_k)$.

However, investment subsidies also increase the issuance of public bonds and accelerate the accumulation of public debt $g^B(\theta_t; \sigma_k)$ for θ_t as given, which we call the short-run effect of σ_k on $g^B(\theta_t; \sigma_k)$. The short-run effect of σ_k on $g^B(\theta_t; \sigma_k)$ dominates the short-run effect of σ_k on $g^Y(\theta_t; \sigma_k)$, shifting $\theta_{t+1} = \Lambda(\theta_t; \sigma_k)$ upward; hence, θ_S^* increases, while θ_U^* decreases. Consequently, in the long run, investment subsidies to firms (i) make public debt less sustainable and (ii) increase the public debt to capital ratio in the steady state S.

[Figure 4]

Figure 4(b) illustrates the case of an increase in σ_k from 0.01 to 0.04, justifying that the short-run effect of σ_k on $g^Y(\theta_t; \sigma_k)$ is smaller than the short-run effect of σ_k on $g^B(\theta_t; \sigma_k)$. The reason for the smaller short-run effect of σ_k on $g^Y(\theta_t; \sigma_k)$ is as follows. First, investment subsidies σ_k lower the barrier to becoming an entrepreneur z^* and increase the number of firms with lower productivity, decreasing aggregate productivity ((33)). Second, the increase in these firms increases aggregate demand for credit in the financial market ((38)) and leads to upward pressure on the interest rate R_{t+1} ((49)). An increase in the interest rate raises the cost of repaying public debt and crowds out private investment ((39)). These two factors have significant negative effects on economic growth. Thus, the positive direct effect of investment subsidies on economic growth becomes small. Conversely, the reason for the larger short-run effect of σ_k on $g^B(\theta_t; \sigma_k)$ is attributable to both a direct effect through the increase in $\sigma_k K_{t+1}$ ((40)) and an indirect effect through the rise in the interest rate ((49)) in the financial market.

Next, we investigate the long-run growth effect of investment subsidies σ_k in the steady state S. In the long run, θ_S^* increases as σ_k rises (see Proposition 4(b)). As shown in Section 5, an increase in θ_t increases the interest rate $\mathcal{R}(\theta_S^*; \sigma_k)$ and decreases the growth rates of K_t and Y_t (recall Proposition 2). Keeping this in mind, we calculate the long-run growth effect of an increase in σ_k evaluated at $\sigma_k = 0$. Using (53), and noting that $\theta_S^* = 0$ for $\sigma_k = 0$ and that $\frac{\partial \ln g^Y(\theta_S^*;\sigma_k)}{\partial \sigma_k}|_{\sigma_k=0} = \frac{1}{g^Y(\theta_S^*;\sigma_k)} \frac{\partial g^Y(\theta_S^*;\sigma_k)}{\partial \sigma_k}|_{\sigma_k=0}$, we obtain

$$\left.\frac{\partial \ln g^Y(\theta_S^*; \sigma_k)}{\partial \sigma_k}\right|_{\sigma_k=0} = \left.\frac{\partial \ln \mathcal{R}(\theta_S^*; \sigma_k)}{\partial \sigma_k}\right|_{\sigma_k=0} - \frac{\beta \frac{\varphi}{\varphi-1}\left(\frac{1-\alpha}{\alpha} + \gamma\right) + (1-\beta\gamma)\frac{\partial \theta_S^*}{\partial \sigma_k}|_{\sigma_k=0}}{\beta \frac{\varphi}{\varphi-1}\left(\frac{1-\alpha}{\alpha} + \gamma\right)}. \quad (55)$$

From (44) and since $\theta_S^* = 0$ for $\sigma_k = 0$, we obtain

$$\left.\frac{\partial \theta_S^*}{\partial \sigma_k}\right|_{\sigma_k=0} = \frac{\beta \frac{\varphi}{\varphi-1}\left(\frac{1-\alpha}{\alpha} + \gamma\right)}{\beta \frac{\varphi}{\varphi-1}\left(\frac{1-\alpha}{\alpha} + \gamma\right) - 1} > 0. \quad (56)$$

Here, note that $\beta \frac{\varphi}{\varphi-1}\left(\frac{1-\alpha}{\alpha} + \gamma\right) - 1 > 0$ from Condition (45). Furthermore, from (49) and the fact that $\theta_S^* = 0$ for $\sigma_k = 0$, we obtain

$$\left.\frac{\partial \ln \mathcal{R}(\theta_S^*; \sigma_k)}{\partial \sigma_k}\right|_{\sigma_k=0} = 1 + \frac{\alpha}{\varphi}\frac{1}{1-\lambda} + \frac{\alpha}{\varphi}\frac{\frac{\partial \theta_S^*}{\partial \sigma_k}|_{\sigma_k=0}}{\beta\frac{\varphi}{\varphi-1}\left(\frac{1-\alpha}{\alpha}+\gamma\right)} > 0. \tag{57}$$

Substituting (56) and (57) into (55) yields the following proposition.

Proposition 5. *Introducing investment subsidies to firms increases (decreases) long-run growth* $\partial \ln g^Y(\theta_S^*; \sigma_k)/\partial \sigma_k|_{\sigma_k=0} > (<)0$ *if and only if*

$$\frac{\alpha}{\varphi}\frac{1}{1-\lambda} - \frac{1-\beta\gamma-\frac{\alpha}{\varphi}}{\beta\frac{\varphi}{\varphi-1}\left(\frac{1-\alpha}{\alpha}+\gamma\right)-1} > (<)0. \tag{58}$$

Investment subsidies encourage investment and increase the marginal productivity of capital ((57)). This works positively for economic growth, as shown in the first term of (58). However, an increase in θ_t decreases the growth rate of $g^Y(\theta_S^*; \sigma_k)$ (Proposition 2), as shown in the second term of (58). The former positive growth effect of σ_k becomes stronger when the financial market is more perfect (λ is larger) by (18) and (49). This is because a larger λ leads to a looser credit constraint (see (7)) and makes borrowing easier (i.e., an increase in the aggregate demand of credit) in the financial market. This increases the interest rate $\mathcal{R}(\theta_S^*; \sigma_k)$. Accordingly, condition (58) shows that when the financial market is close to perfect, $\lambda \approx 1$, the positive growth effects dominate the negative ones. By contrast, when the financial market is somewhat imperfect (λ is low), the negative growth effects dominate the positive ones. This is because a smaller λ makes borrowing more difficult (i.e., a decrease in the aggregate demand of credit) in the financial market and decreases the interest rate $\mathcal{R}(\theta_S^*; \sigma_k)$. Thus, investment subsidies financed by public debt σ_k decrease (increase) long-run economic growth when λ is not so large (very large). Figure 5 uses a numerical exercise to show that $g^Y(\theta_S^*; \sigma_k)$ is decreasing (increasing) in σ_k in wide ranges of λ (in the case of 0.9), allowing us to check the result of Proposition 5. Interestingly, we find an inverted U-shaped relationship between σ_k and growth when $\sigma_k = 0.8$, as shown on the LHS of Figure 5, yielding the growth-maximizing subsidy rate σ_k^{GM}.[15] These results advocate that investment subsidies should not be financed by public debt unless the financial market is not so perfect.

[Figure 5]

[15]This is also likely to happen when λ is around 0.80 and 0.85. Moreover, we find that when λ is larger, so is σ_k^{GM}.

7 Investment subsidies financed through tax increases

The previous section showed that investment subsidies fully financed by public debt issuance hinder economic growth when λ is not so large (i.e., the financial market is not so perfect). The main purpose of this section is to incorporate income tax and show that investment subsidies financed by appropriate increases in income tax can enhance economic growth. Furthermore, we derive numerically how large tax increases associating with a rise in the investment subsidy rate σ_k are necessary to ensure the positive growth effects of investment subsidies.

