The Hidden Adjustment: Re-examining the Capital-Labor-Mix in the Harrod-neutral Growth Model

de la Fonteijne, Marcel R.

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The Hidden Adjustment: Re-examining the Capital-Labor-Mix α in the Harrod-neutral Growth Model

The foundation of economic growth theory

Marcel R. de la Fonteijne

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Abstract

For several decades, it has been acknowledged that the conventional implementation of capital- and labor-augmenting technical progress within CES production functions gives rise to a fundamental paradox: either the production function must be Cobb-Douglas, or technical progress must be labor-augmenting only. Despite this inconsistency—commonly referred to as the “Cobb-Douglas or labor-augmenting-only paradox”—the approach remains widely used in modern growth models.

In this paper, we revisit this theoretical issue through the lens of the Modern Universal Growth Theory (MUGT). MUGT rejects all existing formulations of neutral and non-neutral technical progress and offers a revised implementation that resolves the paradox. Within this framework, economic growth is represented as partially exogenous, through technical change, and partially endogenous, through capital accumulation. We derive explicit expressions to translate total factor productivity (TFP) into measurable output growth, establishing a coherent link between productivity dynamics and long-run economic performance. The central conclusion of MUGT is that no production function can yield a Balanced Growth Path (BGP) unless the capital-labor mix is explicitly adjusted over time. In this sense, MUGT exposes a structural limitation of all traditional growth models and provides a general framework to overcome it.

A key contribution of this paper is the analysis of so-called Harrod-neutral (labor-augmenting) technical progress. We demonstrate that, despite its apparent simplicity, this approach implicitly requires a continuous adjustment of the capital-labor mix—a hidden mechanism that has remained largely unexamined. By revealing this adjustment, we not only explain the inner workings of Harrod's model, but also show that the same hidden mechanism exists across all combinations of capital- and labor-augmenting progress. This insight strengthens the case for adopting the MUGT as a consistent and transparent foundation for growth theory, in which each growth parameter has a clear, consistent, and economically meaningful interpretation.

Keywords: Capital and Labor Augmented Technical Progress, Growth Model, Maximum Profit Condition, Production Functions, General Technical Progress, Capital-Labor-mix, Elasticity of Substitution, DSGE, Total Factor Productivity, Solow model, Hicks, Harrod

JEL Classification E00 · E20 · E23 · E24

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Contact information:

- Marcel R. de la Fonteijne
- DLF Macro Economic Research
- Montgomerylaan 218
- 2625 PV Delft
- The Netherlands
- +31621885101
- marcel@dlfeconomics.com
- https://dlfeconomics.com

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1 Introduction

Economic growth theory has long stood as a central pillar of macroeconomic analysis. Since the postwar era, the models developed by Solow, Harrod, Uzawa, and Hicks have shaped our understanding of how capital accumulation, labor, and technical progress interact to drive long-run development. Each of these thinkers offered critical insights—from Solow's focus on exogenous technical progress, to Harrod's instability concerns, and Uzawa's early steps toward endogenizing innovation.

To mathematically represent growth, economists introduced capital- and labor-augmenting technical progress into production functions—typically Cobb-Douglas or CES types. While this framework gained wide acceptance, it has always contained a hidden flaw: when applied consistently, it leads to the so-called "Cobb-Douglas or labor-augmented-only" paradox. That is, for a balanced growth path (BGP) with constant capital-to-income ratio to exist, the model must either reduce to Cobb-Douglas or allow only for labor-augmenting technical change.

This theoretical inconsistency remained unresolved for decades, despite the model's continued dominance in textbooks and empirical applications. In De la Fonteijne (2018), we showed how this paradox can be resolved by rethinking the implementation of technical progress in a two-factor, homogeneous degree one CES production function. The resulting framework—Modern Universal Growth Theory (MUGT)—offers a corrected, internally consistent theory in which all parameters have clear economic meaning.

The present paper serves both as an application and a deeper clarification of the MUGT.

In Section 2, we introduce the underlying economic system.

Section 3 then details how technical progress is consistently incorporated into the CES framework under the Modern Universal Growth Theory (MUGT).

In Section 4, we discuss the key implications of this implementation for the structure of growth theory.

Finally, Section 5 turns to Harrod's labor-augmented growth model. We show that it contains a hidden adjustment to the capital-labor mix—unacknowledged but structurally equivalent to the one prescribed by MUGT. This comparison makes clear why the MUGT framework is not just an alternative, but a necessary correction.

