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**Abstract**

Existing literature does not capture efficiency losses on the dynamic adjustment path of crime control market from initial to final equilibrium after a shock in order to formulate an optimal crime control policy. Furthermore, number of public service units and crime control rate are major determinants of crimes controlled in a society, and a policy without taking into consideration such vital determinants cannot ensure adjustment of number of crimes controlled as a result of cost movement in desired time, which may lead to extra efficiency losses than those envisaged during policy formulation for an optimal level of crime control in a society. This article designs a comprehensive optimal crime control policy mechanism by modeling a three dimensional crime control system in society capturing number of public service units, crime control rate, and cost, while taking into account efficiency losses during adjustment of crime control market, crime control rate and number of public service units in addition to those which result due to movements from initial to final equilbriums. (JEL A14, H19, H83)

Keywords: Crime, Optimal Policy, Adjustment Path, Equilibrium, Coordination

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## 1 Introduction

When a society is undergoing changes, i.e., cultural, political, and socioeconomic, etc., laws change and may bring certain behaviors under the definition and treatment of a crime, and similarly may free society regarding some behaviors which might have been considered as criminal activities before. Through bringing certain behaviors under the definition and treatment of a crime, a society tends to reduce the anticipated damage. The state tends to reduce the individual liberties after weighing the costs and benefits of those liberties as being potentially harmful to others. State also plays a role in creating public awareness through sharing and presenting data regarding certain activities as causing harm to others. Public reaction to certain activities leads the state to consider the use of law in providing an incentive mechanism to public to behave in a certain manner and shape social norms.

Existing literature does not capture efficiency losses on the dynamic adjustment path of crime control market from initial to final equilibrium after a shock in order to formulate an optimal crime control policy. Furthermore, number of public service units and crime control rate are major determinants of crimes controlled in a society, and a policy without taking into consideration such vital determinants cannot ensure adjustment of number of crimes controlled as a result of cost movement in desired time, which may lead to extra efficiency losses than those envisaged during policy formulation for an optimal level of crime control in a society. This article designs a comprehensive optimal crime control policy mechanism by modeling a three dimensional crime control system in society capturing number of public service units, crime control rate, and cost, while taking into account efficiency losses during adjustment of crime control market, crime control rate and number of public service units in addition to those which result due to movements from initial to final equilbriums.

Research on crime has been pretty extensive, and a lot of theories have been in field from the perspective of sociolgy, biology, economics of crime, etc. Danziger and Wheeler (1975) focuses on the process by which relative welfare comparisons produce one type of conflict–crime. It is hypothesized that shifts toward a greater degree of inequality in the distribution of income and increases in the absolute level of income when the distribution is constant are both accompanied by more crime. Rubin (1978) presents the economic theory of crime as an application of the labor market theory to criminal behavior. Burton Jr and Cullen (1992) attempts to identify issues that allow for a test of strain theory, and prompts criminologists to explore the potentially criminogeists effects of circumstances leading to stress. Benson, Kim and Rasmussen (1994) estimates the impact of deterrence according to theory that deterrence created by police affects crime rate. Bowles and Garoupa (1997) extends model of crime by Becker by allowing for collusion between the criminal and an arresting officer where the cost is borne by police. Agnew (1999) relies on Agnew's general strain theory to explain differences in crime rates across community. Van Winden and Ash (2012) takes into account the behavioral approach, which proposes a decision model comprising cognitive and emotional decision systems. Antonaccio, Smith and Gostjev (2015) confirms that additional clarifications of the concept of anomic strain may be promising. Dippel and Poyker (2019) shows through an empirical study that the private prisons have an impact on criminal sentencing that public ones do not. Bacher-Hicks, Billings and Deming (2019) estimates the net impact of school discipline on student achievement, educational attainment and adult criminal activity. Using data from the New York City Police Department's Stop-and-Frisk program, Lehrer and Lepage (2020) evaluates the impact of a specific terrorist attack threat from Al Qaeda on policing behavior in New York City. Chassang, Del Carpio, Kapon et al. (2020) studies the extent to which divide-and-conquer enforcement strategies can help select a high compliance equilibrium in the presence of realistic compliance frictions. Devi and Fryer Jr (2020) provides an empirical examination of the impact of federal and state "Pattern-or-Practice" investigations on crime and policing. For investigations that were not preceded by "viral" incidents of deadly force, investigations, on average, led to a statistically significant reduction in homicides and total crime. In stark contrast, all investigations that were preceded by "viral" incidents of deadly force have led to a large and statistically significant increase in homicides and total crime.

A comprehensive optimal crime control policy mechanism has been designed by modeling a three dimensional crime control system after bifurcating it into two dimensional panels. In one panel, public service (in terms of number of crimes controlled) and cost of per unit crime controlled is considered (this is a traditional price-quantity panel for depicting supply and demand in a market); whereas, in the other panel, number of public service units and crime control rate are considered. The model in the first panel is based on a market for public service in terms of number of crimes controlled. Four market agents exist in the market, i.e., public sector/government as a supplier of service, consumer of service, government in the role of deciding the cost charged to the private sector for crime control through taxes (tax policy maker) and allocation of budget to the public offices for crime control, and government for exercising crime control policy. The government influences the cost through its roles as a tax policy maker/budget allocator and that of exercising the crime control policy, however, takes the cost as given in the role of a service supplier. For the model in the other panel, there are three types of infinitely-lived agents: public and private sectors which demand a certain number of public service units against each crime control rate, a representative –or a unit mass of– public service units who control crime, and public sector as one entity who supplies certain number of public service units against each crime control rate.

In order to capture a bigger picture of the crime control system in society, we present dynamics and equilibria in panels A and B of figure 1. In order to formulate an optimal crime control policy, the government needs to devise policies for both panels, i.e., A and B. For panel B, this paper develops a dynamic crime control model, and based on that, derives an optimal crime control policy by minimizing the efficiency loss, i.e., excessive or inadequate crime control public service in the final equilibrium as compared to the initial one, i.e., before the implementation of the crime control policy; as well as during the adjustment of the crime control public service system subject to the government's crime control policy cost constraint. The results from panel B decide the constraint in panel A, i.e., an increase/decrease in number of crimes controlled per unit time. For panel A, this paper develops a theory and designs a dynamical model for an optimal number of public service units (a unit can be defined as a cop, a police station, an investigator, or a group of investigators, etc.) and the crime control rate in society based on that theory. For panel A, the optimality is in the sense of having maximum gains possible, i.e., minimizing the social damage in terms of inadequate/excessive number of public service units in initial equilibrium; as well as the social loss in terms of excessive or inadequate number of units on the dynamic adjustment path before arriving at the final equilibrium, subject to a certain increase/decrease in number of crimes controlled per unit time (obtained by deriving an optimal policy for panel B). As soon as a policy to change the number of public service units and crime control rate is adopted, it does not lead to an equilibrium immediately, and rather the number of public service units follow a dynamic adjustment path to come to a point where the number of units demanded in society becomes equal to the number supplied due to both public and private sector's efforts. This paper considers the social damage in the initial equilibrium as well as on the dynamic adjustment path from initial to the final equilibrium after implementation of a panel A policy to find an optimal policy, i.e., to minimize the social damage subject to a certain increase/decrease in number of crimes controlled per unit time.