We first extend our analyses by incorporating income tax. The government issues public bonds ($q_t B_{t+1}$) and collects income tax revenue (T_t) to finance investment subsidies and repay its debt. Specifically, the government imposes a flat rate of tax $\tau \in (0, 1)$ on wage income, the profits of active entrepreneurs, and the asset income of lenders. Therefore, the government budget constraints are given by

$$q_t B_{t+1} + T_t = B_t + \sigma_k \int_{z_t^*}^{\infty} k_{t+1} dF(z_t), \tag{59}$$

$$T_t = \tau \left[w_t N_t + \int_{z_{t-1}^*}^{\infty} \pi_t dF(z_{t-1}) + R_t \int_{1}^{z_{t-1}^*} (l_{t-1} + q_{t-1} b_t) dF(z_{t-1}) \right]. \tag{60}$$

Appendix D fully explains this extension of the model. We select the cases of $\lambda = 0.6, 0.7$ and 0.8 without changing the other parameter values and calculate a marginal increase in the tax rate from $\tau = 0$ in response to a subsidy increase for each value of σ_k to ensure the positive growth effects of σ_k. Here, recall that the growth effect of σ_k is always negative when $\lambda = 0.6$ and 0.7, while it exhibits an inverted U-shape when $\lambda = 0.8$ if investment subsidies are financed entirely by public debt.

The results in Table 2 show the following. Considering a marginal increase in the tax rate from $\tau = 0$ in response to a subsidy increase: $d\tau/d\sigma_k$, we find that the growth effect of σ_k turns positive, $d\hat{g}^Y(\theta_S^*; \sigma_k, \tau)/d\sigma_k|_{\tau=0} > 0$, if and only if $d\tau/d\sigma_k|_{\tau=0}$ is higher than the value in Table 2 for each value of σ_k. In the case of $\lambda = 0.8$, a marginal increase in the tax rate in response to a subsidy increase $d\tau/d\sigma_k|_{\tau=0}$ takes negative values for low values of σ_k because the growth effects of σ_k even without tax increases are already positive for these values of σ_k (as we have seen previously) and no tax increases are necessary. The results here together with those presented in Section 6 indicate that increases in investment subsidies should be financed not only by issuing public bonds, but also through tax increases when λ is not so large (i.e., the financial market is not so perfect).

[Table 2]

The positive effects of income tax on growth are attributed to the following (see Appendix D for more details). In the short run, increases in income tax have a distortionary effect on growth, while they also reduce the issuance of public bonds, decrease the crowding-out effect of public debt on capital accumulation, and promote economic growth. The latter effects dominate the former, resulting in an overall positive impact on growth. Additionally, increases in income tax have a distortionary effect on capital accumulation, which decreases demand for credit in the financial market and lowers interest rates. Lower interest rates decrease borrowing costs and increase the number of active entrepreneurs, which decreases capital intensity and negatively affects growth.

In the long run, increases in income tax rates boost tax revenue and can reduce the issuance of public bonds, thereby lowering the ratio of public debt to capital θ_S^*. This reduces the costs of repaying public debt and issuing new public bonds, which crowds in capital accumulation and enhances economic growth. A lower θ_S^* also decreases the interest rate $\hat{\mathcal{R}}(\cdot)$ because it reduces the costs of repaying public debt and issuing new bonds, increasing the aggregate supply of credit in the financial market. This, in turn, increases the entry of firms, decreases capital intensity, and negatively affects economic growth.

The positive short- and long-run effects on growth outweigh the negative ones. Therefore, the overall effect of income tax on growth is positive.

8 Conclusion

This study examines the effect of public debt on growth, interest rates, and fiscal sustainability using a simple endogenous growth model with financial frictions and firm heterogeneity. An increase in public debt leads to higher real interest rates through financial markets increase the cost of repaying public debt, and reduce private investment, leading to lower long-run growth. Thus, large public debt is less sustainable. This study also examines the effect of investment subsidies financed by public debt and finds that they hamper economic growth in the long run unless the credit market is close to perfect. Therefore, increases in investment subsidies should be financed not only by issuing public bonds, but also through tax increases. This is an important policy implication for many developed countries that have increased investment subsidies while relying heavily on public debt for fiscal management.

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Appendices

A Proof of $U_t^e = U_t^l$ for $z_{i,t} = z_t^*$

Let us denote the items $(c_t^{y,j}, c_{t+1}^{o,j}, x_{t+1}^j, x_t, k_{t+1}, y_{t+1})$, and $j \in \{e, l\}$ of households whose productivity is z_t^* as $(\bar{c}_t^{y,j}, \bar{c}_{t+1}^{o,j}, \bar{x}_{t+1}^j \bar{x}_t, \bar{k}_{t+1}, \bar{y}_{t+1})$, respectively. First, consider the case in which households with $z_{i,t} = z_t^*$ become entrepreneurs.

Substituting $d_{i,t} = \lambda k_{i,t+1}$ and (18) into (4) yields

$$\bar{c}_t^{y,e} = (1-\beta)(w_t + \bar{x}_t), \tag{A.1}$$

while inserting (16) and (17) into (5) yields

$$\bar{c}_{t+1}^{o,e} = (1-\gamma)(\alpha \bar{y}_{i,t+1} - \lambda R_{t+1} \bar{k}_{i,t+1}). \tag{A.2}$$

From (14), $\bar{y}_{t+1} = A z_t^* \bar{k}_{t+1} \left[\frac{(1-\alpha)AK_t}{w_t}\right]^{\frac{1-\alpha}{\alpha}}$. Substituting (15) into this, we obtain $\bar{y}_{t+1} = (1 - \sigma_k) R_{t+1} \bar{k}_{t+1}$. Hence, (A.2) can be rewritten as

$$\begin{aligned} \bar{c}_{t+1}^{o,e} &= (1-\gamma)(1-\sigma_k-\lambda) R_{t+1} \bar{k}_{i,t+1} \\ &= \beta(1-\gamma) R_{t+1}(w_t + \bar{x}_t), \end{aligned} \tag{A.3}$$

where we use (18). Substituting $\bar{y}_{t+1} = (1-\sigma_k) R_{t+1} \bar{k}_{t+1}$ and (17) into (16), we obtain

$$\bar{x}_{t+1}^e = \beta \gamma R_{t+1}(w_t + \bar{x}_t). \tag{A.4}$$

Second, consider the case in which households with $z_{i,t} = z_t^*$ become lenders. Substituting

(21), (22), and (23) into (19) and (20), we obtain

$$\bar{c}_t^{y,l} = (1-\beta)(w_t + \bar{x}_t), \tag{A.5}$$

$$\bar{c}_{t+1}^{o,l} = \beta(1-\gamma)R_{t+1}(w_t + \bar{x}_t). \tag{A.6}$$

From (22) and (23), we obtain

$$\bar{x}_{t+1}^{l} = \beta\gamma R_{t+1}(w_t + \bar{x}_t). \tag{A.7}$$

Owing to $\bar{c}_t^{y,e} = c_{i,t}^{y,l}$, $\bar{c}_{t+1}^{o,e} = c_{t+1}^{o,l}$, and $\bar{x}_{t+1}^{e} = \bar{x}_{t+1}^{l}$ and (3), $U_t^e = U_t^l$ holds for $z_{i,t} = z_t^*$.

B Derivations of (34), (35), and (37)

First, (34) is derived by aggregating (18) as

$$\int_{z_t^*}^{\infty} k_{t+1} dF(z_t) \int_1^{\infty} dF(z_{t-1}) = \frac{\beta}{1-\sigma_k-\lambda} \int_{z_t^*}^{\infty} dF(z_t) \left[w_t \int_1^{\infty} dF(z_{t-1}) + \int_1^{\infty} x_t dF(z_{t-1}) \right],$$

and using (27) and (32), and $X_t \equiv \int_1^{\infty} x_t dF(z_{t-1})$.

