Order of Play in Sequential Network Formation

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Abstract

Research in anthropology and neuroscience has shown that people have a cognitive limit on the number of stable relationships they can maintain. In this spirit, we consider a network formation game in which the cost of link formation is increasing in the agent's degree. In this class of games, as opposed to commonly studied games with a fixed cost of link formation, the order in which agents form the network (order of play) determines its final structure. In particular, we find that only certain orders of play can explain the formation of circle and complete bipartite networks. We also find that there is multiplicity of equilibria only when marginal costs of link formation are intermediate. Our results show as well that some orders of play are better than others for predicting the equilibrium structure when it is not unique, and that playing last is usually harmful.

JEL classification: C72, D85

Keywords: sequential network formation, pairwise stability, order of play, costs of link formation increasing in degree

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"There is a cognitive limit on the number of relationships that an individual can monitor simultaneously."
— Robin Dunbar (1992)

1 Introduction

Research in anthropology and neuroscience has shown that people have a cognitive limit on the number of stable relationships they can maintain. This finding is widely recognized in the social sciences as Dunbar's number, which places the typical upper bound at around 150 stable connections. Network formation models have typically overlooked these empirical findings, often assuming that the cost of forming each link is fixed and independent of an individual's existing number of connections. This assumption has usually been made in both simultaneous link formation games (e.g. Jackson and Wolinsky, 1996; Bala and Goyal, 2000) and sequential ones (e.g. König et al., 2014; Joshi et al., 2025, 2020). When costs of link formation are fixed, the network always converges to the same structure at equilibrium, independently of the order in which agents form links (henceforth order of play).

In the current paper, we show that the order of play determines the equilibrium structure of a network when costs of link formation are increasing in the agent's degree. We consider a sequential game of network formation where agents can *unilaterally* delete a subset of own links, and *bilaterally* form links. Agents have a linear-quadratic utility function, which is very standard in the network economics literature (see, for example, Ballester et al., 2006), so that the incentive of agents to form links comes from the resulting increase in Bonacich centrality. Agents do not necessarily wish to form a link with every other agent, since link formation is costly, and this cost is increasing in the agent's degree.

Our results show that only certain orders of play can explain the formation of circles and complete bipartite networks, which play a key role in various areas like leadership (Cabrales and Hauk, 2022), finance (Gualdi et al., 2016) or scientific competition (Cimini et al., 2014). We find that there is multiplicity of equilibria only when marginal costs of link formation are intermediate. We furthermore find that playing last is usually harmful, because it typically entails that agents who have previously played are "satisfied" with the neighbors they have, and do not wish to form any additional links. Finally, we show that some orders of play are better than others for predicting the final structure of the network when the equilibrium is not unique. The model we present also explains the formation of complete multipartite and ring lattice networks, which hold a central place in topics like public goods (Kinateder and Merlino, 2017), social hierarchy (Baetz, 2015) or trade (Wilhite, 2001). To the best of our knowledge, this paper is the first one to explicitly relate the order of play to the equilibrium structure of the network.

The paper by Jackson and Watts (2002) considers as well a network formation game in which the cost of link formation is increasing in the agent's degree. The incentive of agents to form links comes from the possibility of playing a 2×2 coordination game with their neighbors. However, the authors focus their attention on the strategies of agents in the coordination game, and do not consider how the order of play affects the equilibrium structure of the network. The paper by Sadler (2025) also studies link formation costs that increase with an agent's degree. A key strength of his approach is its broad applicability, as it avoids parametric assumptions on preferences of agents. To guarantee equilibrium existence and uniqueness in such a general setting, he introduces an axiomatic condition known as the *mutual favorite property*. While this assumption is elegant and powerful, it also implies that the order of play does not affect the final network structure, as agents consistently seek to connect with the same preferred partner throughout the game. In contrast, we show that focusing on marginal link formation costs—when either sufficiently high or low—can also ensure the existence of a unique equilibrium, while allowing the network structure to depend on the order of play. This offers a complementary perspective, as it allows the equilibrium network to depend on the order of play.

Although the order in which agents play a game has not yet been explicitly shown to be a determinant of the equilibrium structure of a network, recent work outside the network literature highlights its growing importance. For instance, Le Quement et al. (2025) shows that the order in which one consults experts for advice matters, as it is usually better to ask for advice from less reliable experts first, rather than advice from more reliable ones. As another example, Barberà and Gerber (2025) shows that in games where the order of play is endogenous, it usually determines the outcome of the game. We show, in the current paper, that their conclusion also extends to games with exogenous orders of play. The paper by Hinnosaar (2024) shows, in the context of sequential contests, that earlier movers are better off than later moves, which resonates with our result that playing last is usually harmful.[1]

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 presents the results. Section 4 discusses possible avenues for future research. Section 5 concludes.

2 The model

We next present the model, and introduce notation. We will define additional notation throughout the paper when necessary.

Network definitions. We consider a set of agents $\mathcal{N} = \{1, ..., N\}$. Every pair of agents $i, j \in \mathcal{N}$ can either share a link (in which case, $g_{ij} = 1$) or not (in which case, $g_{ij} = 0$). We call *network*, *adjacency matrix*, or *(network) structure*, the matrix $\boldsymbol{G}$ such that entry of the i^{th} row and j^{th} column corresponds to g_{ij}. We consider that $g_{ij} = g_{ji}$ for any pair of agents $i, j \in \mathcal{N}$, i.e., links are undirected. We denote by $\mathcal{G}_N$ the set which is composed of all possible adjacency matrixes, given N. A *walk* from agent i to agent j is a sequence of agents $\{i, i+1, \cdots, j-1, j\}$ and links $\{g_{i,i+1,}, \cdots, g_{j-1,j}\}$ such that $g_{mn} = 1$ for all $m \in \{i, i+1, \cdots, j-1\}$ and $n = m+1$. A path is a walk in which all nodes are distinct. The length of a walk or path equals the number of nodes in the sequence less 1. The shortest path between nodes i and j is the path which consists of the minimal number of nodes, among all paths between i and j. The *neighborhood* of agent i is $\boldsymbol{N}_i = \{j \in \mathcal{N} \setminus \{i\} : g_{ij} = 1\}$, and its cardinality $|\boldsymbol{N}_i| = d_i$ is the degree of agent i. Agent i is a *neighbor* of agent j if i is in the neighborhood of agent j. Agent i is a *k-distance* neighbor of agent j if the length of the shortest path between i and j is k. A component of a network is a set $\mathcal{C}$ of nodes such that there exists a walk from any node $i \in \mathcal{C}$ to any node

[1]The order of play has also recently gained importance in other fields, such as computer science (e.g. Hu et al., 2024).

$j \in \mathcal{C}$, but not to any node outside $\mathcal{C}$. We denote by $\mathcal{C}_i$ the component to which node i belongs. An *independent set* of $\boldsymbol{G}$ is a non-empty subset of players who are not linked to each other. A *circle* is a component composed of 4 agents or more who all have exactly degree 2.[2] A *complete bipartite component* is a component composed of two independent sets, such that every agent in one independent set is linked to all agents in the other independent set. We denote by $\boldsymbol{G_{+ij}}$ ($\boldsymbol{G_{-ij}}$) the network obtained from $\boldsymbol{G}$ by adding (deleting) link g_{ij}. A *complete* component is a component in which all nodes are linked to each other. A *dyad* is a complete component composed of two nodes, and a *triad* is a complete component composed of three nodes. Network $\boldsymbol{G'}$ is *adjacent* to $\boldsymbol{G}$ if $\boldsymbol{G'} = \boldsymbol{G_{+ij}}$ or $\boldsymbol{G'} = \boldsymbol{G_{-ij}}$ for some ij. We say that two agents i and j are *automorphically equivalent* in network $\boldsymbol{G}$, if there is an automorphism of $\boldsymbol{G}$ that maps i to j and j to i, i.e., if re-labeling i to j and j to i does not alter the structure of the network. We say that $\boldsymbol{G}$ is *denser* than $\boldsymbol{G'}$ if $\boldsymbol{G'} \subseteq \boldsymbol{G}$.

Effort and utility function. Every agent $i \in \mathcal{N}$ exerts effort $x_i \geq 0$ in the activity. The utility of each agent i is defined in (1):

$$U_i(x_i, \boldsymbol{x_{-i}}) = x_i + \alpha \sum_{j \neq i} g_{ij} x_i x_j - \frac{1}{2} x_i^2, \tag{1}$$

given $\boldsymbol{x_{-i}}$ and α, where vector $\boldsymbol{x_{-i}} = \begin{pmatrix} x_1 & \cdots & x_{i-1} & x_{i+1} & \cdots & x_N \end{pmatrix}$ denotes the effort exerted by all agents other than i, and $0 < \alpha < \frac{1}{(N-1)}$ is a parameter that measures strategic complementarity in effort exerted between neighbors.[3] We denote by $U_i(\boldsymbol{G})$ and by $x_i(\boldsymbol{G})$ the utility function of agent i in network $\boldsymbol{G}$, and the effort exerted by agent i in network $\boldsymbol{G}$ respectively. We also denote by $U_i(\boldsymbol{G_{+mn}})$ and by $x_i(\boldsymbol{G_{+mn}})$ the utility of agent i and the effort exerted by agent i, respectively, in network $\boldsymbol{G_{+mn}}$. We define $\Delta U_i(\boldsymbol{G}_{+mn}) = U_i(\boldsymbol{G}_{+mn}) - U_i(\boldsymbol{G})$ and $\Delta x_i(\boldsymbol{G}_{+mn}) = x_i(\boldsymbol{G}_{+mn}) - x_i(\boldsymbol{G})$.

[2] A circle is usually considered to be a component composed of 3 agents or more who all have exactly degree 2. As is explained below, this definition allows for an easier characterization of the results, and does not change the nature of the results.

[3] Condition $0 < \alpha < \frac{1}{(N-1)}$ ensures that Bonacich centrality is well defined (Jackson, 2008).

Nash equilibrium. We denote by x_i^* the equilibrium value of agent i, which is the value x_i that maximizes her utility. Vector $\boldsymbol{x^*} = \begin{pmatrix} x_1^* & \cdots & x_N^* \end{pmatrix}$ is the Nash equilibrium of the game, in which no agent has a profitable unilateral deviation from her equilibrium value. If we let x_i' be any value of x_i, and $\boldsymbol{x^*_{-i}}$ be the set of equilibrium values of all agents other than i, then the Nash equilibrium is such that, for all agents $i \in \mathcal{N}$, $U(x_i^*, \boldsymbol{x^*_{-i}}) \geq U(x_i', \boldsymbol{x^*_{-i}})$ for all $x_i' \in \mathbb{R}_{\geq 0}$.
As follows from Ballester et al. (2006), the Nash equilibrium of the game is given by vector $\boldsymbol{X}$, composed of one column and N rows.

$$\boldsymbol{X} = [\boldsymbol{I}_N - \alpha \boldsymbol{G}]^{-1}\mathbf{1}, \tag{2}$$

where $\boldsymbol{I}_N$ is the identity matrix of dimension N. The i^{th} entry of vector $\boldsymbol{X}$ is commonly called the Bonacich centrality of agent i (Bonacich, 1987).

Cost of link formation. The cost of link formation function $c(d)$ maps a degree $d \in \mathbb{N}$ to a positive real-valued number $c \in \mathbb{R}_{\geq 0}$. Formally, $c : \mathbb{N} \to \mathbb{R}_{\geq 0}$ is a discrete function such that $c(0) = 0$ and $c(d+1) - c(d) > 0$ for all $d \geq 1$.[4]

Network formation process. The game is dynamic and time is discrete. We define function $\mathcal{P} : \mathcal{N} \times \mathcal{G}_N \times \mathbb{R}_{>0} \times (\mathbb{N} \to \mathbb{R}_{>0}) \to \mathcal{N}$, which maps an agent $i \in \mathcal{N}$ who plays at time t, a network structure $\boldsymbol{G} \in \mathcal{G}_N$, a value $\alpha \in \mathbb{R}_{>0}$ and a cost of link formation function $c(d)$ that maps a degree $d \in \mathbb{N}$ to a positive real value $c \in \mathbb{R}_{>0}$, to an agent $j \in \mathcal{N}$ who plays at time $t+1$. After an agent $i \in \mathcal{N}$ plays at time t, function $\mathcal{P}$ selects which agent plays at time $t+1$. We call function $\mathcal{P}$ the *order of play*. Notation $\mathcal{S}[t]$ denotes set $\mathcal{S}$ at time t, where $\mathcal{S}$ can be any set. In particular, notations $\boldsymbol{G}[t]$, $d_i[t]$ and $g_{ij}[t]$ respectively denote adjacency matrix $\boldsymbol{G}$ at time t, degree of agent i at time t and link indicator g_{ij} at time t. Agents are myopic in the sense that they seek to generate the highest immediate incremental utility from the deletion or the formation of a new link. The structure of the network is common knowledge. The timing of events is the following:

1. The network starts as empty, i.e. $g_{ij}[0] = 0$ for all $i, j \in \mathcal{N}$.

[4] We set $c(0) = 0$ to avoid the trivial case in which agents do not form any link.