2 The equations of the economic system

The system under consideration consists of the following equations:

The production function expressed in its base point (Y_0, K_0, L_0) with parameters α and γ

$$Y = Y_0 \left[\alpha \left(\frac{K}{K_0}\right)^{\gamma} + (1-\alpha)\left(\frac{L}{L_0}\right)^{\gamma}\right]^{1/\gamma} \qquad (1)$$

which is a homogeneous degree 1 production function, α is the capital-labor-mix and σ is the elasticity of substitution

$$\sigma = \frac{1}{1-\gamma} \qquad (2)$$

The national income identity:

$$Y = C + I \qquad (3)$$

$$C = c_1 Y \qquad (4)$$

Capital accumulation :

$$\dot{K} = I - \delta K \qquad (5)$$

where δ is the rate of depreciation.

Additionally, we have the equation with the wages, profit and depreciation, i.e., the income distribution:

$$Y = wL + (r+\delta)K \qquad (6)$$

Under maximum profit condition the marginal products equal the factor prices:

$$\frac{\partial Y}{\partial K} = r + \delta \qquad (7)$$

and

$$\frac{\partial Y}{\partial L} = w \qquad (8)$$

Remark:
For an arbitrary value of $c_1 \in (0,1)$, if a solution to the system exists, then that solution is unique and stable.

3 The implementation of TFP growth in a CES production function (MUGT)

Basic idea in the MUGT framework is that we separate the sources of growth into two components:

- Growth in income due to technical progress only
- Growth in income due to capital accumulation

Consider a homogeneous degree 1, CES-type production function. Because the production function is homogeneous degree 1, we can write the production function in the intensive form with a technical progress term

$$y = \xi_{TFP} y_0 \left[\alpha_1 \left(\frac{k}{k_0}\right)^{\gamma} + (1-\alpha_1)\right]^{1/\gamma} \quad (9)$$

$$\alpha_1 = \frac{\alpha_0}{\xi_{TFP}{}^{\gamma}} \quad (10)$$

$$\sigma = \frac{1}{1-\gamma} \quad (11)$$

y denotes income per capita or per hour worked—that is, labor productivity—while k represents capital deepening.

The production function is described in its base point (y_0, k_0).

The Modern Universal Growth Theory (MUGT) introduces a structural distinction between capital deepening and technical progress by placing ξ_{TFP} growth outside the production function.

ξ_{TFP} expresses the factor of total factor productivity, which is the increase of productivity by technical progress only, expressed by moving from point (k_0, y_0) to point $(k_1, y_1) = (k_0, \xi_{TFP} y_0)$. If TFP increases e.g. 2 % then $\xi_{TFP} = 1.02$ (fig. 1).

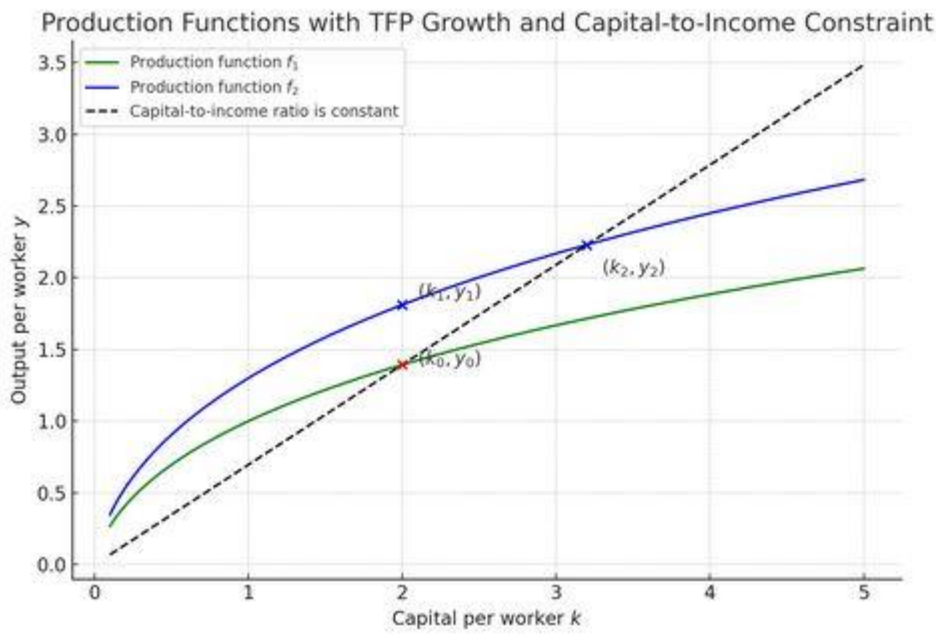

Fig. 1 Technical Progress in the MUGT Framework

In contradiction of its misleading name, the factor ξ_{TFP} expresses only technical progress due to innovations, education, labor improvement, capital improvement, etc. and not through capital increase.