Next, we derive (35). From (16) and (17), we obtain $x_{i,t+1}^e = \gamma(\alpha y_{i,t+1} - R_{t+1}\lambda k_{i,t+1})$. Aggregating this using (27) and (31), we obtain

$$\int_{z_t^*}^{\infty} x_{t+1}^e dF(z_t) = \gamma \left(\frac{\varphi(1-\sigma_k)}{\varphi - 1} - \lambda \right) R_{t+1} K_{t+1}. \tag{B.1}$$

Next, aggregating $x_{i,t+1}^l$ in (23) leads to

$$\int_1^{z_t^*} x_{t+1}^l dF(z_t) = \gamma \left[R_{t+1} \int_1^{z_t^*} l_t dF(z_t) + \int_1^{z_t^*} b_{t+1} dF(z_t) \right].$$

This, together with (2), (25), and (28), yields

$$\int_1^{z_t^*} x_{t+1}^l dF(z_t) = \gamma (R_{t+1}\lambda K_{t+1} + B_{t+1}). \tag{B.2}$$

Associating $X_t \equiv \int_1^{\infty} x_{t+1} dF(z_t) = \int_1^{z_t^*} x_{t+1}^l dF(z_t) + \int_{z_t^*}^{\infty} x_{t+1}^e dF(z_t)$ with (33), (B.1), and (B.2) yields

$$X_{t+1} = \gamma \left[\frac{\varphi(1-\sigma_k)}{\varphi-1} R_{t+1} K_{t+1} + B_{t+1} \right]. \tag{B.3}$$

Thus, we obtain (35).

Finally, we derive (37). By aggregating (22) as

$$\left[\int_1^{z_t^*} l_t dF(z_t) + \int_1^{z_t^*} q_t b_{t+1} dF(z_t)\right] \int_1^{\infty} dF(z_{t-1}) \\ = \beta \int_1^{z_t^*} dF(z_t) \left[w_t \int_1^{\infty} dF(z_{t-1}) + \int_1^{\infty} x_t dF(z_{t-1})\right], \tag{B.4}$$

and substituting (25), (32), and $X_t \equiv \int_1^{\infty} x_t dF(z_{t-1})$ into (B.4), we obtain (37).

C Choice of the parameter values for the numerical example

We set $\beta = 0.3$ because the discount factor should be $\beta/(1-\beta) = 0.97^{30} \approx 0.4$. The parameter α is set to 0.4, which is near the average values of the United States (US) ($\alpha = 0.35$), the EU ($\alpha = 0.38$), and Japan ($\alpha = 0.38$).[16] We select the value of γ to satisfy $\gamma = \beta$ as the benchmark. The scale parameter $A = 5$ yields positive plausible values for the long-run growth rates, as shown in Figures 2, 4, and 5. We select $\lambda = 0.7$ so that the degree of financial market imperfection serves as the benchmark case. Finally, following Diamond and Saez (2011), Jaimovich and Rebelo (2017), and Mino (2015), we set $\varphi = 1.5$. This choice of φ suggests that the right tail of the income distribution implied by the model is the same as that estimated by Diamond and Saez (2011) for the US economy.

D Investment subsidies financed by public debt and income tax

Entrepreneurs' budget constraints with a constant income tax rate τ are given by

$$c_t^{y,e} = (1-\tau) w_t h_t^e + x_t + \sigma_k k_{t+1} - a_{t+1}, \qquad a_{t+1} = k_{t+1} - d_t, \tag{D.1}$$

$$c_{t+1}^{o,e} = (1-\tau)\pi_{t+1} - x_{t+1}^e \qquad \text{with (6).} \tag{D.2}$$

[16] See Trabandt and Uhlig (2011) for the values of the United States and the EU, and Hansen and İmrohoroğlu (2016) for the value of Japan.

The FOC with respect to n_{t+1} is given by (8) and those with respect to d_t and k_{t+1} are replaced by

$$d_t; \qquad \frac{1-\beta}{c_t^{y,e}} = \frac{\beta(1-\gamma)(1-\tau)R_{t+1}}{c_{t+1}^{o,e}} + \tilde{\mu}_t, \tag{D.3}$$

$$k_{t+1}; \qquad \frac{(1-\beta)(1-\sigma_k)}{c_t^{y,e}} = \frac{\beta(1-\gamma)(1-\tau)}{c_{t+1}^{o,e}}\frac{\partial \pi_{t+1}}{\partial k_{t+1}} + \lambda\tilde{\mu}_t, \tag{D.4}$$

$$\tilde{\mu}_t(\lambda k_{i,t+1} - d_t) = 0, \;\; \tilde{\mu}_t \geq 0, \;\; \lambda k_{t+1} - d_t \geq 0, \tag{D.5}$$

$$\tilde{\mu}_t = \frac{\beta(1-\gamma)(1-\tau)}{1-\sigma_k-\lambda}\frac{\alpha(y_{t+1}/k_{t+1}) - (1-\sigma_k)R_{t+1}}{c_{t+1}^{o,e}}. \tag{D.6}$$

Note that (13), (14), and (15) remain unchanged. Maximizing utility with respect to x_{t+1}^e yields

$$x_{t+1}^e = \gamma(1-\tau)\pi_{t+1}. \tag{D.7}$$

From (D.1), (D.2), (D.3), (D.6), (D.7), (17) and $d_t = \lambda k_{t+1}$, we obtain

$$k_{t+1} = \frac{\beta}{1-\sigma_k-\lambda}[(1-\tau)w_t + x_t]. \tag{D.8}$$

The budget constraints of lenders with a balanced budget and without public debt are

$$c_t^{y,l} = (1-\tau)w_t + x_t - l_t - q_t b_{t+1}, \tag{D.9}$$

$$c_{t+1}^{o,l} = (1-\tau)(R_{t+1}l_t + b_{t+1}) - x_{t+1}^l. \tag{D.10}$$

The FOCs of the lenders with respect to $l_t + q_t b_{t+1}$ and x_{t+1}^l result in

$$l_t + q_t b_{t+1} = \beta\left[(1-\tau)w_t + x_t\right], \tag{D.11}$$

$$x_{t+1}^l = \gamma(1-\tau)(R_{t+1}l_t + b_{t+1}). \tag{D.12}$$

The equations (27) to (33) remain unchanged. Aggregating (D.8) and using (32), we obtain

$$K_{t+1} = \frac{\beta}{1-\sigma_k-\lambda}(z_t^*)^{-\varphi}\left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha)(1-\tau)R_t K_t + X_t\right]. \tag{D.13}$$

From (D.7) and (17), we obtain $x_{t+1}^e = \gamma(1-\tau)[\alpha y_{t+1} - R_{t+1}\lambda k_{t+1}]$. Aggregating this using (27) and (31), we obtain

$$\int_{z_t^*}^{\infty} x_{t+1}^e dF(z_t) = \gamma\left(\frac{\varphi(1-\sigma_k)}{\varphi-1} - \lambda\right)(1-\tau)R_{t+1}K_{t+1}. \tag{D.14}$$

Next, aggregating x^l_{t+1} in (D.12) with (2), (25), and (28), we obtain

$$\int_1^{z_t^*} x^l_{t+1} dF(z_t) = \gamma(1-\tau)(R_{t+1}\lambda K_{t+1} + B_{t+1}). \tag{D.15}$$

Associating $X_{t+1} = \int_1^{z_t^*} x^l_{t+1} dF(z_t) + \int_{z_t^*}^{\infty} x^e_{t+1} dF(z_t)$ with (33), $\tilde{B}_{t+1} = q_t B_t$, $R_{t+1} = 1/q_t$, (D.14), and (D.15) yields

$$X_{t+1} = \gamma(1-\tau)R_{t+1}\left[\frac{\varphi(1-\sigma_k)}{\varphi-1}K_{t+1} + \tilde{B}_{t+1}\right]. \tag{D.16}$$

From (D.13) and (D.16), we obtain

$$K_{t+1} = \frac{\beta(1-\tau)}{1-\sigma_k-\lambda}(z_t^*)^{-\varphi}\left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha+\alpha\gamma)R_t K_t + \gamma R_t \tilde{B}_t\right]. \tag{D.17}$$