The natural course of occurrence of panel A, and panel B, and hence equilibria in both panels is simultaneous. There are a certain number of public service units to control crime in a society and they control crime at a certain rate, i.e., the upward sloping curve in panel A; and the areas of rectangles by drawing perpendiculars from points on the supply curve in panel A to $x$, and $y$-axes correspond to the horizontal coordinates or abscissas on the supply curve in panel B. Similarly, society desires/demands a certain number of public service units to control crime against a crime control rate, i.e., the downward sloping curve in panel A; and the areas of rectangles by drawing perpendiculars from points on the demand curve of society in panel A to $x$, and $y$-axes correspond to the abscissas on the demand curve in panel B. The demand and supply of public service measured in terms of number of crimes controlled determines the cost per unit crime controlled in panel B. However, for the government's policy formulation, the government has a cost constraint which must be satisfied for an optimal level of crimes controlled in society, therefore, the natural order for an optimal policy formulation for the government is to find an optimal level of number of crimes controlled subject to the cost constraint, and then for an optimal control to keep the number at the

optimal level, devise a policy for an optimal number of public service units and the crime control rate subject to the constraint determined by the optimal policy in panel B, i.e., the change in number of crimes controlled per unit time. For panel B, the existing literature on crime control policy does not take into consideration the efficiency losses/gains on the dynamic adjustment path as well as the final equilibrium in comparison with the initial equilibrium in the crime control system. When the government exercises a crime control policy, the government's cost as a supplier of public service jumps to the pre-policy cost plus the per crime control cost incurred as a result of the policy, which affects the public service supply in society and disrupts the supply-demand equilibrium. Supply and demand of public service measured as the number of crimes controlled along with the cost adjust over time to bring final equilibrium. The adjustment mechanism of cost is based on the premise that when the crime control system goes out of equilibrium due to crime control policy, the consumers and suppliers of public service do not have coordinated decisions at the current cost. While deriving an optimal crime control policy, it is important to have efficiency considerations both during the adjustment of the system as well as in the final equilibrium as compared to the initial one. For panel B, a dynamic crime control model has been developed and based on that, an optimal crime control policy has been derived by minimizing the efficiency loss, i.e., excessive or inadequate public service in the final equilibrium as compared to the initial one, i.e., before the implementation of the crime control policy; as well as during the adjustment of the demand and supply subject to the government's crime control policy cost constraint.
The remainder of this paper is organized as follows: Section 2 presents the model for panel B. Section 3 solves the model for a crime control policy for panel B. Section 4 presents a dynamic optimal crime control policy for panel B. Section 5 demonstrates how individual components of panel A are joined together to form a dynamic crime control model for panel A. Section 6 provides a solution to model A with a crime control policy. Section 7 derives a dynamic optimal crime control policy for panel A. Section 8 presents the summary of findings and conclusion. Appendix provides the detailed mathematical steps in derivations in the text.

## 2 The Model-Panel B

The model is based on a market for public service in terms of number of crimes controlled. Suppose that the market is in equilibrium in the initial condition. Four market agents exist in the market, i.e., public sector/government as a supplier of service, consumer of service, government in the role of deciding the cost charged to the private sector for crime control through taxes (tax policy maker) and allocation of budget to the public offices for crime control, and government for exercising crime control policy. The government influences the cost through its roles as a tax policy maker/budget allocator and that of exercising the crime control policy, however, takes the cost as given in the role of a service supplier. If the number of crimes controlled changes due to an exogenous shock, the cost cannot jump on its own to bring the public service market in a new equilibrium. Government as a tax policy maker changes the cost/taxes in the public interest to bring the new equilibrium after making the cost follow an adjustment path. In the final equilibrium, it is optimal for tax policy maker to stay put and not to change the cost/taxes further. Supplier of public service receives cost of crime control from another wing of government which relies on tax collection; the tax policy makers keep track of the supply and demand of crime control public service and raise/lower the cost to bring the public service market in equilibrium. The supplier of public service maximizes the public benefit; the tax policy maker maximizes the public benefit as a difference of the public utility due to crime control, and the cost of provision of public service through tax collection subject to the constraints; the consumer maximizes the profit/benefit/utility depending on the type of consumer, i.e., whether the consumer cares more about valuables, security of life, etc.

The mechanism regarding the cost/taxes adjustment is contingent upon the premise that at the current cost, suppliers' and consumers' decisions are not coordinated when an exogenous shock happens to the public service market and pushes it out of equilibrium. Let us consider the following example as an illustration of the working of this market: A public service market is in equilibrium as a starting point/initial condition. An exogenous positive supply shock will result into an expansion of number of crimes controlled as the new total supply does not match the demand of consumers at the current cost, which will be reflected into an increase in cumulative number of crimes controlled. The tax policy maker will reduce the tax rate so that the public service supplier finds it optimal to supply a lower level of public service in terms of number of crimes controlled. A final equilibrum will eventually result with a higher number of crimes controlled and a lower cost/tax rate than those in the initial equilibrium. The equilibrium is defined as follows:

(i) The supplier of public service maximizes the public utility/benefit; the consumer maximizes the profit/benefit/utility; and the tax policy maker maximizes the public benefit as a difference of the public utility due to crime control, and the cost of provision of public service through tax collection subject to the constraints as mentioned in Section 2 in detail.

(ii) The demand of number of crimes controlled equals the supply when the public service market is in equilibrium, and the cumulative number of crimes controlled does not change.

Section 3 mentions the Routh–Hurwitz stability criterion, i.e., the necessary and sufficient equilibrium condition for a linear dynamical system. The tax policy maker is a price taker (takes the cost/tax rate as given) under public service market equilibrium. In an out of equilibrium scenario of the market, the tax policy maker has an incentive to change the cost/tax rate during the adjustment process until the new equilibrium arrives, where the tax policy maker again becomes a price taker. When government exercises a crime control policy, the cost/tax rate adjusts rather than jumping to a new value and gradually brings the new equilibrium. The basis of the adjustment of cost/tax rate is endogenous decision making by public sector/government, the consumer of public service, and the tax policy maker as follows: When public service market is in equilibrium, the number of crimes controlled is equal to the number demanded in each time period. If government exercises an expansionary public service policy, i.e., increases the number of crimes controlled, a wedge is created between the number controlled and the number demanded. If it was possible for the suppliers of public service and the tax policy makers change the service supply and the cost/tax rate immediately; and the tax policy makers had known the new demand and supply patterns after the change in the cost/tax rate, the tax policy makers would set a tax rate such that the public benefit minus their cost through taxes would get maximized and the public service market would clear. This information, however, is not known to the tax policy makers, therefore, they change the tax rate based on their best guess/estimates about the new market scenario, which drives the market to the new equilibrium. When the tax policy maker decreases the cost/tax rate, the supplier supplies a lower quantity of service than before. The tax policy maker will keep decreasing the tax rate until the new equilibrium arrives about which they get an idea through the continuously decreasing number of crimes until eventually a new equilibrium arrives with some efficiency losses during the adjustment. The over employment of resources available with the public service provider/supplier to control number of crimes excessive of the number demanded is the efficiency loss as a result of a crime control policy during the adjustment of the market, and the total loss is equal to the sum of the one during the adjustment period plus/minus the loss/gain in final equilibrium.

Mathematically speaking, the first order derivatives of the objective functions of all agents have been taken to maximize their objectives, and the individual equations are solved simultaneously to get a mathematical expression for their collective response. An assumption for simplification is that the final equilibrium is not too off from the pre-policy equilibrium; this implies that an assumption regarding linearization of demand and supply schedules is reasonable. Figure 2 depicts that linearization is a reasonable assumption for movement of an equilibrium from point $a$ to $b$, however, it does not seem to be reasonable to assume linearity of supply curve when the equilibrium moves from point $a$ to $c$, for which a non-linear dynamical system (beyond the scope of this paper) needs to be considered.

### 2.1 Tax Policy Maker/Budget Allocator (TPM/BA)

Tax policy maker/budget allocator decides the cost of per unit crime controlled after evaluating the existing scenario of demand, and supply of public service, and allocate budget to the public offices to supply public service to control crime. TPM/BA maintain data on cumulative number of uncontrolled crime, i.e., whether the number is increasing, decreasing or staying constant. If the cumulative number of uncontrolled crime does not vary, the public service market is in equilibrium, as neither supply nor demand changes. If the cumulative number of uncontrolled crime is rising, there must be a higher demand from the private sector to control more crimes than before. Similarly, if the number of uncontrolled crime is decreasing, the supply must be higher than the demand. The cumulative number of uncontrolled crime in a public service market is analgous to an inventory between supply and demand in a goods market. If the rate of supply and demand is the same, the inventory does not change. If inventory changes, that implies either a change in the supply rate, demand rate or both (different rates).