The total increase in productivity ξ_g on a BGP, incorporating both technical progress and capital deepening, is as follows:

$$\xi_g = \left(\frac{{\xi_{TFP}}^{\gamma} - \alpha_0}{1-\alpha_0}\right)^{1/\gamma} \qquad (12)$$

In literature the symbol g is used for the increase of productivity, here it is limited to a BGP.

$$g = \frac{\Delta\xi_g}{\xi_g} \qquad (13)$$

The capital to income ratio

$$\beta = \frac{k}{y} \qquad (14)$$

The total increase of productivity due to TFP and capital increase is ξ_g. When we move from point (k_0, y_0) on f_1 to point $(k_1, y_1) = (k_0, \xi_{TFP} y_0)$ and from there along the new production function f_2 to point $(k_2, y_2) = (\xi_g k_0, \xi_g y_0)$, then the capital to income ratio β_2 will obviously remain the same as the original β_0 where we started with (fig. 1).

In the same way you can show that the new production function in its new base point $(k_2, y_2) = (\xi_g k_0, \xi_g y_0)$ has the same α_0 as the production function we started with. Of course, by solving the system under maximum profit conditions.

The new production function in point (k_2, y_2) is

$$y = y_2 \left[\alpha_0 \left(\frac{k}{k_2}\right)^{\gamma} + (1-\alpha_0)\right]^{1/\gamma} \qquad (15)$$

which shows that with this implementation of technical progress the condition of a BGP is fulfilled. The new production function is described in its new basepoint and has the same capital-labor-mix α_0. For a detailed proof, see De la Fonteijne (2025).

It is convenient to write c_1 as a function of ξ_g or g when using the intensive form

$$c_1(g) = 1 - \beta(\delta + g) \qquad (16)$$

In the case of no growth $g = 0$ and a system in equilibrium, the capital to income ratio β_0 is

$$\beta_0 = \frac{1 - c_1(g=0)}{\delta} \qquad (17)$$

This formulation allows the model to generate a BGP with constant capital to income ratio $\beta = \beta_0$ even under time-varying growth, thereby preserving both the structure and economic interpretation of the production function.

The next section turns to the broader implications of this implementation for growth theory as a whole.

4 Implications of the MUGT Framework for CES-Based Growth Models

Within the MUGT framework we employ three core parameters: the technical growth factor ξ_{TFP}, the capital-labor-mix α and the elasticity of substitution σ. When dealing with growth dynamics, it becomes essential to adjust the capital-labor-mix α with the factor $\left(\frac{1}{\xi_{TFP}}\right)^{\gamma}$ in $\alpha_1 = \alpha_0 \left(\frac{1}{\xi_{TFP}}\right)^{\gamma}$ to make a Balanced Growth Path (BGP) with constant capital to income ratio and constant capital-labor-mix possible.

In our view, this adjustment is not just a methodological choice, it is the only consistent way to incorporate technical progress. For formal derivations and proof, see De la Fonteijne (2018,

> **Lemma**
> Within the MUGT framework, achieving a Balanced Growth Path (BGP) requires adapting the capital-labor mix parameter α_0, except in the Cobb-Douglas case. This adjustment is not a modeling choice, but a structural necessity arising from the logic of consistent growth modeling. While alternative formulations may exist, any economically meaningful implementation of technical progress in a CES-type (or similar) production function ultimately leads to the same requirement.

2024). A key implication is that the initial capital-labor-mix α_0 must be actively adapted throughout the growth process. However, the precise mechanisms driving these adjustments remain largely unexplained.

The decision to opt for a final solution that maintains a constant capital-labor-mix might seem somewhat arbitrary, and in a certain way it is. On the other hand, it is the only viable choice possible if you require a BGP with a constant capital-labor-mix.

Relaxing this requirement opens up alternative growth trajectories. One could, for instance, define $\alpha_1 = \alpha_2 \left(\frac{1}{\xi_{TFP}}\right)^{\gamma}$ to express total change due to technical progress, where the difference between α_2 and α_0 is varying around zero, mirroring real-world scenarios. It results in a BGP with constant capital to income ratio and varying capital-labor-mix.