Aggregating (D.11) and using (32), (25), $\tilde{B}_{t+1} = q_t B_{t+1}$, and $R_{t+1} = 1/q_t$, we obtain

$$\int_1^{z_t^*} l_t dF(z_t) = \beta\left[1-(z_t^*)^{-\varphi}\right]\left[\frac{(1-\alpha)\varphi}{\alpha(\varphi-1)}(1-\tau)R_t K_t + X_t\right] - \tilde{B}_{t+1}. \tag{D.18}$$

Substituting (28) and (D.16) into (D.18), we obtain

$$\lambda K_{t+1} = \beta\left[1-(z_t^*)^{-\varphi}\right]\left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha+\alpha\gamma)(1-\tau)R_t K_t + \gamma \tilde{B}_t\right] - \tilde{B}_{t+1}. \tag{D.19}$$

From (D.17) and (D.19), we obtain

$$(1-\sigma_k)K_{t+1} = \beta(1-\tau)R_t\left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha+\alpha\gamma)K_t + \gamma\tilde{B}_t\right] - \tilde{B}_{t+1}. \tag{D.20}$$

Substituting (17) and (29) into (60), we obtain

$$\begin{aligned} T_t =& \tau\Bigg[w_t + \alpha\int_{z_{t-1}^*}^{\infty} y_t dF(z_{t-1}) - R_t\lambda\int_{z_{t-1}^*}^{\infty} k_t dF(z_{t-1}) \\ &+ R_t\int_1^{z_{t-1}^*} l_{t-1} dF(z_{t-1}) + q_{t-1}\int_1^{z_{t-1}^*} b_t dF(z_{t-1})\Bigg]. \end{aligned} \tag{D.21}$$

Substituting (25) with $\tilde{B}_{t+1} = q_t B_t$ and $R_{t+1} = 1/q_t$, (28), $\int_0^1 \int_{z_{t-1} \geq z_{t-1}^*} k_{i,t} dF(z_{t-1}) di = K_t$, $\int_0^1 \int_{z_{t-1} \geq z_{t-1}^*} y_{i,t} dF(z_{t-1}) di = Y_t$ with (31), and (32) into (D.21), we obtain

$$T_t = \tau\left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}R_t K_t + R_t\tilde{B}_t\right]. \tag{D.22}$$

Substituting (27) and (D.22) into (59) and using $\tilde{B}_{t+1} = q_t B_t$ and $R_{t+1} = 1/q_t$, we obtain

$$\tilde{B}_{t+1} = R_t \tilde{B}_t - \tau \left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)} R_t K_t + R_t \tilde{B}_t \right] + \sigma_k K_{t+1}. \tag{D.23}$$

From (D.20) and (D.23), we obtain

$$\frac{K_{t+1}}{K_t} = R_t \left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)} \beta(1-\tau)(1-\alpha+\alpha\gamma) + \tau \frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)} - (1-\tau)(1-\beta\gamma)\theta_t \right], \tag{D.24}$$

$$\frac{\tilde{B}_{t+1}}{\tilde{B}_t} = R_t \left[\sigma_k \beta(1-\tau) \left\{ \tfrac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)} (1-\alpha+\alpha\gamma)\theta_t^{-1} + \gamma \right\} + (1-\sigma_k) \left\{ 1 - \tau - \tau \tfrac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)} \theta_t^{-1} \right\} \right]. \tag{D.25}$$

To ensure the positive growth in capital $K_{t+1}/K_t > 0$, we assume

$$\theta_t < \bar{\theta}' \equiv \frac{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)} [\beta(1-\tau)(1-\alpha+\alpha\gamma)+\tau]}{(1-\tau)(1-\beta\gamma)}. \tag{D.26}$$

From (D.24) and (D.25), we obtain

$$\begin{aligned}
\theta_{t+1} &= \frac{(1-\sigma_k)\left[(1-\tau)\theta_t - \tau \frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\right] + \sigma_k \beta(1-\tau) \left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)} (1-\alpha+\alpha\gamma) + \gamma\theta_t \right]}{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)} \beta(1-\tau)(1-\alpha+\alpha\gamma) + \tau \frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)} - (1-\tau)(1-\beta\gamma)\theta_t} \\
&= \frac{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)} [\sigma_k \beta(1-\tau)(1-\alpha+\alpha\gamma) - (1-\sigma_k)\tau] + (1-\tau)(1-\sigma_k+\sigma_k\gamma\beta)\theta_t}{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)} [\beta(1-\tau)(1-\alpha+\alpha\gamma)+\tau] - (1-\tau)(1-\beta\gamma)\theta_t} \\
&\equiv \hat{\Lambda}(\theta_t; \sigma_k, \tau).
\end{aligned} \tag{D.27}$$

From (D.27), we obtain the following lemma.

Lemma 4. *Two steady states, S and U, as shown in Lemma 2, exist under the following conditions:*

$$\begin{aligned}
&\left[(1-\tau)\{1-\sigma_k(1-\beta\gamma)\} - \frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)} \{\beta(1-\tau)(1-\alpha+\alpha\gamma)+\tau\} \right]^2 \\
&> 4(1-\tau)(1-\beta\gamma) \frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)} [\sigma_k \beta(1-\tau)(1-\alpha+\alpha\gamma) - (1-\sigma_k)\tau],
\end{aligned} \tag{D.28}$$

$$\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)} \{\beta(1-\tau)(1-\alpha+\alpha\gamma)+\tau\} > (1-\tau)\{1-\sigma_k(1-\beta\gamma)\}, \tag{D.29}$$

$$\hat{\Lambda}(\bar{\theta}'; \sigma_k, \tau) > \bar{\theta}'. \tag{D.30}$$

From (D.17) and (D.24), we obtain

$$z_t^* = \left\{ \left(\frac{\beta(1-\tau)}{1-\sigma_k-\lambda} \right) \frac{\frac{\varphi}{\alpha(\varphi-1)}(1-\sigma_k)(1-\alpha+\alpha\gamma)+\gamma\theta_t}{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\left[\beta(1-\tau)(1-\alpha+\alpha\gamma)+\tau\right]-(1-\tau)(1-\beta\gamma)\theta_t} \right\}^{1/\varphi}$$
$$\equiv \hat{z}^*(\theta_t;\sigma_k,\tau). \tag{D.31}$$

Substituting (D.31) into (33) yields

$$R_{t+1} = \frac{\alpha A}{1-\sigma_k}\left(\frac{\varphi-1}{\varphi}\right)^{1-\alpha}\left(\frac{\beta(1-\tau)}{1-\sigma_k-\lambda}\right)^{\frac{\alpha}{\varphi}}$$
$$\times \left[\frac{\frac{\varphi}{\alpha(\varphi-1)}(1-\sigma_k)(1-\alpha+\alpha\gamma)+\gamma\theta_t}{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\left[\beta(1-\tau)(1-\alpha+\alpha\gamma)+\tau\right]-(1-\tau)(1-\beta\gamma)\theta_t} \right]^{\frac{\alpha}{\varphi}}$$
$$\equiv \hat{\mathcal{R}}(\theta_t;\sigma_k,\tau). \tag{D.32}$$

From (D.31) and (D.32), we obtain the same properties as in Lemma 3 in the benchmark model.

Lemma 5. $\hat{z}^{*\prime}(\theta_t;\sigma_k,\tau) > 0$ *and* $\hat{\mathcal{R}}'(\theta_t;\sigma_k,\tau) > 0$.