When the service supply gets a shift to the right while demand stays the same, the supply of service is higher than demand and the cost goes down in the new equilibrium. In the same manner, when the public service demand to control crime shifts to the right while supply stays the same, the cost goes up in the new equilibrium. This implies that summation of differences of supply and demand, i.e., $\Sigma(supply - demand)$ is inversely related to cost change, ceteris peribus. If demand of public service as well as supply both shift in a manner that $\Sigma(supply - demand)$ stays the same, the cost will also not change. The demand and the supply shocks are unified by the term $\Sigma(supply - demand)$, as both are in fact just affecting this $\Sigma$. To depict the model mathematically, the problem of TPM/BA has been considered as follows:

#### 2.1.1 Short-run Problem

In this section, the short-run problem (which means the TPM/BA's objective is myopic and is not doing dynamic optimization) of TPM/BA is considered as follows:

$$\Pi = U_r(r) - \varsigma_B(m_B(r, e_B)), \tag{1}$$

where

$\Pi$ = net social benefit,

$U_r(r)$ = social benefit due to public service of crime control,

$r$ = cost,

$m_B$ = cumulative no. of crimes controlled = $\Sigma(supply - demand)$, which is just $(supply - demand)$ for one time period.

$e_B$ = factors influencing $m_B$ other than cost including the budget allocated to the public service providers which might be different from the cost charged to the private sector in terms of taxes.

$\varsigma_B(m_B(r, e_B))$ = social cost to control crime as a function of $m_B$ (increasing in $m_B$).

Taking the derivative of eq. (1) with respect to cost, we get:

$$U_r'(r) - \varsigma_B'(m_B(r, e_B))m_{B1}'(r, e_B) = 0. \tag{2}$$

If the supply curve shifts to the right on account of a decreased cost per unit crime controlled to the public service providers, say due to an improved computerized database of criminals, the number of crimes controlled is no more in equilibrium. As the number of crimes controlled is higher than before at the current value of $r$, the term $\varsigma'_B(m_B(r, e_B))$ is higher at the existing $r$ for TPM/BA. As the term, $m'_{B1}(r, e_B)$ is a function of $r$, therefore, it is the same as before because the value of $r$ has not yet changed. The implication is that at the existing value of $r$, the TPM/BA now faces the following inequality:

$$\frac{\partial \Pi}{\partial r} = U'_r(r) - \varsigma'_B(m_B(r, e_B))m'_{B1}(r, e_B) < 0, \tag{3}$$

which implies that in order to have an extra crime controlled, the TPM/BA must decrease the cost to the private sector in the form of taxes after supply shock to satisfy the net social benefit maximization condition. Now the short term gains accrued from a reduced marginal cost are being reaped by the public service provider, as the marginal cost of public service has decreased but their market cost is the same as before until changed by the TPM/BA in the next budget allocation. A plot of net social benefit maximizing pairs of $m_B$ and the respective cost is a downward sloping curve with cost on the $y$-axis and $m_B$ on the $x$-axis. The concept is analogous to *demand* and *supply curves*.

### 2.1.2 Dynamic Problem

This section discusses the dynamic problem of TPM/BA. Present discounted value of future stream of net social benefits are maximized in a dynamic environment, and the present value at time zero is given below:

$$V(0) = \int_0^\infty \left[U_r(r) - \varsigma_B(m_B(r, e_B))\right] e^{-\sigma t} dt, \tag{4}$$

$\sigma$ denotes the discount rate. $r(t)$ is the *control variable* and $m_B(t)$ the *state variable.* Maximization problem is as follows:

$$\underset{\{r(t)\}}{Max} V(0) = \int_0^\infty \left[U_r(r) - \varsigma_B(m_B(r, e_B))\right] e^{-\sigma t} dt,$$

subject to the constraints that

$\dot{m_B}(t) = m'_{B1}(r(t), e_B(r(t), z_B))\dot{r}(t) + m'_{B2}(r(t), e_B(r(t), z_B))e'_{B1}(r(t), z_B)\dot{r}(t)$ (state equation, describing how the state variable changes with time; $z_B$ are exogenous factors),

$m_B(0) = m_{Bs}$ (initial condition),

$m_B(t) \geq 0$ (non-negativity constraint on state variable),

$m_B(\infty)$ free (terminal condition).

The current-value Hamiltonian is as follows:

$$\widetilde{H} = U_r(r\,(t)) - \varsigma_B(m_B(r(t), e_B(r(t), z_B))) + \mu_B(t)\dot{r}(t) \left[ \begin{array}{c} m'_{B1}(r(t), e_B(r(t), z_B)) + m'_{B2}(r(t), e_B(r(t), z_B)) * \\ e'_{B1}(r(t), z_B) \end{array} \right]. \tag{5}$$

Now the maximizing conditions are as follows:
$(i)$ $r^*(t)$ maximizes $\widetilde{H}$ for all $t$: $\frac{\partial \widetilde{H}}{\partial r} = 0$,
$(ii)$ $\dot{\mu_B} - \sigma \mu_B = -\frac{\partial \widetilde{H}}{\partial m_B}$,
$(iii)$ $\dot{m_B}^* = \frac{\partial \widetilde{H}}{\partial \mu_B}$ (this just gives back the state equation),
$(iv)$ $\lim_{t \to \infty} \mu_B(t) m_B(t) e^{-\sigma t} = 0$ (the transversality condition).
The first two conditions are as follows:

$$\frac{\partial \widetilde{H}}{\partial r} = 0, \tag{6}$$

and

$$\dot{\mu_B} - \sigma \mu_B = -\frac{\partial \widetilde{H}}{\partial m_B} = \varsigma'_B(m_B(r(t), e_B(r(t), z_B))). \tag{7}$$

In equilibrium, $\dot{r}(t) = 0$, and the expression $\frac{\partial \widetilde{H}}{\partial r}$ boils down to the following (see appendix):

$$U'_r(r\,(t)) - \varsigma'_B(m_B(r(t), e_B(r(t), z_B))) \left\{ \begin{array}{c} m'_{B1}(r(t), e_B(r(t), z_B)) + m'_{B2}(r(t), e_B(r(t), z_B)) * \\ e'_{B1}(r(t), z_B) \end{array} \right\} = 0.$$

If supply curve shifts to the right, then the number of crimes controlled is higher at the existing cost, and the term $\varsigma'_B(m_B(r(t), e_B(r(t), z_B)))$ is higher at the existing cost at that time. The term multiplying $\varsigma'_B(m_B(r(t), e_B(r(t), z_B)))$, i.e., $m'_{B1}(r(t), e_B(r(t), z_B)) + m'_{B2}(r(t), e_B(r(t), z_B)) e'_{B1}(r(t), z_B)$ is a function of cost and has not changed as the cost is the same as before. Therefore, the TPM/BA now faces the following inequality at the existing cost:

$$\frac{\partial \widetilde{H}}{\partial r} < 0.$$

The TPM/BA must decrease the cost for satisfying the net social benefit optimization condition after the shock. This implies that there is a negative relationship between the cumulative number of crimes controlled in society and the cost. If the rate of supply of public service in terms of number of crimes controlled is equal to the demand rate, the number of crimes controlled is in equilibrium. If a difference of a finite magnitude comes into force between the supply and demand rates, and the public and the private sector do not react to a change in the cost caused by a difference in the supply and demand rates, the cost will continue changing until the saturation point of society comes. The response of TPM/BA can be depicted by the following formulation:

$$\begin{aligned}
\textit{Cost rate change} &\propto \textit{change in cumulative no. of controlled crime.}\\
R &= \textit{cost rate change.}\\
M_B &= m_B - m_{Bs} = \textit{change in cumulative no. of controlled crime,}\\
m_B &= \textit{cumulative no. of controlled crime at time } t,\\
m_{Bs} &= \textit{cumulative no. of controlled crime in steady state equilibrium.}\\
\textit{Input} - \textit{output} &= \frac{dm_B}{dt} = \frac{d(m_B - m_{Bs})}{dt} = \frac{dM_B}{dt},\\
\text{or } M_B &= \int (\textit{input} - \textit{output})\, dt.\\
\textit{Cost rate change} &\propto \int (\textit{supply rate} - \textit{demand rate})\, dt, \text{ or}\\
R &= -K_m \int (\textit{supply rate} - \textit{demand rate})\, dt,
\end{aligned}$$

where $K_m$ is the proportionality constant. A negative sign indicates that when $(\textit{supply rate} - \textit{demand rate})$ is positive, $R$ is negative, i.e., the cost decreases. The above expression can also be written as:

$$\int (\textit{supply rate} - \textit{demand rate})\, dt = -\frac{R}{K_m}, \text{ or}$$

$$\int (w_{Bi} - w_{B0})\, dt = -\frac{R}{K_m}, \tag{8}$$

$$\begin{aligned}
w_{Bi} &= \textit{supply rate},\\
w_{B0} &= \textit{demand rate},\\
K_m &= \textit{dimensional constant.}
\end{aligned}$$