Moreover, if predictive or policy tools exist to anticipate or influence the evolution of either the capital-labor-mix α or the elasticity of substitution, you can leverage these insights to shape the economic growth trajectory accordingly.

For foundational perspectives, see Jones (2013) and Acemoglu (2009).

For further implications of the MUGT framework, refer to De la Fonteijne (2023).

The MUGT framework challenges the validity of an estimated 40% of the existing growth theory, which will need to be reconsidered or reformulated in light of the theoretical inconsistency it resolves.

5 The Hidden Adjustment of the Capital-Labor Mix α in Harrod's Model: A Comparison with the MUGT

To re-express Solow's growth process through the lens of the MUGT, we separate the sources of growth into two components, as done earlier:

- Growth in income due to technical progress only
- Growth in income due to capital accumulation

We begin with the standard per capita CES production function, incorporating capital- and labor-augmenting technical progress

$$y = y_0 \left[\alpha_0 {\xi_K}^{\gamma} \left(\frac{k}{k_0}\right)^{\gamma} + (1-\alpha_0){\xi_{LT}}^{\gamma}\right]^{1/\gamma} \quad (18)$$

To align this with the MUGT formulation, we normalize the technical progress term and rewrite the production function in the form

$$y = y_0 \xi_{TFP} \left[\alpha_1 \left(\frac{k}{k_0}\right)^{\gamma} + (1-\alpha_1)\right]^{1/\gamma} \quad (19)$$

where

$$\xi_{TFP} = [\alpha_0 {\xi_K}^{\gamma} + (1-\alpha_0){\xi_{LT}}^{\gamma}]^{1/\gamma} \quad (20)$$

and

$$\alpha_1 = \frac{\alpha_0 {\xi_K}^{\gamma}}{\alpha_0 {\xi_K}^{\gamma} + (1-\alpha_0){\xi_{LT}}^{\gamma}} = \alpha_0 \left(\frac{\xi_K}{\xi_{TFP}}\right)^{\gamma} \quad (21)$$

This reveals that technical progress—when modeled via separate capital and labor augmenting terms—implicitly alters the capital-labor mix α, even though this adjustment is not visible in the formulation. In the special case of Harrod-neutral growth ($\xi_K = 1$), this simplifies to:

$$\alpha_1 = \alpha_0 \left(\frac{1}{\xi_{TFP}}\right)^{\gamma} \quad (22)$$

This matches exactly the adjustment required in the MUGT to achieve a BGP, revealing that Harrod's approach—despite its apparent simplicity—carries a hidden shift in the capital-labor-mix. While Harrod interprets technical progress as purely labor-augmenting, the formula in fact embeds joint capital and labor contributions, obscured by the shift in the capital-labor-mix parameter α.

The corresponding TFP term in this special case becomes:

$$\xi_{TFP} = [\alpha_0 + (1 - \alpha_0){\xi_{LT}}^{\gamma}]^{1/\gamma} \qquad (23)$$

ξ_{TFP} expresses technical progress of both capital and labor at the initial capital level $k = k_0$.

Even if ξ_{LT} is varying in time (and so is ξ_{TFP}) this will result in a BGP.

This explains the relationship between the MUGT and Harrod.

We turn to the general case where both ξ_K and ξ_{LT} will vary in time.

Also, in this case it is possible to create a growing economy with constant capital to income ratio, albeit that the capital-labor-mix is additional adapted with a term ${\xi_K}^{\gamma}$ in

$$\alpha_1 = \alpha_0 \left(\frac{\xi_K}{\xi_{TFP}}\right)^{\gamma} \qquad (24)$$

You can use ξ_K to move to another level of the capital-labor mix. Although the process of adapting the capital-labor-mix is not understood yet.

It is important to note that this framework is not limited to CES functions. The principle of capital-labor-mix adjustment under technical progress holds more generally for any well-defined production function with similar structure.

This comparison reveals that traditional implementations of technical progress—including Harrod-neutral models—implicitly rely on hidden adjustments to the capital-labor mix (except for the Cobb-Douglas case). These adjustments are not derived from first principles but emerge as side effects of the functional form.

This reinforces the need for adopting the Modern Universal Growth Theory (MUGT), which offers a consistent framework for modeling growth. The MUGT not only avoids theoretical paradoxes but also provides mathematically rigorous formulations. More importantly, it ensures that all growth parameters—such as the capital-labor-mix α, total factor productivity ξ_{TFP}, and the elasticity of substitution σ—retain clear and sound economic interpretations. This makes the MUGT a more transparent and robust foundation for future growth modeling.

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