2. A randomly chosen agent, denoted by 1, plays first at time $t = 1$. Agent 1 can delete any existing links with neighbors in $\boldsymbol{N}_1[1]$ (however, at time $t = 1$, the neighborhood of agent 1 is empty). Agent 1 deletes a subset of own links if the incremental utility generated by the link deletion is strictly positive.

3. All agents then adjust the effort they exert in the activity, so that a new Nash equilibrium is achieved given the new network structure.

4. Agent 1 proposes at most one link to one agent $j \notin \boldsymbol{N}_1[1]$. Agent 1 will only propose a link to another agent if the incremental utility generated by the newly formed link is strictly larger than some cost she incurs from link formation. The cost she incurs from link formation is given by function $c(d)$, where d is her degree before the proposition of the link. If a link is formed between two agents, the cost corresponding to their degree before the formation of the link is incurred by both agents, and sunk.
   Out of all the agents j who i can select to propose a link to, i will select the agent with whom the formation of a new link generates the highest incremental utility $\Delta U_i(\boldsymbol{G_{+ij}})$. If there are two or more such agents, agent i will randomly select whom to propose the link to among these agents. The agent who proposes the link (in this case, agent i) is called the *sender*, and the agent to whom the link is proposed to (in this case, agent j) is called the *receiver*.

5. Receiver j can either accept or decline the link proposed by sender i. If j accepts, then the link between i and j is formed. If j declines, then it is not formed. Agent j accepts if $\Delta U_j(\boldsymbol{G}_{+ij}) \geq c(d_j)$, and declines if $\Delta U_j(\boldsymbol{G}_{+ij}) < c(d_j)$. In equilibrium, a sender only proposes a link to a receiver if the receiver will accept the formation of the link.

6. All agents adjust again the effort they exert in the activity, so that a new Nash equilibrium is achieved given the new network structure.

7. Function $\mathcal{P}$ selects which agent plays next, at time $t = 2$.

8. The process is recursive and starts again from bullet point number 2, where it is the agent selected by $\mathcal{P}$ that plays at the next period. We call any sequence of networks generated between times r and s, $\{\boldsymbol{G}[r], ..., \boldsymbol{G}[s]\}$, an *improving path*. When no agent has an incentive to *(i)* delete any subset of own links, and *(ii)* form any new link, the game ends. We denote by $\boldsymbol{G}^*$ such a resulting network.

Network $\boldsymbol{G}^*$ is commonly defined as the *pairwise stable (Nash) equilibrium*. Formally, $\boldsymbol{G}^*$ is a pairwise stable equilibrium if

$$U_i(\boldsymbol{G}^*) \geq U_i(\boldsymbol{G}^*_{-i\lambda_1 - i\lambda_2 \cdots - i\lambda_r}), \tag{3}$$

for each $i \in \mathcal{N}$ and any $\{\lambda_1, \lambda_2, \cdots, \lambda_r\} \subseteq \boldsymbol{N}_i(\boldsymbol{G}^*)$, and

$$U_i(\boldsymbol{G}^*_{+ij}) - c(d) > 0 \implies U_j(\boldsymbol{G}^*_{+ij}) - c(d_j) < 0, \tag{4}$$

for each $i, j \in \mathcal{N}$. Equation (3) indicates that, in $\boldsymbol{G}^*$, no agent has an incentive to delete any subset of own links, and equation (4) indicates that, in $\boldsymbol{G}^*$, no agent has an incentive to form a new link.

Notation related to the order of play. The following notation will be used to define specific orders of play, and will be recalled throughout the paper when necessary. We denote by $\mathbf{GL}[t]$ the set of agents that have an incentive to form a link at time $t+1$, given the network structure at time t. We denote by $\boldsymbol{GLC}_i[t] = \boldsymbol{GL}[t] \cap \{j \in \mathcal{C}_i[t]\}$ the subset of $\boldsymbol{GL}[t]$ in which all nodes belong to the component of i. We denote by $\boldsymbol{GLNC}_i[t] = \boldsymbol{GL}[t] \setminus \boldsymbol{GLC}_i[t]$ the subset of $\boldsymbol{GL}[t]$ in which all nodes do not belong to the component of i, i.e. $\boldsymbol{GLNC}_i[t]$ is the complementary set of $\boldsymbol{GLC}_i[t]$. Time τ is the first time in which $\mathcal{P} = j \in \boldsymbol{GLNC}_i[t]$ for some agent i, i.e., the first time in which a node from a different component plays. We denote by $\boldsymbol{GP}[t]$ the set of agents who have played at least once in the time interval $\{1, ..., t\}$. We denote by $\mathcal{U}(\cdot)$ the discrete uniform distribution. Binary variable $L^t_{i \to j}$ equals 1 if agent i proposes a link to agent j at time t and the link is formed at time t, and equals 0 otherwise.

3 Network formation analysis

We first provide some results regarding the incentives of players and the existence of a pairwise stable (Nash) equilibrium, in Section 3.1. We next characterize network structures in equilibrium by considering two orders of play, which will be formally defined below, in Section 3.2 (additional orders of play are considered in Appendix A). We show that complete bipartite networks and circles can only be explained by one order of play each, out of the ones we consider. We then provide some additional results, in Section 3.3. We first give conditions for specific structures to emerge, which are complete multipartite and ring lattice networks. We next show that playing last is usually harmful. Finally, we look at which orders of play are better for predicting the equilibrium structure of the network.

3.1 Preliminary results

The incremental utility that an agent i receives when she forms a new link with agent j is:

$$\Delta U_i(\boldsymbol{G}_{+ij}) = \Delta x_i(\boldsymbol{G}_{+ij}) \cdot (\frac{1}{2}\Delta x_i(\boldsymbol{G}_{+ij}) + x_i(\boldsymbol{G})). \tag{5}$$

Recall that $\Delta x_i(\boldsymbol{G}_{+ij})$ is the incremental effort that agent i exerts after the addition of link ij in network $\boldsymbol{G}$. Equation (5) can be retrieved from the equality $U_i = \frac{1}{2}x_i^2$, which holds at equilibrium (Belhaj et al., 2016). Equation (5) is particularly important because it allows us to know whether the formation of a link is profitable for agent i. We now state Lemma 1, and give an intuition of it in Example 1 and Figures 1a and 1b below.

Lemma 1. *[Joshi et al. 2025] Suppose $\{\mathcal{C}_i \subseteq \boldsymbol{G}\} \subseteq (\subset)\{\tilde{\mathcal{C}}_i \subseteq \tilde{\boldsymbol{G}}\}$ and $ij \notin \tilde{\mathcal{C}}_i$. Then, for all $\boldsymbol{G}$:*

$$\Delta U_i(\tilde{\boldsymbol{G}}_{+ij}) \geq (>)\Delta U_i(\boldsymbol{G}_{+ij}) \text{ and } \Delta U_j(\tilde{\boldsymbol{G}}_{+ij}) \geq (>)\Delta U_j(\boldsymbol{G}_{+ij}).$$

Example 1. Suppose that it is profitable for agent 2 to form a link with agent 4 in network $\boldsymbol{G}$, i.e., $\Delta U_2(\boldsymbol{G}_{+24}) > c(2)$ (see Figure 1a). Because component $\tilde{\mathcal{C}}_2$ is denser than component $\mathcal{C}_2$, the Bonacich centrality of agent 2 in network $\tilde{\boldsymbol{G}}$, $x_2(\tilde{\boldsymbol{G}})$,

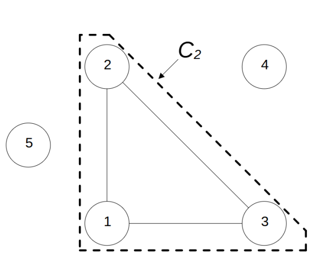

Figure 1a: Network $\boldsymbol{G}$

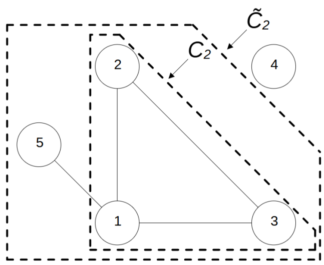

Figure 1b: Network $\tilde{\boldsymbol{G}}$

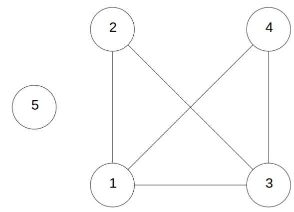

Figure 1c: Network $\boldsymbol{G'}$

is strictly larger than her Bonacich centrality in network $\boldsymbol{G}$, $x_2(\boldsymbol{G})$ (see Figure 1b). The incremental Bonacich centrality of agent 2 generated by linking with agent 4 in network $\tilde{\boldsymbol{G}}$, $\Delta x_2(\tilde{\boldsymbol{G}}_{+24})$, is also strictly larger than the incremental Bonacich centrality of agent 2 generated by linking with agent 4 in network $\boldsymbol{G}$, $\Delta x_2(\boldsymbol{G}_{+24})$. It follows, from equation (5), that the incremental utility of agent 2 generated by linking with agent 4 in network $\tilde{\boldsymbol{G}}$, $\Delta U_2(\tilde{\boldsymbol{G}}_{+24})$, is strictly larger than the incremental utility of agent 2 generated by linking with agent 4 in network $\boldsymbol{G}$, $\Delta U_2(\boldsymbol{G}_{+24})$.$\square$

Lemma 1 is very helpful because the formation of one link between some agent i in a component $\mathcal{C}_i$ (e.g., agent 1 in the network of Figure 1a) and some agent j (e.g., agent 5 in the network of Figure 1a), implies that the formation of one link between some agent $\tilde{i}$ in a component $\tilde{\mathcal{C}}_2$ denser than $\mathcal{C}_2$ (e.g., agent 2 in the network of Figure 1b), who is automorphically equivalent with agent i in component $\mathcal{C}_2$, and some agent who is as least as Bonacich central as j (e.g., agent 4 in the network of Figure 1b, who is at least as Bonacich central as agent 5 in the network of Figure 1a), generates at least as much incremental utility. It is important that agents i and $\tilde{i}$ are automorphically equivalent in $\mathcal{C}_2$. Otherwise, we cannot infer the profitability of the formation of one link from the profitability of the formation of another link. For instance, agents 1 and 3 are automorphically equivalent in component $\mathcal{C}_1$ in network $\boldsymbol{G'}$ of Figure 1c. If it is profitable for agent 1 to form a link with agent 5 at time t, then we can infer that it is profitable as well for agent 3 to form a link with agent 5 at time t. We cannot infer, however, that it is profitable for agents 2 and 4 to form a link with agent 5 at time t, since their Bonacich centralities $x_2(\boldsymbol{G'})$ and $x_4(\boldsymbol{G'})$ are lower than the Bonacich centralities

$x_1(\boldsymbol{G'})$ and $x_3(\boldsymbol{G'})$ of agents 1 and 3.
We next show that a pairwise stable (Nash) equilibrium exists. For this, we first state the following lemma, which directly follows from the fact that deleting a subset of own links reduces one's Bonacich centrality.

Lemma 2. *It is never profitable for any agent to delete a subset of own links.*

Since it is never profitable for agents to delete links, they never enter cycles in which links are repeatedly formed and deleted indefinitely.