Substituting (D.32) into (D.24), we obtain the following growth rate of GDP at the steady state:

$$\frac{Y_{t+1}}{Y_t} \equiv \hat{g}^Y(\theta_S^*;\sigma_k,\tau)\left(= \frac{K_{t+1}}{K_t} \equiv \hat{g}^K(\theta_S^*;\sigma_k,\tau)\right)$$
$$= \hat{\mathcal{R}}(\theta_S^*;\sigma_k,\tau)\left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\left[\beta(1-\tau)(1-\alpha+\alpha\gamma)+\tau\right]-(1-\tau)(1-\beta\gamma)\theta_S^*\right]. \tag{D.33}$$

Before proceeding, considering the balanced budget case that does not incorporate government bonds ($\tilde{B}_t = 0$ for any period) helps explain the effects on economic growth when investment subsidies are funded through tax increases. Applying $\tilde{B}_t = 0$ (or $\theta_t = 0$) for any $t \geq 0$ induces (D.23) and (D.24) into

$$\tau\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}R_tK_t = \sigma_k K_{t+1}, \tag{D.34}$$
$$\frac{K_{t+1}}{K_t} = \beta\frac{\varphi}{\alpha(\varphi-1)}(1-\alpha+\alpha\gamma)(1-\tau)R_t, \tag{D.35}$$

both of which lead to

$$\tau = \frac{\beta\sigma_k(1-\alpha+\alpha\gamma)}{1-\sigma_k+\beta\sigma_k(1-\alpha+\alpha\gamma)}. \tag{D.36}$$

Because (D.31) is reduced to $z_t^* = z^* = \left(\frac{\beta(1-\sigma_k)}{1-\sigma_k-\lambda}\right)^{\frac{1}{\varphi}}$, (D.32) and (D.33) become

$$R_t = R = \frac{\alpha A}{1-\sigma_k}\left(\frac{\varphi-1}{\varphi}\right)^{1-\alpha}\left(\frac{\beta(1-\sigma_k)}{1-\sigma_k-\lambda}\right)^{\frac{\alpha}{\varphi}},$$
$$\frac{Y_{t+1}}{Y_t} = \frac{\beta A}{1-\sigma_k}\left(\frac{\varphi}{\varphi-1}\right)^{\alpha}\frac{1-\alpha+\alpha\gamma}{1-\sigma_k[1-\beta(1-\alpha+\alpha\gamma)]}\left(\frac{\beta(1-\sigma_k)}{1-\sigma_k-\lambda}\right)^{\frac{\alpha}{\varphi}},$$

where we use (D.36) to derive the long-run growth rate. As $\varphi > 1$, $\partial(Y_{t+1}/Y_t)/\partial\sigma_k > 0$ for all $\sigma_k > 0$. Therefore, investment subsidies fully financed by income tax enhance economic growth. Even with public debt, the increase in σ_k with income tax financing enhances economic growth.

To see this, we return to the original topic. The total differentials of (D.33), (D.32), and (D.27) are given by

$$d\hat{g}^Y = \frac{\partial\hat{g}^Y(\cdot)}{\partial\hat{\mathcal{R}}}d\hat{\mathcal{R}} + \frac{\partial\hat{g}^Y(\cdot)}{\partial\tau}d\tau + \frac{\partial\hat{g}^Y(\cdot)}{\partial\sigma_k}d\sigma_k + \frac{\partial\hat{g}^Y(\cdot)}{\partial\theta_S^*}d\theta_S^*, \tag{D.37}$$
$$d\hat{\mathcal{R}} = \frac{\partial\hat{\mathcal{R}}(\cdot)}{\partial\tau}d\tau + \frac{\partial\hat{\mathcal{R}}(\cdot)}{\partial\sigma_k}d\sigma_k + \frac{\partial\hat{\mathcal{R}}(\cdot)}{\partial\theta_S^*}d\theta_S^*, \tag{D.38}$$
$$d\theta_S^* = \frac{\partial\theta_S^*}{\partial\tau}d\tau + \frac{\partial\theta_S^*}{\partial\sigma_k}d\sigma_k. \tag{D.39}$$

Substituting (D.38) and (D.39) into (D.37), we obtain

$$\begin{aligned} d\hat{g}^Y = &\left(\frac{\partial\hat{g}^Y(\cdot)}{\partial\tau} + \frac{\partial\hat{g}^Y(\cdot)}{\partial\hat{\mathcal{R}}}\frac{\partial\hat{\mathcal{R}}(\cdot)}{\partial\tau} + \frac{\partial\hat{g}^Y(\cdot)}{\partial\theta_S^*}\frac{\partial\theta_S^*}{\partial\tau} + \frac{\partial\hat{g}^Y(\cdot)}{\partial\hat{\mathcal{R}}}\frac{\partial\hat{\mathcal{R}}}{\partial\theta_S^*}\frac{\partial\theta_S^*}{\partial\tau}\right)d\tau \\ &+ \underbrace{\left(\frac{\partial\hat{g}^Y(\cdot)}{\partial\sigma_k} + \frac{\partial\hat{g}^Y(\cdot)}{\partial\hat{\mathcal{R}}}\frac{\partial\hat{\mathcal{R}}(\cdot)}{\partial\sigma_k} + \frac{\partial\hat{g}^Y(\cdot)}{\partial\theta_S^*}\frac{\partial\theta_S^*}{\partial\sigma_k} + \frac{\partial\hat{g}^Y(\cdot)}{\partial\hat{\mathcal{R}}}\frac{\partial\hat{\mathcal{R}}}{\partial\theta_S^*}\frac{\partial\theta_S^*}{\partial\sigma_k}\right)}_{(-)}d\sigma_k. \end{aligned} \tag{D.40}$$

The second term on the LHS of (D.40) is negative because we consider the case in which $d\hat{g}^Y(\cdot)/d\sigma_k < 0$ for $d\tau = 0$. Thus, $d\hat{g}^Y(\cdot)/d\sigma_k > 0$ if and only if

$$\frac{d\tau}{d\sigma_k} > -\frac{\frac{\partial\hat{g}^Y(\cdot)}{\partial\sigma_k} + \frac{\partial\hat{g}^Y(\cdot)}{\partial\hat{\mathcal{R}}}\frac{\partial\hat{\mathcal{R}}(\cdot)}{\partial\sigma_k} + \frac{\partial\hat{g}^Y(\cdot)}{\partial\theta_S^*}\frac{\partial\theta_S^*}{\partial\sigma_k} + \frac{\partial\hat{g}^Y(\cdot)}{\partial\hat{\mathcal{R}}}\frac{\partial\hat{\mathcal{R}}}{\partial\theta_S^*}\frac{\partial\theta_S^*}{\partial\sigma_k}}{\frac{\partial\hat{g}^Y(\cdot)}{\partial\tau} + \frac{\partial\hat{g}^Y(\cdot)}{\partial\hat{\mathcal{R}}}\frac{\partial\hat{\mathcal{R}}(\cdot)}{\partial\tau} + \frac{\partial\hat{g}^Y(\cdot)}{\partial\theta_S^*}\frac{\partial\theta_S^*}{\partial\tau} + \frac{\partial\hat{g}^Y(\cdot)}{\partial\hat{\mathcal{R}}}\frac{\partial\hat{\mathcal{R}}}{\partial\theta_S^*}\frac{\partial\theta_S^*}{\partial\tau}}, \tag{D.41}$$

since the denominator on the RHS (the growth effects of income tax) takes a positive value for any λ numerically. (D.41) indicates that a marginal increase in the tax rate in response to a subsidy increase $d\tau/d\sigma_k$ must be larger than the the value in the RHS to ensure the positive growth effect of income tax rate. To see more, we divide the growth effects of income tax into the short-run effects $\frac{\partial \hat{g}^Y(\cdot)}{\partial \tau} + \frac{\partial \hat{g}^Y(\cdot)}{\partial \hat{\mathcal{R}}} \frac{\partial \hat{\mathcal{R}}(\cdot)}{\partial \tau}$ and the long-run effects (through changes in θ_S^*) $\frac{\partial \hat{g}^Y(\cdot)}{\partial \theta_S^*} \frac{\partial \theta_S^*}{\partial \tau} + \frac{\partial \hat{g}^Y(\cdot)}{\partial \hat{\mathcal{R}}} \frac{\partial \hat{\mathcal{R}}}{\partial \theta_S^*} \frac{\partial \theta_S^*}{\partial \tau}$.