When $t = 0$, $\textit{supply rate} = \textit{demand rate}$, i.e., public service market is in equilibrium and eq. (8) can be expressed as:

$$\int (w_{Bis} - w_{B0s})\, dt = 0. \tag{9}$$

The subscript $s$ denotes steady state equilibrium and $R = 0$ in steady state. Subtracting eq. (9) from eq. (8), we get:

$$\int (w_{Bi} - w_{Bis})\, dt - \int (w_{B0} - w_{B0s})\, dt = -\frac{R}{K_m}, \text{ or}$$

$$\int (W_{Bi} - W_{B0})\, dt = -\frac{R}{K_m}, \tag{10}$$

$$\text{where } w_{Bi} - w_{Bis} = W_{Bi} = \textit{change in supply rate},$$
$$w_{B0} - w_{B0s} = W_{B0} = \textit{change in demand rate.}$$

$R$, $W_{Bi}$ and $W_{B0}$ are deviation variables, i.e., deviation from steady state equilibrium and have zero initial values. Eq. (10) can also be expressed as:

$$R = -K_m \int W_B dt = -K_m M_B, \tag{11}$$

where $W_B = W_{Bi} - W_{B0}$. If $R$ gets a jump as a result of some factor other than a change in cumulative number of crimes controlled, that is another input which can be added to eq. (11) as follows:

$$R = -K_m \int W_B dt + E_B = -K_m M_B + E_B. \tag{11a}$$

There can also be an exogenous shock in cumulative number of crimes controlled other than the feedback of cost.

### 2.2 Private Sector/Consumer of Public Service

The private sector/people living in a society maximize the present discounted value of the future stream of net benefits, and their present value at time zero is as follows:

$$V(0) = \int_0^\infty \left[Z_p\left(n_p(t)\right) - \varsigma_p(r\left(n_p\left(t\right)\right))\right] e^{-r_p t} dt. \tag{12}$$

$Z_p\left(n_p\right)$ is a concave downward (decreasing in slope) increasing function of the number of crimes controlled, the higher the number, the higher the private sector's utility. $\varsigma_p(r\left(n_p\right))$ is the cost to the private sector for consumption of public service to control crime, the higher the number of crimes controlled, the higher is the cost. The cost curve with respect to $r(t)$ is concave upward, i.e., increasing in slope.

$r_p$ denotes the discount rate. $n_p(t)$ is the *control variable*, and $r(t)$ is the *state variable*. The maximization problem can be written as

$$\underset{\{n_p(t)\}}{Max}\, V(0) = \int_0^\infty \left[Z_p\left(n_p(t)\right) - \varsigma_p(r\left(n_p\left(t\right)\right))\right] e^{-r_p t} dt,$$

subject to the constraints that:

$\dot{r}(t) = r'(n_p\left(t\right))\dot{n}_p\left(t\right)$ (state equation, describing how the state variable changes with time),

$r(0) = r_s$ (initial condition),

$r(t) \geq 0$ (non-negativity constraint on state variable),

$r(\infty)$ free (terminal condition).
The current-value Hamiltonian for this case is

$$\widetilde{H} = Z_p\left(n_p(t)\right) - \varsigma_p(r\left(n_p\left(t\right)\right)) + \mu_p(t)\ r'(n_p\left(t\right))\dot{n}_p\left(t\right). \tag{13}$$

Now the maximizing conditions are as follows:
$(i)$ ${n_p}^*(t)$ maximizes $\widetilde{H}$ for all $t$: $\frac{\partial \widetilde{H}}{\partial n_p} = 0$,
$(ii)$ $\dot{\mu_p} - r_p\mu_p = -\frac{\partial \widetilde{H}}{\partial r}$,
$(iii)$ $\dot{r}^* = \frac{\partial \widetilde{H}}{\partial \mu_p}$ (this just gives back the state equation),
$(iv)$ $\lim\limits_{t\to\infty} \mu_p(t) r(t) e^{-r_p t} = 0$ (the transversality condition).
The first two conditions can be expressed as follows:

$$\frac{\partial \widetilde{H}}{\partial n_p} = Z'_p\left(n_p(t)\right) - \varsigma'_p(r\left(n_p\left(t\right)\right))\ r'\left(n_p\left(t\right)\right) + \mu_p(t)\ r''(n_p\left(t\right))\dot{n}_p\left(t\right) = 0. \tag{14}$$

and

$$\dot{\mu_p} - r_p\mu_p = -\frac{\partial \widetilde{H}}{\partial r} = \varsigma'_p(r\left(n_p\left(t\right)\right)). \tag{15}$$

In equilibrium, $\dot{n}_p\left(t\right) = 0$, and the expression $\frac{\partial \widetilde{H}}{\partial n_p}$ boils down to the following:

$$Z'_p\left(n_p(t)\right) - \varsigma'_p(r\left(n_p\left(t\right)\right))\ r'\left(n_p\left(t\right)\right) = 0.$$

If $r(t)$ goes up, the term $\varsigma'_p(r\left(n_p\left(t\right)\right))$ goes up, and the private sector now faces the following inequality:

$$\frac{\partial \widetilde{H}}{\partial n_p} < 0. \tag{16}$$

The demand of private sector regarding number of crimes to be controlled goes down to satisfy the dynamic optimization problem of the private sector. If change in demand of private sector is proportional to a change in per unit cost i.e., $R$, or linearization of the *demand curve* around the steady state equilibrium leads to the following:

$$W_d(t) = -K_d R(t), \tag{17}$$

where $W_d(t)$ is change in demand of the private sector with respect to the initial steady state equilibrium value. As it is a deviation variable, i.e., deviation from the steady state, it has a zero initial value. There is a time lag between the change in cost of per unit crime controlled and change in demand of number of crimes to be controlled, therefore, a dead time element needs to be incorporated in the above expression which results in the following:

$$W_d(t) = -K_d R(t - \tau_{d1}). \tag{18}$$

### 2.3 Public Service Provider/Supplier

Although there are a variety of public/private service providers to control crime, such as journalists, security guards, etc., however, we limit our focus to public authorities, such as police. Public authorities maximize the present discounted value of future stream of net benefits for society, and the present value at time zero is as follows:

$$V(0) = \int_0^\infty \left[Z_c\left(n_c(t)\right) - \varsigma_c(r\left(n_c(t)\right))\right] e^{-r_c t} dt. \tag{19}$$

$Z_c\left(n_c(t)\right)$ is the public service benefit for society, and increasing in number of crimes controlled, i.e., $n_c(t)$. $\varsigma_c(r\left(n_c(t)\right))$ is the public service cost to society, the higher the number of crimes controlled, the higher is the cost. The cost curve with respect to $r(t)$ is concave downward, i.e., decreasing in slope.

$r_c$ denotes the discount rate. $n_c(t)$ is the *control variable*, and $r(t)$ is the *state variable*. The maximization problem is as follows:

$$\underset{\{r(t)\}}{Max} V(0) = \int_0^\infty \left[Z_c\left(n_c(t)\right) - \varsigma_c(r\left(n_c(t)\right))\right] e^{-r_c t} dt,$$

subject to the constraints that:

$\dot{r}(t) = r'(n_c\,(t))\dot{n_c}(t)$ (state equation which describes how the state variable changes with time),

$r(0) = r_s$ (initial condition),

$r(t) \geq 0$ (non-negativity constraint on state variable), and

$r(\infty)$ free (terminal condition).

The current-value Hamiltonian is expressed as follows:

$$\widetilde{H} = Z_c\left(n_c(t)\right) - \varsigma_c(r\left(n_c(t)\right)) + \mu_c(t) r'(n_c(t))\dot{n_c}(t). \tag{20}$$

The maximizing conditions can be expressed in the following form:

$(i)$ ${n_c}^*(t)$ maximizes $\widetilde{H}$ for all $t$: $\frac{\partial \widetilde{H}}{\partial n_c} = 0$,

$(ii)$ $\dot{\mu_c} - r_c \mu_c = -\frac{\partial \widetilde{H}}{\partial r}$,

$(iii)$ $\dot{r}^* = \frac{\partial \widetilde{H}}{\partial \mu_c}$ (this just gives back the state equation), and

$(iv)$ $\lim_{t \to \infty} \mu_c(t) r(t) e^{-r_c t} = 0$ (the transversality condition).