Proposition 1. *Given any order of play $\mathcal{P}$, a pairwise stable equilibrium exists.*

3.2 Main results

3.2.1 Equilibrium uniqueness

A recent and very useful approach to addressing equilibrium multiplicity in network formation games—when link formation costs increase with the agent's degree—is the application of the *mutual favorite property* (Sadler, 2025). This property requires that agent i prefers agent j as a neighbor over all agents k such that $ik \in \boldsymbol{G}$, and simultaneously, j prefers i over all ℓ such that $j\ell \in \boldsymbol{G}$, throughout the entire linking process. Under this assumption, the order of play becomes irrelevant to the equilibrium structure of the resulting network. Because we study the influence of the order of play on the equilibrium structure, we do not adopt this method. Instead, we propose an alternative way to address equilibrium multiplicity—one that retains the relevance of the order of play—by considering either low or high marginal costs of link formation. As we will show, it is typically intermediate marginal costs that generate multiple equilibria. We will first define the order of play $\mathcal{P}_{RO}$ (which stands for receiver-outside). Then, we will show by means of an example that there can be equilibrium multiplicity. Finally, we will use the same example and vary the marginal costs of link formation to show that multiplicity of equilibria can only happen when marginal costs of link formation are intermediate.
When the order of play is $\mathcal{P}_{RO}$, the receiver of the link is the one playing next, and the agents who do not belong to the component of the receiver play afterwards.

Recall that $L_{i\to j}^{t}$ is a binary variable which equals 1 if agent i proposes a link to agent j at time t and the link is formed at that period, and equals 0 otherwise. Set $\mathbf{GL}[t]$ comprises the agents that have an incentive to form links at time $t+1$, given the network structure at time t. Set $\mathbf{GLC}[t]$ ($\mathbf{GLNC}[t]$) is a subset of $\mathbf{GL}[t]$ that comprises the agents that do (not) belong to the component of agent i, and have an incentive to form a link at time time $t+1$. Formally,

$$\mathcal{P}_{RO} = \begin{cases} j \in \boldsymbol{G}, & \text{if } L_{i\to j}^{t} = 1 \text{ and } j \in \boldsymbol{GL}[t] \\ l \sim \mathcal{U}(\boldsymbol{GLNC}_i[t]) & \text{if } L_{i\to j}^{t} = 1, j \notin \boldsymbol{GL}[t] \text{ and } \exists l \in \boldsymbol{GLNC}_i[t] \\ l \sim \mathcal{U}(\boldsymbol{GLC}_i[t]) & \text{if } L_{i\to j}^{t} = 1, j \notin \boldsymbol{GL}[t] \text{ and } \nexists l \in \boldsymbol{GLNC}_i[t] \end{cases}.$$

We distinguish between three cases which lead to different identities of agent j:

- If a sender i successfully links with an agent j at time t, i.e. $L_{i\to j}^{t} = 1$, and agent j can propose a link at time $t+1$, i.e. $j \in \boldsymbol{GL}[t]$, then it is agent j that plays at time $t+1$, i.e. $\mathcal{P}_{RO} = j$.

- If a sender i successfully links with an agent j at time t, i.e. $L_{i\to j}^{t} = 1$, but agent j is not able to propose a link at time $t+1$, i.e. $j \notin \boldsymbol{GL}[t]$, and there exists at least another agent l who does not belong to the component of i who can propose a link at time $t+1$, i.e. $\exists l \in \boldsymbol{GLNC}_i[t]$, then an agent l is randomly selected out of these agents who do not belong to the same component of i at time t and who can propose a link at time $t+1$, i.e. $\mathcal{P}_{RO} = l \sim \mathcal{U}(\boldsymbol{GLNC}_i[t])$.

- If a sender i successfully links with an agent j at time t, i.e. $L_{i\to j}^{t} = 1$, but agent j is not able to propose a link at time $t+1$, i.e. $j \notin \boldsymbol{GL}[t]$, and there is no other agent l who does not belong to the component of i who can propose a link at time $t+1$, i.e. $\nexists l \in \boldsymbol{GLNC}_i[t]$, then an agent l is randomly selected out of the agents who do belong to the component of agent i at time t and can propose a link at time $t+1$, i.e. $\mathcal{P}_{SO} = l \sim \mathcal{U}(\boldsymbol{GLC}_i[t])$.

By means of an example, we next show that there can be multiplicity of equilibria. Figures 2a, 2b, 2c, 2d, 3a, 3b, 3c and Table 1 illustrate the example.

Example 2. Suppose $\alpha = 0.01$, $c(1) = 0.01$, $c(2) = 0.0105$, $c(3) = 0.0107985$ and $c(4) = 5$.

Periods 1-6. At time $t = 1$, agent 1 links with agent 2. Given the value of α and the shape of $c(d)$, agent 2 plays at time $t = 2$ and links with agent 3. Agent 3 plays at time $t = 3$ and links with agent 1. Agent 6 plays at time $t = 4$, and a second triad is formed, with agents 5 and 6, at time $t = 6$ (see Figure 2a).

Periods 7-9. At time $t = 7$, agent 6 plays and links with agent 3 (chosen randomly among agents 1, 2 and 3). It is not profitable for the receiver of the link, agent 3, to form a link at time $t = 8$, and hence agent 4 is randomly selected to play at time $t = 8$. It is too costly for agent 3 to accept a link from agent 4, and so agent 4 forms a link with agent 1 (chosen randomly among agents 1 and 2). It is not profitable for the receiver of the link, agent 1, to form a link at time $t = 9$, and hence agent 2 is selected to play at time $t = 9$. It is too costly for agents 4 and 6 to accept a link from agent 2, and so agent 2 forms a link with agent 5 (see Figure 2b). At time $t = 10$, agent 5 plays and links with agent 3 (chosen randomly among agents 1 and 3 —see Figure 2c).

***Path A* - Period 11.** It is not profitable for agent 3 to form a link at time $t = 11$, and so agent 6 is selected to play at time $t = 11$ and links with agent 2, who generates the highest incremental utility. At time $t = 12$, agents 1 and 4 only have three links, but they cannot form any other links, because it is not profitable for the other agents to form a fifth link. Because no agent has any incentive to form a link at time $t = 12$, $\boldsymbol{G'}[11]$ is the pairwise stable equilibrium (see Figure 2d).

Path B* - Periods 11,12.** At time $t = 11$, it is not profitable for agent 3 to form a link, and so agent 4 is randomly selected to play (instead of agent 6, as in ***Path A). Agent 4 is already linked to all agents, except 2 and 3, and would like to link with agent 3. However, agent 3 would decline the link proposed by agent 4 because a fifth link is too costly. Therefore, agent 4 links with agent 2 at time $t = 11$ (see Figure 3b). At time $t = 11$, all agents have four links, except agents 1 and 6 who have three links. Since it is too costly to form a fifth link, agent 1 (chosen randomly among agents 1 and 6) is selected to play at time $t = 12$ and links with agent 6. Therefore, network $\boldsymbol{G''}[12]$ forms, and since it is too costly to form a fifth link, $\boldsymbol{G''}[12]$ is the pairwise stable equilibrium (see Figure 3c). $\square$

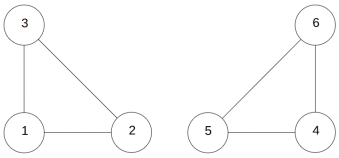

Figure 2a: $\boldsymbol{G'}[6]$

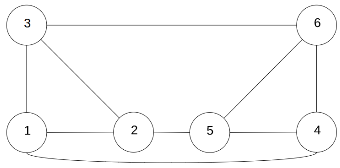

Figure 2b: $\boldsymbol{G'}[9]$

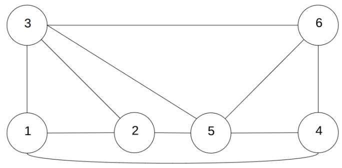

Figure 2c: $\boldsymbol{G'}[10]$

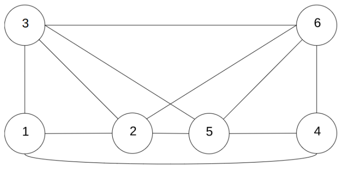

Figure 2d: $\boldsymbol{G'}[11]$

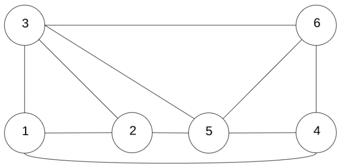

Figure 3a: $\boldsymbol{G''}[10]$

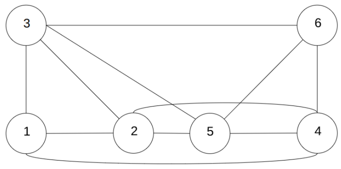

Figure 3b: $\boldsymbol{G''}[11]$

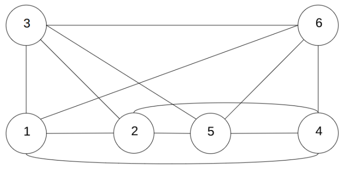

Figure 3c: $\boldsymbol{G''}[12]$

(a) *Example 2 (**Path A**)*

Path A		
	Sender	Receiver
Period 1	1	2
Period 2	2	3
Period 3	3	1
Period 4	6	4
Period 5	4	5
Period 6	5	6
Period 7	6	3
Period 8	4	1
Period 9	2	5
Period 10	5	3
Period 11	6	2

(b) *Example 2 (**Path B**)*

Path B		
	Sender	Receiver
Period 1	1	2
Period 2	2	3
Period 3	3	1
Period 4	6	4
Period 5	4	5
Period 6	5	6
Period 7	6	3
Period 8	4	1
Period 9	2	5
Period 10	5	3
Period 11	4	2
Period 12	1	6

*Table 1: Linking process of Example 2: **Path A** and **Path B**.*

Let us now increase the value of $c(2)$. As we show in the next example, the equilibrium is now unique.

Example 3. Suppose $\alpha = 0.01$, $c(1) = 0.01$, $c(2) = 5$.
Periods 1-6. The linking process is the same as the one presented in Example 2, up until period 6 (see Figure 2a). Since it is too costly to form a third link, $\boldsymbol{G'}[6]$ is the pairwise stable equilibrium. □

When marginal costs of link formation are large, they prevent agents from entering network structures which can lead to equilibrium multiplicity.

Proposition 2. *If there exists low enough d for which $c(d)$ is large enough, then the pairwise stable equilibrium is unique.*

Marginal costs of link formation need to be large enough for a low enough degree, because otherwise, agents can still enter a network structure which allows for equilibrium multiplicity. If, for instance, the structure which allows for equilibrium multiplicity is attained when nodes reach degree 5, but costs of link formation can only prevent the formation of degree 7, then large marginal costs do not ensure equilibrium uniqueness.
It is straightforward that, when marginal costs of link formation are low enough, the unique pairwise stable equilibrium is a complete network. The complete network is taken as an intuitive example to show that low marginal costs of link formation ensure equilibrium uniqueness, but low marginal costs of link formation can also ensure the existence of a unique non-complete network. Figure 4 summarizes Section 3.2.1.

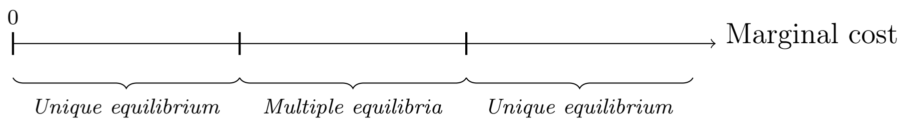

Figure 4: Network equilibrium structure by marginal cost of link formation

3.2.2 The receiver-outside case

Order of play $\mathcal{P}_{RO}$ has been defined in Section 3.2.1. We next show that $\boldsymbol{G}^*$ can be a circle of any size when the order of play is $\mathcal{P}_{RO}$, whereas it cannot be a circle composed of 6 or more agents when the order of play is any of the other ones we consider (including those in Appendix A).

An example is shown in Figures 5a, 5b and 5c which represent a network $\boldsymbol{G}$ composed of 6 agents at times 2, 4 and 6 respectively.

Example 4. Suppose $\alpha = 0.01$, $c(1) = 0.01$ and $c(2) = 5$.

Periods 1,2. At time $t = 1$, agent 1 plays and links with agent 2. Given the values of α and $c(1)$, it is not profitable for agent 2 to form a link at time $t = 2$, and therefore, agent 5 plays next. At time $t = 2$, agent 5 links with agent 6, and so network $\boldsymbol{G}[2]$, represented in Figure 5a, arises.