In the short run, $\frac{\partial \hat{g}^Y(\cdot)}{\partial \tau} > 0$ from (D.43), whereas $\frac{\partial \hat{g}^Y(\cdot)}{\partial \hat{\mathcal{R}}} \frac{\partial \hat{\mathcal{R}}(\cdot)}{\partial \tau} < 0$ from (D.42) and (D.48). The reasons for this are as follows. First, increases in income tax have a distortionary effect on growth, while they also reduce the issuance of public bonds, decrease the crowding-out effect of public debt on capital accumulation, and promote economic growth. The latter effects dominate the former, leading to $\frac{\partial \hat{g}^Y(\cdot)}{\partial \tau} > 0$. Second, increases in income tax have a distortionary effect on capital accumulation, which decreases demand for credit in the financial market and lowers interest rates. Lower interest rates decrease borrowing costs and increase the number of active entrepreneurs, which decreases capital intensity and negatively affects growth $\frac{\partial \hat{g}^Y(\cdot)}{\partial \hat{\mathcal{R}}} \frac{\partial \hat{\mathcal{R}}(\cdot)}{\partial \tau} < 0$.

In the long run, $\frac{\partial \hat{g}^Y(\cdot)}{\partial \theta_S^*} \frac{\partial \theta_S^*}{\partial \tau} > 0$ ($\frac{\partial \hat{g}^Y(\cdot)}{\partial \theta_S^*} < 0$ from (D.45) and $\frac{\partial \theta_S^*}{\partial \tau} < 0$ numerically) and $\frac{\partial \hat{g}^Y(\cdot)}{\partial \hat{\mathcal{R}}} \frac{\partial \hat{\mathcal{R}}}{\partial \theta_S^*} \frac{\partial \theta_S^*}{\partial \tau} < 0$ ($\frac{\partial \hat{g}^Y(\cdot)}{\partial \hat{\mathcal{R}}} \frac{\partial \hat{\mathcal{R}}}{\partial \theta_S^*} > 0$ from (D.42) and (D.47), and $\frac{\partial \theta_S^*}{\partial \tau} < 0$ numerically). The reasons for this are as follows. First, an increase in the income tax rate boosts tax revenue and can reduce the issuance of public bonds, which lowers the ratio of public debt to capital θ_S^*. This reduction in θ_S^* decreases the costs of repaying public debt and issuing new public bonds, thereby crowding in capital accumulation and enhancing economic growth. Second, a lower θ_S^* decreases the interest rate $\hat{\mathcal{R}}(\cdot)$ because it reduces both the costs of repaying public debt and the issuance of new public bonds, increasing the aggregate supply of credit in the financial market. This increase in the supply of credit promotes the entry of firms, decreases capital intensity, and lowers economic growth.

The positive short- and long-run effects on growth dominate the negative ones. Therefore, the total effect of income tax on growth is positive.

The remainder of this appendix provides the partial derivatives that consist of (D.41). First, from (D.33), we obtain

$$\frac{\partial \hat{g}^Y(\cdot)}{\partial \hat{\mathcal{R}}} = \frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)} [\beta(1-\tau)(1-\alpha+\alpha\gamma)+\tau] - (1-\tau)(1-\beta\gamma)\theta_S^* > 0 \text{ by (D.26)}, \tag{D.42}$$

$$\frac{\partial \hat{g}^Y(\cdot)}{\partial \tau} = \hat{\mathcal{R}}(\theta_S^*; \sigma_k, \tau) \left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)} (1-\beta(1-\alpha+\alpha\gamma)) + (1-\beta\gamma)\theta_S^* \right] > 0, \tag{D.43}$$

$$\frac{\partial \hat{g}^Y(\cdot)}{\partial \sigma_k} = -\frac{\varphi}{\alpha(\varphi-1)} \hat{\mathcal{R}}(\theta_S^*; \sigma_k, \tau)[\tau+\beta(1-\tau)(1-\alpha+\alpha\gamma)] < 0, \tag{D.44}$$

$$\frac{\partial \hat{g}^Y(\cdot)}{\partial \theta_S^*} = (1-\tau)\hat{\mathcal{R}}(\theta_S^*; \sigma_k, \tau)(1-\beta\gamma) > 0. \tag{D.45}$$

Second, from (D.32), we obtain

$$\begin{aligned}\frac{\partial \ln \hat{\mathcal{R}}(\cdot)}{\partial \theta_S^*} = & \frac{\alpha}{\varphi}\frac{\gamma}{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha+\alpha\gamma)+\gamma\theta_S^*} \\ & + \frac{\alpha}{\varphi}\frac{(1-\tau)(1-\beta\gamma)}{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\left[\beta(1-\tau)(1-\alpha+\alpha\gamma)+\tau\right]-(1-\tau)(1-\beta\gamma)\theta_S^*}.\end{aligned} \tag{D.46}$$

Because of $\frac{\partial \ln \hat{\mathcal{R}}(\cdot)}{\partial \theta_S^*} = \frac{1}{\hat{\mathcal{R}}}\frac{\hat{\mathcal{R}}(\cdot)}{\partial \theta_S^*}$ and (D.46), we obtain

$$\begin{aligned}\frac{\partial \hat{\mathcal{R}}(\cdot)}{\partial \theta_S^*} = & \frac{\alpha}{\varphi}\hat{\mathcal{R}}(\theta_S^*;\sigma_k,\tau)\Bigg[\frac{\gamma}{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha+\alpha\gamma)+\gamma\theta_S^*} \\ & + \frac{(1-\tau)(1-\beta\gamma)}{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\left[\beta(1-\tau)(1-\alpha+\alpha\gamma)+\tau\right]-(1-\tau)(1-\beta\gamma)\theta_S^*}\Bigg] > 0 \text{ by (D.26).}\end{aligned} \tag{D.47}$$

Similarly, we have

$$\begin{aligned}\frac{\partial \hat{\mathcal{R}}(\cdot)}{\partial \tau} = & -\frac{\alpha}{\varphi}\hat{\mathcal{R}}(\theta_S^*;\sigma_k,\tau) \\ & \times\left[\frac{1}{1-\tau}+\frac{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\beta(1-\alpha+\alpha\gamma))+(1-\beta\gamma)\theta_S^*}{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\left[\beta(1-\tau)(1-\alpha+\alpha\gamma)+\tau\right]-(1-\tau)(1-\beta\gamma)\theta_S^*}\right] < 0 \text{ by (D.26),}\end{aligned} \tag{D.48}$$

$$\begin{aligned}\frac{\partial \hat{\mathcal{R}}(\cdot)}{\partial \sigma_k} = & \hat{\mathcal{R}}(\theta_S^*;\sigma_k,\tau)\Bigg[\frac{1}{1-\sigma_k}+\frac{\alpha}{\varphi}\frac{1}{1-\sigma_k-\lambda}-\frac{\alpha}{\varphi}\frac{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha+\alpha\gamma)}{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha+\alpha\gamma)+\gamma\theta_S^*} \\ & + \frac{\alpha}{\varphi}\frac{\frac{\varphi}{\alpha(\varphi-1)}\left[\tau+\beta(1-\tau)(1-\alpha+\alpha\gamma)\right]}{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\left[\beta(1-\tau)(1-\alpha+\alpha\gamma)+\tau\right]-(1-\tau)(1-\beta\gamma)\theta_S^*}\Bigg].\end{aligned} \tag{D.49}$$

