Double-Hopf bifurcation in an extended Goodwin model with Mechanization, Independent Investment, and Disequilibrium: Toward a Marxian-Keynesian Synthesis

Cajas Guijarro, John

Central University of Ecuador

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John Cajas-Guijarro[1]

Abstract

This paper proposes an extended Goodwin model that synthesizes Marxian and Keynesian dynamics into a unified four-dimensional framework. The model integrates endogenous technical change via mechanization, investment behavior driven by effective demand, and goods market disequilibrium. We develop two three-dimensional closures–a Classical-Marxian and a Keynesian-Kaleckian formulation–each capable of generating persistent endogenous cycles through Hopf bifurcations. These are then combined into a Marxian-Keynesian (MK) system, which exhibits complex dynamics including quasi-periodicity and, under specific parameter values, a double-Hopf bifurcation. This result, to our knowledge not previously identified in extended Goodwin models, points to the potential for interacting oscillatory modes and long-run fluctuations even with relatively simple behavioral rules. Numerical simulations suggest that the MK synthesis captures rich endogenous fluctuations without relying on exogenous shocks and may exhibit chaotic dynamics under future extensions. These findings lay the groundwork for a more comprehensive Mars-Keynes-Schumpeter synthesis of capital instability, as suggested in the conclusion section.

Keywords: Extended Goodwin model. Mechanization. Effective demand. Marxian-Keynesian synthesis. Double-Hopf bifurcation.

JEL Classification: B51, C61, E12, E32, O41.

[1] School of economic sciences. Central University of Ecuador. Email: jcajasg@uce.edu.ec. Personal site: https://sites.google.com/view/johncajasguijarro

1. Introduction

The two-dimensional model of distributive cycles formulated by Richard Goodwin (1967) is one of the most parsimonious representations of Marx's (2010a, chap. 25) intuition that the distributive struggle between workers and capitalists, mediated through the labor market, generates persistent oscillations around the trend of capitalist accumulation.[2] These oscillations take the concrete form of clockwise cycles in the wage share–employment rate plane, with the employment rate leading and the wage share lagging, symbolizing the ongoing contest between labor and capital. Subsequent literature has extended Goodwin's framework, introducing additional elements to capture more complex economic dynamics. Particularly, two strands of literature have emerged: one incorporating endogenous technical change and another integrating effective demand fluctuations.

In the realm of technical change, Shah and Desai (1981) pioneered an extension by embedding endogenous technical progress driven by cost-optimizing innovations. Their three-dimensional model introduced the output-capital ratio as a new state variable, reflecting interactions between labor productivity and mechanization (capital-labor ratio). While their model suggested that cycles may dampen over time, subsequent studies have proposed alternative formulations capable of generating persistent cycles as well as other new dynamics. Some examples include van der Ploeg (1987), Foley (2003), Julius (2005), Ryzhenkov (2009, 2022), Rada (2012), Tavani (2012), Zamparelli (2015), Tavani and Zamparelli (2015, 2021), Cajas Guijarro (2024), among others.

Parallelly, efforts to integrate effective demand into the Goodwin model have been notable. Glombowski and Krüger (1988) and Skott (1989) are some of the initial works that assumed production expands due to excess demand while investment decisions are decoupled from savings. These models incorporated the output-capital ratio as an indicator of capacity utilization, subject to demand-driven fluctuations. Further advancements have enriched this perspective, incorporating goods market disequilibrium, as well as other components associated with effective demand. Some examples here include Dutt (1992), Barbosa-Filho and Taylor (2006), Velupillai (2006), Flaschel (2009), Sasaki (2013), Sordi and Vercelli (2014), von Arnim and Barrales (2015), Skott (2015, 2023), Sportelli and De Cesare (2022), among others.[3]

Recognizing the complementary nature of these extensions, Rada et al. (2023) proposed a preliminary synthesis wherein endogenous technical change and effective demand fluctuations can be seriously compared. In their formulation, the Classical perspective assumes full capacity utilization with technological forces driving output-capital ratio fluctuations, while the Keynesian view holds the technical output-capital ratio constant,

[2] Other Marxian models exploring the emergence of endogenous cycles include Eagly (1972), Laibman (1978, 1988), Sherman (1971, 1979), Glombowski (1982), Dupont (2014), Cajas Guijarro and Vera (2022), Nikolaos (2022), among others. For detailed literature reviews, see Cámara Izquierdo (2022) and Cajas Guijarro (2023).

[3] It should be mentioned that some works combine elements of endogenous technical change and effective demand like, for instance, Barbosa-Filho and Taylor (2006), Velupillai (2006), Sasaki (2013). Therefore, the classification proposed in the Introduction should be taken only in referential terms.

attributing variations in the effective output-capital ratio to demand-side factors. However, their formulations do not focus on unifying these dynamics within a single framework where both technical change and demand fluctuations simultaneously influence output-capital ratio and capacity utilization.

Building upon this insight, the present paper introduces an extended Goodwin model that integrates: (i) endogenous technical change via mechanization, reflecting capital-labor substitution as a class strategy and cost-saving response rooted in Classical and Marxian thought; (ii) investment dynamics driven by effective demand, aligning with Keynesian and Kaleckian principles that decouple investment from aggregate savings; and (iii) goods market disequilibrium, introducing feedback from excess demand into capital accumulation and output expansion. This integration aims to forge a Marxian-Keynesian (MK) synthesis capable of generating rich endogenous dynamics.

To operationalize this synthesis, we first construct two three-dimensional relatively parsimonious closures of the model: a Classical-Marxian (M) closure focusing on mechanization and its relationship with employment and distribution, and a Keynesian-Kaleckian (K) closure emphasizing the interplay between income distribution, demand-led investment, and capacity utilization. Each closure independently exhibits persistent endogenous cycles via Hopf bifurcations. Subsequently, we integrate these into a four-dimensional MK system, wherein the interaction of the two mechanisms leads to complex cyclical dynamics, including quasi-periodicity and, under certain parameter configurations, indications of a double-Hopf bifurcation.

This paper contributes to the specialized literature in three ways. First, it presents a tractable yet structurally rich model synthesizing Classical-Marxian and Keynesian-Kaleckian dynamics within an endogenous cycle framework. Second, it analytically establishes the conditions under which each closure and the combined MK system exhibit persistent fluctuations. While earlier attempts have been made to merge Marxian, Keynesian, and even Schumpeterian perspectives (Glombowski and Krüger 1987; Flaschel 2015), to the best of our knowledge this is the first study to identify a double-Hopf bifurcation in an extended Goodwin model. Third, through numerical simulations, the paper makes a preliminary exploration of the MK system's dynamics, laying the groundwork for future research into capitalist fluctuations without reliance on exogenous shocks, and with the potential to uncover chaotic dynamics.

The remainder of the paper is structured as follows. Section 2 outlines the core theoretical framework underpinning the various closures of the model. Sections 3 and 4 detail the Classical-Marxian and Keynesian-Kaleckian closures, respectively, including their dynamic properties and simulations. Section 5 develops the integrated MK system and explores its dynamic complexity. Section 6, offering insights into future research and model extensions.

2. The Core Theoretical Framework

We consider a closed economy without government, producing a single good used for both consumption and investment. This economy employs fixed capital and labor, applying fixed-coefficients technology. Society comprises wage-earning workers who do not save, and profit-earning capitalists who save a constant fraction ($0 < s \leq 1$) of their profits. Define q as the effective output of the economy, growing at a rate $\hat{q}$.[4] This output is produced using l hours of labor with productivity $a = q/l$. Given the labor supply n, the employment rate $v = l/n$ can be rewritten as $v = q/na$, and its growth rate is:

$$\hat{v} = \hat{q} - \hat{n} - \hat{a} \quad (1)$$

From equation (1), under a hypothetical scenario of full employment ($v = 1$) where the employment rate remains constant ($\hat{v} = 0$), the output growth rate would match:

$$\hat{q}_N = \hat{n} + \hat{a} \quad (2)$$

Here, $\hat{q}_N$ denotes the natural growth rate, defined as the maximum sustainable output growth in the long term while maintaining a constant employment rate, given the growth rates of labor supply ($\hat{n}$) and labor productivity ($\hat{a}$) (Harrod 1939, p. 31).

For the capital stock, let k indicate the installed fixed capital, which depreciates at a constant rate ($0 < \delta < 1$). The portion of this capital actively employed in production is defined as effective capital, $k^{ef} = xk$, where x is the capacity utilization rate ($0 < x < 1$). From these terms, we define the (effective) capital-output ratio as $\sigma = k^{ef}/q = xk/q$ (inverse of effective capital productivity), with its growth rate equal to:

$$\hat{\sigma} = \hat{k}^{ef} - \hat{q} = \hat{x} + \hat{k} - \hat{q} \quad (3)$$

Regarding income distribution, w represents the real wage per hour of labor, while $u = wl/q$ is the wage share, with $1 - u$ being the profit share. Since capitalists save a constant fraction s of their profits, total savings in the economy are $sq(1 - u)$. In terms of market behavior, we assume disequilibrium in the goods market, characterized by excess demand q^{ED}, which is the difference between real demand (q^D) and effective output (q):

$$q^{ED} = q^D - q = (q^C + q^I) - q$$

Here, q^C and q^I represent the real demand for consumption and capital goods, respectively. The consumption demand q^C includes all wages spent by workers (wl) and the non-saved portion of capitalist profits $(1 - s)q(1 - u)$:

$$q^C = wl + (1 - s)q(1 - u)$$

Meanwhile, investment demand q^I includes capital depreciation (δk) and desired net investment ($\dot{k}_d$):

$$q^I = \delta k + \dot{k}_d$$

Applying the definitions of q^{ED}, q^D, q^C, q^I, and u, excess demand can be formulated as:

$$q^{ED} = \left(\dot{k}_d + \delta k\right) - sq(1 - u) \quad (4)$$

Equation (4) indicates that excess demand is equal to the gap between gross desired investment (desired net investment $\dot{k}_d$ and depreciation δk) and total savings $sq(1 - u)$.

[4] For any variable z, its time derivative is denoted as $\dot{z} = dz/dt$, and its growth rate is given by $\hat{z} = \dot{z}/z$.

Next, consider an alternative hypothetical scenario (denoted as IS) where the goods market achieves equilibrium, signifying that savings equal investment. In this equilibrium state, excess demand vanishes ($q_{IS}^{ED} = 0$), resulting in the following relationship:

$$\dot{k}_{IS} + \delta k = sq(1-u)$$

where $\dot{k}_{IS}$ is the potential net investment that could be financed entirely through capitalist savings, after accounting for capital depreciation. From this expression we can obtain the hypothetical growth rate of capital under conditions of savings-investment equilibrium:

$$\frac{\dot{k}_{IS}}{k} = \hat{k}_{IS} = \frac{sq(1-u)}{k} - \delta \quad (5)$$

In this hypothetical equilibrium scenario (IS), we further assume that firms' expectations regarding production and technical change are completely met. Consequently, firms can align the growth of installed capital with the growth rates of production and capital-output ratio, without needing to adjust the usage of installed capacity ($\hat{x}_{IS} = 0$). This indicates that expectations are met in such a manner that the capacity of firms to respond to demand fluctuations remains intact. Linked with equation (3), this assumption leads to:

$$\hat{k}_{IS}^{ef} = \hat{k}_{IS} = \hat{q}_{IS} + \hat{\sigma} \quad (6)$$

Here, $\hat{q}_{IS}$ is the warranted rate, a hypothetical output growth rate that assures both savings-investment equilibrium and the fulfillment of firms' expectations for production and technical change (Harrod 1939, p. 16).[5] Therefore, if firms produce at the growth rate $\hat{q}_{IS}$, they achieve the precise quantity of goods needed to meet market demands without altering their capacity utilization.[6] By combining equations (5) and (6) with the definition of σ, the warranted rate can be expressed as:

$$\hat{q}_{IS} = \frac{sx(1-u)}{\sigma} - \delta - \hat{\sigma} \quad (7)$$

By combining equations (4) and (7) with the definition of σ, and defining the desired growth rate of capital as $\hat{k}_d = \dot{k}_d/k$, we derive the following expression for excess demand as a proportion of effective output:

$$\frac{q^{ED}}{q} = \frac{\sigma\left(\hat{k}_d - \hat{q}_{IS} - \hat{\sigma}\right)}{x} = \frac{\sigma\left(\hat{k}_d + \delta\right)}{x} - s(1-u) \quad (8)$$

Following Skott (2015, p. 376), we assume that excess demand and the employment rate induce firms to adjust their effective production, causing fluctuations around the goods market equilibrium. This assumption is captured by the following output expansion function:[7]

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[5] Harrod (1939) defined the warranted growth rate under the assumption of a constant capital-output ratio ($\hat{\sigma} = 0$). In contrast, our formulation relaxes this assumption by allowing for technical change that can induce variations in the capital-output ratio ($\hat{\sigma} \neq 0$).

[6] For a discussion of the warranted growth rate, specifically in the context of savings-investment equilibrium and fulfillment of expectations under the assumption of a constant capital-output ratio, see Blecker and Setterfield (2019, pp. 106–108).

[7] Equation (9) resembles the output expansion functions found in Glombowski and Krüger (1988, p. 427), Chiarella and Flaschel (2000, pp. 285–286), and Sordi and Vercelli (2014, p. 331), with a distinction: those

$$\hat{q} = \hat{q}_{IS} + \phi\left(\frac{q^{ED}}{q}\right), \qquad \phi > 0 \quad (9)$$

Equation (9) integrates two components: one reflecting the trend defined by the warranted rate ($\hat{q}_{IS}$) and another accounting for fluctuations induced by excess demand. In this formulation, ϕ quantifies how capitalist firms adjust their production in response to goods market disequilibrium. Thus, positive excess demand ($q^{ED} > 0$) induces firms to accelerate production beyond the warranted rate ($\hat{q} > \hat{q}_{IS}$), whereas negative excess demand ($q^{ED} < 0$) leads to the opposite effect ($\hat{q} < \hat{q}_{IS}$). In contrast, when the goods market is in equilibrium ($q^{ED} = 0$), the effective and warranted rates converge ($\hat{q} = \hat{q}_{IS}$) (Glombowski and Krüger 1988).[8]

The dynamics of effective output significantly influence the employment rate, as illustrated by the following expression derived from combining equations (1), (3), and (9):

$$\hat{v} = (\hat{q}_{IS} - \hat{q}_N) + \phi\left(\frac{q^{ED}}{q}\right) \quad (10)$$

According to this equation, the growth rate of the employment rate, $\hat{v}$, is influenced by the disparity between the warranted rate and the natural rate ($\hat{q}_{IS} - \hat{q}_N$) and by excess demand expressed as a proportion of effective output (q^{ED}/q). Note that, even when the goods market is in equilibrium ($q^{ED} = 0$), $\hat{v}$ still fluctuates due to the gap between $\hat{q}_{IS}$ and $\hat{q}_N$. By combining equations (2), (7), (8), and (10), we can reformulate $\hat{v}$ as:

$$\hat{v} = \left[\frac{sx(1-u)}{\sigma} - (\hat{a} + \hat{n} + \delta) - \hat{\sigma}\right] + \phi\left[\frac{\sigma(\hat{k}_d + \delta)}{x} - s(1-u)\right] \quad (11)$$

Expression (11) is an extended version of Goodwin's (1967) first dynamical equation, incorporating effects on the employment rate due to goods market disequilibrium and fluctuations in the capital-output ratio driven by different forms of technical change.[9]

With respect to real wage dynamics, we consider the following Phillips curve:

$$\hat{w} = -\gamma + \rho v, \qquad \gamma, \rho > 0 \quad (12)$$

Consistent with Goodwin (1967), γ symbolizes an autonomous downward pressure on the real wage, while ρ captures the sensitivity of wage growth to the employment rate. A higher γ or a lower ρ indicates diminished bargaining power of the working class at a given employment rate v. To keep the model simple, in equation (12) we assume that excess demand does not have a direct influence on the growth rate of the real wage.

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authors employ the natural growth rate ($\hat{q}_N$) rather than the warranted rate ($\hat{q}_{IS}$). We diverge from this approach because using $\hat{q}_N$ in equation (9) implicitly assumes a constant growth rate whenever the goods market is in equilibrium ($q^{ED} = 0$), an assumption that seems to contradict the spirit of the original Goodwin model, where output fluctuations are not driven by savings-investment disequilibrium.

[8] Glombowski and Krüger (1988, p. 427) describe the excess demand component in equation (9) as a continuous-time analogue of a 'discrete-time dynamic multiplier with an exponentially distributed lag.'

[9] Under the assumptions of savings-investment equilibrium ($q^{ED} = 0 \rightarrow \hat{k}_d = \hat{q}_{IS} + \hat{\sigma} = sx(1-u)/\sigma - \delta$), full capacity utilization ($x = 1$), no capital depreciation ($\delta = 0$), capitalists saving all their profits ($s = 1$), a constant capital-output ratio ($\hat{\sigma} = 0$), and constant growth rates of labor productivity and labor supply ($\hat{a} = \alpha, \hat{n} = \beta$), equation (11) becomes equivalent to equation (1) in Goodwin's (1967) model.