Periods 3,4. At time $t = 3$, agent 6 links with agent 1. At time $t = 4$, it is not profitable for agent 1 to form a link, and so agent 4 (randomly chosen among agents 3 and 4) plays next. At time $t = 4$, agent 4 links with agent 5, and so network $\boldsymbol{G}[4]$, represented in Figure 5b, arises.

Periods 5,6. At time $t = 5$, it is not profitable for agent 5 to form a new link, and so agent 3 is selected to play next. At time $t = 5$, agent 3 links with agent 4 (randomly chosen among agents 2 and 4). At time $t = 6$, it is not profitable for agent 4 to form a link, and so agent 2 plays at time $t = 6$ and links with agent 3. Network $\boldsymbol{G}[6]$, represented in Figure 5c, arises. At time $t = 7$, no agent has an incentive to form any other link, and so $\boldsymbol{G}[6]$ is the pairwise stable equilibrium. □

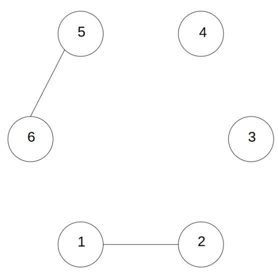

Figure 5a: $\boldsymbol{G}[2]$

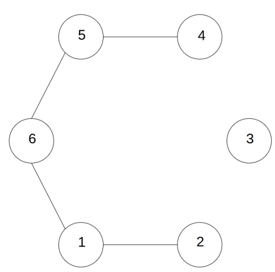

Figure 5b: $\boldsymbol{G}[4]$

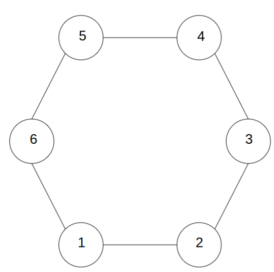

Figure 5c: $\boldsymbol{G}[6]$

Example 4		
	Sender	Receiver
Period 1	1	2
Period 2	5	6
Period 3	6	1
Period 4	4	5
Period 5	3	4
Period 6	2	3

Table 2: Linking process in Example 4.

Circle networks are widely studied in a variety of contexts within the networks literature (see, for example, Cabrales and Hauk (2022) in the context of leadership in networks). Given that the order of play is $\mathcal{P}_{RO}$, $\boldsymbol{G^*}$ is a circle if and only if *(i)* $c(1)$ is large enough so that it is not profitable for an agent in a dyad to link with a singleton, and low enough so that it is profitable for an agent in a dyad to link with another agent in another dyad, and *(ii)* $c(2)$ is large enough so that a third link is not profitable.

We define the *circle* as a component *composed of 4 or more agents* who all have degree 2. This way, we exclude the trivial case of the triad, which necessitates a different linking process to form than circles of any other size. We can thus give *sufficient* and *necessary* conditions on the shape of $c(d)$ for the formation of a circle at equilibrium, which would not be *necessary* if we included the special case of the triad. We denote by $\underline{\underline{c(1)}}$, $\underline{c(1)}$ and $\underline{c(2)}$ the threshold values of $c(1)$ and $c(2)$ that satisfy points *(i)* and *(ii)* above.

Proposition 3. *Suppose $\mathcal{P} = \mathcal{P}_{RO}$. $\boldsymbol{G^*}$ is a circle if and only if $\underline{\underline{c(1)}} \leq c(1) < \underline{c(1)}$ and $c(2) \geq \underline{c(2)}$ for some $\underline{\underline{c(1)}}$, $\underline{c(1)}$ and $\underline{c(2)}$.*

As explained below, the order of play that is presented in the next section, 3.2.3, cannot explain the formation of circles of any size.

3.2.3 The sender-outside case

We next consider an order of play in which the sender of a link is the one playing next, and in which agents who do not belong to the component of the sender play afterwards. Formally,

$$\mathcal{P}_{SO} = \begin{cases} i \in \boldsymbol{G} & \text{if } L^t_{i\to j} = 1 \text{ and } i \in \mathbf{GL}[t] \\ j \sim \mathcal{U}(\boldsymbol{GLNC}[t]) & \text{if } L^t_{i\to j} = 1, i \notin \mathbf{GL}[t] \text{ and } \exists j \in \boldsymbol{GLNC}[t] \\ j \sim \mathcal{U}(\boldsymbol{GLC}[t]) & \text{if } L^t_{i\to j} = 1, i \notin \mathbf{GL}[t] \text{ and } \nexists j \in \boldsymbol{GLNC}[t] \end{cases}.$$

We distinguish between three cases which lead to different identities of agent j.

- If a sender i successfully links with an agent j at time t, i.e. $L^t_{i\to j} = 1$, and agent i is able to propose another link at time $t+1$, i.e. $i \in \boldsymbol{GL}[t]$, then it is agent i that plays at period $t+1$, i.e. $\mathcal{P}_{SO} = i$.
- If a sender i successfully links with an agent j at time t, i.e. $L^t_{i\to j} = 1$, but agent i is not able to propose another link at time $t+1$, i.e. $i \notin \boldsymbol{GL}[t]$, and there exists at least another agent j who does not belong to the component of i who can propose a link at time $t+1$, i.e. $\exists j \in \boldsymbol{GLNC}_i[t]$, then an agent j is randomly selected out of these agents who do not belong to the same component of i at time t and who can propose a link at time $t+1$, i.e. $\mathcal{P}_{SO} = j \sim \mathcal{U}(\boldsymbol{GLNC}_i[t])$.
- If a sender i successfully links with an agent j at time t, i.e. $L^t_{i\to j} = 1$, but agent i is not able to propose another link at time $t+1$, i.e. $i \notin \boldsymbol{GL}[t]$, and there is no other agent j who does not belong to the component of i who can propose a link at time $t+1$, i.e. $\nexists j \in \boldsymbol{GLNC}_i[t]$, then an agent j is randomly selected out of the agents who do belong to the component of agent i at time t and can propose a link at time $t+1$, i.e. $\mathcal{P}_{SO} = j \sim \mathcal{U}(\boldsymbol{GLC}_i[t])$.

One can easily verify that $\mathcal{P}_{SO}$ can explain the formation of circles composed of up to 5 agents, but not more. Indeed, two agents in different dyads can link with each other, after which a singleton plays and links with either agent with degree 1 in the line. Since this agent plays again, she necessarily links with the other agent with degree 1 in the line and closes the circle.

While $\mathcal{P}_{SO}$ cannot explain the formation of circles of any size, it can explain the formation of complete bipartite networks, which $\mathcal{P}_{RO}$ cannot explain (nor the other two orders of play presented in Appendix A). An example is shown in Figures 6a,

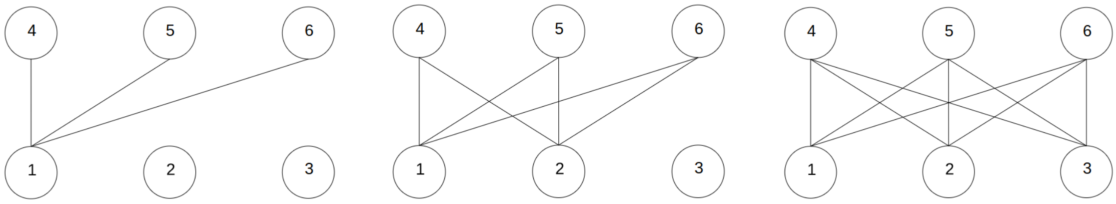

Figure 6a: $\boldsymbol{G}[3]$ *Figure 6b:* $\boldsymbol{G}[6]$ *Figure 6c:* $\boldsymbol{G}[9]$

6b and 6c which represent a network $\boldsymbol{G}$ composed of 6 agents at times 3, 6 and 9 respectively.

Example 5. Suppose $\alpha = 0.01$, $c(1) = 0.01$, $c(2) = 0.0103$ and $c(3) = 5$.
Periods 1-3. At time $t = 1$, agent 1 plays and links with agent 4. Given the values of α, $c(1)$, $c(2)$ and $c(3)$, agent 1 links with agents 5 and 6 at times $t = 2$ and $t = 3$ respectively.
Periods 4-6. At time $t = 4$, it is not profitable for agent 1 to form a link, and so agent 2 plays at time $t = 4$. It is profitable for agent 2 to link with agents 4, 5 and 6 as well, at times $t = 4$, $t = 5$ and $t = 6$ respectively.
Periods 7-9. At time $t = 7$, it is not profitable for agent 2 to form a link, and so agent 3 plays at time $t = 7$. It is profitable for agent 3 to link with agents 4, 5 and 6, at times $t = 7$, $t = 8$ and $t = 9$ respectively. A fourth link is too costly to form. Since no agent has an incentive to form a new link at time $t = 10$, $\boldsymbol{G}[9]$ is the pairwise stable equilibrium. □

Bipartite networks are central in understanding matters such as European integration (Di Clemente et al., 2022), trade networks (Saracco et al., 2015), finance networks (Gualdi et al., 2016) or scientific competition of countries (Cimini et al., 2014). The order in which nodes form links determines whether these networks arise or not. We now state Proposition 4. Note that $t = (\tau - 1)^2$ corresponds to the period in which the first complete bipartite forms, and $t = 2(\tau - 1)^2$ corresponds the period in which the second one forms.

Proposition 4. *Suppose* $\mathcal{P} = \mathcal{P}_{SO}$. $\boldsymbol{G}^*$ *is composed of complete bipartite components if and only if*

Example 5		
	Sender	Receiver
Period 1	1	4
Period 2	1	5
Period 3	1	6
Period 4	2	4
Period 5	2	5
Period 6	2	6
Period 7	3	4
Period 8	3	5
Period 9	3	6

Table 3: Linking process in Example 5.

(i) $N \geq 2(\tau - 1)$, *and*
(ii) $g_{ij}[t] = 0$ *for* $\tau \leq t \leq 2(\tau - 1)^2$ *for all* $i, j \in \boldsymbol{GP}[t]$.

Condition (i) ensures that at least one complete bipartite network arises. Recall that τ is the first time in which an agent that does not belong to the component of the sender plays, and hence, $(\tau - 1)$ corresponds to number of nodes in one of the two independent sets of the bipartite component. If $N < 2(\tau - 1)$, then a complete bipartite network will temporarily arise, after which agents in the independent set of larger size will link with each other. Condition (ii) ensures that agents in different bipartite components do not link with each other, and that agents in the same partition of a bipartite component do not link with each other either.[5]

One can easily check as well that when the order of play is $\mathcal{P}_{RO}$, complete bipartite components composed of 5 or more agents cannot arise. If the order of play is $\mathcal{P}_{RO}$, a dyad first forms and either agent of the dyad links with a singleton, then a triad necessarily forms, which is by definition absent from a bipartite network. If two dyads form first, a line of four agents forms at the following period, and $N \geq 5$, then a circle composed of 5 agents necessarily forms, as mentioned earlier.

[5]Suppose that the conditions of Proposition 4 hold. One can easily check that $\boldsymbol{G}[\tau - 1]$ is a star, $\boldsymbol{G}[(\tau - 1)^2]$ is a complete bipartite component and $\boldsymbol{G}[2(\tau - 1)^2]$ is a network composed of 2 complete bipartite components, given that N is large enough.

3.3 Additional results

3.3.1 Multipartite and ring lattice components

As discussed in the previous sections, complete bipartite and circle networks can only be explained by one order of play each, out of the four we consider (including those in Appendix A). These structures are special cases of complete multipartite and ring lattice networks respectively. Multipartite and ring lattice networks are essential in understanding matters such as public goods (Kinateder and Merlino, 2017), social hierarchy (Baetz, 2015), trade (Wilhite, 2001) and small-world networks (Watts and Strogatz, 1998). A natural follow-up question to the previous analysis is whether these structures can also arise at equilibrium, and if they are also exclusive to a certain order of play. We find that these structures can arise, but that multiple orders of play can explain their formation. We next define these structures.

A complete multipartite component is composed of two or more independent sets, such that every node in each independent set is linked to every node in every other independent set, whereas a complete bipartite component is a complete multipartite component with only two independent sets. A k-ring lattice is a regular component which consists of a circle in which all nodes are also linked to their distance-l neighbors, for all $l \in \{2, ...k\}$.