The first two conditions can be expressed as follows:

$$\frac{\partial \widetilde{H}}{\partial n_c} = Z_c'\left(n_c(t)\right) - \varsigma_c'(r\left(n_c(t)\right))\ r'\left(n_c\,(t)\right) + \mu_c(t)\ r''(n_c\,(t))\dot{n_c}\,(t) = 0. \tag{21}$$

and

$$\dot{\mu_c} - r_c \mu_c = -\frac{\partial \widetilde{H}}{\partial r} = \varsigma_c'(r\left(n_c\left(t\right)\right)). \tag{22}$$

In equilibrium $\dot{n_c}\left(t\right) = 0$, and the expression $\frac{\partial \widetilde{H}}{\partial n_c}$ boils down to the following:

$$Z_c'\left(n_c(t)\right) - \varsigma_c'(r\left(n_c(t)\right))\; r'\left(n_c\left(t\right)\right).$$

If $r(t)$ goes up, the term $\varsigma_c'(r\left(n_c\left(t\right)\right))$ goes down, and the public service provider faces the following inequality:

$$Z_c'\left(n_c(t)\right) - \varsigma_c'(r\left(n_c(t)\right))\; r'\left(n_c\left(t\right)\right) > 0.$$

The number of crimes controlled by the public service provider will go up to satisfy the dynamic optimization problem. If the change in the number of crimes controlled by the public service provider is proportional to a change in $r(t)$, i.e., $R$, or linearization of the *supply curve* around the steady state equilibrium leads to the following:

$$W_m = -K_s\left(C_c - R\right) = -K_s \epsilon(t), \tag{23}$$

where $C_c$ is the change in the cost of public service provider per unit crime controlled, which might get affected due to various factors in society. The decision to change the number of crimes controlled depends on the difference of $R$, and $C_c$. $K_s$ is the proportionality constant; $W_m, C_c$ and $R$ are deviation variables. There is a time lag between the change in cost of per unit crime controlled and change in number of crimes controlled by the public service provider, therefore, a dead time element needs to be incorporated in the above expression which results in the following:

$$W_m = -K_s \epsilon(t - \tau_{d2}).$$

## 3 Solution of the Model-Panel B with a Crime Control Policy

Expressions from eqs. (11$a$), (18),and (23) respectively along with $\tau_{d1} = 0$ are as follows:

$$\begin{aligned}\frac{dR(t)}{dt} &= -K_m W_B(t),\\ W_d(t) &= -K_d R(t),\\ W_m &= -K_s\left(C_c - R\right),\end{aligned}$$

and

$$W_B(t) = W_m(t) - W_d(t),$$

if no exogenous demand or supply shock happens. $W_m(t)$ is number of crimes controlled. Combining the above expressions together, we can write:

$$\begin{aligned}\frac{dR(t)}{dt} &= -K_m \left[W_m(t) - W_d(t)\right] \\ &= -K_m \left[-K_s \left\{C_c(t) - R(t)\right\} + K_d R(t)\right] \\ &= -K_m \left[-K_s C_c(t) + (K_s + K_d) R(t)\right].\end{aligned}$$

Rearranging the above expression gives:

$$\frac{dR(t)}{dt} + K_m(K_s + K_d)R(t) = K_m K_s C_c(t). \tag{24}$$

The Routh-Hurwitz stability criterion (which provides a necessary and sufficient condition for stability of a linear dynamical system) for the above differential equation's stability is $K_m(K_s + K_d) > 0$; and as $K_m$,$K_s$, and $K_d$ are all defined as positive numbers, this criterion holds. This ensures that, away from a given initial equilibrium, every adjustment mechanism will lead to another equilibrium.

Suppose government reduces the per crime controlled cost of the public service provider by $B$, say through provision of some funds to buy an advanced technology for crime control, the above equation can be written as:

$$\frac{dR(t)}{dt} + K_m(K_s + K_d)R(t) = -K_m K_s B. \tag{25}$$

The solution is given by the following expression:

$$R(t) = -\frac{K_s B}{(K_s + K_d)} + \frac{K_s B}{(K_s + K_d)} e^{-[K_m(K_s+K_d)]t}. \tag{26}$$

$R(0) = 0$ (the initial condition), and $R(\infty) = -\frac{K_s B}{(K_s+K_d)}$ (the final steady state equilibrium value). In response to a policy, the per unit crime controlled cost dynamics depends on the parameters $K_s, K_d$, $K_m$ and $B$.

## 4 A Dynamic Optimal Crime Control Policy-Panel B

After a crime control policy, there are efficiency gains in post policy equilibrium in comparison with the initial/pre-policy equilibrium. However, there are also some efficiency losses during the adjustment period of crime control market until new equilibrium arrives. As soon as crime control policy is implemented, supply of public service expands, whereas the demand remains the same at the initial per unit cost, pushing the crime control market out of equilibrium. Now the adjustment of per unit cost begins to equalize the supply and demand to bring the new crime control market equilibrium. The post policy equilibrium cost is a function of demand and supply elasticities. From the previous section, change in supply as a result of a crime control policy is as follows:

$$W_m(0) = -K_s\left[C_c(0) - R(0)\right] = K_s B, \tag{27}$$
$$\text{as } R(0) = 0.$$

As a result of crime control policy, supply of public service goes up by $K_sB$. As demand does not change, therefore, cumulative number of crimes controlled also goes up by $K_sB$. Now the market is out of equilibrium, and the market forces push the crime control market toward a new equilibrium through the movement in per unit cost. As the per unit cost changes, the demand and supply of public service also change through feedback. If cumulative number of crime controlled goes up, it indicates a higher supply than demand and vice versa. There is no efficiency loss if crime control market is in equilibrium, and demand and supply are same. If market is out of equilibrium, either supply or demand is excessive at that point in time. Therefore, the total efficiency loss during the adjustment of crime control market is a sum of the differences in supply and demand at all points in time. Total efficiency loss for a crime control policy can be expressed as:

$$\begin{aligned} EL &= \int_{-\infty}^{0} W_m(\infty)dt + \int_{0}^{\infty} \left[W_m(t) - W_d(t)\right] dt \\ &= \int_{-\infty}^{0} W_m(\infty)dt + M_B(t). \end{aligned} \tag{28}$$

**With Crime Control Policy Cost Constraint:**

According to eq. (23), public service supply change due to change in per unit cost is:

$$W_m(t) = -K_s\left[C_c(t) - R(t)\right].$$

It can also be written as:

$$w_{nm}(t) - w_{im}(0) = -K_s\left[C_c(t) - R(t)\right],$$

where $w_{im}(0)$ is the initial public service supply, and $w_{nm}(t)$ is the new supply after government exercises crime control policy. $W_m(t) = w_{nm}(t) - w_{im}(0)$, as $W_m(t)$ is a deviation variable, i.e., deviation from the initial steady state equilibrium value. The crime control policy cost ($CCPC$) can be expressed as:

$$CCPC = B\left[w_{im}(0) + K_s\left\{B + R(t)\right\}\right]. \tag{29}$$

Our problem of minimising efficiency loss subject to crime control policy cost constraint is as follows:

$$\min_B EL \quad \text{s.t.} \quad CCPC \leq G_B.$$

$G_B$ is the government's cost for exercising crime control policy. The choice variable is crime control policy, i.e., $B$, and the constraint is binding at $t = 0$. The Lagrangian for the above problem can be written as follows:

$$\begin{aligned}
\mathcal{L} &= \int_{-\infty}^{0} W_m(\infty)dt + M_B(t) + \lambda\left[G_B - B\left[w_{im}(0) + K_s\left\{B + R(t)\right\}\right]\right] \\
&= \int_{-\infty}^{0}\left[K_sB - \frac{K_s^2B}{(K_s+K_d)}\right]dt \\
&- \frac{1}{K_m}\left[-\frac{K_sB}{(K_s+K_d)} + \frac{K_sB}{(K_s+K_d)}e^{-[K_m(K_s+K_d)]t} - K_mK_sB\right] \\
&+ \lambda\left[G_B - B\left[w_{im}(0) + K_s\left\{B - \frac{K_sB}{(K_s+K_d)} + \frac{K_sB}{(K_s+K_d)}e^{-[K_m(K_s+K_d)]t}\right\}\right]\right] \\
&= \int_{-\infty}^{0}\frac{K_sK_dB}{(K_s+K_d)}dt - \frac{1}{K_m}\left[-\frac{K_sB}{(K_s+K_d)} + \frac{K_sB}{(K_s+K_d)}e^{-[K_m(K_s+K_d)]t} - K_mK_sB\right] \\
&+ \lambda\left[G_B - B\left[w_{im}(0) + K_s\left\{B - \frac{K_sB}{(K_s+K_d)} + \frac{K_sB}{(K_s+K_d)}e^{-[K_m(K_s+K_d)]t}\right\}\right]\right].
\end{aligned}$$