Since the wage share can be written as $u = w/a$, with its growth rate expressed as $\hat{u} = \hat{w} - \hat{a}$, then combining these results with equations (7), (8), and (12) yields:

$$\hat{u} = -\gamma + \rho v - \hat{a} \quad (13)$$

This formulation almost coincides with the second dynamical equation from Goodwin's (1967) model, emphasizing the wage share dynamics.[10]

The presence of disequilibrium in the goods market impacts not only the dynamics of the employment rate v and, indirectly, the dynamics of the wage share u. Disequilibrium also influences the dynamics of the rate of capacity utilization x. To elucidate this, we combine equations (3), (7), (8), and (9) to derive the following dynamic equation for x:

$$\hat{x} = \frac{sx(1-u)}{\sigma} - \delta - \hat{k} + \phi\left[\frac{\sigma(\hat{k}_d + \delta)}{x} - s(1-u)\right] \quad (14)$$

Additionally, by defining (effective) mechanization as the ratio between effective capital and labor employed in production, $m = k^{ef}/l = xk/l$, and combining this with the definitions of a and σ, we note that $\sigma = m/a$. Applying logarithmic differentiation yields:

$$\hat{\sigma} = \hat{m} - \hat{a} \quad (15)$$

This expression indicates that potential fluctuations in the capital-output ratio ($\hat{\sigma}$) depend crucially on how technical change influences the growth rates of mechanization ($\hat{m}$) and labor productivity ($\hat{a}$).[11] Finally, combining equations (11), (13), (14), (15), and assuming as in Goodwin (1967) that labor productivity and labor supply grow at constant rates ($\hat{a} = \alpha, \hat{n} = \beta$), we obtain the following four differential equations:[12]

$$\hat{u} = -(\alpha + \gamma) + \rho v \quad (16)$$

$$\hat{v} = \frac{sx(1-u)}{\sigma} - (\beta + \delta) + \phi\left[\frac{\sigma(\hat{k}_d + \delta)}{x} - s(1-u)\right] - \hat{m} \quad (17)$$

$$\hat{x} = \frac{sx(1-u)}{\sigma} - \delta - \hat{k} + \phi\left[\frac{\sigma(\hat{k}_d + \delta)}{x} - s(1-u)\right] \quad (18)$$

$$\hat{\sigma} = \hat{m} - \alpha \quad (19)$$

Equations (16) to (19) constitute the core framework of the dynamic model outlined in this paper. From this framework it is possible to generate three different versions of the model depending on the type of 'closure' applied to obtain a complete dynamical system: a Classical-Marxian (M) closure if we consider a scenario with equilibrium in the goods market and endogenous mechanization dynamics; a Keynesian-Kaleckian (K) closure if we consider a scenario with excess demand, investment independent of saving, and Harrod-neutral technical

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[10] Assuming labor productivity growing at a constant rate ($\hat{a} = \alpha$), equation (13) aligns with equation (2) from Goodwin's (1967) model.

[11] The original Goodwin model assumes a constant capital-output ratio ($\hat{\sigma} = 0$), implying that labor productivity and mechanization grow at identical rates ($\hat{m} = \hat{a}$).

[12] All mathematical derivations and simulations presented in this paper were performed using a Mathematica notebook, which is available as supplementary material upon request to the author.

change; and a Marxian-Keynesian synthesis (MK) when we combine both closures. These different versions of the model are obtained in the following sections.

3. The Classical-Marxian Closure

4.1. Assumptions and formulation

In his 'general law of capitalist accumulation' Marx (2010a, chap. 25) proposed several intuitions about how distributive class struggle, the role of the reserve army of labor, and capitalist technical change strongly focused on the mechanization of production can generate endogenous cycles without requiring disequilibrium in the goods market.[13] Following this perspective, we propose a Classical-Marxian closure of the model presented in this paper, where the emphasis is in the interaction between distributive struggle and endogenous technical change through mechanization.[14]

Regarding mechanization-driven technical change, we consider the following mechanization function:

$$\hat{m} = \mu_0 + \mu_1 u + \mu_2 (v - v^n) - \mu_3 x, \qquad \mu_0 \in R, \qquad 0 < \mu_i < 1 \quad (20)$$

Following Ryzhenkov (2006, 2009), equation (21) specifies μ_0 as an autonomous component of mechanization growth. The parameter μ_1 captures the assumption that capitalists are incentivized to accelerate mechanization–substitution of capital for labor–in response to the relative cost of labor, proxied by the wage share (u). The term μ_2 reflects the tendency of capitalists to further promote mechanization and restrict employment growth as the employment rate (v) exceeds its 'normal' level (v^n).[15] Together, these components formalize the Marxian intuition that mechanization functions both as a means to weaken the bargaining position of labor when employment is high and as a strategy to minimize production costs (Cajas Guijarro 2024).

In addition to μ_1 and μ_2, equation (21) introduces the term μ_3 to capture the potential tendency of capacity utilization to slow down mechanization, based on the following reasoning. Since labor is the productive factor, with a productivity that grows at a constant rate ($\hat{a} = \alpha$), and it is more easily adjustable–through hiring or dismissal–than the investment and installation of new capital, an expansion in demand, reflected in a higher capacity utilization rate ($\uparrow x$), tends to induce firms to increase production by employing additional labor per 'unit' of installed capital. This process effectively decelerates mechanization ($\downarrow \hat{m}$). This assumption aligns with

[13] See Cajas Guijarro and Vera (2022) and Cajas Guijarro (2024) for a synthesis of Marxian intuitions and their formalization through three-dimensional dynamical systems incorporating endogenous technical change and capable of generating stable limit cycles.

[14] The role of mechanization in capitalist dynamics–particularly its impact on labor–was addressed by Ricardo in his chapter 'On machinery' (1821, chap. 31), and by Marx in 'Machinery and modern industry' (2010a, chap. 15) and 'The General Law of Capitalist Accumulation' (2010a, chap. 25), especially in his discussion of the 'organic composition of capital' and its relation with the reserve army of labor.

[15] For models that posit a direct positive effect of the wage share and the employment rate on mechanization growth, see Glombowski and Krüger (1987, pp. 263–265), Eagly (1972, p. 531), Ryzhenkov (2006, p. 31, 2009, pp. 353–355), Cajas Guijarro (2024, p. 704).

Schoder's (2014) argument that, during periods of strong demand, firms may expand production without immediately increasing capital investment.[16]

Regarding capacity utilization and demand, Marx proposed a complex critique of the existence of equilibrium in the goods market, particularly through his concept of 'overproduction' (Marx 2010b, c). However, for the sake of simplicity, the Classical-Marxian closure adopted in this paper assumes that excess demand is zero ($q^{ED} = 0$) and the economy operates continuously at its 'normal' level of capacity utilization ($x = x^n$), which is further assumed to be constant ($\hat{x} = 0$).[17] Combining these assumptions with equations (7) and (8) allow us to express desired investment as:

$$\hat{k}_d = \hat{k}_{IS} = \frac{s(1-u)}{\sigma} - \delta \quad (21)$$

This formulation implies that desired investment coincides with the capital growth rate associated with the hypothetical IS scenario, in which the goods market clear and firms' expectations are fully realized. Moreover, combining this result with equation (18) and the previous assumptions establishes that desired investment necessarily equals the effective growth rate of installed capital ($\hat{k} = \hat{k}_d$), implying that effective capital accumulation is fully financed through savings and is completely realized.

Combining equations (16), (17), (19), (20), and (21) yields the following three-dimensional autonomous dynamical system, where the wage share (u), the employment rate (v), and the capital-output ratio (σ) are the state variables:

$$\hat{u} = -(\alpha + \gamma) + \rho v \quad (22)$$

$$\hat{v} = \frac{s x^n (1-u)}{\sigma} - (\beta + \delta + \mu_0) - \mu_1 u - \mu_2 (v - v^n) + \mu_3 x^n \quad (23)$$

$$\hat{\sigma} = \mu_0 + \mu_1 u + \mu_2 (v - v^n) - \mu_3 x^n - \alpha \quad (24)$$

Under the simplifying assumption that the 'normal' employment rate coincides with its equilibrium value ($v^n = v^*$), the steady state ($\hat{u} = \hat{v} = \hat{\sigma} = 0$) is characterized by the following non-trivial equilibrium point (u^*, v^*, σ^*):

$$u^* = \frac{H_1}{\mu_1}, \qquad v^* = \frac{H_2}{\rho}, \qquad \sigma^* = \frac{s x^n (\mu_1 - H_1)}{\mu_1 H_3} \quad (25)$$

where:

$$H_1 = \alpha - \mu_0 + \mu_3 x^n, \qquad H_2 = \alpha + \gamma, \qquad H_3 = \alpha + \beta + \delta$$

[16] In a four-dimensional structuralist model of endogenous cycles, Schoder (2014, p. 16) assumes that a higher capacity utilization rate ($\uparrow x$) accelerates the 'capacity-capital ratio,' which, in our formulation, corresponds to a deceleration of the effective output-capital ratio ($\downarrow \hat{\sigma}$). Given that labor productivity grows at a constant rate ($\hat{a} = \alpha$), equation (19) implies that the only way a higher x can reduce $\hat{\sigma}$ is by slowing the grow rate of mechanization ($\downarrow \hat{m}$).

[17] These assumptions follow the interpretation of the Classical-Marxian framework proposed by Blecker and Setterfield (2019, chap. 2), with one exception: the 'normal' level of capacity utilization is not necessarily assumed to coincide with full capacity utilization ($x^n < 1$). Analytically, we allow for varying levels of 'normal' utilization to facilitate the construction of the broader model developed in subsequent sections.

This equilibrium is economically meaningful (i.e., $0 < u^*, v^* < 1,\ \sigma^* > 0$) if the following conditions hold:

$$\mu_0 - \mu_3 x^n < \alpha, \qquad \mu_1 > H_1, \qquad \rho > H_2 \quad (26)$$

From equation (25), it can be observed that the equilibrium employment rate (v^*) mirrors the result in Goodwin's original model: it increases when the bargaining power of the working class weakens in the real wage Phillips curve ($\uparrow \gamma, \downarrow \rho$) or when labor productivity growth rises ($\uparrow \alpha$). In contrast to Goodwin's model, however, the equilibrium wage share (u^*) now depends on parameters governing the mechanization process: it decreases with a stronger autonomous tendency to mechanize production ($\uparrow \mu_0$) and with a stronger sensitivity of mechanization to the wage share ($\uparrow \mu_1$). Conversely, u^* increases with a higher 'normal' utilization rate ($\uparrow x^n$), a stronger influence of utilization on mechanization ($\uparrow \mu_3$), and a faster labor productivity growth rate ($\uparrow \alpha$). The latter result contrasts with the standard Goodwin framework, suggesting that when labor productivity growth outpaces the autonomous growth of mechanization, it supports a higher share of income accruing to workers. Regarding the equilibrium capital-output ratio (σ^*), it increases with stronger mechanization parameters ($\uparrow \mu_0, \uparrow \mu_1, \downarrow \mu_3, \downarrow x^n$), with a higher savings rate ($\uparrow s$), and decreases with higher labor productivity growth ($\uparrow \alpha$), a higher rate of labor supply growth ($\uparrow \beta$), and a higher depreciation rate (δ).

4.2. Stability, Simulations, and Sensitivity Analysis

Regarding the dynamic properties of the system proposed within the Classical-Marxian closure, it can be analytically demonstrated that its equilibrium point is stable, as stated in Proposition 1.

Proposition 1. *The dynamic system (22), (23), and (24) is locally asymptotically stable provided that, in addition to condition (26), the following inequalities are satisfied:*

$$\gamma < \frac{\rho^2 H_1 H_3}{\mu_2(\rho H_1 + \mu_2 H_3)} - \alpha \quad (27)$$

$$\mu_1 < \mu_1^c = H_1\left[1 + \frac{\rho \mu_2 H_2 H_3}{\rho^2 H_1 H_3 - \mu_2 H_2(\rho H_1 + \mu_2 H_3)}\right] \quad (28)$$

That is, the system is stable when the tendency of the real wage to fall (γ) is sufficiently low, and the firms' incentive to accelerate mechanization in response to labor costs (μ_1) remains below a critical threshold ($\mu_1 < \mu_1^c$).

Proof. See Appendix A.1.

It is also possible to establish the existence of persistent cyclical behavior:

Proposition 2. *When conditions (26) and (27) are satisfied, and the influence of the wage share on mechanization growth approaches the critical value in (28) ($\mu_1 \approx \mu_1^c$), the system (22), (23), and (24) undergoes a Hopf bifurcation.*

Proof. See Appendix A.2.

It is important to emphasize that Proposition 2 is based solely on the existence part of the Hopf bifurcation theorem for three-dimensional dynamical systems. Accordingly, these results do not provide a theoretical characterization of the stability or direction of the limit cycles generated by the system. To address this limitation, we rely on numerical simulations to illustrate the main dynamics of the Classical-Marxian closure, using parameter values primarily estimated for the French economy.[18]

Specifically, we adopt the values for labor productivity growth (α), labor supply growth (β), depreciation rate (δ), capitalist savings rate (s), the parameters of the real wage Phillips curve (γ, ρ), and the 'normal' and equilibrium employment rate ($v^* = v^n$) estimated by Grasselli and Maheshwari (2018, p. 630) for the case of France from 1960 to 2010, based on a two-dimensional Goodwin model with a general capital accumulation rate. Regarding the mechanization function, due to a lack of direct estimates for France, we adopt the value for the influence of the wage share on mechanization (μ_1) estimated by Ryzhenkov (2009, p. 368) for Italy over 1980-2007. We further assume that the 'normal' capacity utilization rate ($x^* = x^n$) matches the estimate provided by Charles (2024, p. 757) in his empirical analysis of investment functions in France over 1979-2020. As a baseline, we assume $\mu_3 = \mu_1$.

For the autonomous component of mechanization growth (μ_0), we use equation (25) to calibrate a value that ensures the model's equilibrium wage share (u^*) matches the estimate by Grasselli and Maheshwari (2018). Finally, for the sensitivity of mechanization to deviations of the employment rate from its 'normal' level (μ_2), we use equation (28) to indirectly determine a value that guarantees the existence of persistent cycles.

The resulting values derived through the outlined parametrization process are summarized in Table 1. The obtained equilibrium point (u^*, v^*, σ^*) closely matches the estimates reported by Grasselli and Maheshwari (2018) for France, and all parameter values satisfy the conditions formulated in Propositions 1 and 2. Using these values, we simulate the dynamical system (22)-(24), with the resulting trajectories illustrated in Figure 1.

[18] We emphasize that the primary aim of our numerical simulations is to illustrate the dynamic behavior of the different models proposed in this paper. Calibrating the different models to accurately reflect real-world economies using econometric techniques is reserved for future research.

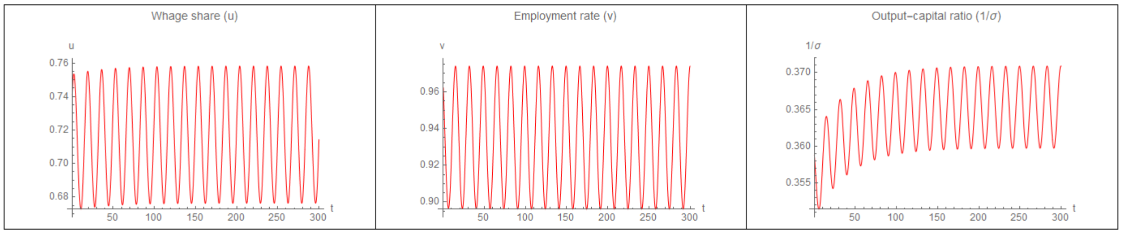
Table 1. Parameter Values for the Classical-Marxian Closure (Base Case Simulation)

Parameter	Description	Economic role	Value (Base case)	Effect on stability	Effect on periodicity
β	Labor supply growth rate	Labor market	0.008	Destabilizing	Decrease
γ	Autonomous tendency of the real wage to fall	Labor market	0.491	Stabilizing	Decrease
ρ	Effect of the employment rate on real wage growth	Labor market	0.549	Destabilizing	Increase
α	Labor productivity growth rate	Technology	0.022	Stabilizing	Decrease
μ_0	Autonomous component of mechanization growth	Technology	0.0375	Destabilizing	Increase
μ_1	Effect of wage share on mechanization growth (bifurcation parameter)	Technology	0.138	Destabilizing	Increase
μ_2	Effect of employment gap on mechanization growth	Technology	0.0262	Stabilizing	Decrease
μ_3	Effect of capacity utilization on mechanization growth	Technology	0.138	Stabilizing	Decrease
s	Capitalist savings rate	Savings	0.792	Stabilizing	Decrease
δ	Depreciation rate	Capital accumulation	0.038	Destabilizing	Decrease

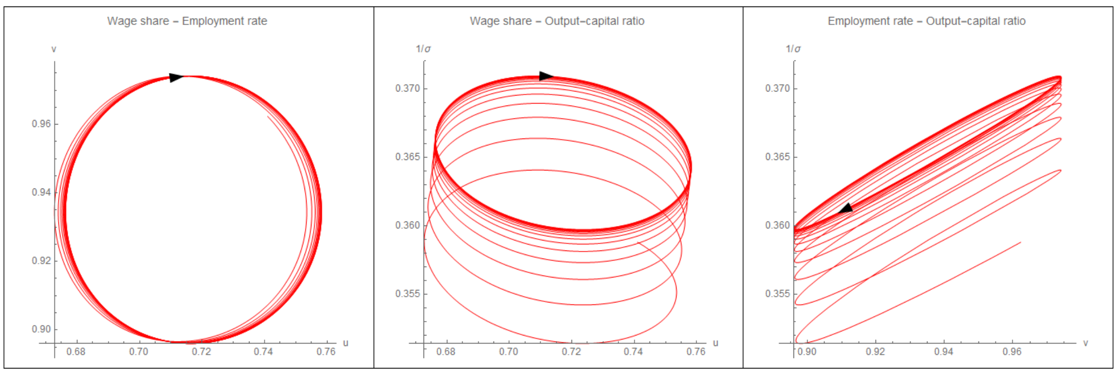
Note: Stability and periodicity effects are assessed by individually increasing each parameter while holding all others at their base-case values. Equilibrium: $u^* = 0.7165, v^* = 0.9344, \sigma^* = 2.7373, \sigma^{-1*} = 0.3653$. Periodicity of the limit cycle: $P = 16.7593$ years. See Figure 3 for a visual representation of parameter effects on system stability.