Remark 1. *Consider any order of play.*
(i) If two complete components $\mathcal{C}_{C1}$ and $\mathcal{C}_{C2}$, both composed of x agents, arise at some period t, and
(ii) if it is profitable for every agent in $\mathcal{C}_{C1}[t]$ to link with exactly $x-1$ agents in $\mathcal{C}_{C2}[t]$ and no agent outside $\mathcal{C}_{C2}[t]$, and
(iii) if it is profitable for every agent in $\mathcal{C}_{C2}[t]$ to link with exactly $x-1$ agents in $\mathcal{C}_{C1}[t]$ and no agent outside $\mathcal{C}_{C1}[t]$,
then $\boldsymbol{G^*}$ is composed of components which are both complete multipartite and ring lattice.

Example 2 (***Path B***) shows how conditions (i), (ii) and (iii) lead to a network composed of components which are both complete multipartite and lattice at equilibrium, when the order of play is $\mathcal{P}_{SI}$. At period $t = 6$, two triads arise (condition *(i)*), and at period $t = 12$, every agent in each triad links with exactly two agents of the other triad (conditions *(ii)* and *(iii)*). The different independent sets of network $\boldsymbol{G''}[12]$ in Figure 3c are {1,5}, {2,6} and {3,4}. To see why network $\boldsymbol{G''}[12]$ is also a ring lattice, consider network $\boldsymbol{G''}[12]$, such that $g_{13} = g_{32} = g_{25} = g_{54} = g_{46} = g_{61} = 1$ and all other link indicators take value 0, which is a circle. It is then easy to see that all agents are linked to their distance-2 neighbors in the circle.

3.3.2 Social inequalities

We have characterized the formation of networks composed of complete bipartite, complete multipartite and ring lattice components, as well as circle networks. The question of welfare is natural in this context. Are there agents who benefit from, or are harmed by the formation of these structures? Social inequalities is a central topic in public economics, and has also drawn attention in the field of social networks (DiMaggio and Garip, 2012). We find that the number of agents in the network, N, is a determinant of inequalities. Furthermore, playing last is usually harmful. We next state new definitions. We denote by $\mathcal{C}_K$ the component in which the last link is formed. We say that a component $\mathcal{C}$ is of *degree* d if $d_i = d$ for all $i \in \mathcal{C}$. We denote by $|\mathcal{C}_1^*|$ the number of nodes in component $\mathcal{C}_1$ at equilibrium.

Proposition 5. *Suppose that $\boldsymbol{G}^*$ is unique and composed of K components, among which $K-1$ are either complete bipartite, complete multipartite or ring lattice components of degree d.*
(i) If N is a multiple of $|\mathcal{C}_1^*|$, then $U_i(\boldsymbol{G}^*) = U_j(\boldsymbol{G}^*)$ for all $i, j \in \mathcal{N}$.
(ii) If N is not a multiple of $|\mathcal{C}_1^*|$ and all agents $i \in \mathcal{C}_K$ have degree $d_i \leq d$ in $\boldsymbol{G}^*$ with at least one strict inequality, then $U_i(\boldsymbol{G}^*) < U_j(\boldsymbol{G}^*)$ for all $i \in \mathcal{C}_K$ and $j \notin \mathcal{C}_K$.

When N is a multiple of $|\mathcal{C}_1^*|$, all components of $\boldsymbol{G}^*$ have the same structure. Furthermore, all agents in this component that is formed and replicated exert the same effort. Since utility of agent i is given by $U_i^* = \frac{1}{2}x_i^{*2}$ (Belhaj et al., 2016), all agents have the same utility. When N is not a multiple of $|\mathcal{C}_1^*|$, the component that is formed last is composed of less agents, and agents in this component usually have a lower Bonacich centrality, and hence a lower utility. Playing last is usually harmful, because it forces the player to form a component with a lower number of nodes. Even when N is not a multiple of $|\mathcal{C}_1^*|$, it is not always the case that agents in component $\mathcal{C}_K$ have a lower Bonacich centrality. If, for instance, $\boldsymbol{G^*}$ is composed of $K-1$ complete bipartite components composed of 4 agents each, and one triad, then N is not a multiple of $|\mathcal{C}_1^*|$. However, all agents have the same degree, and hence the same Bonacich centrality, and hence the same utility.[6]

3.3.3 Time of viability

We have shown that there often is multiplicity of pairwise stable equilibria. We next study whether some orders of play allow us to know at an earlier period the final structure of the network. We find that, under some conditions, orders of play $\mathcal{P}_{RI}$ and $\mathcal{P}_{SI}$ (defined in Appendix A) are better for predicting the equilibrium structure than $\mathcal{P}_{RO}$ and $\mathcal{P}_{SO}$. Informally speaking, orders of play $\mathcal{P}_{RI}$ and $\mathcal{P}_{SI}$ are identical to orders of play $\mathcal{P}_{RO}$ and $\mathcal{P}_{SO}$, with the exception that when it is not profitable for the sender or receiver of the link to play next, it is an agent *inside* the component—and not *outside*—who plays next. We next make some definitions.
We call the selection of some agent i by function $\mathcal{P}$ that ensures the existence of some pairwise stable equilibrium $\boldsymbol{G^*}$, the *viability* of $\boldsymbol{G^*}$ as the unique pairwise stable network. We call the period $\gamma_{\boldsymbol{G^*}} \in \boldsymbol{T}$ at which the viability of $\boldsymbol{G^*}$ as the unique pairwise stable network happens, the *time of viability* of $\boldsymbol{G^*}$ as the unique pairwise stable network. We denote by $\gamma^j_{\boldsymbol{G^*}}$, for $j \in \{SI, SO, RI, RO\}$, the time of viability of $\boldsymbol{G^*}$ as the unique pairwise stable network, given that the order of play is $\mathcal{P}_j$, for $j \in \{SI, SO, RI, RO\}$ respectively. We denote by $\boldsymbol{C^*_K} = \{\mathcal{C}_1^*, ..., \mathcal{C}_K^*\}$ the set of components that can form at equilibrium. We denote by $\eta(\mathcal{C}_k^*)$ the number of periods needed to know with certainty the structure of component $\mathcal{C}_k^*$, if $\boldsymbol{G^*}$ were only composed of $\mathcal{C}_k^*$. We denote by $\theta(\mathcal{C}_k^*)$ the maximal number of periods needed for component $\mathcal{C}_k^*$ to form, if $\boldsymbol{G^*}$ were only composed of $\mathcal{C}_k^*$.[7] We

[6] When $\boldsymbol{G^*}$ is a circle, all agents have the same Bonacich centrality, and hence the same utility, regardless of the value of N.

[7] We want to capture the number of periods needed for a component to form. If we do not assume that $\boldsymbol{G^*}$ is only composed of $\mathcal{C}_k^*$, then $\eta(\mathcal{C}_k^*)$ and $\theta(\mathcal{C}_k^*)$ may capture periods in which

define $V = \sum_{k=1}^{K} \theta(\mathcal{C}_k^*) - \eta(\mathcal{C}_k^*)$. For example, in Example 2, the structure of the component at equilibrium is known with certainty at period $t = 11$ (when Paths $\boldsymbol{A}$ and $\boldsymbol{B}$ start), and so, $\eta(\mathcal{C}_1^*) = 11$. The maximal number of periods needed for the component to form is $\theta(\mathcal{C}_1^*) = 12$. Since there is only one component at equilibrium, we have $V = 12 - 11 = 1$.

Proposition 6. *There exists a threshold of V under which $\gamma_{\boldsymbol{G^*}}^{SI}$ ($\gamma_{\boldsymbol{G^*}}^{RI}$) $\leq \gamma_{\boldsymbol{G^*}}^{SO}$ ($\gamma_{\boldsymbol{G^*}}^{RO}$).*

We next give an intuition for Proposition 6. Suppose that the order of play is $\mathcal{P}_{SI}$, that some network $\boldsymbol{G^*}$ is composed of 3 components $\mathcal{C}_1^*$, $\mathcal{C}_2^*$ and $\mathcal{C}_3^*$, such that it takes up to 12 periods for $\mathcal{C}_1^*$ to form, up to 12 periods for $\mathcal{C}_2^*$ to form, and 4 periods for $\mathcal{C}_3^*$ to form. Suppose further that $\mathcal{C}_3^*$ only has one possible structure, and that $\eta(\mathcal{C}_1^*) = \eta(\mathcal{C}_2^*) = 12$, i.e., $V = \sum_{k=1}^{2} \theta(\mathcal{C}_k^*) - \eta(\mathcal{C}_k^*) = 0$ is low. In that case, it takes 12 periods for $\boldsymbol{C_1^*}$ to form, and 12 additional periods to know with certainty the structure of $\mathcal{C}_2^*$. Thus, $\gamma_{\boldsymbol{G^*}}^{SI} = 12 + 12 = 24$. If the order of play is $\mathcal{P}_{SO}$ instead, then the formation of component $\mathcal{C}_3^*$ can start before we know with certainty the structure of $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$ at the pairwise stable equilibrium, and thus, $\gamma_{\boldsymbol{G^*}}^{SI} \leq \gamma_{\boldsymbol{G^*}}^{SO}$.
Suppose instead that $\eta(\mathcal{C}_1^*) = \eta(\mathcal{C}_2^*) = 7$, i.e., $V = \sum_{k=1}^{2} \theta(\mathcal{C}_k^*) - \eta(\mathcal{C}_k^*) = 5 + 5 = 10$ is larger. If $\mathcal{P} = \mathcal{P}_{SO}$, then it is possible that $\mathcal{C}_1^*$ is known with certainty at period $t = 7$, that $\mathcal{C}_2^*$ starts forming at $t = 8$ so that it is known with certainty at period $t = 14$, and thus, that $\gamma_{\boldsymbol{G^*}}^{SO} = 14$, which is lower than $\gamma_{\boldsymbol{G^*}}^{SI} = 12 + 7 = 19$. The same reasoning applies to orders of play $\mathcal{P}_{RI}$ and $\mathcal{P}_{RO}$.
Even though it is not always possible, *ex ante*, to know with certainty which network structure arises in the long run, it is possible to know it after the network starts forming, and it is possible to know it earlier for some orders of play than for others. Whether it is possible to know it earlier for orders of play $\mathcal{P}_{SI}$ and $\mathcal{P}_{RI}$ or $\mathcal{P}_{SO}$ and $\mathcal{P}_{RO}$ depends on V.

4 Discussion

We believe that this model opens up promising avenues for future research. In particular, there are two important directions worth exploring, which pose major

other components are being formed, which are irrelevant to the structure of $\mathcal{C}_k^*$.

challenges within the current framework, but may be more tractable when approached through a different, simpler model. *First*, we consider *non-simultaneous* formation and deletion of links, where agents first decide whether to delete a subset of own links, and only then decide to propose the formation of a link. In this sense, we follow the approach of papers like Hiller (2017), which, applied to our paper, implies that agents never find it profitable to delete links, as is the case in papers like Joshi et al. (2025, 2020).[8] Considering *simultaneous* deletion and formation of links would prevent us from ensuring the existence of a pairwise stable equilibrium, since agents could enter a cycle where some links are repeatedly formed and deleted. *Second*, agents are *myopic*, in the sense that they seek to generate the highest immediate incremental utility from the deletion or the formation of a new link. If agents are allowed to be farsighted, then a set of farsightedly pairwise stable equilibria exists (see Theorem 2 of Herings et al. (2009)). However, the model becomes intractable, making it difficult to characterize the equilibrium. These modeling challenges remain particularly compelling, offering interesting directions for future research.

5 Conclusion

We consider a network formation model in which sunk costs of link formation are increasing in the agent's degree. In this class of models, the order in which agents sequentially form the network (order of play) determines its final structure. Our main result is that only certain orders of play can explain the formation of complete bipartite and circle networks. We also find that the equilibrium is unique when marginal costs of link formation are intermediate, and that some orders of play are better than others for predicting the final structure of the network when the equilibrium is not unique. We furthermore show that playing last is usually harmful. The model we present also explains the formation of complete multipartite and ring lattice networks.

[8] There is no consensus on whether simultaneous or non simultaneous deletion and formation of links should be used (Jackson, 2008, p. 375).