Finally, from (D.27), we obtain

$$
\begin{aligned}
\frac{\partial\theta_S^*}{\partial\tau} &= \frac{\Theta(\theta_S^*;\sigma_k,\tau)}{\Omega(\theta_S^*;\sigma_k,\tau)}, \quad \frac{\partial\theta_S^*}{\partial\sigma_k} = \frac{\Upsilon(\theta_S^*;\sigma_k,\tau)}{\Omega(\theta_S^*;\sigma_k,\tau)}, \\
\Omega(\theta_S^*;\sigma_k,\tau) &\equiv -\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}[\beta(1-\tau)(1-\alpha+\alpha\gamma)+\tau] + (1-\tau)(1-\beta\gamma)\theta_S^* \\
&\quad + (1-\tau)[1-\sigma_k(1-\beta\gamma)], \\
\Theta(\theta_S^*;\sigma_k,\tau) &\equiv \theta_S^*\left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\beta(1-\alpha+\alpha\gamma)) + (1-\beta\gamma)\theta_S^*\right] + [1-\sigma_k(1-\beta\gamma)]\theta_S^* \\
&\quad + \frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}[1-\sigma_k+\sigma_k\beta(1-\alpha+\alpha\gamma)], \\
\Upsilon(\theta_S^*;\sigma_k,\tau) &\equiv -\frac{\varphi}{\alpha(\varphi-1)}[\tau+\beta(1-\tau)(1-\alpha+\alpha\gamma)]\theta_S^* + (1-\tau)(1-\beta\gamma)\theta_S^* \\
&\quad - 2\tau\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)} - \beta\frac{\varphi}{\alpha(\varphi-1)}(1-\tau)(1-2\sigma_k)(1-\alpha+\alpha\gamma).
\end{aligned}
\tag{D.50}
$$

Table 1: The benchmark numerical settings of the parameters

	Benchmark	Source
α	0.4	Average velues of the US, the EU, and Japan
β	0.3	$\beta/(1-\beta) = 0.97^{30}$
γ	0.3	$\beta = \gamma$
φ	1.5	Diamond and Saez (2011), Jaimovich and Rebelo (2017), and Mino (2015)
λ	0.7	Set
A	5	Set to yield positive plausible values for the long-run growth rates
σ_k	0.01	Set

Table 2: The magnitude of tax increases relative to investment subsidy increases for $d\hat{g}^Y(\cdot)/d\sigma_k > 0$

			$\lambda = 0.8$			
σ_k	0	0.01	0.02	0.03	0.04	0.05
$d\tau/d\sigma_k\|_{\tau=0}$	-0.0411	-0.0282	-0.0120	0.0091	0.0375	0.0806
			$\lambda = 0.7$			
σ_k	0	0.01	0.02	0.03	0.04	0.05
$d\tau/d\sigma_k\|_{\tau=0}$	0.0207	0.0328	0.0471	0.0646	0.0865	0.1174
			$\lambda = 0.6$			
σ_k	0	0.01	0.02	0.03	0.04	0.05
$d\tau/d\sigma_k\|_{\tau=0}$	0.0515	0.0625	0.0750	0.0900	0.1083	0.1332

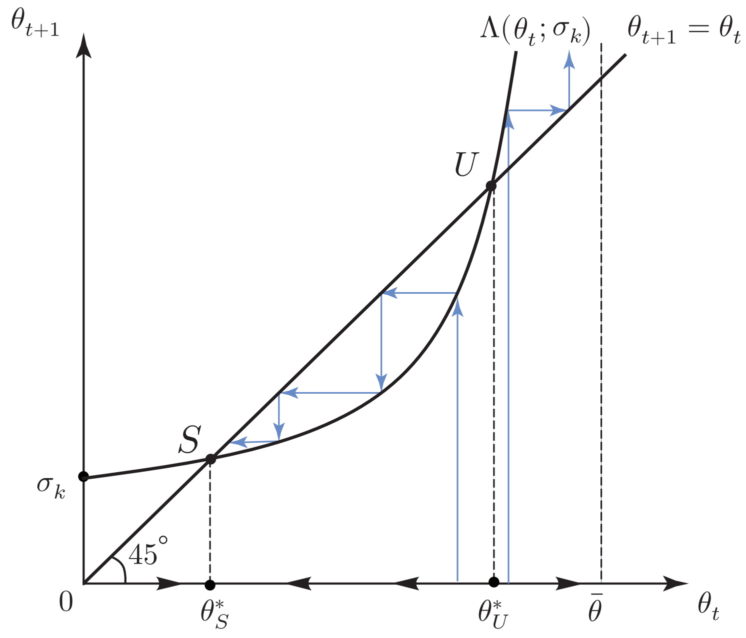

Figure 1: The dynamics of θ_t

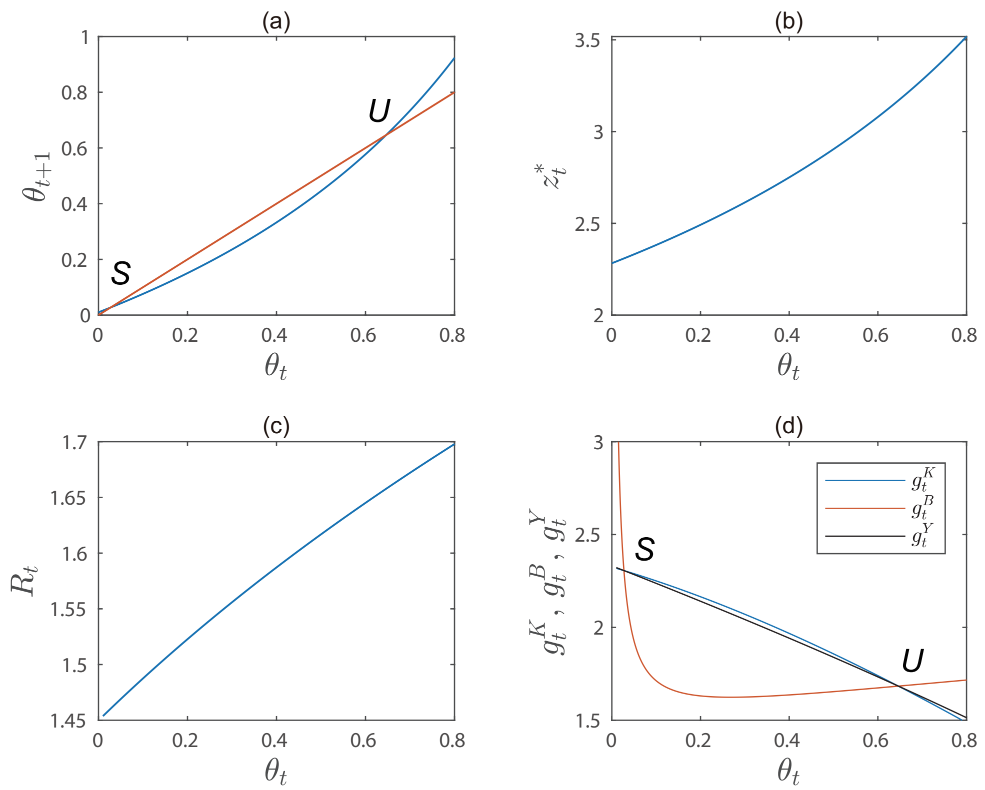

Figure 2: $\theta_{t+1} = \Lambda(\theta_t, \sigma_k)$, $R_t = \Psi(\theta_t, \sigma_k)$, $z(\theta_t, \sigma_k)$, $g^K(\theta_t, \sigma_k)$, $g^B(\theta_t, \sigma_k)$, and $g^Y(\theta_t, \sigma_k)$ in the numerical example