Figure 1. Simulation of the Classical-Marxian Closure (Base-Case)

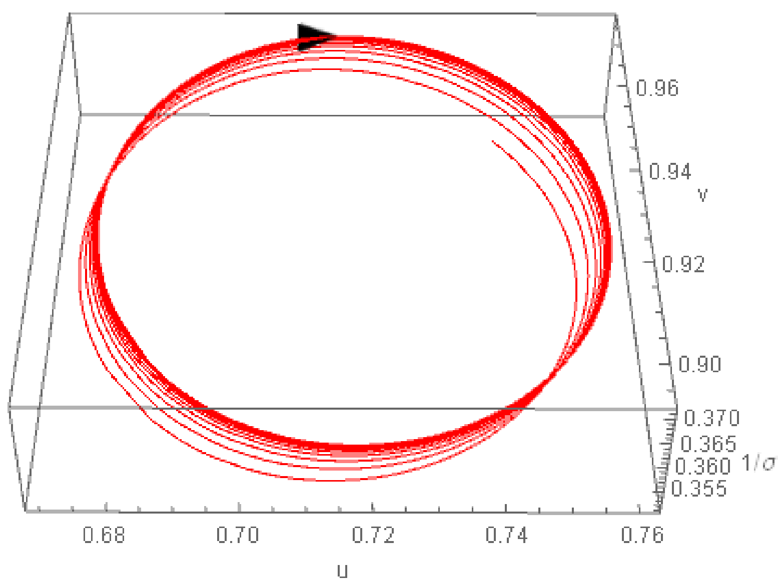
1A. Time Series



1B. Two-Dimensional Phase Trajectories



1C. Three-Dimensional Phase Trajectory



Note: Simulation of system (22), (23), and (24) using the parameter values defined in Table 1 and the following initial conditions: $u_0 = 0.75, v_0 = 0.95, \sigma_0 = 2.8, \sigma_0^{-1} \approx 0.3571$. Equilibrium: $u^* = 0.7165, v^* = 0.9344, \sigma^* = 2.7373, \sigma^{-1*} = 0.3653$. Periodicity of the limit cycle: $P = 16.7593$ years. $t \in [0,300]$. See Appendix A.3 for additional details.

From Figure 1, it can be observed that after some initial fluctuations, the trajectories converge to stable, clockwise-oriented limit cycles in both the $u - v$ and $u - \sigma^{-1}$ planes. This dynamic implies that the economic activity indicators–the employment rate (v) and the output-capital ratio (σ^{-1})[19]–act as leading variables within the cycle, while the wage share (u) is the lagging variable. This sequency of variables is characteristic of the so-called 'Goodwin pattern' (Setterfield 2023).[20] Additionally, clockwise oscillation in the $v - \sigma^{-1}$ plane are identified, suggesting that movements in the output-capital ratio lead those in the employment rate. These oscillatory patterns align with the direction of oscillations observed in both theoretical models and empirical studies on distributive cycles (Zipperer and Skott 2011; Barrales-Ruiz et al. 2022; Rada et al. 2023).

The persistence of cycles is confirmed not only visually but also numerically. Specifically, the Jacobian of the system (22)-(24) evaluated at the equilibrium point (u^*, v^*, σ^*) defined in (25) has two purely imaginary eigenvalues ($\lambda_{1,2}^M = \pm\omega i$) and one real negative eigenvalue ($\lambda_3^M < 0$) under the base-case parameter values presented in Table 1 (see Appendix A.3 for details). These eigenvalues imply that the limit cycles generated by the Classical-Marxian closure have an approximate periodicity of $P = 2\pi/\omega \approx 16.75$ years. Although this result exceeds the typical duration of business cycles observed in European economies such as France (approximately 11 years),[21] it remains below the periodicity of 21.25 years estimated by Grasselli and Maheshwari (2018) for the standard Goodwin model. In any case, it should be emphasized that the cycles identified within the Classical-Marxian closure emerge from a highly stylized model that abstracts from fluctuations in aggregate demand–a feature that significantly affects cycle periodicity, as has been discussed in the literature and will be illustrated when analyzing the Keynesian-Kaleckian closure.[22]

Concerning the sensitivity of the system to parameter changes, and following Proposition 2, we treat the influence of the wage share on mechanization growth (μ_1) as a bifurcation parameter. Simulations reveal a critical value at $\mu_1 = 0.138$ that guarantees the emergence of limit cycles. Furthermore, additional simulations suggest that when μ_1 crosses this threshold–from values below the critical level ($\mu_1 < \mu_1^c$) to values above ($\mu_1 > \mu_1^c$)–the system transitions from generating stable cycles to unstable ones. However, it may take several years for the long-run stability or instability to become evident, as illustrated in Figure 2, where μ_1 is varied by $\pm 5\%$ around the base value.

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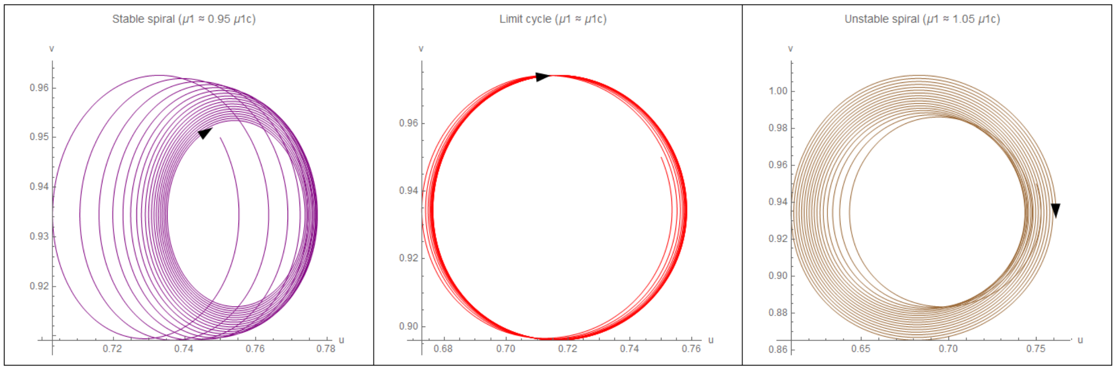
[19] We use the inverse of σ to facilitate interpretation and comparison with other results in the literature, where the output-capital ratio (i.e., the inverse of σ) is commonly used as an indicator of economic activity.

[20] More specifically, Setterfield (2023) associates the 'Goodwin pattern' with counterclockwise oscillations in the plane defined by real activity indicators (such as the employment rate, output-capital ratio, or capacity utilization rate) and the wage share. These correspond to the clockwise oscillations shown in Figure 1, due to differences in variable ordering and axis orientation.

[21] According to Aviat et al. (2023, p. 41), the average duration of economic cycles in the French economy since 1970 is approximately 11 years.

[22] For discussion on the periodicity of cycles in the Goodwin model and the potential influence of demand, see Atkinson (1969) and Glombowski and Krüger (1988).

Figure 2. Transition from Stable to Unstable Cycles on the Classical-Marxian Closure



Note: Simulation of system (22), (23), and (24) using the parameter values defined in Table 1 and the following initial conditions: $u_0 = 0.75, v_0 = 0.95, \sigma_0 = 2.8, \sigma_0^{-1} = 0.3571$. $t \in [0,300]$. See Appendix A.3 for additional details.

The transition from stable to unstable cycles due to changes in the bifurcation parameter (μ_1) is also verified numerically since the Jacobian matrix of the system (22)-(24), evaluated at (u^*, v^*, σ^*) exhibits two complex eigenvalues with negative real parts when $\mu_1 < \mu_1^c$, purely imaginary eigenvalues when $\mu_1 = \mu_1^c$, and complex eigenvalues with positive parts when $\mu_1 > \mu_1^c$ (see Appendix A.3). Thus, a pair of eigenvalues crosses the imaginary axis, indicating that the model undergoes a Hopf bifurcation at μ_1^c, consistent with Proposition 2. The Hopf bifurcation appears to be supercritical, as trajectories initially starting outside the limit cycle are eventually attracted toward it when μ_1 is sufficiently close to μ_1^c. Moreover, it can be classified as a 'direct' bifurcation, given that stability deteriorates as μ_1 increases beyond the critical threshold.

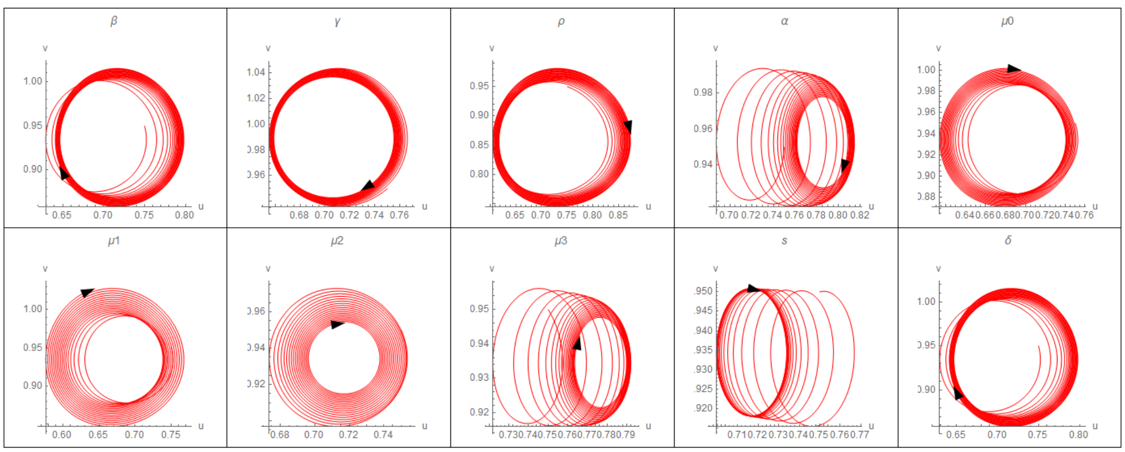
Based on these results, we conclude that the capitalist firms' propensity to accelerate mechanization in response to the relative labor cost, represented by the wage share, is a destabilizing force within the system, increasing instability as μ_1 rises beyond its critical value. This result contrasts with the intuition advanced by Shah and Desai (1981) in their three-dimensional extension of the Goodwin model. In their framework, endogenous technical change acts as an 'additional weapon' for the capitalist class, enabling stabilization and the avoidance of permanent cyclical struggle with the working class. By contrast, within the Classical-Marxian closure developed here, the possibility of permanent cycles and instability is not ruled out. Thus, the 'additional weapon' provided by endogenous technical change is not without limits: when the incentive to accelerate mechanization in response to labor costs becomes excessive, it can instead generate system instability.[23] This conclusion is consistent with other extensions of the Goodwin model incorporating endogenous technical change

[23] Cajas Guijarro (2024) argues that, even within a framework of optimizing capitalist firms, endogenous technical change induced by labor costs does not necessarily discards the existence of persistent limit cycles–provided that labor productivity growth is positively related to the growth rate of the employment rate. This relationship can be economically justified by incorporating a Kaldor-Verdoorn effect into the specification of labor productivity growth.

capable of producing persistent limit cycles, such as van der Ploeg (1987), Julius (2005), Ryzhenkov (2009), Cajas Guijarro (2024).[24]

Beyond the destabilizing role of μ_1, further simulations reveal that the sensitivity of mechanization to the employment gap (μ_2) and to capacity utilization (μ_3) are stabilizing influences, while the autonomous component of mechanization growth (μ_0) is destabilizing, as summarized in Table 1 and illustrated in Figure 3. These parameters also exert mixed effects on cycle periodicity: increases in μ_0 and μ_1 tend to lengthen the cycle, while increases in μ_2 and μ_3 tend to shorten it. Labor productivity growth (α) is stabilizing and reduces periodicity. Concerning the parameters of the real wage Phillips curve, a stronger bargaining power of the working class ($\downarrow \gamma, \uparrow \rho$) tends to destabilize the system and lengthen cycle duration. The growth rate of labor supply (β) similarly exerts a destabilizing influence. Finally, the capitalist savings rate (s) is stabilizing, whereas depreciation (δ) is destabilizing but reduces periodicity.

Figure 3. Sensitivity to Parameter Variations in the Classical-Marxian Closure



Note: Simulations of system (22), (23), and (24) obtained by individually increasing each parameter while holding all others at their base-case values defined in Table 1. Initial conditions: $u_0 = 0.75, v_0 = 0.95, \sigma_0 = 2.8, \sigma_0^{-1} = 0.3571$. $t \in [0,300]$.

From these interpretations, we conclude that within the Classical-Marxian closure proposed in this paper, two 'arenas' of class struggle–the wage bargaining process, as in the original Goodwin model, and technological change, particularly through endogenous mechanization–jointly determine the stability and periodicity of cycles. The interaction between these two mechanisms gives rise to distributive cycles, manifested in fluctuations of the wage share and the employment rate, alongside 'technological cycles,' characterized by fluctuations in the capital-output ratio driven by mechanization dynamics. Importantly, all these cyclical behaviors emerge endogenously, without requiring explicit fluctuations in aggregate demand–a dimension that will be introduced in the next section.

[24] Indeed, the presence of instability when $\mu_1 > \mu_1^c$ aligns with the findings of Ryzhenkov (2009, p. 366) in his extension of the Goodwin model that incorporates endogenous mechanization.

4. The Keynesian-Kaleckian Closure

5.1. Assumptions and Formulation

Building on Marx's (2010c) insights on credit money, the theory of investment financing though the banking sector and the creation of purchasing power formulated by Kalecki (1971), the concept of 'monetary theory of production' elaborated by Keynes (1933), and some post-Keynesian perspectives on money and credit (Hein 2023), we introduce a Keynesian-Kaleckian closure of the model.[25] Within this framework, desired net investment ($\hat{k}_d$)–and thus effective demand–is treated as being determined independently of aggregate savings.

Within the Keynesian-Kaleckian closure proposed in this paper, the emphasis lies on the interaction between distributive struggle and fluctuations in effective demand driven by desired investment. In this setting, we model desired net investment ($\hat{k}_d$) through an investment function that encapsulates the principal factors motivating firms to invest, independently of the total level of savings. The function is specified as follows:

$$\hat{k}_d = -\theta_0 + \theta_1(x - x^n) + \theta_2(1 - u) - \theta_3(v - v^n), \qquad 0 < \theta_i < 1 \quad (29)$$

In this formulation, θ_0 captures an exogenous tendency for investment to stabilize,[26] θ_1 represents the positive impact on desired investment resulting from deviations of actual capacity utilization (x) from its 'desired' or 'normal' level (x^n), reflecting investment incentives generated by buoyant market conditions (Flaschel 2009, chap. 4); and θ_2 measures the positive effect of expected profitability, proxied by the profit share $(1 - u)$ (Bhaduri and Marglin 1990).

Beyond these elements, which are commonly incorporated into post-Keynesian investment functions, similarly to Flaschel and Skott (2006), we introduce an additional term θ_3 to capture the negative effect of an employment rate exceeding a 'normal' level ($v > v^n$) on desired investment. The underlying idea is that a high employment rate enhances the bargaining power of the working class relative to capitalists. This scenario implies that capitalists require 'more surveillance' to maintain the work effort, thereby increasing supervision costs and raising workforce recruitment and turnover expenses. Consequently, when the employment rate is elevated, investment and capital accumulation become less attractive to firms. This mechanism is also consistent with Kalecki's (2021) argument that high employment diminishes the political power of capitalists and constrains their expansion plans.[27]

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[25] For simplicity, the Keynesian-Kaleckian closure presented in this paper implicitly assumes that prices are determined by an endogenous markup that adjusts in response to changes in income distribution. A formal treatment of inflation dynamics is left for future discussion.

[26] Some authors interpret the term $-\theta_0$ as reflecting the influence of 'animal spirits' on investment, encompassing historical, political, and psychological factors that shape investment decisions (Hein 2014, p. 248). However, as Blecker and Setterfield (2019, p. 136) note, this influence may not be limited to $-\theta_0$ but could also be implicitly embedded in other parameters of the investment function. Moreover, empirical evidence suggests that investment functions may indeed exhibit a negative autonomous component like $-\theta_0$, depending on their specific form (see, e.g., Charles, 2024).

[27] Other works that assume a negative influence of the employment rate on desired investment include Ryoo and Skott (2008), Skott and Zipperer (2012), Skott (2023), the latter of which explicitly models this effect using the gap between the employment rate and its average value. Similar intuitions–though

Another hey assumption in the Keynesian-Kaleckian closure is the persistent existence of excess demand ($q^{ED} \neq 0$). Within this context, the concepts of desired net investment ($\hat{k}_d$) and effective growth of installed capital ($\hat{k}$) exhibit distinctions that are not present in the Classical-Marxian closure. In particular, Kalecki (1935) emphasizes that there is typically a time delay between the intention of a capitalist firm to invest and the subsequent production, delivery, and installation of capital goods.[28] Furthermore, the realization of the entire volume of desired investment into an effective increase in installed capital depends critically on adequate financing; even partial financial constraints can hinder the complete execution of planned investments.[29]

These differences are relevant for analyzing the dynamic nature of capital accumulation within economic cycles that incorporate fluctuations in effective demand. Nevertheless, a full incorporation of these complexities into the model would substantially increase its analytical burden.[30] To maintain tractability, we adopt a simplified approach by defining the relationship between effective capital growth and desired net investment as follows:

$$\hat{k} = \varepsilon \hat{k}_d, \qquad 0 < \varepsilon \leq 1 \quad (30)$$

In this formulation, ε represents the efficiency with which desired investment is transformed into installed capital. Specifically, when $\varepsilon = 1$, it is assumed that the entire volume of desired investment is seamlessly realized as installed capital, implying the absence of credit constraints and minimal delays between capital acquisition and installation (Marglin and Bhaduri 1991, p. 126). Conversely, when $\varepsilon < 1$, only a portion of the planned investment materializes. For simplicity, we assume that unrealized investment is lost rather than carried over or accumulated for future periods.

In addition to these assumptions regarding desired investment and effective capital accumulation, since the primary purpose of the Keynesian-Kaleckian closure is to illustrate the interaction between distributive cycles and fluctuations of effective demand, we further assume the presence of Harrod-neutral technical change. Under this assumption, the capital-

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applied to the output expansion function rather than investment–can be found, for instance, in Skott (1989, 2015) and Rada et al. (2023).