Appendix A Other orders of play

A.1 The sender-inside case

We consider an order of play in which the sender of a link is the one playing next, and in which agents belonging to the component of the sender play afterwards. Formally,

$$\mathcal{P}_{SI} = \begin{cases} i \in \boldsymbol{G} & \text{if } L^t_{i\to j} = 1 \text{ and } i \in \mathbf{GL}[t] \\ j \sim \mathcal{U}(\boldsymbol{GLC}_i[t]) & \text{if } L^t_{i\to j} = 1, i \notin \mathbf{GL}[t] \text{ and } \exists j \in \boldsymbol{GLC}_i[t] \\ j \sim \mathcal{U}(\boldsymbol{GLNC}_i[t]) & \text{if } L^t_{i\to j} = 1, i \notin \mathbf{GL}[t] \text{ and } \nexists j \in \boldsymbol{GLC}_i[t] \end{cases}.$$

We distinguish three cases which lead to different identities of agent j:

- If a sender i successfully links with an agent j at time t, i.e. $L^t_{i\to j} = 1$, and agent i is able to propose another link at time $t+1$, i.e. $i \in \boldsymbol{GL}[t]$, then it is agent i that plays at time $t+1$, i.e. $\mathcal{P}_{SI} = i$.
- If a sender i successfully links with an agent j at time t, i.e. $L^t_{i\to j} = 1$, but agent i is not able to propose another link at time $t+1$, i.e. $i \notin \boldsymbol{GL}[t]$, and there exists some agent j belonging to the component of i who can propose a link at time $t+1$, i.e. $\exists j \in \boldsymbol{GLC}_i[t]$, then an agent j is randomly selected out of these agents who belong to the same component of i at time t and can propose a link at time $t+1$, i.e. $\mathcal{P}_{SI} = j \sim \mathcal{U}(\boldsymbol{GLC}_i[t])$.
- If a sender i successfully links with an agent j at time t, i.e. $L^t_{i\to j} = 1$, but agent i is not able to propose another link at time $t+1$, i.e. $i \notin \boldsymbol{GL}[t]$, and there is no other agent j belonging to the component of i who can propose a link at time $t+1$, i.e. $\nexists j \in \boldsymbol{GLC}_i[t]$, then an agent j is randomly selected out of the agents who do not belong to the component of agent i at time t and who can propose a link at time $t+1$, i.e. $\mathcal{P}_{SI} = j \sim \mathcal{U}(\boldsymbol{GLNC}_i[t])$.

When the order of play is $\mathcal{P}_{SI}$, the pairwise stable equilibrium is usually composed of complete components. An example is shown in Figures 7a, 7b, 7c, 7d, 7e and 7f which represent a network $\boldsymbol{G}$ composed of 4 agents at times 1, 2, 3, 4, 5 and 6

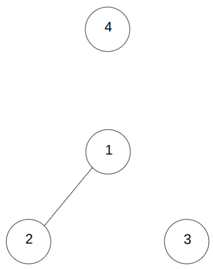

Figure 7a: $\boldsymbol{G}[1]$

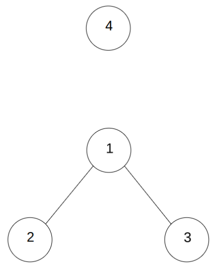

Figure 7b: $\boldsymbol{G}[2]$

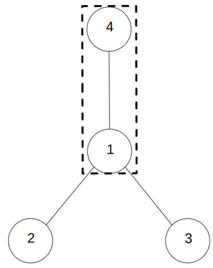

Figure 7c: $\boldsymbol{G}[3]$

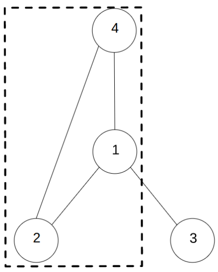

Figure 7d: $\boldsymbol{G}[4]$

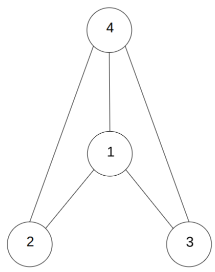

Figure 7e: $\boldsymbol{G}[5]$

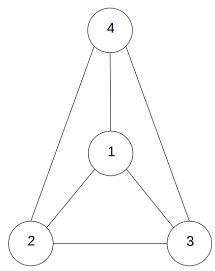

Figure 7f: $\boldsymbol{G}[6]$

respectively.

Example 6. Suppose $\alpha = \frac{1}{4}$, $c(1) = 0.2$ and $c(2) = 0.5$.
Periods 1-3. At time $t = 1$, agent 1 plays first, and forms a link with agent 2. Given the values of α, $c(1)$ and $c(2)$, agent 1 forms a link with agent 3 at time $t = 2$, and with agent 4 at time $t = 3$.
Periods 4,5. Agent 4 is randomly selected to play at $t = 4$. Notice that the dashed frame in Figure 7c has the same structure as $\boldsymbol{G}[1]$, from which follows $\boldsymbol{G}[1] \subseteq \boldsymbol{G}[3]$. Agent 4 therefore forms a link with agent 2 at time $t = 4$, by Lemma 1. Notice that the dashed frame in Figure 7d, has a denser structure than $\boldsymbol{G}[2]$, from which follows $\boldsymbol{G}[2] \subseteq \boldsymbol{G}[4]$. Agent 4 therefore forms a link with agent 3 at time $t = 5$, by Lemma 1.
Period 6. By following the same reasoning, we deduce that link 23 is profitable as well. Because no agent can form another link at time $t = 7$, $\boldsymbol{G}[6]$ is the pairwise stable equilibrium. □

Example 6		
	Sender	Receiver
Period 1	1	2
Period 2	1	3
Period 3	1	4
Period 4	4	2
Period 5	4	3
Period 6	2	3

Table 4: Linking process in Example 6.

When the order of play is $\mathcal{P}_{SI}$, a complete component, such as $\boldsymbol{G}[6]$ presented in Example 6, always arises. If it is never profitable for any two agents in different non-singleton components, then $\boldsymbol{G}^*$ is composed of complete components. If it can be profitable for agents in different non-singleton components to form a link with each other, then there is multiplicity of equilibria, and thus the characterization of equilibria becomes more challenging.

Proposition 7. *Suppose $\mathcal{P} = \mathcal{P}_{SI}$. If $g_{ij}[t] = 0$ for $\tau \leq t \leq 2\tau$ for all $i \in \mathcal{C}_1[\tau - 1]$ and all $j \notin \mathcal{C}_1[\tau - 1]$, then $\boldsymbol{G}^*$ is composed of complete components.*

Recall that τ, defined in Section 2, is the first time in which some player outside $\mathcal{C}_1$ is selected to play next. Hence, condition $g_{ij}[t] = 0$ for $\tau \leq t \leq 2\tau$ for all $i \in \mathcal{C}_1[\tau - 1]$ and all $j \notin \mathcal{C}_1[\tau - 1]$ means that, between the time in which the first component has finished forming (period τ) and the time in which the second component has finished forming (period 2τ), agents in the first component have not formed a link with agents in other non-singleton components ($g_{ij}[t] = 0$ for all $i \in \mathcal{C}_1[\tau - 1]$ and all $j \notin \mathcal{C}_1[\tau - 1]$). One can easily show that if this condition is fulfilled, then $\boldsymbol{G}^*$ is composed of 2 identical complete components. If it has not been profitable for agents in the first component to form a link with agents in the second component between periods τ and 2τ, then it immediately follows that is not profitable either for agents in components formed at later periods to form a link with agents in preivously formed components. Hence, $\boldsymbol{G}^*$ is composed of complete components in this case.

A.2 The receiver-inside case

We consider an order of play in which the receiver of a link is the one playing next, and in which agents who belong to the component of the receiver play afterwards. Formally,

$$\mathcal{P}_{RI} = \begin{cases} j \in \boldsymbol{G}, & \text{if } L^t_{i\to j} = 1 \text{ and } j \in \boldsymbol{GL}[t] \\ l \sim \mathcal{U}(\boldsymbol{GLC}_i[t]) & \text{if } L^t_{i\to j} = 1, j \notin \boldsymbol{GL}[t] \text{ and } \exists l \in \boldsymbol{GLC}_i[t] \\ l \sim \mathcal{U}(\boldsymbol{GLNC}_i[t]) & \text{if } L^t_{i\to j} = 1, j \notin \boldsymbol{GL}[t] \text{ and } \nexists l \in \boldsymbol{GLC}_i[t] \end{cases},$$

We distinguish between three cases which lead to different identities of agent j:

- If a sender i successfully links with an agent j at time t, i.e. $L^t_{i\to j} = 1$, and j is able to form a new link at time $t+1$, i.e. $j \in \boldsymbol{GL}[t]$, then it is agent j that plays at time $t+1$, i.e. $\mathcal{P}_{RI} = j$.
- If a sender i successfully links with an agent j at time t, i.e. $L^t_{i\to j} = 1$, but agent j is not able to form a new link at time $t+1$, i.e. $j \notin \boldsymbol{GL}[t]$, and there exists at least another agent who belongs to the component of i who can form a link at time $t+1$, i.e. $\exists l \in \boldsymbol{GLC}_i[t]$, then an agent l is randomly selected out of these agents who belong to the same component of i at time t and can form a new link at time $t+1$, i.e. $\exists l \in \boldsymbol{GLC}_i[t]$.
- If a sender i successfully links with an agent j at time t, i.e. $L^t_{i\to j} = 1$, but agent j is not able to form a new link at time $t+1$, i.e. $j \notin \boldsymbol{GL}[t]$, and there does not exist at least another agent who belongs to the component of i who can form a link at time $t+1$, i.e. $\nexists l \in \boldsymbol{GLC}_i[t]$, then an agent l is randomly selected out of the agents who do not belong to the same component of i at time t and who can form a new link at time $t+1$, i.e. $\exists l \in \boldsymbol{GLNC}_i[t]$.

When the order of play is $\mathcal{P}_{RI}$, the pairwise stable equilibrium is usually also composed of complete components. An example is shown in Figures 8a, 8b, 8c, 8d and 8e which represent a network $\boldsymbol{G}$ composed of 5 agents at times 1, 2, 3, 5 and 7 respectively.

Example 8. Suppose $\alpha = 0.01$, $c(1) = 0.01$, $c(2) = 0.0103$ and $c(3) = 0.0104$.
Periods 1-7. At time $t = 1$, agent 1 plays first and forms a link with agent 2. Given the values of α and $c(1)$, agent 2 links with agent 3 at time $t = 2$. Notice that the dashed frame in Figure 8b has the same structure as $\boldsymbol{G}[1]$, from which follows $\boldsymbol{G}[1] \subseteq \boldsymbol{G}[2]$. Agent 3 therefore forms a link with agent 1 at time $t = 3$, by Lemma 1. By following the same reasoning, one can easily check that agents 1 and 2 form triads with agents 4 and 5 at times $t = 5$ and $t = 7$ respectively. $\square$

Example 11		
	Sender	Receiver
Period 1	1	2
Period 2	2	3
Period 3	3	1
Period 4	1	4
Period 5	4	2
Period 6	2	5
Period 7	5	1
Period 8	3	4
Period 9	4	5
Period 10	5	3

Table 5: Linking process in Example 11.

Once it is not profitable for the two agents that played first to form any other triad, it is the turn of agents inside the formed component to play. These agents link together until the component becomes complete. An example is shown in Figures 9a, 9b, 9c and 9d which represent network $\boldsymbol{G}$ at times 7, 8, 9 and 10 respectively.