The first order condition with respect to $B$ leads to the following:

$$B = -\frac{\lambda w_{im}(0) - \left[\int_{-\infty}^{0}\frac{K_sK_d}{(K_s+K_d)}dt - \frac{1}{K_m}\left[-\frac{K_s}{(K_s+K_d)} + \frac{K_s}{(K_s+K_d)}e^{-[K_m(K_s+K_d)]t} - K_mK_s\right]\right]}{2\lambda K_s\left[1 - \frac{K_s}{(K_s+K_d)} + \frac{K_s}{(K_s+K_d)}e^{-[K_m(K_s+K_d)]t}\right]}. \tag{30}$$

The derivative with respect to $\lambda$, is as follows:

$$G_B - B\left[w_{im}(0) + K_s\left\{B - \frac{K_sB}{(K_s+K_d)} + \frac{K_sB}{(K_s+K_d)}e^{-[K_m(K_s+K_d)]t}\right\}\right] = 0. \tag{31}$$

Putting eq. (30) into (31), we get:

$$\lambda = \frac{J_B}{\sqrt{w_{im}^2(0) + 4Q_B G_B}}.$$

$$\text{where } Q_B = K_s \left[ 1 - \frac{K_s}{(K_s + K_d)} + \frac{K_s}{(K_s + K_d)} e^{-[K_m(K_s+K_d)]t} \right],$$

$$J_B = \int_{-\infty}^{0} \frac{K_s K_d}{(K_s + K_d)} dt - \frac{1}{K_m} \left[ -\frac{K_s}{(K_s + K_d)} + \frac{K_s}{(K_s + K_d)} e^{-[K_m(K_s+K_d)]t} - K_m K_s \right].$$

$\lambda$ is a positive number because when $G_B$ increases, the minimum efficiency loss also increases. From eq. (30):

$$B = -\frac{\lambda w_{im}(0) - J_B}{2\lambda Q_B}. \tag{32}$$

By replacing $\lambda$ with its value in the above expression, we get:

$$B = -\frac{w_{im}(0) - \sqrt{w_{im}^2(0) + 4Q_B G_B}}{2Q_B}. \tag{33}$$

The second order condition for minimization has been checked (see appendix). Suppose that the government has \$1000 available to be spent as crime control policy cost. The initial value of number of crimes controlled is 100, and the value of each one of the variables, $K_m$, $K_s$ and $K_d$ is equal to one. After plugging in these values in eq. (33), we get:

$$B = -\frac{100 - \sqrt{10000 + 4000}}{2} = 9.161,$$

where $Q_B = 1 - 0.333 + 0.333e^{-3t}$, and at $t = 0$, $Q_B = 1$. The crime control policy cost is $CCPC = B\left[w_{im}(0) + Q_B B\right] = 1000$. Therefore the optimal crime control policy is that the government provides an extra remuneration of \$9.161 per crime controlled to public service provider.

An interesting question would be whether it was possible for the government to follow a policy such that all adjustments could be made instantaneously, such that market clearing was never an issue; and the answer is that it could have been possible only if the government would instantaneously fix the cost and the number of crimes controlled at the new equilibrium values, i.e., after the public service market had cleared as a result of the policy change. For this, the government would have to know the exact equilibrium point, as well as the exact patterns of supply and demand of number of crimes controlled, which seems impracticable.

## 5 The Model-Panel A

Theoretical arguments are built upon Figure 3, where the $y$-axis represents the *crime control rate*, i.e., *number of crimes controlled per public service unit*, and the $x$-axis, represents the *number of public service units per unit time* in society. The upward sloping supply curve represents total number of public service units in society due to efforts of public sector. If number of crimes controlled per public service unit is higher, the public sector has an incentive to invest more in public service units as their objective is to maximize social benefit. The downward sloping demand curve shows the relationship between crime control rate and demand of number of public service units. As crime control rate goes up, demand of number of public service units decreases. The point where both curves intersect is an equilibrium point representing the equilibrium crime control rate and number of public service units in society. At a crime control rate, where demand of number of public service units is higher than supply, the crime control rate will increase until the number of public service units on both curves equal. Similarly, at a crime control rate where supply is higher than demand, the crime control rate will go down until the equilibrium arrives.

Let us assume that number of public service units on supply curve equals the number on demand curve and there is an equilibrium crime control rate. There are three types of infinitely-lived agents: public and private sectors which demand a certain number of public service units against each crime control rate, a representative –or a unit mass of– public service units who control crime, and public sector as one entity who supplies certain number of public service units against each crime control rate. The mechanism for adjustment of crime control rate is based on lack of coordination between agents in society regarding supply and demand of number of public service units at existing crime control rate when either supply or demand curve shifts and pushes the crime control rate and the number of public service units out of equilibrium. Suppose that number of public service units are in equilibrium, and an upward shift in the demand curve increases the number of public service units on demand curve at the existing crime control rate. Now, the number of public service units on demand curve are greater than their number on supply curve. Public service providers will increase crime control rate, and public sector will find it optimal to have a higher number of public service units in new equilibrium. This will result in a higher crime control rate, and a higher number of public service units when new equilibrium arrives. The equilibrium is defined as follows:

(i) Private sector as consumer of public service maximize their benefit, public service providers maximize the net benefit of public service for society, and the public sector maximizes the social benefit, subject to the constraints they face (mentioned in their individual dynamic optimization problems in Section 5).

(ii) The number of public service units on upward sloping supply curve equals the number on downward sloping demand curve, and the crime control rate does not change during equilibrium.

The conditions for existence of equilibrium (Routh–Hurwitz stability criterion, which provides a necessary and sufficient condition for the stability of a linear dynamical system) are mentioned in Section 6.

The crime control rate is given for both private and public sector as consumer and supplier of public service units respectively. Public service provider does not have an incentive to change crime control rate during equilibrium. They have an incentive to change crime control rate only during the state of disequilibrium. The government formulates and implements a policy to increase/decrease number of public service units in society, either by increasing or decreasing the public supply or demand; or by influencing the private sector demand. A new equilibrium does not result instantaneously as soon as the policy gets implemented, and rather the crime control rate, and number of public service providers adjust over time to lead to a new equilibrium. The adjustment takes place as a result of an endogenous decision making by agents to maximize their objective functions subject to constraints, i.e., both the public and private sector in their roles regarding supply and demand and public service providers. There is some social damage during the adjustment process, which is defined as the sum of too many or too few public service providers. The total social damage includes the damage during the adjustment process as well as that in the initial equilibrium. This is the total loss for the purpose of minimizing it subject to constraints. There might still be some social damage in the final equilibrium, however, that is not part of the objective function which needs to be minimized as that cannot be improved upon due to constraints.

In order to derive the results mathematically, the objectives of the agents have been maximized subject to their respective constraints through the first order conditions, which are solved simultaneously to get the collective outcome of their decisions. An important assumption is that after the implementation of the policy to have an optimal number of public service providers, the new equilibrium arrived at is not too off the equilibrium in the initial state. On account of this, the linearization of supply and demand curves seems reasonable.

### 5.1 Public Service Provider

A public service provider controls crimes. A state of equilibrium implies that demand of number of public service providers in society equals supply. Any change in number of public service provider is on account of a change in supply, demand or both due to private sector, government or both at a different rate.