[28] Sportelli and De Cesare (2022) propose a model that emphasizes the time delay between firms' invest intentions and the actual realization of installed capital as a key factor in generating economic cycles. Their analysis also explores how such delays can contribute to instability and even chaotic dynamics.

[29] According to Palley (2002, pp. 175–176), within an endogenous money framework, actual expenditure on capital goods is determined by the minimum of desired investment and the sum of available credit and firms' internal funds (free cash flows).

[30] On one hand, like Sportelli and De Cesare (2022, p. 96), we might posit that the effective increase in installed capital equates to the desired investment but incorporates a time delay (e.g., $\hat{k}_t = \hat{k}_{d,t-t_1}$), necessitating the use of delay-differential equations. On the other hand, following Dutt (2006, p. 436), it could be assumed that discrepancies between the effective increase of installed capital and desired investment prompt dynamic adjustments in future capital growth (e.g., $d\hat{k}/dt = \Lambda(\hat{k}_d - \hat{k}), \Lambda > 0$), alongside dynamic effects linked to credit. Such assumptions would expand the number of state variables and dimensions of the model formulated in this paper. Therefore, these concepts and others are left for future discussion focused on the details of investment dynamics.

output ratio remains constant ($\hat{\sigma} = 0$), implying that mechanization and labor productivity grow at the same rate ($\hat{m} = \hat{\alpha}$).

Combining these assumptions with equations (16)-(29) and (30), we obtain the following three-dimensional autonomous dynamical system, with the wage share (u), the employment rate (v), and the rate of capacity utilization (x) as the state variables:

$$\hat{u} = -(\alpha + \gamma) + \rho v \quad (31)$$

$$\hat{v} = \frac{sx(1-u)}{\sigma} - (\alpha + \beta + \delta) + \phi G(u, v, x) \quad (32)$$

$$\hat{x} = (1-u)\left(\frac{sx - \varepsilon\sigma\theta_2}{\sigma}\right) - \delta - \varepsilon[-\theta_0 + \theta_1(x - x^n) - \theta_3(v - v^n)] + \phi G(u, v, x) \quad (33)$$

where:

$$G(u, v, x) = \frac{\sigma[-\theta_0 + \theta_1(x - x^n) - \theta_3(v - v^n) + \delta] + (1-u)(\sigma\theta_2 - sx)}{x}$$

To study the behavior of this system, we consider a scenario in which all desired investment is fully converted into installed capital ($\varepsilon = 1$).[31] Under this assumption, and at the steady state ($\hat{u} = \hat{v} = \hat{x} = 0$), the dynamic system (31)-(33) admits a non-trivial equilibrium point (u^*, v^*, x^*). By further assuming that the 'normal' levels of capacity utilization and employment coincide with their equilibrium values ($x^n = x^*, v^n = v^*$), the equilibrium can be expressed as:

$$u^* = \frac{\theta_2 - H_4}{\theta_2}, \qquad v^* = \frac{H_2}{\rho}, \qquad x^* = \sigma H_5 \quad (34)$$

where:

$$H_2 = \alpha + \gamma, \qquad H_3 = \alpha + \beta + \delta, \qquad H_4 = \alpha + \beta + \theta_0, \qquad H_5 = \frac{\theta_2 H_3}{s H_4},$$

This equilibrium is economically meaningful (i.e., $0 < u^*, v^*, x^* < 1$) if the following conditions are satisfied:

$$H_4 < \theta_2 < \frac{sH_4}{\sigma H_3}, \qquad \alpha + \gamma < \rho \quad (35)$$

From equation (34), we note that the growth rate of labor productivity (α) negatively affects the equilibrium wage share (u^*), while it positively influences the equilibrium employment rate (v^*) and capacity utilization rate (x^*). This pattern is consistent with the original Goodwin model, where higher labor productivity growth strengthens the profit share and boosts employment.

Moreover, an expansion of investment–represented by a lower autonomous tendency to stabilize (↓ θ_0) or a stronger influence of the profit share on investment (↑ θ_2)–positively impacts u^* and x^*, but does not affect v^*. Economically, this suggests that exogenous increases in investment stimulate aggregate demand and raise the degree of capacity utilization, a result

[31] We assume $\varepsilon = 1$ for simplicity, as allowing $\varepsilon < 1$ significantly increases the complexity of the steady-state equilibrium. Nonetheless, the influence of ε on the stability and periodicity of cycles is examined in Table 2.

that aligns with post-Keynesian models of distribution and growth incorporating an autonomous investment function (Blecker and Setterfield 2019, chap. 4).

Additionally, the equilibrium employment rate (v^*) increases when the bargaining power of the working class weakens in the real wage Phillips curve ($\uparrow\gamma, \downarrow\rho$), while u^* and x^* remain unaffected–again reflecting patterns found in the original Goodwin model. We also note that the capitalist savings rate (s) negatively influences x^* through its depressive effect on aggregate demand, illustrating the so-called 'paradox of thrift,' which is only possible when investment is independent of savings. Lastly, the growth rate of labor supply (β) reduces both the equilibrium wage share (u^*) and capacity utilization (x^*), while the depreciation rate (δ) increases x^*.

5.2. Stability, Simulations, and Sensitivity Analysis

Similarly to the Classical-Marxian closure, it is possible to analytically demonstrate the stability of the equilibrium point for the Keynesian-Kaleckian closure proposed in this section, as summarized in Proposition 3.

Proposition 3. *The dynamic system (31), (32), and (33) is locally asymptotically stable provided that, in addition to condition (35), the following conditions are satisfied:*

$$\phi > H_5, \qquad \theta_0 > \delta, \qquad \frac{sH_4}{\sigma\theta_2} < \theta_1 < \frac{sH_4^2}{\sigma\theta_2 H_3}, \qquad \theta_3 > \frac{\rho H_3(\phi - H_5)(\sigma\theta_1\theta_2 - sH_4)}{s\phi H_2 H_4} \quad (36)$$

$$H_7^2 < 4H_6H_8 \ \text{ or } \ \phi > \frac{H_7 + \sqrt{H_7^2 - 4H_6H_8}}{2H_6} \quad (37)$$

$$\theta_1 < \theta_1^c = \frac{Z_1}{Z_2} \quad (38)$$

where:

$$Z_1 = H_6\phi^2 - H_7\phi + H_8$$

$$H_6 = (\theta_3H_2 + \rho H_3)[H_3H_4\theta_3 + \rho(\theta_0 - \delta)(\theta_2 - H_4)]$$

$$H_7 = H_3H_5\{H_4\theta_3(\theta_3H_2 + 2\rho H_3) - \rho(\theta_2 - H_4)[H_2\theta_3 - 2\rho(\theta_0 - \delta)]\}$$

$$H_8 = \rho H_3H_5^2[H_3H_4\theta_3 + \rho(\theta_0 - \delta)(\theta_2 - H_4)]$$

$$Z_2 = \rho\sigma H_5H_9(\phi - H_5)$$

$$H_9 = \rho\phi(\theta_0 - \delta)(\theta_2 - H_4) + \theta_3H_3H_4(\phi - H_5)$$

In other words, the system is locally stable when the reaction of capitalist firms to excess demand (ϕ), the autonomous tendency of investment to stabilize (θ_0), and the sensitivity of desired investment to the employment gap (θ_3) are sufficiently large, while sensitivity to the capacity utilization gap (θ_1) falls within a specific range.

Proof. See Appendix A.4.

It is also possible to demonstrate the existence of persistent cyclical behavior:

Proposition 4. *When conditions (35), (36), and (37) are satisfied, and the sensitivity of investment to deviations in capacity utilization approaches the critical value defined in (38) ($\theta_1 \approx \theta_1^c$), the system (31), (32), and (33) undergoes a Hopf bifurcation.*

Proof. See Appendix A.5.

As in the Classical-Marxian closure, the result in Proposition 4 relies only on the existence part of the Hopf bifurcation theorem. Consequently, these results do not offer a complete characterization regarding the stability or direction of the limit cycles generated by the model. Given this limitation, we again turn to numerical simulations to illustrate the dynamics of the Keynesian-Kaleckian closure, using parameter values estimated primarily for France.

Thus, we assume that all planned investment is fully realized as installed capital ($\varepsilon = 1$). Regarding the values for labor productivity growth (α), labor supply growth (β), depreciation rate (δ), capitalist savings rate (s), the parameters of the real wage Phillips curve (γ, ρ), and the 'normal' and equilibrium employment rate ($v^* = v^n$), we adopt the estimates provided by Grasselli and Maheshwari (2018, p.630). Additionally, we assume that the 'normal' and equilibrium capacity utilization rate ($x^* = x^n$) corresponds to the estimate by Charles (2024, p. 757). For the capital-output ratio, we use the equilibrium value (σ^*) obtained from our Classical-Marxian closure.

Concerning the investment function specified in equation (29), the parameter reflecting the influence of the profit share (θ_2) is adopted from Charles (2024, p. 752). The sensitivity of investment to the employment rate gap is set at $\theta_3 = 0.03$.[32] Regarding the autonomous stabilization of investment (θ_0), we use equation (34) to calibrate a value ensuring that the equilibrium wage share (u^*) matches the estimate by Grasselli and Maheshwari (2018). We normalize the reaction to excess demand to $\phi = 1$, and we determine the value of θ_1 using equation (38) to ensure the existence of limit cycles.

The values obtained through this procedure are summarized in Table 2, and the simulated trajectories are presented in Figure 4. From this figure we observe that, after some initial fluctuations, the trajectories converge toward stable, clockwise-oriented limit cycles in the $u - v$ and $u - x$ planes. This dynamic implies, once again, that the wage share (u) acts as the lagging variable within the cycle, while the economic activity indicators–the employment rate (v) and the capacity utilization rate (x)–serve as the leading variables. Thus, the Keynesian-Kaleckian closure also replicates the 'Goodwin pattern' (Setterfield, 2023). Additionally, we observe clockwise oscillations in the $v - x$ plane, suggesting that movements in capacity utilization lead movements in the employment rate. Overall, these patterns appear consistent with available empirical evidence (Zipperer and Skott 2011; Barrales-Ruiz et al. 2022; Rada et al. 2023).

[32] Although the value chosen for θ_3 is relatively low compared to the other parameters of the investment function, it was selected to ensure the emergence of limit cycles with a plausible periodicity, while simultaneously satisfying all the conditions stated in Propositions 3 and 4.

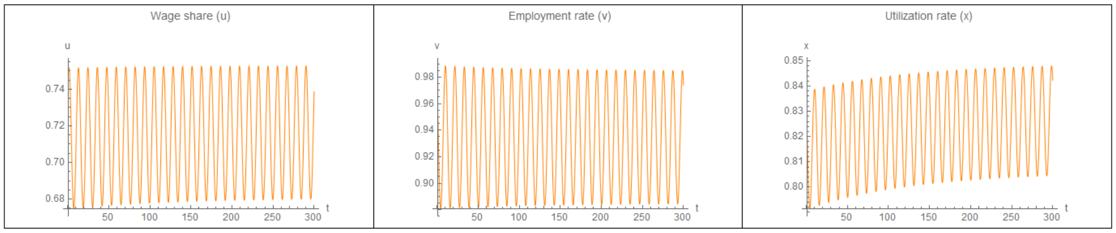
Table 2. Parameter Values for the Keynesian-Kaleckian Closure (Base Case Simulation)

Paramete r	Description	Economic role	Value (Base case)	Effect on stability	Effect on periodicity
β	Labor supply growth rate	Labor market	0.008	Destabilizing	Increase
γ	Autonomous tendency of the real wage to fall	Labor market	0.491	Stabilizing	Decrease
ρ	Effect of the employment rate on real wage growth	Labor market	0.549	Destabilizing	Increase
ϕ	Influence of excess demand on effective growth	Output expansion	1	Destabilizing	Decrease
α	Labor productivity growth rate	Technology	0.022	Destabilizing	Increase
σ	Effective capital-output ratio	Technology	2.7373	Destabilizing	Increase
s	Capitalist savings rate	Savings	0.792	Stabilizing	Decrease
δ	Depreciation rate	Investment	0.038	Destabilizing	Increase
ε	Efficiency of converting desired investment into installed capital	Investment	1	Stabilizing	Increase
θ_0	Autonomous tendency of investment to stabilize	Investment	0.0851	Stabilizing	Decrease
θ_1	Effect of capacity utilization (gap) on desired investment (bifurcation parameter)	Investment	0.1269	Destabilizing	Increase
θ_2	Effect of profit share on desired investment	Investment	0.406	Destabilizing	Decrease
θ_3	Effect of employment rate gap on desired investment	Investment	0.03	Stabilizing	Decrease

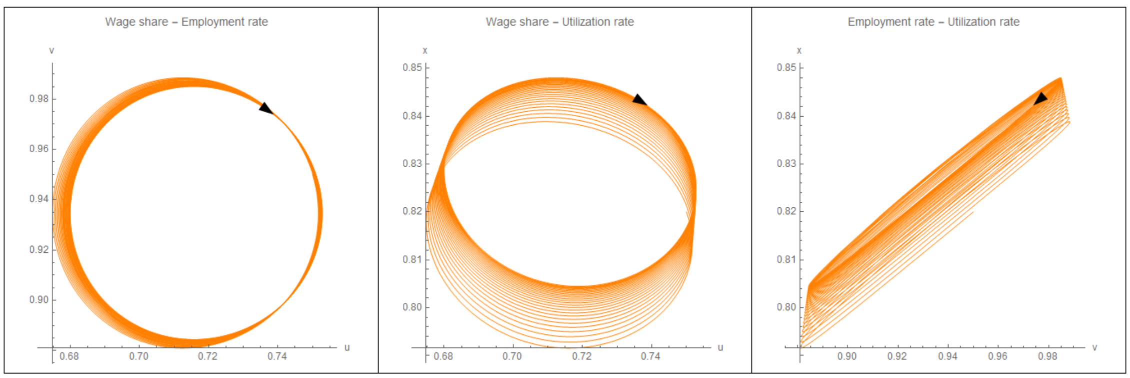
Note: Stability and periodicity effects are assessed by individually increasing each parameter while holding all others at their base-case values. Equilibrium: $u^* = 0.7165, v^* = 0.9344, x^* = 0.829$. Periodicity of the limit cycle: $P = 11.5841$ years. See Figure 6 for a visual representation of parameter effects on system stability.

Figure 4. Simulation of the Keynesian-Kaleckian Closure (Base-Case)

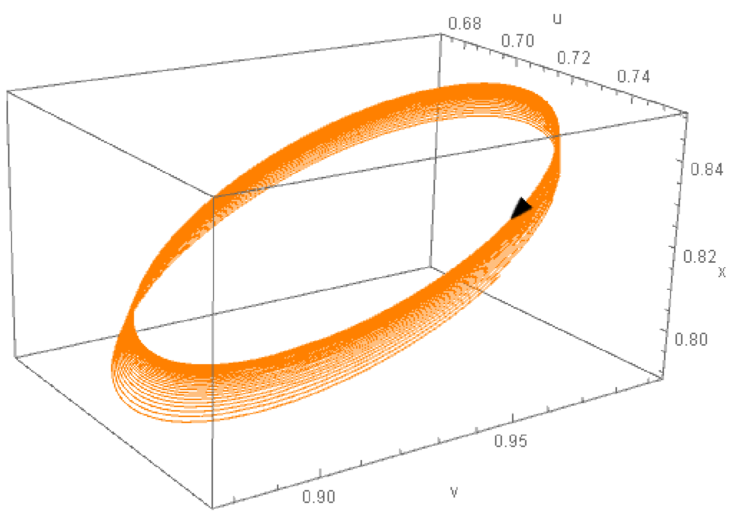
4A. Time Series



4B. Two-Dimensional Phase Trajectories



4C. Three-Dimensional Phase Trajectory



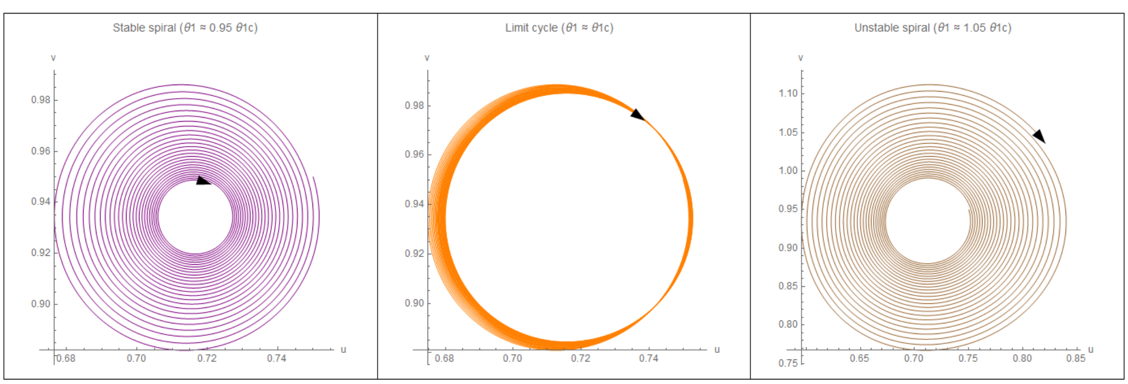
Note: Simulation of system (31), (32), and (33) using the parameter values defined in Table 2 and the following initial conditions: $u_0 = 0.75, v_0 = 0.95, x_0 = 0.82$. Equilibrium: $u^* = 0.7165, v^* = 0.9344, x^* = 0.829$. Periodicity of the limit cycle: $P = 11.5841$ years. $t \in [0,300]$. See Appendix A.6 for additional details.