Example 8 (continuation). Periods 8-10. Suppose that, at $t = 8$, agent 3 is selected to play. Notice that the dashed frame in Figure 9a has the same structure as $\boldsymbol{G}[3]$, from which follows $\boldsymbol{G}[3] \subseteq \boldsymbol{G}[7]$. Agent 3 therefore forms a link with agent 4 at time $t = 8$, by Lemma 1. Notice that the dashed frame in Figure 9b has a denser structure than $\boldsymbol{G}[5]$, from which follows $\boldsymbol{G}[5] \subseteq \boldsymbol{G}[8]$. Agent 4 therefore forms a link with agent 5 at $t = 9$, by Lemma 1. By following the same reasoning, we can deduce that it is profitable for agent 5 to form a link with agent 3 at time

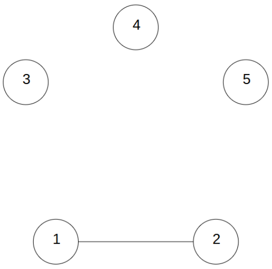

Figure 8a: $\boldsymbol{G}[1]$

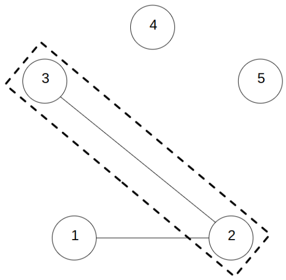

Figure 8b: $\boldsymbol{G}[2]$

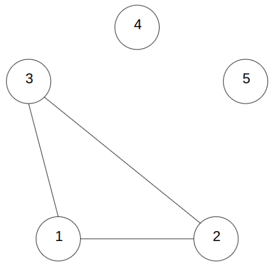

Figure 8c: $\boldsymbol{G}[3]$

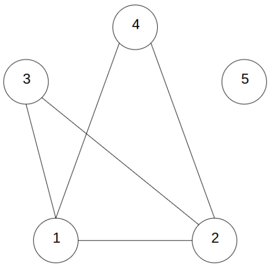

Figure 8d: $\boldsymbol{G}[5]$

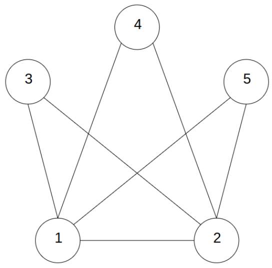

Figure 8e: $\boldsymbol{G}[7]$

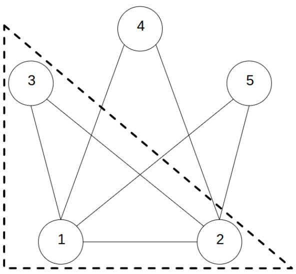

Figure 9a: $\boldsymbol{G}[7]$

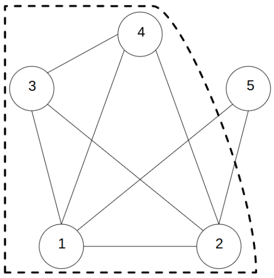

Figure 9b: $\boldsymbol{G}[8]$

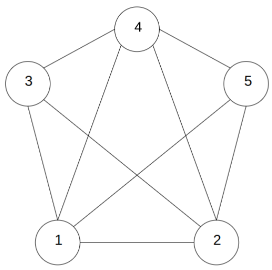

Figure 9c: $\boldsymbol{G}[9]$

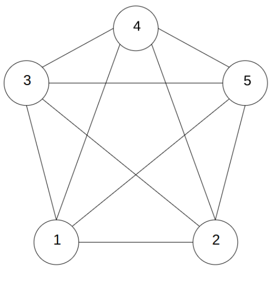

Figure 9d: $\boldsymbol{G}[10]$

$t = 10$. At time $t = 11$, no agent has an incentive to form a link, and so the complete network $\boldsymbol{G}[10]$ is the pairwise stable equilibrium. $\square$

Order of play $\mathcal{P}_{RI}$ is similar to $\mathcal{P}_{RI}$: when two components form and it is not profitable for agents in either component to form a link with agents in the other components, $\boldsymbol{G^*}$ is composed of complete components.

Proposition 8. *Suppose $\mathcal{P} = \mathcal{P}_{RI}$. If $g_{ij}[t] = 0$ for $\tau \leq t \leq 2\tau$ for all $i \in \mathcal{C}_1[\tau - 1]$ and all $j \notin \mathcal{C}_1[\tau - 1]$, then $\boldsymbol{G^*}$ is composed of complete components.*

Appendix B Proofs

Proof of Lemma 2. Suppose, *ad absurdum*, that it is profitable for an agent to delete a subset of own links with agents in set $\lambda = \{\lambda_1, ..., \lambda_k\}$. It follows that $\Delta x_i(\boldsymbol{G}_{-i\lambda_1,...,-i\lambda_k}) < 0$. It directly follows from $x_i(\boldsymbol{G}) \geq 0$ for any i and $\boldsymbol{G}$ that $\Delta x_i(\boldsymbol{G}_{-i\lambda_1,...,-i\lambda_k}) + x_i(\boldsymbol{G}) \geq 0$, and hence that $\frac{1}{2}\Delta x_i(\boldsymbol{G}_{-i\lambda_1,...,-i\lambda_k}) + x_i(\boldsymbol{G}) \geq 0$. It follows that $\Delta U_i(\boldsymbol{G}_{-i\lambda_1,...,-i\lambda_k}) = \Delta x_i(\boldsymbol{G}_{-i\lambda_1,...,-i\lambda_k})\cdot(\frac{1}{2}\Delta x_i(\boldsymbol{G}_{-i\lambda_1,...,-i\lambda_k})+x_i(\boldsymbol{G})) \leq 0$. A contradiction arises. Since set λ is chosen without loss of generality, a contradiction arises for any λ. $\square$

Proof of Proposition 1. Suppose, *ad absurdum*, that a pairwise stable equilibrium does not exist. Because it is never profitable for agents to delete links (Lemma 2), we deduce that at every period $t \in [1, \infty)$, an agent forms a link. It follows that there exists a period t in which $\boldsymbol{G}[t]$ is a complete network, and a period $t + 1$ in which $\boldsymbol{G}[t + 1] = \boldsymbol{G_{+ij}}[t]$ for some pair of agents $i, j \in \mathcal{N}$. A contradiction arises, since no additional link can be formed in the complete network $\boldsymbol{G}[t]$. $\square$

Proof of Proposition 2. Suppose that, given α and $c(d)$, the maximum degree in some structure $\boldsymbol{G_1}$, which allows for multiplicity of equilibria, is m. Increase the value of $c(m - 1)$ so that $\boldsymbol{G_1}$ cannot arise. If the pairwise stable equilibrium is unique, then the proof is complete. If some structure $\boldsymbol{G_2}$ with maximum degree n and which allows for multiplicity of equilibria can still arise, increase $c(n - 1)$ so that $\boldsymbol{G_2}$ cannot arise. By induction of the argument, we arrive to network $\boldsymbol{G}$ composed of dyads and at most one singleton, which is necessarily a unique pairwise stable equilibrium. $\square$

Proof of Proposition 3. Suppose that the order of play is $\mathcal{P}_{RO}$. Threshold values $\underline{\underline{c(1)}}$, $\underline{c(1)}$ and $\underline{c(2)}$ are defined as follows. It is profitable for an agent in a dyad to link with a singleton if and only if $c(1) < \underline{\underline{c(1)}}$. It is profitable for an agent in a dyad to link with another agent in a dyad if and only if $c(1) < \underline{c(1)}$. It is profitable for an agent in a circle network to form a link with a distance-2 neighbor if and only if $c(2) < \underline{c(2)}$.

Step 1: Prove that, if $\boldsymbol{\underline{\underline{c(1)}} \leq c(1) < \underline{c(1)}}$ and $\boldsymbol{c(2) \geq \underline{c(2)}}$, then $\boldsymbol{G^*}$ is a circle.

At time $t = 1$, the agent who plays first, denoted by 1, links with some singleton, denoted by 2. At time $t = 2$, it is not profitable for agent 2 to link with a singleton, and so some agent, denoted by 3, plays and links with some singleton, denoted by 4. At time $t = 3$, agent 4 links with agent 2, chosen among agent 1 and agent 2 without loss of generality. It is not profitable for agent 2 to form a third link, and so some singleton, denoted by 5, plays at time $t = 4$, and links with agent 4, chosen among agent 1 and agent 4 without loss of generality. By induction of the argument, every agent who does not belong to the formed line component links with either of the two agents with degree 1 in the line, until $\boldsymbol{G}$ becomes a line. When $\boldsymbol{G}$ is a line, both agents with degree 1 in the line link with each other, and so $\boldsymbol{G^*}$ is a circle. Since $c(2) \geq \underline{c(2)}$, it is not profitable for any agent in the circle to form third link, and so the pairwise stable equilibrium is a circle network.

Step 2: Prove that, if $\boldsymbol{G^*}$ is a circle, then $\boldsymbol{\underline{\underline{c(1)}} \leq c(1) < \underline{c(1)}}$ and $\boldsymbol{c(2) \geq \underline{c(2)}}$.

Suppose that $\boldsymbol{G^*}$ is a circle, and, *ad absurdum*, that $c(2) < \underline{c(2)}$. Because $c(2) < \underline{c(2)}$, it is profitable for agents in the circle network $\boldsymbol{G^*}$ to link between themselves. Network $\boldsymbol{G^*}$ is not a circle, and so, a contradiction arises.

Let us suppose that $\boldsymbol{G^*}$ is a circle, and, *ad absurdum*, that $c(1) < \underline{\underline{c(1)}}$ and $c(2) \geq \underline{c(2)}$. Because $c(1) < \underline{\underline{c(1)}}$, agents will form triads. Because $c(2) \geq \underline{c(2)}$, it is not profitable for the agents in the triads to link between themselves. Network

$\boldsymbol{G}^*$ is not a circle, and so, a contradiction arises—recall that we define a *circle* as a component composed of 4 agents or more in which all agents have degree 2.
Let us suppose that $\boldsymbol{G}^*$ is a circle, and, *ad absurdum*, that $c(1) \geq \underline{c(1)}$ and $c(2) \geq \underline{c(2)}$. Because $c(1) \geq \underline{c(1)}$, agents form dyads, and it is not profitable for the agents in the dyads to link between themselves. Network $\boldsymbol{G}^*$ is not a circle, and so, a contradiction arises. $\square$

Proof of Proposition 4. Suppose that the order of play is $\mathcal{P}_{SO}$.
Step 1: Prove that if conditions (*i*) **and** (*ii*) **are fulfilled, then $\boldsymbol{G}^*$ is composed of complete bipartite components.**
Let us suppose that $N \geq 2(\tau-1)$ and that it is not profitable for any pair of agents who have played during time interval $\tau \leq t \leq 2(\tau-1)^2$ to form a link with each other during time interval $\tau \leq t \leq 2(\tau-1)^2$. Agent 1 successively links with $\tau-1$ agents until it is not profitable for her to form a new link, so that a star component forms. Agent 2, chosen without loss of generality among the players which do not belong to the newly formed star component, successively links with all agents in the neighborhood of agent 1. Once agent 2 has linked with all the neighborhood of agent 1, it is not profitable for agent 2 to link with a singleton, since it is not profitable for agent 2 to link with agent 1. By induction of the argument, agents who do not belong to the component of agent 1 successively link with all agents in the neighborhood of agent 1, until a bipartite component forms. Because all agents in the formed bipartite component have the same Bonacich centrality, and because it is not profitable for agents outside the formed bipartite component to link with agents who have already played, it is not profitable for them either to link with any other agent inside the formed bipartite component. Therefore, a second complete bipartite component forms, and it forms at $t = 2(\tau-1)^2$. By induction of the argument, complete bipartite components form until the network attains a pairwise stable equilibrium.
Step 2: Prove that if $\boldsymbol{G}^*$ is composed of complete bipartite components, then conditions (*i*) **and** (*ii*) **are fulfilled.**
Let us suppose that $\boldsymbol{G}^*$ is composed of complete bipartite components, and, *ad absurdum*, that $N < 2(\tau-1)$. Because $N < 2(\tau-1)$, there exists a period in

which $\boldsymbol{G}$ is a complete bipartite network, and in which the independent set s_1 of the agent who played first is composed of less agents than the other independent set s_2. Because, at this period, all agents are in the same component, it is some agent inside the component who plays next. Agents in set s_2 next link with each other and, a contradiction arises, since we assumed that $\boldsymbol{G^*}$ is composed of complete bipartite components.
Ssuppose that $\boldsymbol{G^*}$ is composed of complete bipartite components, and, *ad absurdum*, that there exists at least one period t in which $g_{ij}[t] = 1$, where $\tau \leq t \leq 2(\tau-1)^2$ for a pair of agents $i, j \in \boldsymbol{GP}[t]$. It follows that there exists some agent i who played and successively linked with singletons, after which a singleton j linked with at least one neighbor of agent i, and up to all neighbors of agent i, and then linked with agent i, which implies that a triad formed. A contradiction arises, since there are no triads in bipartite components, by definition. $\square$