The link between number of public service providers, supply, and demand can be illustrated as follows: When demand curve shifts to the right, while supply stays the same, the cumulative number of public service provider is unable to meet new societal demand at the existing crime control rate, and the crime control rate goes up to equalize demand and supply in new equilibrium. If supply curve shifts rightwards, whereas the demand does not shift, cumulative number of public service provider goes up at the existing crime control rate, therefore crime control rate goes down in new equilibrium. This discussion implies that there exists a negative relationship between change in cumulative number of public service providers and change in crime control rate. The horizontal axis in Figure 3 reflects the rate of supply and demand both by private and public sectors, and not the cumulative number of public service providers in society. Supply and demand rates are flow variables, whereas cumulative number of public service providers is a stock variable.

The following mechanism is involved in bringing about such changes: Suppose that the number of public service providers demanded is in equilibrium with the supply, and the crime control rate stays the same over time. Now suppose that the demand curve does not shift whereas the supply curve shifts to the right due to a decrease in the marginal cost of having another unit of public service provider by public sector. As the number of provider units increases, the crime control rate decreases, and the feedback of private sector is to increase their demand of public service providers along the demand curve. The adjustment path to the new equilibrium is dependent on the direction of shock and how public service providers react to that shock. In order to depict the behavior of the public service provider mathematically, let us consider the utility/benefit maximization problem of public service provider as follows:

#### 5.1.1 Short Run Problem

The short run problem of public service provider is myopic in the sense that no dynamic optimization is being done on their part. A discrete analog is a one period problem, and the objective is to make the intuition clear and simple so that author is ready to grasp the more complicated dynamic problem in next section. The objective function of public service provider is as follows:

$$\Theta = U_c(c) - \varsigma_A(m_A(c, e_A)), \tag{34}$$

where

$\Theta$ = net benefit of public service for society,

$U_c(c)$ = benefit of public service as a positive function of crime control rate,

$c$ = number of crimes controlled per public service provider (crime control rate in a dynamic setting),

$m_A$ = cumulative number of public service providers in society = $\Sigma(supply - demand)$, which is just $(supply - demand)$ for one time period.

$e_A$ = other factors which affect the total number of public service providers in society,

$\varsigma_A(m_A(c, e_A))$ = cost as a function of total number of public service providers in society (increasing in number).

The first order condition of $\Theta$ with respect to $c$ is as follows:

$$U_c'(c) - \varsigma_A'(m_A(c, e_A))m_{A1}'(c, e_A) = 0. \tag{35}$$

If supply curve shifts to the right, say on account of a decreased cost to public sector for establishing a public service provider unit, the number of public service providers is no more in equilibrium. As number of units is higher than before at the current value of $c$, the term $\varsigma_A'(m_A(c, e_A))$ is higher at the existing $c$. As the term, $m_{A1}'(c, e_A)$ is a function of $c$, therefore, it is the same as before because the value of $c$ has not yet changed. The implication is that at the existing value of $c$, the public service provider now faces the following inequality:

$$\frac{\partial \Theta}{\partial c} = U_c'(c) - \varsigma_A'(m_A(c, e_A))m_{A1}'(c, e_A) < 0, \tag{36}$$

which implies that the public service provider chooses to decrease crime control rate to satisfy the condition of maximization of net benefit for society after the supply shock. If various net benefit maximizing pairs of values of cumulative number of public service providers and the respective crime control rate chosen by public service provider are plotted together, a downward sloping curve results with number of units on $x$-axis, and the crime control rate on $y$-axis.

#### 5.1.2 Dynamic Problem

The public service provider maximizes present discounted value of future stream of net benefits of public service in a dynamic setting, and the present value at time zero is as follows:

$$V(0) = \int_0^\infty \left[U_c(c) - \varsigma_A(m_A(c, e_A))\right] e^{-\varpi t} dt, \tag{37}$$

$\varpi$ denotes the discount rate. $c(t)$ is the *control variable*, and $m_A(t)$ is the *state variable*. The maximization problem can be written as

$$\underset{\{c(t)\}}{Max} V(0) = \int_0^\infty \left[U_c(c) - \varsigma_A(m_A(c, e_A))\right] e^{-\varpi t} dt,$$

subject to the constraints that
$\dot{m}_A(t) = m_{A1}^{'}(c(t), e_A(c(t), z_A))\dot{c}(t) + m_{A2}^{'}(c(t), e_A(c(t), z_A))\ e_{A1}^{'}(c(t), z_A)\dot{c}(t)$ (state equation, describing how the state variable changes with time; $z_A$ are exogenous factors),
$m_A(0) = m_{As}$ (initial condition),
$m_A(t) \geq 0$ (non-negativity constraint on state variable),
$m_A(\infty)$ free (terminal condition).
The current-value Hamiltonian for this case is

$$\widetilde{H} = U_c(c(t)) - \varsigma_A(m_A(c(t), e_A(c(t), z_A))) + \mu_A(t)\dot{c}(t) \left[ \begin{array}{c} m'_{A1}(c(t), e_A(c(t), z_A)) + m'_{A2}(c(t), e_A(c(t), z_A)) * \\ e'_{A1}(c(t), z_A) \end{array} \right]. \tag{38}$$

Now the maximizing conditions are as follows:
$(i)$ $c^*(t)$ maximizes $\widetilde{H}$ for all $t$: $\frac{\partial \widetilde{H}}{\partial c} = 0$,
$(ii)$ $\dot{\mu_A} - \varpi \mu_A = -\frac{\partial \widetilde{H}}{\partial m_A}$,
$(iii)$ $\dot{m_A}^* = \frac{\partial \widetilde{H}}{\partial \mu_A}$ (this just gives back the state equation),
$(iv)$ $\lim_{t \to \infty} \mu_A(t) m_A(t) e^{-\varpi t} = 0$ (the transversality condition).
The first two conditions are as follows:

$$\begin{aligned} \frac{\partial \widetilde{H}}{\partial c} &= U'_c(c(t)) - \varsigma'_A(m_A(c(t), e_A(c(t), z_A))) \left\{ \begin{array}{c} m'_{A1}(c(t), e_A(c(t), z_A)) + m'_{A2}(c(t), e_A(c(t), z_A)) * \\ e'_{A1}(c(t), z_A) \end{array} \right\} \\ &+ \mu_A(t)\dot{c}(t) * \left[ \begin{array}{c} m''_{A11}(c(t), e_A(c(t), z_A)) + m''_{A12}(c(t), e_A(c(t), z_A)) e'_{A1}(c(t), z_A) + \\ m''_{A21}(c(t), e_A(c(t), z_A)) e'_{A1}(c(t), z_A) + m''_{A22}(c(t), e_A(c(t), z_A)) e'^2_{A1}(c(t), z_A) + \\ m'_{A2}(c(t), e_A(c(t), z_A)) e''_{11}(c(t), z_A) \end{array} \right] \\ &= 0. \end{aligned} \tag{39}$$

and

$$\dot{\mu_A} - \varpi \mu_A = -\frac{\partial \widetilde{H}}{\partial m_A} = \varsigma'_A(m_A(c(t), e_A(c(t), z_A))). \tag{40}$$

In equilibrium, $\dot{c}(t) = 0$, and the expression $\frac{\partial \widetilde{H}}{\partial c}$ boils down to the following:

$$U'_c(c(t)) - \varsigma'_A(m_A(c(t), e_A(c(t), z_A))) \left\{ \begin{array}{c} m'_{A1}(c(t), e_A(c(t), z_A)) + m'_{A2}(c(t), e_A(c(t), z_A)) * \\ e'_{A1}(c(t), z_A) \end{array} \right\} = 0.$$