Numerically, we verify that the Jacobian matrix of the system (31)-(33) evaluated at the equilibrium point (u^*, v^*, x^*) defined in equation (34) exhibits two purely imaginary eigenvalues ($\lambda_{1,2}^K = \pm\omega i$) and one real negative eigenvalue ($\lambda_3^K < 0$) when using the parameter values presented in Table 2 (see Appendix A.6 for details). Under these values, the Keynesian-Kaleckian closure generates limit cycles with a periodicity of approximately $P_K = 2\pi/\omega \approx 11.58$ years, which is close to the potential duration of business cycles observed in France. Nevertheless, we emphasize that the model remains highly simplified, and this periodicity should be interpreted for illustrative purposes only.

Regarding the sensitivity of the model to parameter changes, and based on Proposition 4, we treat the influence of the capacity utilization gap on desired investment (θ_1) as a bifurcation parameter, with a critical value $\theta_1^c \approx 0.1269$. Additional simulations suggest that when θ_1 is lower than the critical threshold ($\theta_1 < \theta_1^c$), the model produces stable oscillations; as θ_1 approaches the critical value ($\theta_1 \approx \theta_1^c$), the system transitions to limit cycles; and when θ_1 exceeds the critical value ($\theta_1 > \theta_1^c$), unstable oscillations emerge. Numerically, we observe a corresponding transition in the eigenvalues of the Jacobian matrix evaluated at (u^*, v^*, x^*): the relevant pair of eigenvalues displays negative real parts when $\theta_1 < \theta_1^c$, zero real parts at $\theta_1 = \theta_1^c$, and positive real parts when $\theta_1 > \theta_1^c$. This behavior is summarized in Figure 5 and detailed in Appendix A.6, where θ_1 is varied in $\pm 5\%$. These results confirm that the system (31)-(33) undergoes a Hopf bifurcation at θ_1^c, which appears to be supercritical, as trajectories initially starting outside the limit cycle are attracted toward it when θ_1 approaches θ_1^c. Furthermore, the bifurcation can be characterized as 'direct,' since stability is lost as θ_1 increases.

Comparing this sensitivity analysis with that of the Classical-Marxian closure, we note that the Keynesian-Kaleckian closure appears structurally less stable with respect to variations in the bifurcation parameter. Particularly, when θ_1 is increased, the resulting unstable spirals in the Keynesian-Kaleckian closure reach or exceed the full employment boundary ($v = 1$) more rapidly than in the Classical-Marxian case (compare Figures 2 and 5).

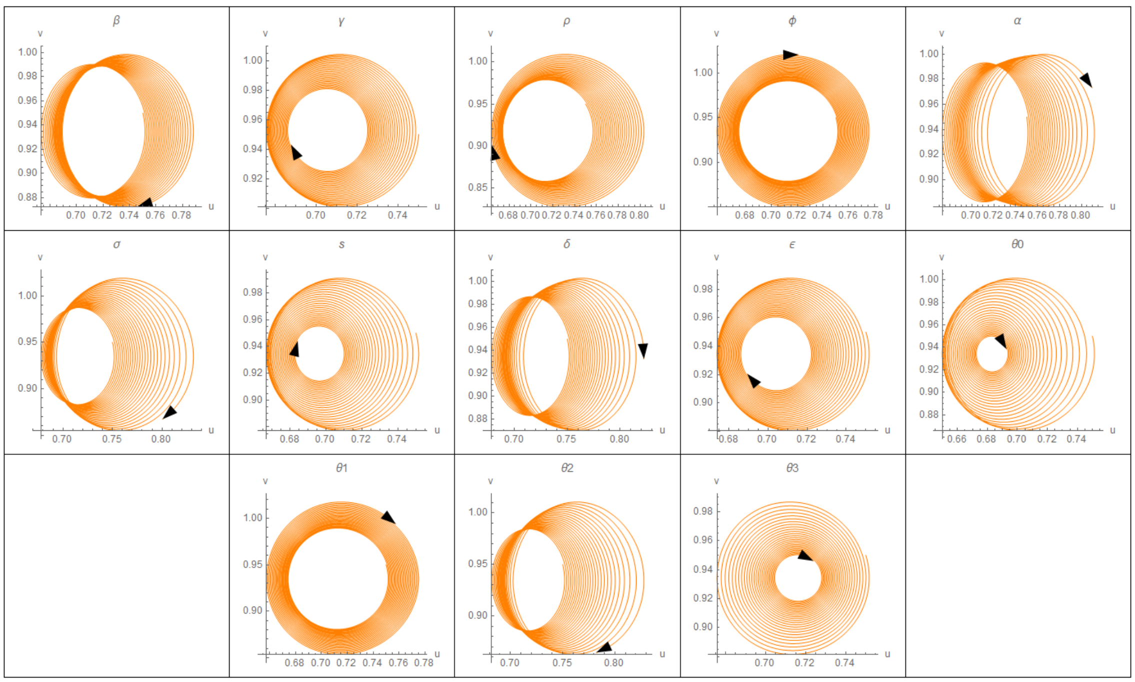
Figure 5. Transition from Stable to Unstable Cycles in the Keynesian-Kaleckian Closure



Note: Simulation of system (31), (32), and (33) using the parameter values defined in Table 2 and the following initial conditions: $u_0 = 0.75, v_0 = 0.95, x_0 = 0.82$. $t \in [0,300]$. See Appendix A.6 for additional details.

Additional simulations allow us to examine the sensitivity of the stability and periodicity of cycles to variations in specific parameters of the system (31)-(33), as summarized in Table 2 and illustrated in Figure 6. From these results we observe that, in addition to the sensitivity of desired investment to capacity utilization (θ_1), the sensitivity to the profit share (θ_2) also exerts a destabilizing influence on cycles. However, while θ_1 reduces the periodicity, θ_2 increases it. In contrast, the autonomous tendency of investment to stabilize (θ_1) and the influence of the employment rate gap on desired investment (θ_3) are both stabilizing effects and reduce the duration of cycles. In summary, parameter changes associated with stronger investment ($\downarrow \theta_0, \uparrow \theta_1, \uparrow \theta_2, \downarrow \theta_3$) tend to destabilize cycles. This result appears consistent with the notion of 'Keynesian stability,' according to which investment must exhibit relatively low sensitivity–particularly to the profit share–compared to savings, to ensure a stable equilibrium (Blecker and Setterfield, 2019).

Figure 6. Sensitivity to Parameter Variations in the Keynesian-Kaleckian Closure



Note: Simulations of system (31), (32), and (33) obtained by individually increasing each parameter while holding all others at their base-case values defined in Table 2. Initial conditions: $u_0 = 0.75, v_0 = 0.95, x_0 = 0.82.\ t \in [0{,}300]$.

Another relevant result from the sensitivity analysis is that the reaction of firms to disequilibrium in the goods market (ϕ) has a destabilizing effect on cycles and reduces their periodicity. This finding aligns with the concept of 'Harrodian instability,' in the sense that, when confronted with disequilibrium and unmet expectations, capitalists tend to respond in ways that amplify economic fluctuations. This behavior is consistent with other three-dimensional extensions of the Goodwin model that address Harrodian instability, such as von Armin and Barrales (2015) and Skott (2015, 2023).[33] In the present model, this source of instability is

[33] When comparing the numerical values of the Jacobian matrix from our Keynesian-Kaleckian closure (see Appendix A.6) with the sign pattern proposed by von Armin and Barrales (2015, p. 369), we find close alignment in nearly all entries. The only exceptions are $\partial \dot{u}/\partial u$ and $\partial \dot{u}/\partial x$, which we have intentionally set to zero in the simplified Phillips curve defined in equation (12) to facilitate the analytical derivation of

mitigated primarily through the inclusion of the negative influence of the employment rate on desired investment (θ_3).

For the parameters related to the labor market (β, γ, ρ), the capitalist savings rate (s), and the depreciation rate (δ), their qualitative effects on cycle stability are similar to those found in the Classical-Marxian closure, although their effects on periodicity differ somewhat. A particularly noteworthy case is the growth rate of labor productivity (α), which exhibits a destabilizing influence within the Keynesian-Kaleckian closure, whereas it had a stabilizing role within the Classical-Marxian formulation (and in both closures it affects periodicity in distinct ways). This contrast highlights how the role of labor productivity can undergo significant qualitative shifts depending on whether mechanization has its own dynamics. Finally, a higher efficiency in the conversion of desired investment into installed capital (ε) exerts a stabilizing influence on the system, although it tends to increase the periodicity of cycles.

From these interpretations, we conclude –similarly to the Classical-Marxian closure–that the Keynesian-Kaleckian closure is shaped by two distinct 'arenas' that determine the stability and periodicity of capitalist cycles. Once again, we identify the role of distributive class conflict, as captured by the structure of the original Goodwin model, combined with the influence of effective demand fluctuations driven by an autonomous investment function and by the response of firms to disequilibrium. When these two 'arenas' interact, distributive cycles–reflected in the oscillations of the wage share and the employment rate–combine with 'demand cycles,' particularly visible in the oscillation of the capacity utilization rate. Notably, all these cyclical dynamics emerge in the absence of endogenous technical change, which was instead a key mechanism in the Classical-Marxian closure through the channel of mechanization.

5. The MK System. A First Approximation

In the previous sections, we have examined the Classical-Marxian (M) and the Keynesian-Kaleckian (K) closures of the model proposed in this paper. We have shown that both closures can generate persistent endogenous cycles, each through its own internal assumptions and structural features. These formulations have been developed in a relatively parsimonious manner, in line with the main objective of this paper: to integrate them into a Marxian-Keynesian (MK) synthesis in which the cyclical dynamics associated with distributive class conflict, endogenous mechanization, and effective demand can coexist within a coherent dynamical framework.

To construct the MK synthesis proposed in this paper, we combine the general framework defined by equations (16) to (19) with the mechanization function introduced in equation (20), the investment function defined in equation (29), the relationship between desired investment and effective capital accumulation in equation (30), and the assumptions that labor productivity and labor supply grow at constant rates ($\hat{a} = \alpha, \hat{n} = \beta$). This procedure yields the following four-dimensional autonomous dynamical system, hereafter referred to as the MK system:

Propositions 3 and 4. As noted by von Armin and Barrales, the potentially destabilizing effects associated with these terms may be mitigated when trajectories deviate from the equilibrium point if nonlinearities are introduced into the output expansion function.

$$\hat{u} = -(\alpha + \gamma) + \rho v \quad (39)$$

$$\hat{v} = \frac{sx(1-u)}{\sigma} - (\beta + \delta + \mu_0) - \mu_1 u - \mu_2(v - v^n) + \mu_3 x + \phi G(u, v, x, \sigma) \quad (40)$$

$$\hat{x} = (1-u)\left(\frac{sx - \varepsilon\sigma\theta_2}{\sigma}\right) - \delta - \varepsilon[-\theta_0 + \theta_1(x - x) - \theta_3(v - v^n)] + \phi G(u, v, x, \sigma) \quad (41)$$

$$\hat{\sigma} = f_\sigma^M(u, v, x) = \mu_0 + \mu_1 u + \mu_2(v - v^n) - \mu_3 x - \alpha \quad (42)$$

where:

$$G(u, v, x, \sigma) = \frac{\sigma[-\theta_0 + \theta_1(x - x^n) - \theta_3(v - v^n) + \delta] + (1-u)(\sigma\theta_2 - sx)}{x}$$

At the steady state ($\hat{u} = \hat{v} = \hat{x} = \hat{\sigma} = 0$), and under the simplifying assumptions that the 'natural' levels of the employment rate and capacity utilization coincide with their respective equilibrium values ($v^n = v^*, x^n = x^*$) and all desired investment is fully converted into installed capital ($\varepsilon = 1$), the MK system admits the following equilibrium:

$$u^* = \frac{\theta_2 - H_4}{\theta_2}, \qquad v^* = \frac{H_2}{\rho}, \qquad x^* = \frac{H_{10}}{\mu_3\theta_2}, \qquad \sigma^* = \frac{H_{10}}{\mu_3\theta_2 H_5} \quad (43)$$

where:

$$H_2 = \alpha + \gamma, \qquad H_3 = \alpha + \beta + \delta, \qquad H_4 = \alpha + \beta + \theta_0, \qquad H_5 = \frac{\theta_2 H_3}{s H_4}$$

$$H_{10} = \mu_1(\theta_2 - H_4) - \theta_2(\alpha - \mu_0)$$

This equilibrium is economically meaningful (i.e., $0 < u^*, v^*, x^* < 1$ and $\sigma^* > 0$) if the following conditions hold:

$$\theta_2 > H_4, \qquad \alpha + \gamma < \rho, \qquad \mu_1 > \frac{\theta_2(\alpha - \mu_0)}{\theta_2 - H_4}, \qquad \mu_3 > \frac{H_{10}}{\theta_2} \quad (44)$$

By inspecting equation (43), we note that the equilibrium employment rate (v^*) coincides with the original formulation of the Goodwin model and the equilibrium wage share (u^*) coincides with the value derived in the Keynesian-Kaleckian closure (equation (34)). Meanwhile, the equilibrium capacity utilization (x^*) and capital-output ratio (σ^*) reflect a combination of the equilibrium σ^* obtained in the Classical-Marxian closure (equation (25)) and the equilibrium x^* from the Keynesian-Kaleckian closure. This structure reinforces the argument that the two closures are compatible and can be coherently integrated within a single dynamical framework.

Given the complexity of the four-dimensional MK system, we rely on numerical simulations to analyze its dynamic behavior. For this purpose, we use the parameter values employed in the simulations of the Classical-Marxian and Keynesian-Kaleckian closures (Tables 1 and 2), except for the parameters μ_2 and θ_1, which are now set at $\mu_2 \approx 0.0361$ and $\theta_1 \approx 0.1482$. The procedure used to identify these specific values is detailed in Appendix A.7.

The simulated trajectories of the MK system under these conditions are presented in Figure 7, which suggests a potential coexistence of both 'short-run' and 'long-run' cyclical oscillations in the time series of state variables. Concerning two and three-dimensional trajectories, Figure 7 points to the existence of quasi-periodic dynamics: the trajectories fluctuate persistently around the equilibrium without damping, but also without producing a strictly regular, repeating cycle. Instead, the paths appear to form a toroidal structure–indicative of the interaction between to incommensurate frequencies.

Intuitively, this dual oscillatory behavior can be interpreted as the interplay of two distinct cyclical mechanisms. The first corresponds to periodic interactions between distributive class struggle and endogenous mechanization, as described in the Classical-Marxian closure. The second arises from the oscillatory relationship between distributive conflict and effective demand fluctuations through investment and output expansion. As outlined in the Keynesian-Kaleckian closure. While each closure on its own generates regular cycles, their combination in the MK system appears to produce a more complex and quasi-periodic structure–even though the underlying model does not include relevant non-linear components.

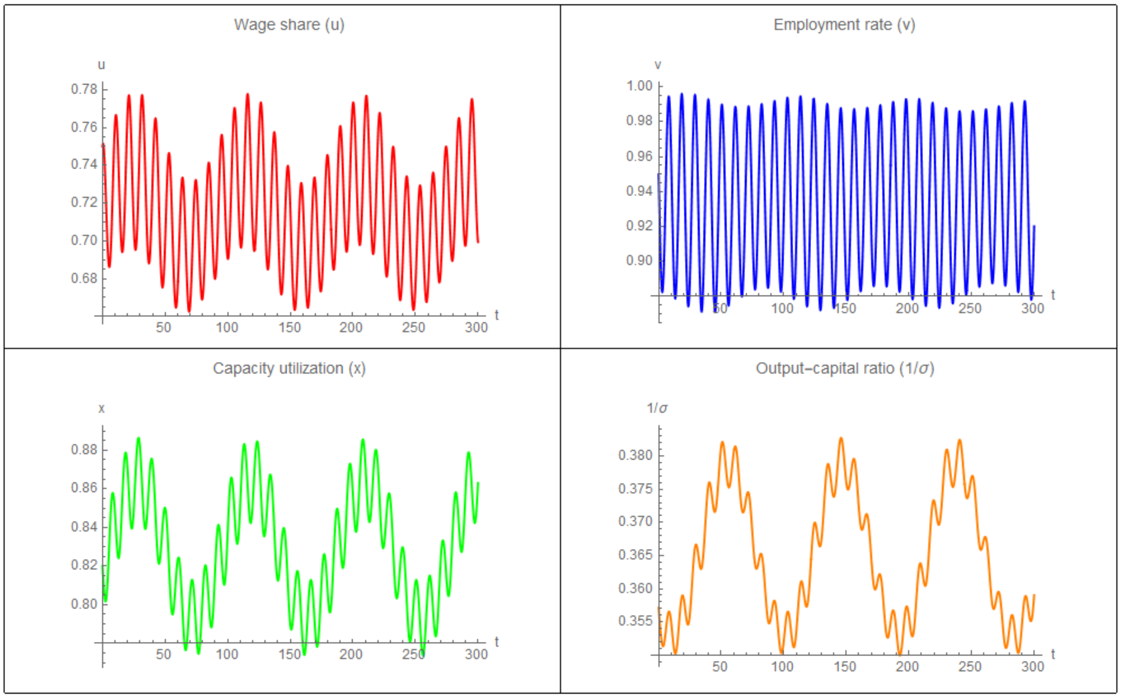
The quasi-periodic nature of the MK system is also confirmed numerically. As shown in Appendix A.7, under the given parameter values, the Jacobian matrix evaluated at the equilibrium displays two distinct pairs of purely imaginary eigenvalues: $\lambda_{1,2}^{MK} = \pm\omega_1 i$ and $\lambda_{3,4}^{MK} = \pm\omega_2 i$, suggesting the coexistence of two independent oscillatory modes. The corresponding approximate periodicities are:

$$P_1 = \frac{2\pi}{\omega_1} \approx 10.53 \text{ years}, \qquad P_2 = \frac{2\pi}{\omega_2} \approx 91.00 \text{ years}$$