Proof of Proposition 5. Suppose that $\boldsymbol{G^*}$ is unique and composed of K components, among which $K-1$ are either complete bipartite, complete multipartite or ring lattice components of degree d.
Prove part (i)**.**
Let us suppose that N is a multiple of $|\mathcal{C}_1^*|$, and, *ad absurdum*, that there is a pair $i, j \in \mathcal{N}$ such that $U_i(\boldsymbol{G^*}) \neq U_j(\boldsymbol{G^*})$. Since N is a multiple of $|\mathcal{C}_1^*|$, all components in the network have the same structure, which is either complete bipartite, complete multipartite or ring lattice of degree d. In any of these components, every agent has the same Bonacich centrality. Because all agents have the same Bonacich centrality, they also have the same utility. A contradiction arises, since we assumed that there exist at least two agents who have different utilities.
Prove part (ii)**.**
Let us suppose that N is not a multiple of $|\mathcal{C}_1^*|$, that all agents $i \in \mathcal{C}_K$ have degree $d_i \leq d$ in $\boldsymbol{G^*}$ with at least one strict inequality, and, *ad absurdum*, that $U_i(\boldsymbol{G^*}) \geq U_j(\boldsymbol{G^*})$ for at least one $i \in \mathcal{C}_K$ and one $j \notin \mathcal{C}_K$. Because $d_i \leq d$ for all $i \in \mathcal{C}_K$, with at least one strict inequality, we have $x_i(\boldsymbol{G^*}) < x_j(\boldsymbol{G^*})$ for all $i \in \mathcal{C}_K$ and $j \notin \mathcal{C}_K$. It follows that $U_i(\boldsymbol{G^*}) < U_j(\boldsymbol{G^*})$ for all $i \in \mathcal{C}_K$ and $j \notin \mathcal{C}_K$. A contradiction arises, since we assumed that $U_i(\boldsymbol{G^*}) \geq U_j(\boldsymbol{G^*})$ for all $i \in \mathcal{C}_K$ and

$j \notin \mathcal{C}_K$. $\square$

Proof of Proposition 6. We prove that, if $V = \sum_{k=1}^{K} \theta(\mathcal{C}_k^*) - \eta(\mathcal{C}_k^*)$ is low enough and there exists a network $\boldsymbol{G}$ of viability of $\boldsymbol{G^*}$ as the unique pairwise stable network, then $\gamma_{\boldsymbol{G^*}}^{SI} \leq \gamma_{\boldsymbol{G^*}}^{SO}$. The same reasoning suffices to show $\gamma_{\boldsymbol{G^*}}^{RI} \leq \gamma_{\boldsymbol{G^*}}^{RO}$. We denote by $\mathcal{K}$ the set of agents who belong to components whose structure in $\boldsymbol{G^*}$ is known with certainty at period $t = 0$.
Let us suppose that, for a given α and $c(d)$, network $\boldsymbol{G^*}$ is composed of K components, among which $L < K$ are known with certainty at period $t = 0$. Suppose further that $\eta(\mathcal{C}_i) = \theta(\mathcal{C}_i)$ for all $i \notin \mathcal{K}$, and, *ad absurdum*, that $\gamma_{\boldsymbol{G^*}}^{SI} > \gamma_{\boldsymbol{G^*}}^{SO}$. The earliest period at which the structure of $\boldsymbol{G^*}$ can be known with certainty is $\sum_{i \notin \mathcal{K}} \theta(\mathcal{C}_i)$, which coincides with $\gamma_{\boldsymbol{G^*}}^{SI}$. A contradiction arises.
Let us now suppose that $L = K$, i.e. all components can have multiple structures at equilibrium, that $\eta(\boldsymbol{C}_i) = \theta(\boldsymbol{C}_i)$ for all $i \in \mathcal{N}$, and, *ad absurdum*, that $\gamma_{\boldsymbol{G^*}}^{SI} > \gamma_{\boldsymbol{G^*}}^{SO}$. The period at which the structure of $\boldsymbol{G^*}$ is known with certainty is $\sum_{i \in \mathcal{N}} \theta(\mathcal{C}_i)$, which coincides with the period at which the last link is formed, which thus coincides with both $\gamma_{\boldsymbol{G^*}}^{SI}$ and $\gamma_{\boldsymbol{G^*}}^{SO}$. A contradiction arises. $\square$

Proof of Proposition 7. Suppose that the order of play is $\mathcal{P}_{SI}$, that it is not profitable for any pair of agents $i \in \mathcal{C}_1[\tau - 1]$ and $j \notin \mathcal{C}_1[\tau - 1]$ to form a link during time interval $\tau \leq t \leq 2\tau$, and, *ad absurdum*, that $\boldsymbol{G^*}$ is not composed of complete components. Agent 1 successively links with x singletons until it is not profitable for her to form a new link, so that a star component $\mathcal{C}_1$ forms. It follows that it is profitable for any agent $i \in \mathcal{C}_1$ with degree $d_i \leq (x - 1)$ to link with any agent $j \in \mathcal{N}$.
Thus, all agents inside $\mathcal{C}_1$ form a link between themselves until the component becomes complete. After the complete component has formed, if it is profitable for some agent in the component to successively form links with singletons, and attain some degree $\tilde{d}_i$. Then, it is profitable as well for all agents inside the component to attain degree $\tilde{d}_i$, and hence form a complete component again. By following the same reasoning and because $g_{ij}[t] = 0$ for $\tau \leq t \leq 2\tau$ for all $i \in \mathcal{C}_1[\tau - 1]$ and all $j \notin \mathcal{C}_1[\tau - 1]$, we can deduce that a second complete component forms in the

time interval $\tau \leq t \leq 2\tau$. It is therefore not profitable either for any complete component formed afterwards to link with any other previously formed complete component. By induction of the argument, complete components form until the network attains a pairwise stable equilibrium, and a contradiction arises since we assumed that $\boldsymbol{G^*}$ is not composed of complete components. $\square$

Proof of Proposition 8. Suppose that the order of play is $\mathcal{P}_{RI}$, that it is not profitable for any pair of agents $i \in \mathcal{C}_1[\tau - 1]$ and $j \notin \mathcal{C}_1[\tau - 1]$ to form a link during time interval $\tau \leq t \leq 2\tau$, and, *ad absurdum*, that $\boldsymbol{G^*}$ is not composed of complete components. At time $t = 1$, agent 1 links with some agent 2. At time $t = 2$, agent 2 plays. If it is not profitable for agent 2 to link with a singleton, then $\boldsymbol{G^*}$ is composed of dyads and at most a singleton, and so, a contradiction arises. If, at time $t = 2$, agent links with some agent 3, then agent 3 links with agent 1 at time $t = 3$, by Lemma 1, and so a triad forms. By induction of the argument, agents 1 and 2 successively form y triads with other agents until it is not profitable for them to form a new triad, and reach degree $y + 1$. We denote this time by ζ. It follows that it is profitable for any agent who is part of $x < y$ triads and with degree $(x + 1)$ to form a link with any agent. At time $t = \zeta$, agent 3 is part of 1 triad, and has degree 2, which implies that it is profitable for her to form a link with any agent inside the component. Agent 3, chosen among players inside the formed component without loss of generality, links with agent 4 at time $t = \zeta + 1$, chosen among players inside the formed component without loss of generality. At time $t = \zeta + 1$, agent 4 is part of $x < y$ triads and has degree $(x + 1)$, and thus links with some agent 5 at time $t = \zeta + 2$, chosen among players inside the formed component without loss of generality. By following the same reasoning, we find that 5 links with agent 3 at time $t = \zeta + 3$, so that agent 3 is part of $x < y$ triads and has degree $(x + 1)$ at time $t = \zeta + 3$. By induction of the argument, agents 3 and 4 successively form triads with agents inside the component until they reach degree $y + 1$. By induction of the argument, all agents inside the component successively form triads until they reach degree $y + 1$. Once all agents reach degree $y + 1$, the component is complete.

If, after the component becomes complete, it is profitable for some agents $i, j \in \mathcal{C}_1$

to successively form triads with singletons, and attain some degree $\tilde{d}_i$, then it is profitable as well for all agents inside $\mathcal{C}_1$ to attain degree $\tilde{d}_i$, and hence form a complete component again.

By following the same reasoning and because $g_{ij}[t] = 0$ for $\tau \leq t \leq 2\tau$ for all $i \in \mathcal{C}_1[\tau - 1]$ and all $j \notin \mathcal{C}_1[\tau - 1]$, we can deduce that a second complete component forms in the time interval $\tau \leq t \leq 2\tau$. It is therefore not profitable either for any complete component formed afterwards to link with any other previously formed complete component. By induction of the argument, complete components form until the network attains a pairwise stable equilibrium, and a contradiction arises since we assumed that $\boldsymbol{G^*}$ is not composed of complete components. $\square$

References

Baetz, O. (2015). Social activity and network formation. *Theoretical Economics 10*(2), 315–340.

Bala, V. and S. Goyal (2000). A noncooperative model of network formation. *Econometrica 68*(5), 1181–1229.

Ballester, C., A. Calvó-Armengol, and Y. Zenou (2006). Who's who in networks. wanted: The key player. *Econometrica 74*(5), 1403–1417.

Barberà, S. and A. Gerber (2025). On the endogenous order of play in sequential games. *Available at SSRN 4810792*.

Belhaj, M., S. Bervoets, and F. Deroïan (2016). Efficient networks in games with local complementarities. *Theoretical Economics 11*(1), 357–380.

Bonacich, P. (1987). Power and centrality: A family of measures. *American Journal of Sociology 92*, 1170 – 1182.

Cabrales, A. and E. Hauk (2022). Norms and the evolution of leaders' followership.

Cimini, G., A. Gabrielli, and F. Sylos Labini (2014). The scientific competitiveness of nations. *PloS one 9*(12), e113470.

Di Clemente, R., B. Lengyel, L. F. Andersson, and R. Eriksson (2022). Understanding european integration with bipartite networks of comparative advantage. *PNAS nexus 1*(5), pgac262.

DiMaggio, P. and F. Garip (2012). Network effects and social inequality. *Annual review of sociology 38*(1), 93–118.

Gualdi, S., G. Cimini, K. Primicerio, R. Di Clemente, and D. Challet (2016). Statistically validated network of portfolio overlaps and systemic risk. *Scientific reports 6*(1), 39467.

Herings, P. J.-J., A. Mauleon, and V. Vannetelbosch (2009). Farsightedly stable networks. *Games and Economic Behavior 67*(2), 526–541.

Hiller, T. (2017). Peer effects in endogenous networks. *Games and Economic Behavior 105*, 349–367.

Hinnosaar, T. (2024). Optimal sequential contests. *Theoretical Economics 19*(1), 207–244.

Hu, H., G. Dragotto, Z. Zhang, K. Liang, B. Stellato, and J. F. Fisac (2024). Who plays first? optimizing the order of play in stackelberg games with many robots. *arXiv preprint arXiv:2402.09246*.

Jackson, M. O. (2008). *Social and Economic Networks*. Princeton University Press.

Jackson, M. O. and A. Watts (2002). On the formation of interaction networks in social coordination games. *Games and Economic Behavior 41*(2), 265–291.

Jackson, M. O. and A. Wolinsky (1996). A strategic model of social and economic networks. *Journal of Economic Theory 71*(1), 44–74.

Joshi, S., A. S. Mahmud, and S. Sarangi (2020). Network formation with multigraphs and strategic complementarities. *Journal of Economic Theory 188*, 105033, https://doi.org/10.1016/j.jet.2020.105033.

Joshi, S., A. S. Mahmud, and H. Tzavellas (2025). Multiplex formation. *Available at SSRN 4499938*. Mimeo.

Kinateder, M. and L. P. Merlino (2017). Public goods in endogenous networks. *American Economic Journal: Microeconomics 9*(3), 187–212.

König, M. D., C. J. Tessone, and Y. Zenou (2014). Nestedness in networks: A theoretical model and some applications. *Theoretical Economics 9*(3), 695–752.

Le Quement, M., J. Rivas, and K. Kawamura (2025). Whom to ask first? a model of sequential consultation. *Available at https://drive.google.com /file/d/1P41UKTAecQ0b2Lea9m6uq3VcuMyGonEk/view?usp=sharing*.

Sadler, E. (2025). Making a swap: network formation with increasing marginal costs. *Available at SSRN*.

Saracco, F., R. Di Clemente, A. Gabrielli, and T. Squartini (2015). Randomizing bipartite networks: the case of the world trade web. *Scientific reports 5*(1), 10595.

Watts, D. J. and S. H. Strogatz (1998). Collective dynamics of 'small-world' networks. *Nature 393*(6684), 440–442.

Wilhite, A. (2001). Bilateral trade and 'small-world'networks. *Computational economics 18*, 49–64.