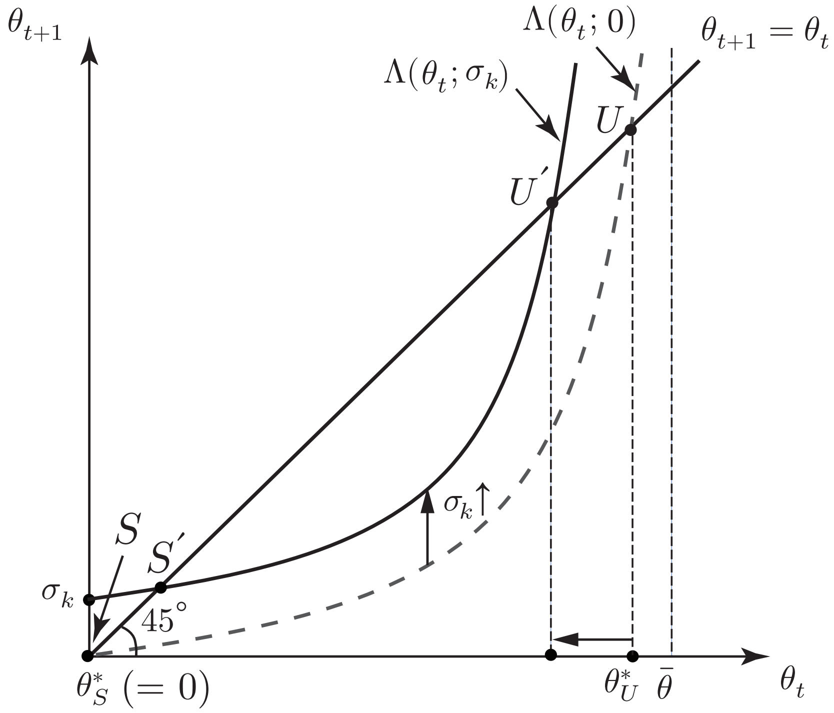

Figure 3: The effect of an increase in σ_k on θ_S^* and fiscal sustainability

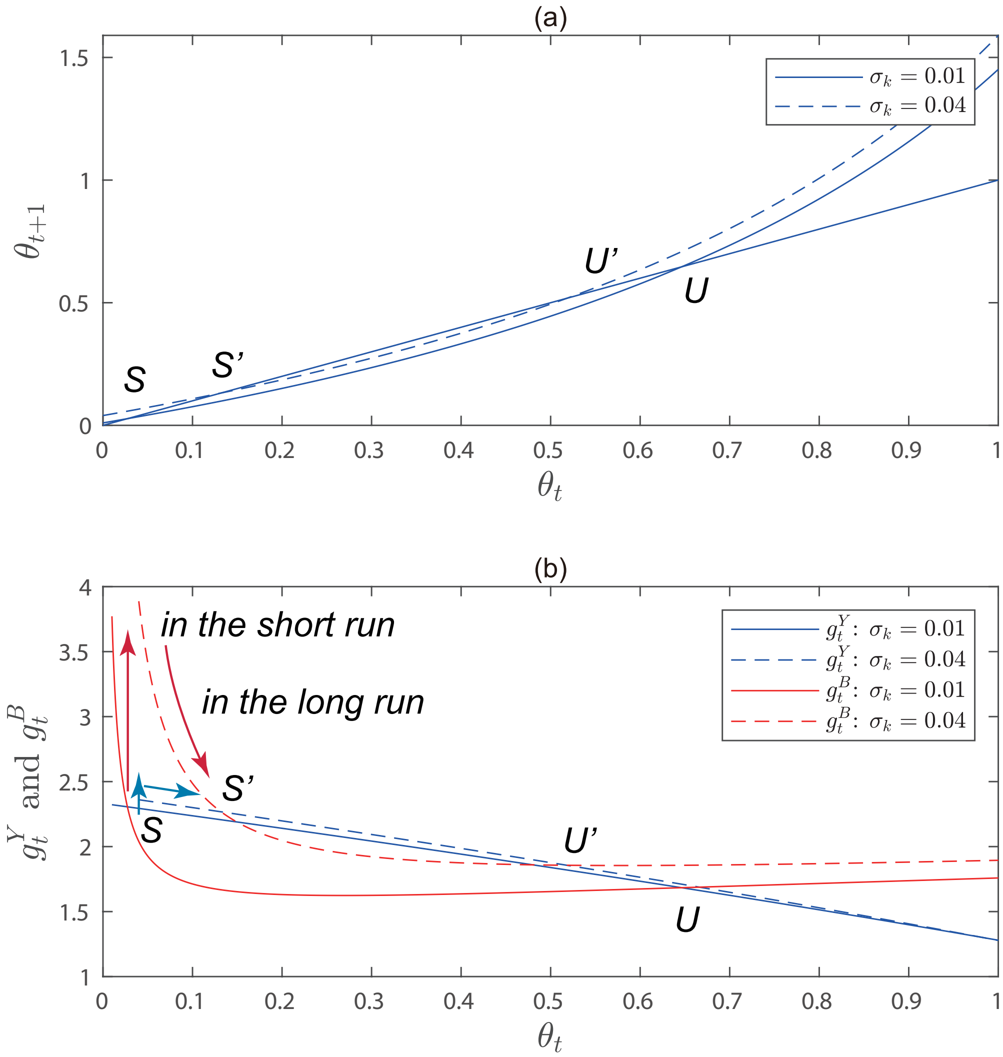

Figure 4: The effect of an increase in σ_k from 0.01 to 0.04 on θ_S^*, θ_U^*, $g^Y(\theta_t, \sigma_k)$, and $g^B(\theta_t, \sigma_k)$

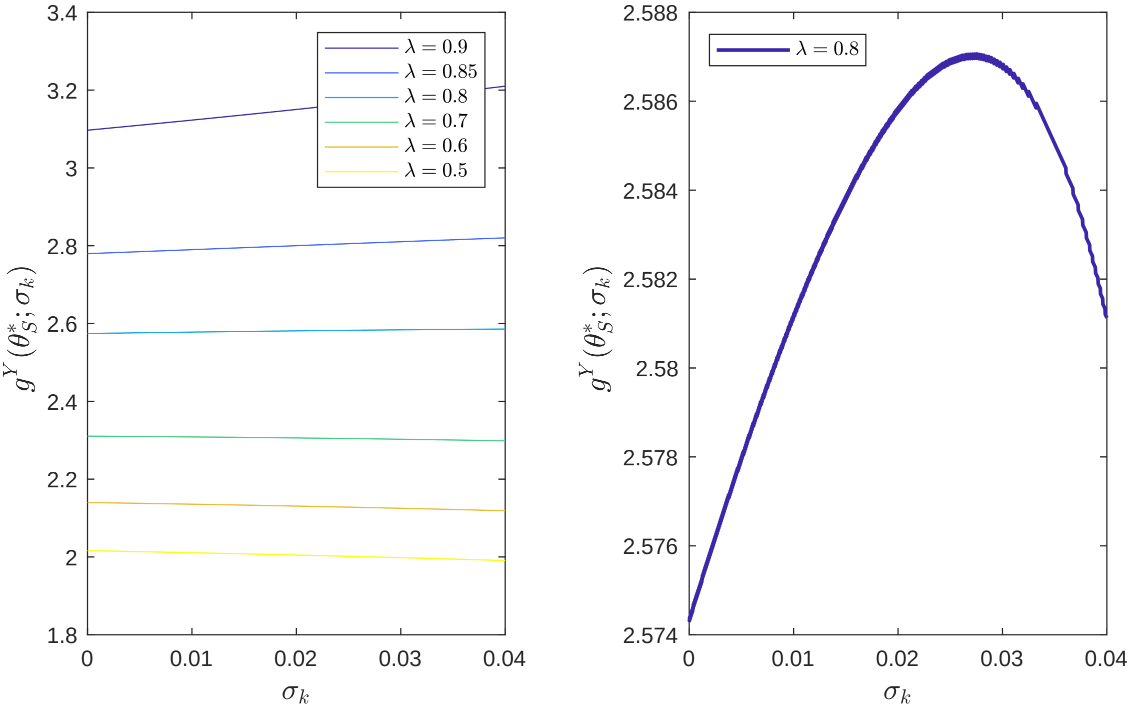

Figure 5: The relationship between λ and the growth effect of σ_k