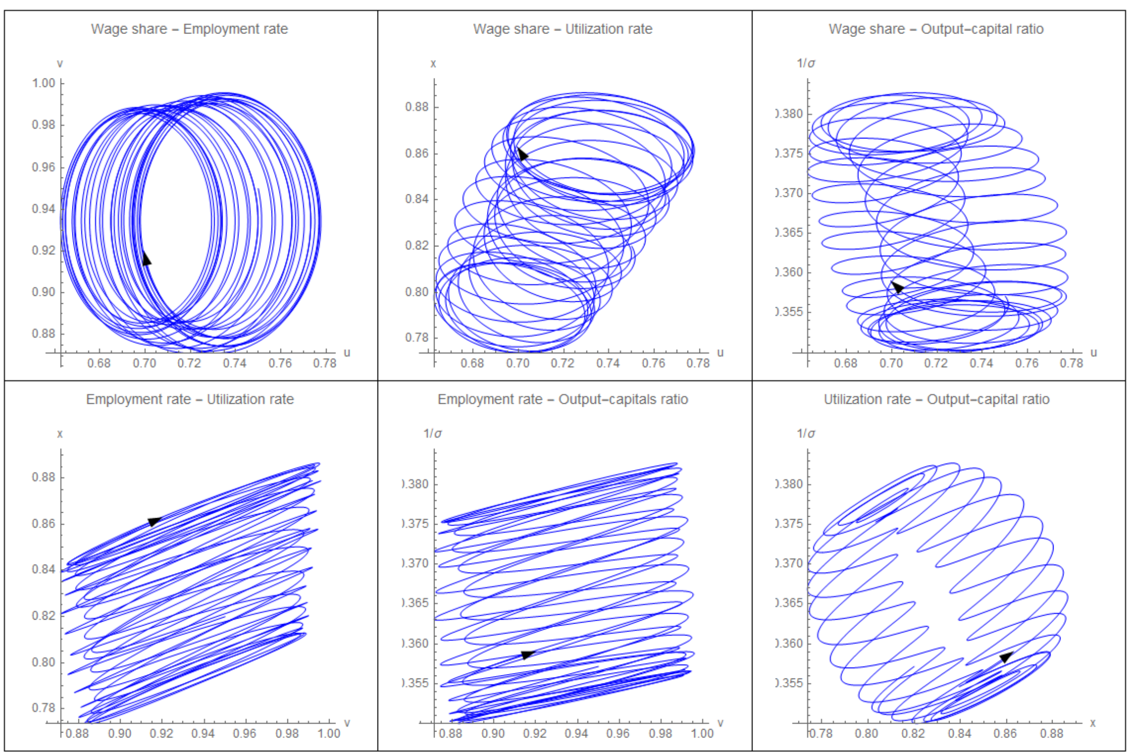
These results confirm the earlier intuition that the time series–depicted in Figure 7–combine 'short-run' and 'long-run' cyclical components. This is a noteworthy outcome, since neither the Classical-Marxian nor the Keynesian-Kaleckian closure, when considered independently with the parameter values in Tables 1 and 2, produces cycles with large periodicities. The implication is that the synthesis of both closures potentially reinforces their respective oscillatory mechanisms in a way that amplifies the time required for trajectories to complete full rotations around the equilibrium, although more research may be required for a definitive response.

Figure 7. Simulation of the MK System (Base Case)

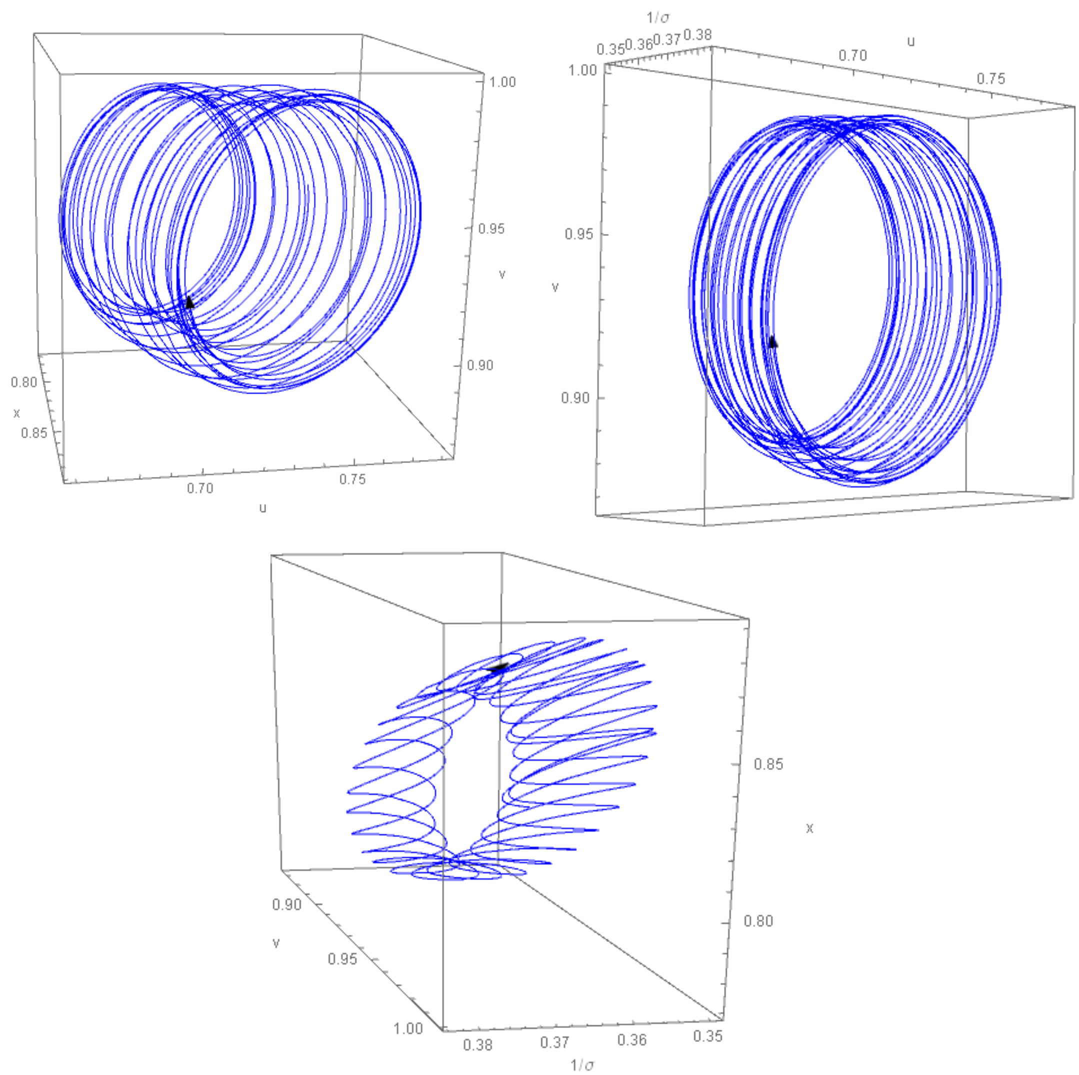
7A. Time Series



7B. Two-Dimensional Phase Trajectories



7C. Three-Dimensional Phase Trajectories



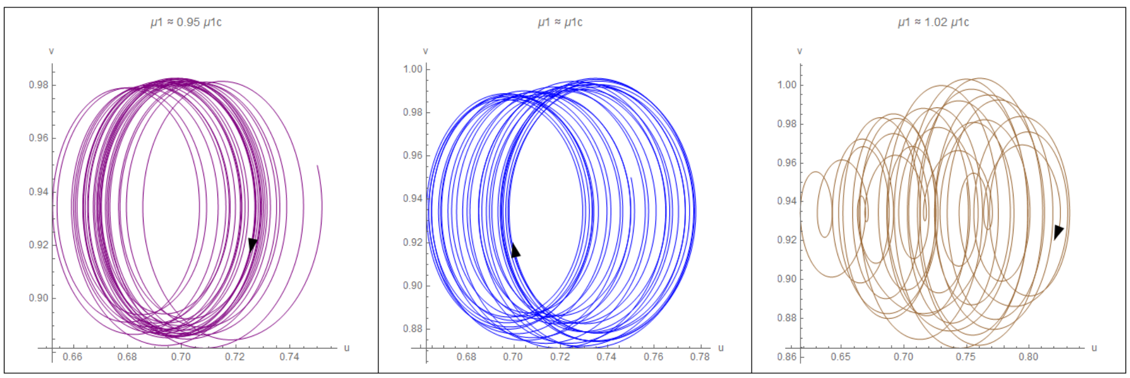
Note: Simulation of system (39) to (42) using the parameter values defined in Tables 1 and 2 except for the terms $\mu_2 = 0.0361$ and $\theta_1 = 0.1482$. Initial conditions: $u_0 = 0.75, v_0 = 0.95, x_0 = 0.82, \sigma_0 = 2.8, \sigma_0^{-1} = 0.3571$. Equilibrium: $u^* = 0.7165, v^* = 0.9344, x^* = 0.829, \sigma^* = 2.7373, \sigma^{-1*} = 0.3653$. Periodicity 1: $P_1 = 10.5325$ years. Periodicity 2: $P_2 = 91.0007$ years. $r = P_2/P_1 = 8.6399$. $t \in [0,300]$. See Appendix A.7 for additional details.

In fact, additional simulations indicate that, if we trat the terms μ_1 and θ_1 as bifurcations parameters, with critical values $\mu_1^c = 0.138$ and $\theta_1^c = 0.1482$, the system undergoes a significant qualitative change in its local dynamics. When these parameters are set below their critical thresholds, the four complex eigenvalues of the Jacobian matrix of the MK system, evaluated at the steady-state equilibrium $(u^*, v^*, x^*, \sigma^*)$, have negative real parts. When the parameters reach their critical values, the real parts of all four eigenvalues become zero. And when the parameters exceed the thresholds, the real parts turn positive (see Appendix A.7 for numerical details).

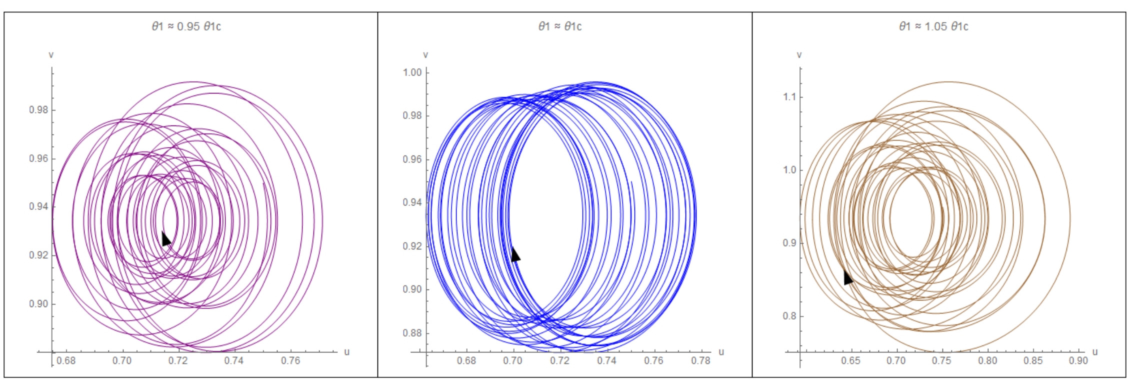
This behavior suggests that the four eigenvalues of the Jacobian matrix cross the imaginary axis as μ_1 and θ_1 vary, indicating the occurrence of a double-Hopf bifurcation–a bifurcation involving the simultaneous emergence of two interacting oscillatory modes (Kuznetsov 2023, chap. 8). Figure 8 illustrates the presence of this bifurcation: when the parameters are below their critical thresholds, the system exhibits either regular or irregular stable trajectories; as the critical value is approaches, the trajectories converge to a closed torus, signaling quasi-periodic behavior; and when the parameters exceed their respective critical values, the system produces increasingly complex and unstable oscillations.

Figure 8. Bifurcation Dynamics of the MK System

8A. With respect to μ_1



8B. With respect to θ_1



Note: For the parameter μ_2, only a 2% was applied to generate visible trajectories over the simulated period considered. Simulation of system (39) to (42) using the parameter values defined in Tables 1 and 2 except for the terms $\mu_2 = 0.0361$ and $\theta_1 = 0.1482$. Initial conditions: $u_0 = 0.75, v_0 = 0.95, x_0 = 0.82, \sigma_0 = 2.8, \sigma_0^{-1} = 0.3571$. $t \in [0,300]$. See Appendix A.7 for additional details.

Four-dimensional dynamical systems undergoing a double-Hopf bifurcation often give rise to trajectories where two distinct oscillatory dynamics not only coexist but can also become mutually reinforcing and, in some cases, synchronize. To illustrate this possibility, we conduct a new numerical simulation of the MK system, using the same parameter values as those presented in Tables 1 and 2, but now setting $\theta_3 = 0.01$.[34] Applying the procedure detailed in Appendix A.7, we obtain new values for $\mu_2 = 0.0922$ and $\theta_1 = 0.1433$. Using these updated values, se simulate the MK system, with the results shown in Figure 9.

While the time series in this figure does not indicate any abrupt qualitative transition, the two- and three-dimensional phase trajectories exhibit a peculiar pattern of synchronization–a phenomenon known as resonance. According to the theory of double Hopf bifurcations (Knobloch et al. 1997; Kuznetsov 2023, chap. 8), resonance occurs when the ratio of the two periodicities $r = P_2/P_1$ approximates a rational number. This behavior is consistent with our simulations. In the case illustrated in Figure 9, the estimated periodicities are $P_1 = 10.877$ and $P_2 = 87.2846$, giving a ratio of $r = 8.0233$, which is notably closer to a rational number compared to the base-case simulation (Figure 7), where $r = 8.6399$. This shift suggests that under certain parameter values, the MK system may approach complex but synchronized persistent cyclical behavior.

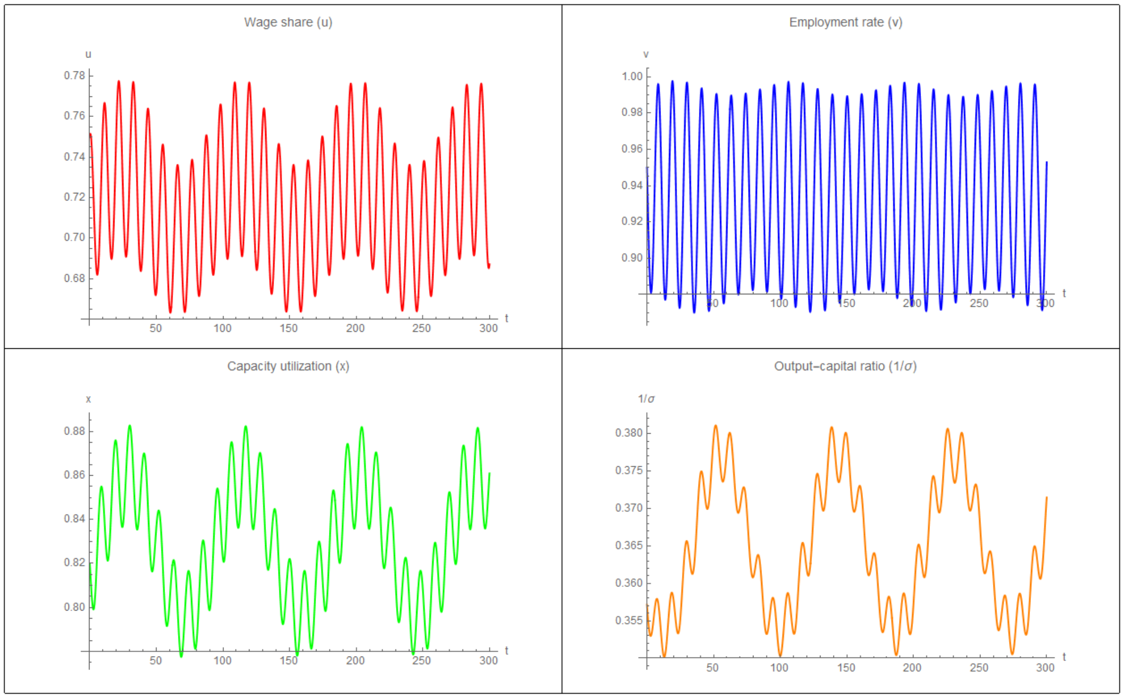
In summary, the simulations presented in this paper suggest that the MK system can generate rich nonlinear dynamics–including quasi-periodicity and resonance–emerging from the interaction of oscillatory mechanisms linked to distributive class struggle, endogenous mechanization, and demand-led accumulation. Moreover, since the system features a double-Hopf bifurcation, it is theoretically plausible that, under alternative parameter values or functional specifications, it could transition into chaotic behavior through mechanisms such as torus breakdown or period-doubling cascades (Kuznetsov 2023).

Exploring the possibility of chaotic dynamics seems promising for future research. In particular, introducing nonlinearities into core behavioral equations–such as the Phillips curve, the investment and mechanization functions, the output expansion function, or the conversion efficiency of desired investment into effective installed capital–may significantly enrich the complexity of the model. Investigating whether such modifications can produce bounded chaotic attractors would not only deepen the theoretical scope of the MK synthesis but also improve our ability to understand the irregular and turbulent dynamics observed in capitalist economies.

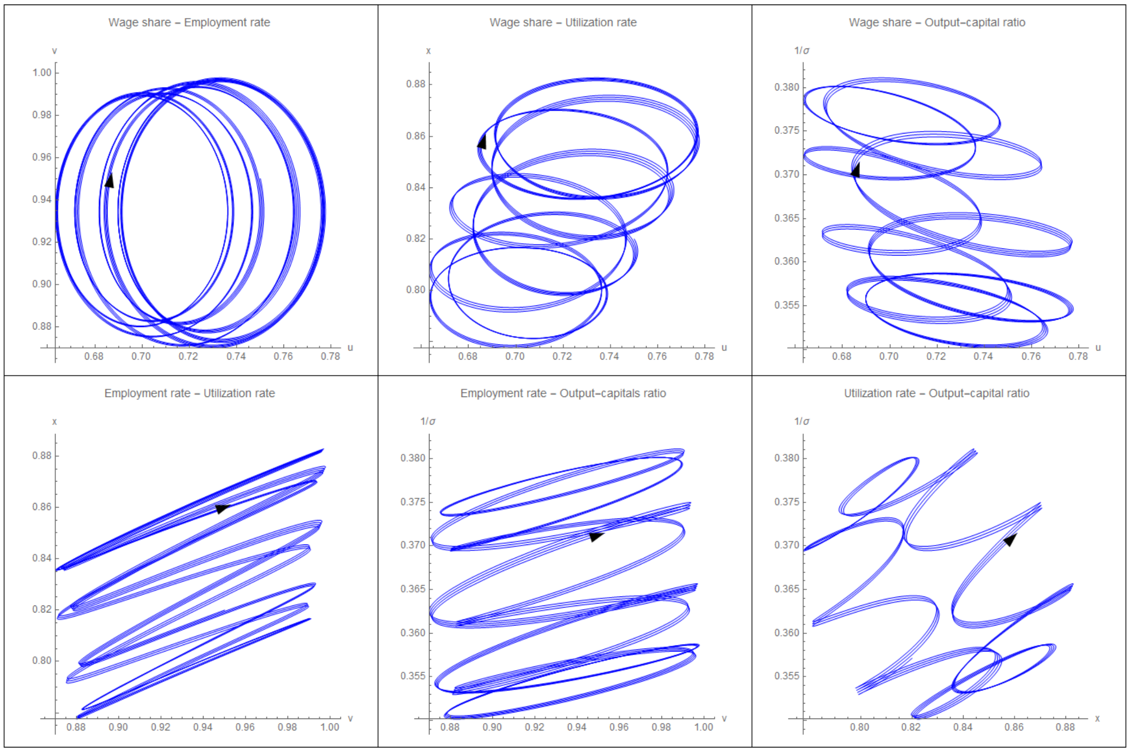
[34] This adjustment is convenient since it does not alter the equilibrium point defined in equation (43).

Figure 9. Simulation of Resonant Dynamics in the MK System

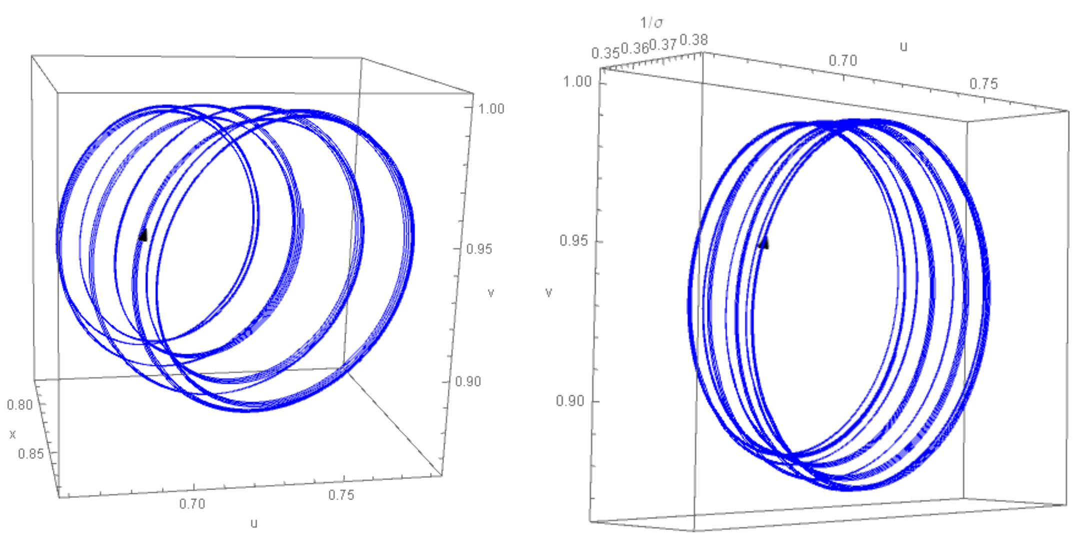
9A. Time Series

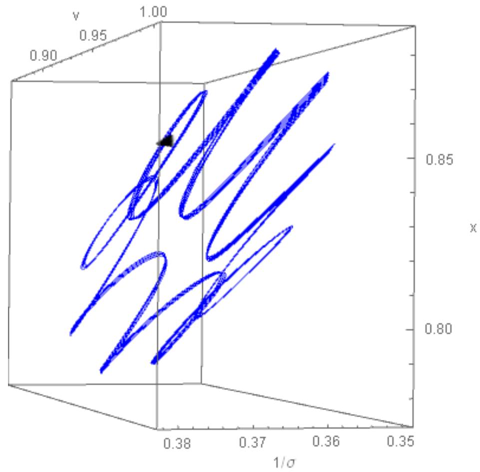


9B. Two-Dimensional Phase Trajectories



9C. Three-Dimensional Phase Trajectories





Note: Simulation of system (39) to (42) using the parameter values defined in Tables 1 and 2 except for the terms $\theta_3 = 0.01$, $\mu_2 = 0.0922$ and $\theta_1 = 0.1433$ (values obtained applying the process detailed in Appendix A.7). Initial conditions: $u_0 = 0.75, v_0 = 0.95, x_0 = 0.82, \sigma_0 = 2.8, \sigma_0^{-1} = 0.3571$. Equilibrium: $u^* = 0.7165, v^* = 0.9344, x^* = 0.829, \sigma^* = 2.7373, \sigma^{-1*} = 0.3653$. Periodicity 1: $P_1 = 10.8788$ years. Periodicity 2: $P_2 = 87.2846$ years. $r = P_2/P_1 = 8.0233$. $t \in [0,300]$.

6. Conclusion

Motivated by Goodwin's (1967) model of endogenous cycles, this paper has developed and analyzed a Marxian-Keynesian (MK) macroeconomic model that integrates key dynamic forces of capitalist economies: distributive class struggle, endogenous technical change through mechanization, investment independent of savings, and goods market disequilibrium. We first constructed and compared two parsimonious three-dimensional closures–a Classical-Marxian closure focused on mechanization, and a Keynesian-Kaleckian closure centered on effective demand. Each of these formulations can generate persistent endogenous cycles through a Hopf bifurcation mechanism.

Building on these results, we proposed an integrated four-dimensional MK system that unifies both closures into a coherent dynamical framework. Within this system, we identified a double Hopf-bifurcation, allowing the coexistence of two interacting oscillatory modes with distinct periodicities. Under certain parameter configurations, these modes synchronize–producing resonance effects and long-term cyclical fluctuations–even when the behavioral equations employed remain relatively simple. These findings demonstrate the capacity of the MK synthesis to generate complex macroeconomic dynamics endogenously, without relying on exogenous shocks.

In addition to these results, the MK framework opens several avenues for future research. One direction involves exploring the emergence of chaotic dynamics through the introduction of nonlinearities into the Phillips curve, the investment and mechanization functions, the output expansion function, or the effectiveness of investment implementation. Such modifications could give rise to bound chaotic attractors, further enriching the system's dynamical complexity.[35]

Another important extension concerns the endogenization of labor productivity, in line with Schumpeterian and post-Keynesian approaches. For instance, drawing on insights from Schumpeter (1939), Flaschel (2015) proposes a formulation where productivity evolver through changes in the proportion of workers engaged in skill enhancement, which itself evolves with the employment rate. Under the assumption that high employment incentivizes firms to invest in innovation, the proportion of 'innovation workforce' becomes an endogenous state variable. Additionally, it would be fruitful to integrate Kaldor's (1957) cumulative causation framework in which both mechanization and output expansion can stimulate labor productivity growth. These elements can be complemented with Kalecki's (1971) view that technological progress–reflected in rising labor productivity–has a positive influence on investment demand, particularly when newer capital-embodied technologies are available.[36]

Together, these extensions suggest a promising path toward the development of a five-dimensional Marx-Keynes-Schumpeter (MKS) synthesis, in which the wage share, employment rate, capacity utilization, capital-output ratio, and the proportion of workers engaged in skill enhancement evolve as endogenous state variables. Such a framework–deeply inspired by Goodwin's (1989) vision of a unified MKS system–would offer a more comprehensive representation of the cyclical dynamics of capitalist economies, grounded in class conflict, innovation, and effective demand.

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Declaration of generative AI and AI-assisted technologies in the writing process. During the preparation of this work the author used ChatGPT in order to improve the readability and language of the manuscript. After using this tool/service, the author reviewed and edited the content as needed and takes full responsibility for the content of the publication.

Data availability. Data, code and supporting documentation will be made available upon request.

[35] See Bella (2025) for an example of the identification of chaos in the extended Goodwin model with disequilibrium in the goods market originally formulated by Sordi and Vercelli (2014).
[36] Some post-Keynesian formulations that extend the investment function with a positive effect of labor productivity include Cassetti (2003), Hein (2014), Nah and Lavoie (2019).

Appendix

A.1. Proof of Proposition 1

The Jacobian matrix of the system (22), (23), and (24), evaluated at the equilibrium point defined in equation (25), is given by:

$$J = \begin{bmatrix} 0 & \frac{\rho H_1}{\mu_1} & 0 \\ -\frac{\mu_1 H_2 (H_3 + \mu_1 - H_1)}{\rho(\mu_1 - H_1)} & -\frac{\mu_2 H_2}{\rho} & -\frac{\mu_1 H_2 H_3^2}{s x^n \rho(\mu_1 - H_1)} \\ \frac{s x^n (\mu_1 - H_1)}{H_3} & \frac{s x^n (\mu_1 - H_1)\mu_2}{H_3 \mu_1} & 0 \end{bmatrix}$$

where:

$$H_1 = \alpha - \mu_0 + \mu_3 x^n, \qquad H_2 = \alpha + \gamma, \qquad H_3 = \alpha + \beta + \delta$$

The characteristic polynomial of this matrix is:

$$\lambda^3 + b_1 \lambda^2 + b_2 \lambda + b_3 = 0$$

with coefficients:

$$b_1 = -\mathrm{Tr}(J) = \frac{\mu_2 H_2}{\rho} \quad (A.1)$$

$$b_2 = \begin{vmatrix} J_{22} & J_{23} \\ J_{32} & J_{33} \end{vmatrix} + \begin{vmatrix} J_{11} & J_{13} \\ J_{31} & J_{33} \end{vmatrix} + \begin{vmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{vmatrix} = H_2 \left(H_1 + \frac{H_1 H_3}{\mu_1 - H_1} + \frac{\mu_2 H_3}{\rho} \right) \quad (A.2)$$

$$b_3 = -\mathrm{Det}(J) = H_1 H_2 H_3 \quad (A.3)$$

Furthermore, define:

$$B = b_1 b_2 - b_3 = \frac{H_2}{\rho^2} \left\{ \frac{\rho \mu_2 H_1 H_2 H_3}{\mu_1 - H_1} - [\rho^2 H_1 H_3 - \mu_2 H_2 (\rho H_1 + \mu_2 H_3)] \right\} \quad (A.4)$$

According to the Routh-Hurwitz criterion, when b_1, b_2, b_3, and $B > 0$, the characteristic polynomial has three roots with negative real parts, implying local asymptotic stability. From equations (A.1)-(A.3), it follows that b_1, b_2, and b_3 are strictly positive as long as a positive equilibrium exists, which is guaranteed under condition (26).

To ensure $B > 0$, we additionally impose the condition:

$$H_2 = \alpha + \gamma < \frac{\rho^2 H_1 H_3}{\mu_2 (\rho H_1 + \mu_2 H_3)} \quad (A.5)$$

Under this condition, along with (26), we also require:

$$\mu_1 < \mu_1^c = H_1 \left[1 + \frac{\rho \mu_2 H_2 H_3}{\rho^2 H_1 H_3 - \mu_2 H_2 (\rho H_1 + \mu_2 H_3)} \right] \quad (A.6)$$

In summary, if conditions (26), (A.5), and (A.6) are satisfied, the system (22), (23), and (24) is locally asymptotically stable.

A.2. Proof of Proposition 2

Consider μ_1 as a bifurcation parameter of the system defined by equations (22), (23), and (24). Following Liu (1994), the existence of a Hopf bifurcation can be demonstrated by verifying that μ_1 has a critical value μ_1^c such that the following condition hold:

$$b_1|_{\mu_1^c}, b_2|_{\mu_1^c}, b_3|_{\mu_1^c} > 0, \qquad B|_{\mu_1^c} = 0, \qquad \left.\frac{dB}{d\mu_1}\right|_{\mu_1^c} \neq 0$$

Since b_1, b_2, and b_3 remain strictly positive under condition (26), the existence of a Hopf bifurcation depends on identifying a critical value μ_1^c that satisfies $B|_{\mu_1^c} = 0$ and $\left.\frac{dB}{d\mu_1}\right|_{\mu_1^c} \neq 0$.

First, assuming that condition (A.5) is satisfied, direct substitution of the critical value defined in expression (A.6) into (A.4) yields $B|_{\mu_1^c} = 0$.

Second, consider the derivative of B with respect to μ_1. From expression (A.4), we obtain:

$$\frac{dB}{d\mu_1} = -\frac{\mu_2 H_1 H_2^2 H_3}{\rho(\mu_1 - H_1)^2}$$

Evaluating this derivative at $\mu_1 = \mu_1^c$ gives:

$$\left.\frac{dB}{d\mu_1}\right|_{\mu_1^c} = -\frac{[\rho^2 H_1 H_3 - H_2\mu_2(\rho H_1 + \mu_2 H_3)]^2}{\rho^3 \mu_2 H_1 H_3} < 0 \qquad (A.7)$$

Since this expression is always negative under the assumed conditions, we conclude that the system undergoes a Hopf bifurcation as μ_1 approaches μ_1^C.

A.3. Numerical Details of the Simulation of the Classical-Marxian Closure

For the simulation presented in Figure 1, which corresponds to the case exhibiting limit cycles, we evaluate the Jacobian matrix at the critical value $\mu_1 = \mu_1^c = 0.138$, obtaining:

$$J(\mu_1^c) = \begin{bmatrix} 0 & 0.3933 & 0 \\ -0.3530 & -0.0245 & -0.0232 \\ 0.3777 & 0.07188 & 0 \end{bmatrix}$$

The corresponding eigenvalues are:

$$\lambda_{1,2}(\mu_1^c) = \pm\omega i = \pm 0.3749i, \qquad \lambda_3(\mu_1^c) = -0.0245$$

From this, the associated periodicity is:

$$P(\mu_1^c) = \frac{2\pi}{\omega(\mu_1^c)} = 16.7593 \text{ years}$$

For the simulation of the stable spirals shown in Figure 2, the Jacobian matrix is evaluated at $\mu_1 = 0.95\mu_1^c \approx 0.1311$, yielding:

$$J(0.95\mu_1^c) = \begin{bmatrix} 0 & 0.4140 & 0 \\ -0.3810 & -0.0245 & -0.0267 \\ 0.3111 & 0.0623 & 0 \end{bmatrix}$$

The associated eigenvalues are:

$$\lambda_{1,2}(0.95\mu_1^c) = -0.0014 \pm 0.3992i, \qquad \lambda_3(0.95\mu_1^c) = -0.0216$$

In contrast, for the simulation of unstable spirals also shown in Figure 2, the Jacobian matrix is evaluated at $\mu_1 = 1.05\mu_1^c = 0.1449$, giving:

$$J(1.05\mu_1^c) = \begin{bmatrix} 0 & 0.3746 & 0 \\ -0.3354 & -0.0245 & -0.0207 \\ 0.4443 & 0.0805 & 0 \end{bmatrix}$$

The corresponding eigenvalues are:

$$\lambda_{1,2}(1.05\mu_1^c) = 0.0012 \pm 0.3569i, \qquad \lambda_3(1.05\mu_1^c) = -0.0270$$

These results reinforce the intuition that the Classical-Marxian closure undergoes a Hopf bifurcation near μ_1^c, with the transition from stable to unstable oscillatory behavior.

A.4. Proof of Proposition 3

The Jacobian matrix of the system (31), (32), and (33), evaluated at the equilibrium point defined in equation (34), is given by:

$$J = \begin{bmatrix} 0 & \dfrac{\rho(\theta_2 - H_4)}{\theta_2} & 0 \\ -\dfrac{H_2\theta_2[\phi H_4 - H_3(\phi - H_5)]}{\rho H_4 H_5} & -\dfrac{H_2\theta_3\phi}{\rho H_5} & \dfrac{H_2H_3[\phi\sigma\theta_1\theta_2 - sH_4(\phi - H_5)]}{\rho\sigma s H_4(H_5 + \phi)^2} \\ -\dfrac{\sigma\theta_2(\theta_0 - \delta)(\phi - H_5)}{H_4} & -\sigma\theta_3(\phi - H_5) & \dfrac{H_3(\phi - H_5)(\sigma\theta_1\theta_2 - sH_4)}{sH_4(H_5 + \phi)} \end{bmatrix}$$

where:

$$H_2 = \alpha + \gamma, \qquad H_3 = \alpha + \beta + \delta, \qquad H_4 = \alpha + \beta + \theta_0, \qquad H_5 = \frac{\theta_2 H_3}{sH_4}$$

The characteristic polynomial of this matrix takes the form:

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$$

with the coefficients defined as:

$$b_1 = \frac{\phi s\theta_3 H_2 H_4 - \rho H_3(\phi - H_5)(\sigma\theta_1\theta_2 - sH_4)}{\rho\theta_2 H_3} \qquad (A.8)$$

$$b_2 = \frac{sH_2\{\rho(\theta_2 - H_4)[\phi(\theta_0 - \delta) + H_3H_5] + \theta_3 H_3 H_4(\phi - H_5)\}}{\rho\theta_2 H_3} \qquad (A.9)$$

$$b_3 = \frac{H_2H_3(\theta_2 - H_4)(\phi - H_5)(sH_4^2 - \sigma\theta_1\theta_2 H_3)}{\theta_2 H_3 H_4} \qquad (A.10)$$

The terms b_1, b_2, and b_3 are all positive provided that, in addition to condition (35), the following inequalities also hold:[37]

$$\phi > H_5, \qquad \theta_0 > \delta, \qquad \frac{sH_4}{\sigma\theta_2} < \theta_1 < \frac{sH_4^2}{\sigma\theta_2 H_3}, \qquad \theta_3 > \frac{\rho H_3(\phi - H_5)(\sigma\theta_1\theta_2 - sH_4)}{s\phi H_2 H_4} \qquad (A.11)$$

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[37] Since $\theta_0 > \delta$, it follows that $H_4 > H_3$, making it feasible to identify the admissible interval for θ_1 in expression (A.11).

Additionally, we define the term:

$$B = b_1 b_2 - b_3 = \frac{s^2 H_4(\alpha + \gamma)(Z_1 - Z_2\theta_1)}{\rho^2 \theta_2^2 H_3^2} \quad (A.12)$$

where:

$$Z_1 = H_6\phi^2 - H_7\phi + H_8$$
$$H_6 = (\theta_3 H_2 + \rho H_3)[H_3 H_4 \theta_3 + \rho(\theta_0 - \delta)(\theta_2 - H_4)]$$
$$H_7 = H_3 H_5\{H_4\theta_3(\theta_3 H_2 + 2\rho H_3) - \rho(\theta_2 - H_4)[H_2\theta_3 - 2\rho(\theta_0 - \delta)]\}$$
$$H_8 = \rho H_3 H_5^2[H_3 H_4 \theta_3 + \rho(\theta_0 - \delta)(\theta_2 - H_4)]$$
$$Z_2 = \rho\sigma H_5 H_9(\phi - H_5)$$
$$H_9 = \rho\phi(\theta_0 - \delta)(\theta_2 - H_4) + \theta_3 H_3 H_4(\phi - H_5)$$

Given conditions (35) and (A.11), Z_1 is a convex quadratic function of ϕ, and it will positive if:

$$H_7^2 < 4H_6H_8 \quad \text{or} \quad \phi > \frac{H_7 + \sqrt{H_7^2 - 4H_6H_8}}{2H_6} \quad (A.13)$$

Therefore, the term B becomes positive if, in addition to conditions (35), (A.11), and (A.13), we also require:

$$\theta_1 < \theta_1^c = \frac{Z_1}{Z_2} \quad (A.14)$$

where θ_1^c must lie within the range:

$$\frac{sH_4}{\sigma\theta_2} < \theta_1^c < \frac{sH_4^2}{\sigma\theta_2 H_3}$$

As the numerical simulations in section 4.2 suggest, this critical value exists for economically reasonable parameter configurations. Hence, the system (31), (32), and (33) is locally stable under conditions (35), (A.11), (A.13), and (A.14).

A.5. Proof of Proposition 4

Consider θ_1 as a bifurcation parameter of the system (31), (32), and (33). Following Liu (1994), we establish the existence of a Hopf bifurcation by showing that θ_1 has a critical value θ_1^c such that the following conditions are satisfied:

$$b_1|_{\theta_1^c}, b_2|_{\theta_1^c}, b_3|_{\theta_1^c} > 0, \qquad B|_{\theta_1^c} = 0, \qquad \left.\frac{dB}{d\theta_1}\right|_{\theta_1^c} \neq 0$$

Since b_1, b_2, and b_3 are positive under conditions (35) and (A.11), it remains to verify that a critical value θ_1^c exists such that $B|_{\theta_1^c} = 0$ and $\left.\frac{dB}{d\theta_1}\right|_{\theta_1^c} \neq 0$.

Substituting the critical value defined in expression (A.14) into expression (A.12) verifies that $B|_{\theta_1^c} = 0$, if condition (A.13) is satisfied. Finally, taking the derivative of expression (A.12) with respect to θ_1, we obtain:

$$\frac{dB}{d\theta_1} = \left.\frac{dB}{d\theta_1}\right|_{\theta_1^c} = -\frac{s^2 H_4(\alpha+\gamma)Z_2}{\rho^2\theta_2^2 H_3^2} < 0 \quad (A.15)$$

This derivative is negative under conditions (35) and (A.11). Therefore, the critical value θ_1^c, as defined in expression (A.14), ensures that the system (31), (32), and (33) undergoes a Hopf bifurcation as θ_1 approaches θ_1^C, provided that conditions (35), (A.11), and (A.13) are satisfied.

A.6. Numerical Details of the Simulation of the Keynesian-Kaleckian Closure

For the simulation presented in Figure 4, which corresponds to the case exhibiting limit cycles, we evaluate the Jacobian matrix at the critical value $\theta_1 = \theta_1^c \approx 0.1269$, obtaining:

$$J(\theta_1^c) = \begin{bmatrix} 0 & 0.3933 & 0 \\ -0.7367 & -0.0925 & 0.2153 \\ -0.3170 & -0.0572 & 0.0857 \end{bmatrix}$$

The corresponding eigenvalues are:

$$\lambda_{1,2}(\theta_3^c) = \pm\omega i = \pm 0.5423i, \qquad \lambda_3(\theta_3^c) = -0.0067$$

From this, the associated periodicity is:

$$P(\theta_1^c) = \frac{2\pi}{\omega(\theta_1^c)} = 11.5841$$

For the simulation of stable spirals presented in Figure 5, the Jacobian matrix is evaluated at $\theta_1 = 0.95\theta_1^c = 0.1206$, yielding:

$$J(0.95\theta_1^c) = \begin{bmatrix} 0 & 0.3933 & 0 \\ -0.7367 & -0.0925 & 0.1957 \\ -0.3170 & -0.0572 & 0.0736 \end{bmatrix}$$

The corresponding eigenvalues are:

$$\lambda_{1,2}(0.95\theta_1^c) = -0.0042 \pm 0.5422i, \qquad \lambda_3(0.95\theta_1^c) = -0.0104$$

In contrast, for the simulation of unstable spirals also presented in Figure 5, the Jacobian is evaluated at $\theta_1 = 1.05\theta_1^c = 0.1333$, resulting in:

$$J(1.05\theta_1^c) = \begin{bmatrix} 0 & 0.3933 & 0 \\ -0.7367 & -0.0925 & 0.2349 \\ -0.3170 & -0.0572 & 0.0978 \end{bmatrix}$$

The associated eigenvalues are:

$$\lambda_{1,2}(1.05\theta_1^c) = 0.0042 \pm 0.5424i, \qquad \lambda_3(1.05\theta_1^c) = -0.0031$$

These results reinforce the intuition that the Keynesian-Kaleckian closure undergoes a Hopf bifurcation near θ_1^c, with the transition from stable to unstable oscillatory behavior.

A.7. Numerical Identification of Double-Hopf Bifurcation in the MK system

The Jacobian matrix of the MK system defined by expressions (39)-(42), evaluated at the equilibrium point given in equation (43), is structured as:

$$J = \begin{bmatrix} 0 & J_{12} & 0 & 0 \\ J_{21} & J_{22} & J_{23} & J_{24} \\ J_{31} & J_{32} & J_{33} & J_{34} \\ J_{41} & J_{42} & J_{43} & 0 \end{bmatrix}$$

with entries defined as:

$$J_{12} = \frac{\rho(\theta_2 - H_4)}{\theta_2}, \qquad J_{21} = -\frac{H_2}{\rho H_4}\left[\frac{H_{11}H_4}{\theta_2 - H_4} + H_3\theta_2 + \frac{\theta_2\phi(\theta_0 - \delta)}{H_5}\right]$$

$$J_{22} = -\frac{H_2(H_5\mu_2 + \theta_3\phi)}{\rho H_5}, \qquad J_{23} = \frac{H_2[H_5\mu_3(H_{10} + H_3\theta_2) + \phi(H_{10}\theta_1 - H_3\theta_2\mu_3)]}{\rho H_5 H_{10}}$$

$$J_{24} = \frac{H_2H_3\theta_2\mu_3(\phi - H_5)}{\rho H_{10}}, \qquad J_{31} = -\frac{H_{10}(\theta_0 - \delta)(\phi - H_5)}{H_4H_5\mu_3}, \qquad J_{32} = -\frac{H_{10}\theta_3(\phi - H_5)}{H_5\theta_2\mu_3}$$

$$J_{33} = \frac{(H_{10}\theta_1 - H_3\theta_2\mu_3)(\phi - H_5)}{H_5\theta_2\mu_3}, \qquad J_{34} = H_3(\phi - H_5), \qquad J_{41} = \frac{H_{10}H_{11}}{H_5\theta_2\mu_3(\theta_2 - H_4)}$$

$$J_{42} = \frac{H_{10}\mu_2}{H_5\theta_2\mu_3}, \qquad J_{43} = -\frac{H_{10}}{H_5\theta_2}$$

where:

$$H_2 = \alpha + \gamma, \qquad H_3 = \alpha + \beta + \delta, \qquad H_4 = \alpha + \beta + \theta_0, \qquad H_5 = \frac{\theta_2 H_3}{sH_4}$$

$$H_{10} = \mu_1(\theta_2 - H_4) - \theta_2(\alpha - \mu_0), \qquad H_{11} = \mu_1(\theta_2 - H_4)$$

The characteristic polynomial of the Jacobian takes the form:

$$\lambda^4 + b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4 = 0$$

with the coefficients defined as:

$$b_1 = -\mathrm{Tr}(J) = \frac{H_2\theta_2\mu_3(H_5\mu_2 + \theta_3\phi) - \rho(H_{10}\theta_1 - H_3\theta_2\mu_3)(\phi - H_5)}{\rho H_5\theta_2\mu_3} \qquad (A.16)$$

$$b_2 = \begin{vmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{vmatrix} + \begin{vmatrix} J_{11} & J_{13} \\ J_{31} & J_{33} \end{vmatrix} + \begin{vmatrix} J_{11} & J_{14} \\ J_{41} & J_{44} \end{vmatrix} + \begin{vmatrix} J_{22} & J_{23} \\ J_{32} & J_{33} \end{vmatrix} + \begin{vmatrix} J_{22} & J_{24} \\ J_{42} & J_{44} \end{vmatrix} + \begin{vmatrix} J_{33} & J_{34} \\ J_{43} & J_{44} \end{vmatrix}$$

$$b_2 = \frac{H_{10}H_4(\phi - H_5)H_{12} + H_2\mu_3[H_3\theta_2(\phi - H_5)H_{13} + \rho H_4H_{14}]}{\rho\mu_3H_4H_5\theta_2} \qquad (A.17)$$

$$b_3 = -\begin{vmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{vmatrix} - \begin{vmatrix} J_{11} & J_{12} & J_{14} \\ J_{21} & J_{22} & J_{24} \\ J_{41} & J_{42} & J_{44} \end{vmatrix} - \begin{vmatrix} J_{11} & J_{13} & J_{14} \\ J_{31} & J_{33} & J_{34} \\ J_{41} & J_{43} & J_{44} \end{vmatrix} - \begin{vmatrix} J_{22} & J_{23} & J_{24} \\ J_{32} & J_{33} & J_{34} \\ J_{42} & J_{43} & J_{44} \end{vmatrix}$$

$$b_3 = \frac{H_2(\phi - H_5)\{\rho[\mu_3\theta_2(\theta_2 - H_4)H_{15} - \theta_1H_{10}H_{16}] + H_{10}H_3H_4\theta_2(\theta_3\mu_3 - \theta_1\mu_2)\}}{\rho\mu_3H_4H_5\theta_2^2} \qquad (A.18)$$

$$b_4 = \text{Det}(J) = \frac{H_{10}H_2H_3(\phi - H_5)[\mu_3\theta_2(\theta_2 - H_4) - H_{11}\theta_1]}{\mu_3\theta_2^2H_5} \quad (A.19)$$

where:

$$H_{12} = H_3\mu_3\rho + H_2(\theta_3\mu_3 - \theta_1\mu_2), \qquad H_{13} = H_4\theta_3 - \rho(\theta_2 - H_4)$$

$$H_{14} = H_{11}H_5 + \phi\theta_2(\theta_2 - H_4), \qquad H_{15} = H_{10}(\theta_0 - \delta) + H_3H_4\theta_2$$

$$H_{16} = H_{11}H_4 + H_3\theta_2(\theta_2 - H_4)$$

Despite the complexity of these coefficients, we can numerically identify critical values for a pair of parameters that satisfy:

$$b_1 = b_3 = 0, \qquad b_2, b_3 > 0$$

This detail is crucial to prove the existence of a double-Hopf bifurcation, characterized by the simultaneous presence of two pair of purely imaginary eigenvalues.

For instance, consider the following parameter values, common to both Classical-Marxian and Keynesian-Kaleckian closures of the model:

$$\varepsilon = 1, \quad \alpha = 0.022, \quad \beta = 0.008, \quad \delta = 0.038, \quad s = 0.792, \quad \gamma = 0.491$$

$$\rho = 0.549, \quad \theta_2 = 0.406, \quad \theta_3 = 0.03, \quad \mu_1 = 0.138, \quad \mu_3 = 0.138, \quad \phi = 1$$

$$\theta_0 = 0.0851, \quad \mu_0 = 0.0375$$

Substituting these values into equation (43), we obtain the following equilibrium point:

$$u^* = 0.7165, \quad v^* = 0.9344, \quad x^* = 0.829, \quad \sigma^* = 2.7373$$

This equilibrium point is consistent with the two closures of the model.

By substituting the parameter values into expressions (A.16) and (A.18) and setting $b_1 = b_3 = 0$, we get a system of two equations. Solving numerically the system for μ_2 and θ_1, we get:[38]

$$\mu_2 = 0.0361, \quad \theta_1 = 0.1482$$

At these values, the Jacobian matrix becomes:

$$J = \begin{bmatrix} 0 & 0.3933 & 0 & 0 \\ -0.8656 & -0.1263 & 0.4099 & 0.0534 \\ -0.3170 & -0.0572 & 0.1263 & 0.0474 \\ 0.3777 & 0.0990 & -0.3777 & 0 \end{bmatrix}$$

Its eigenvalues are:

$$\lambda_{1,2} = \pm\omega_1 i = 0.5965i, \qquad \lambda_{3,4} = \pm\omega_2 i = 0.0690i$$

[38] Equations (A.16) and (A.18) can be solved numerically for various parameters, not only for μ_2 and θ_1. However, we have selected these two since they do not affect the equilibrium point of the MK system.

corresponding to two different pairs of purely imaginary eigenvalues, suggesting the interaction of two oscillatory dynamics, as illustrated in Figure 7. These interacting oscillatory dynamics have approximate periodicities:

$$P_1 = \frac{2\pi}{\omega_1} = 10.5325 \text{ years}, \qquad P_2 = \frac{2\pi}{\omega_2} = 91.0007 \text{ years}$$

Now, consider the terms μ_1 and θ_1 as bifurcation parameters with critical values $\mu_1^c = 0.138$ and $\theta_1^c = 0.1482$. To further illustrate the behavior around the bifurcation, we evaluate the Jacobian matrix at slightly lower and higher values of the bifurcation parameters. When:

$$\mu_1 = 0.95\mu_1^c = 0.1311, \qquad \theta_1 = 0.95\theta_1^c = 0.1407$$

we get the following Jacobian matrices:

$$J(0.95\mu_1^c) = \begin{bmatrix} 0 & 0.3933 & 0 & 0 \\ -0.8592 & -0.1263 & 0.4019 & 0.0558 \\ -0.3033 & -0.0547 & 0.1141 & 0.0474 \\ 0.3433 & 0.0947 & -0.3614 & 0 \end{bmatrix}$$

$$J(0.95\theta_1^c) = \begin{bmatrix} 0 & 0.3933 & 0 & 0 \\ -0.8656 & -0.1263 & 0.3870 & 0.0534 \\ -0.3170 & -0.0572 & 0.1122 & 0.0474 \\ 0.3777 & 0.0990 & -0.3777 & 0 \end{bmatrix}$$

Here, all the associated eigenvalues have negative real parts:

$$\lambda_{1,2}(0.95\mu_1^c) = -0.0037 \pm 0.5938i, \qquad \lambda_{3,4}(0.95\mu_1^c) = -0.0023 \pm 0.0687i$$

$$\lambda_{1,2}(0.95\theta_1^c) = -0.0042 \pm 0.5967i, \qquad \lambda_{3,4}(0.95\theta_1^c) = -0.0027 \pm 0.0699i$$

In contrast, when $\mu_1 = 1.02\mu_1^c = 0.1407$ and $\theta_1 = 1.05\theta_1^c = 0.1556$, we get the following Jacobian matrices:

$$J(1.02\mu_1^c) = \begin{bmatrix} 0 & 0.3933 & 0 & 0 \\ -0.868 & -0.1263 & 0.4129 & 0.0525 \\ -0.3225 & -0.0582 & 0.1312 & 0.0474 \\ 0.3919 & 0.1007 & -0.3842 & 0 \end{bmatrix}$$

$$J(1.05\theta_1^c) = \begin{bmatrix} 0 & 0.3933 & 0 & 0 \\ -0.8656 & -0.1263 & 0.4328 & 0.0534 \\ -0.3170 & -0.0572 & 0.1405 & 0.0474 \\ 0.3777 & 0.0990 & -0.3777 & 0 \end{bmatrix}$$

All four corresponding eigenvalues have positive real parts in this case:

$$\lambda_{1,2}(1.02\mu_1^c) = 0.0014 \pm 0.5976i, \qquad \lambda_{3,4}(1.02\mu_1^c) = 0.0009 \pm 0.0691i$$

$$\lambda_{1,2}(1.05\theta_1^c) = 0.0043 \pm 0.5962i, \qquad \lambda_{3,4}(1.05\theta_1^c) = 0.0027 \pm 0.0680i$$

These results indicate that, as μ_1 and θ_1 cross their respective critical values, the real parts of all four complex eigenvalues cross the imaginary axis. This behavior confirms the presence of a double-Hopf bifurcation, as discussed in Kuznetsov (2023, chap. 8).

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