If supply curve shifts to the right, then the number of public service providers is higher at the existing crime control rate, and the term $\varsigma'_A(m_A(c(t), e_A(c(t), z_A)))$ is higher at the existing crime control rate at that time. The term multiplying $\varsigma'_A(m_A(c(t), e_A(c(t), z_A)))$, i.e., $m'_{A1}(c(t), e_A(c(t), z_A)) + m'_{A2}(c(t), e_A(c(t), z_A)) e'_{A1}(c(t), z_A)$ is a function of crime control rate and has not changed as the crime control rate is the same as before. Therefore, the public service provider now faces the following inequality at the existing crime control rate:

$$\frac{\partial \widetilde{H}}{\partial c} < 0.$$

The public service unit must decrease the crime control rate for satisfying the dynamic optimization condition after the shock. This implies that there is a negative relationship between cumulative number of public service units in society and the crime control rate. If the rate of supply of units in society is equal to the demand rate, the number of units is in equilibrium. If a difference of a finite magnitude comes into force between the supply and demand rates, and the public and the private sector do not react to a change in the crime control rate caused by a difference in the supply and demand rates, the crime control rate will continue changing until the saturation point of the society comes. The behavior of public service unit can be depicted by the following formulation:

$$\begin{aligned}
\textit{Crime control rate change} &\propto \textit{change in cumulative number of public service units.}\\
C &= \textit{crime control rate change.}\\
M_A &= m_A - m_{As} = \textit{change in cumulative number of units,}\\
m_A &= \textit{cumulative number of units at time } t,\\
m_{As} &= \textit{cumulative number of units in steady state equilibrium.}\\
\textit{Input} - \textit{output} &= \frac{dm_A}{dt} = \frac{d(m_A - m_{As})}{dt} = \frac{dM_A}{dt},\\
\text{or } M_A &= \int (\textit{input} - \textit{output})\, dt.\\
\textit{Crime control rate change} &\propto \int (\textit{supply rate} - \textit{demand rate})\, dt, \text{ or}\\
C &= -K_c \int (\textit{supply rate} - \textit{demand rate})\, dt,
\end{aligned}$$

$K_c$ is the constant of proportionality; *supply* and *demand rates* are number of public service providers per unit time. When (*supply rate* − *demand rate*) is positive, $C$ is negative, and hence a negative sign, i.e., the crime control rate goes down. Rearranging the above expression gives:

$$\int (\textit{supply rate} - \textit{demand rate})\, dt = -\frac{C}{K_c}, \text{ or}$$

$$\int (w_{Ai} - wA_0)\, dt = -\frac{C}{K_c}, \tag{41}$$

$$\begin{aligned}
w_{Ai} &= \textit{supply rate},\\
w_{A0} &= \textit{demand rate},\\
K_c &= \textit{dimensional constant.}
\end{aligned}$$

In the initial steady state equilibrium at $t = 0$, *supply rate* = *demand rate*, and eq. (41) can be written as

$$\int (w_{Ais} - w_{A0s})\, dt = 0. \tag{42}$$

Due to the condition of the steady state equilibrium, the subscript $s$ has been added. In the steady state, $C = 0$, and subtracting eq. (42) from eq. (41) leads to:

$$\int (w_{Ai} - w_{Ais})\, dt - \int (w_{A0} - wA_{0s})\, dt = -\frac{C}{K_c}, \text{ or}$$

$$\int (W_{Ai} - W_{A0})\, dt = -\frac{C}{K_c}, \tag{43}$$

$$\text{where } w_{Ai} - w_{Ais} = W_{Ai} = \textit{change in supply rate},$$
$$w_{A0} - w_{A0s} = W_{A0} = \textit{change in demand rate.}$$

The capital letters denote the deviation variables, i.e., deviation from the initial equilibrium. $C$, $W_{Ai}$ and $W_{A0}$ are all deviation variables, and their initial values are zero. Eq. (43) can be rearranged as:

$$C = -K_c \int W_A dt = -K_c M_A, \tag{44}$$

where $W_A = W_{Ai} - W_{A0}$. If $C$ gets affected by an input other than $M_A$, then an input must be added to the right hand side of eq. (44) which changes to the following:

$$C = -K_c \int W_A dt + E_A = -K_c M_A + E_A. \tag{44a}$$

$M_A$ can also get an exogenous input other than the feedback of the crime control rate.

### 5.2 Public Sector/Supplier of Public Service Units

As a supplier of public service units to control crime, the public sector maximizes the present discounted value of the future stream of net benefits, and their present value at time zero is as follows:

$$V(0) = \int_0^\infty \left[U_{pr}\left(n_{pr}\right) - \varsigma_{pr}(c\left(n_{pr}\right))\right] e^{-r_{pr}t} dt. \tag{45}$$

$U_{pr}\left(n_{pr}\right)$ is social benefit and an increasing function of number of public service units, the higher the number, the higher the social benefit. $\varsigma_{pr}(c\left(n_{pr}\right))$ is cost to society for establishment of public service units, the higher the crime control rate, the higher is the cost. The cost curve with respect to crime control rate is concave downward, i.e., decreasing in slope.

$r_{pr}$ denotes the discount rate. $n_{pr}(t)$ is the *control variable*, and $c(t)$ is the *state variable.* The maximization problem can be written as

$$\underset{\{n_{pr}(t)\}}{Max}\, V(0) = \int_0^\infty \left[U_{pr}\left(n_{pr}\right) - \varsigma_{pr}(c\left(n_{pr}\right))\right] e^{-r_{pr}t} dt,$$

subject to the constraints that:

$\dot{c}(t) = c'(n_{pr}\left(t\right))\dot{n}_{pr}\left(t\right)$ (state equation, describing how the state variable changes with time),

$c(0) = c_s$ (initial condition),

$c(t) \geq 0$ (non-negativity constraint on state variable),

$c(\infty)$ free (terminal condition).

The current-value Hamiltonian for this case is

$$\widetilde{H} = U_{pr}\left(n_{pr}(t)\right) - \varsigma_{pr}(c\left(n_{pr}\left(t\right)\right)) + \mu(t)\ c'(n_{pr}\left(t\right))\dot{n}_{pr}\left(t\right). \tag{46}$$

Now the maximizing conditions are as follows:

$(i)$ $n_{pr}{}^*(t)$ maximizes $\widetilde{H}$ for all $t$: $\frac{\partial \widetilde{H}}{\partial n_{pr}} = 0$,

$(ii)$ $\dot{\mu_{pr}} - r_{pr}\mu_{pr} = -\frac{\partial \widetilde{H}}{\partial c}$,

$(iii)$ $\dot{c}^* = \frac{\partial \widetilde{H}}{\partial \mu_{pr}}$ (this just gives back the state equation),

$(iv)$ $\lim_{t\to\infty} \mu_{pr}(t)c(t)e^{-r_{pr}t} = 0$ (the transversality condition).

The first two conditions are as follows:

$$\frac{\partial \widetilde{H}}{\partial n_{pr}} = U'_{pr}\left(n_{pr}(t)\right) - \varsigma'_{pr}(c\left(n_{pr}\left(t\right)\right))\ c'\left(n_{pr}\left(t\right)\right) + \mu_{pr}(t)\ c''(n_{pr}\left(t\right))\dot{n}_{pr}\left(t\right) = 0, \tag{47}$$

and

$$\dot{\mu_{pr}} - r_{pr}\mu_{pr} = -\frac{\partial \widetilde{H}}{\partial c} = \varsigma'_{pr}(c\left(n_{pr}\left(t\right)\right)). \tag{48}$$

In equilibrium, $\dot{n}_{pr}\left(t\right) = 0$, and the expression $\frac{\partial \widetilde{H}}{\partial n_{pr}}$ boils down to the following:

$$U'_{pr}\left(n_{pr}(t)\right) - \varsigma'_{pr}(c\left(n_{pr}\left(t\right)\right))\ c'\left(n_{pr}\left(t\right)\right) = 0.$$

If crime control rate goes up, the term $\varsigma'_{pr}(c\left(n_{pr}\left(t\right)\right))$ goes down, and the public sector now faces the following inequality:

$$\frac{\partial \widetilde{H}}{\partial n_{pr}} > 0.$$

The number of public service units due to public sector's efforts will go up to satisfy the dynamic optimization problem of the public sector. If change in number of public service units is proportional to a change in crime control rate, i.e., $C$, or linearization of the supply curve around the steady state equilibrium leads to the following:

$$W_{pr}(t) = K_{pr}C(t), \tag{49}$$

where $W_{pr}(t)$ is change in number of public service units with respect to initial steady state equilibrium value. As it is a deviation variable, i.e., deviation from the steady state, it has a zero initial value. There is a time lag between the change in crime control rate and the change in number of public service units, therefore, a dead time element needs to be incorporated in the above expression which results in the following:

$$W_{pr}(t) = K_{pr}C(t - \tau_{d1}). \tag{50}$$

### 5.3 Private Sector/Demander of Public Service Units

Both public and private sectors in a society demand public service units to control crime in society. However, we just present the private sector as a demander to economize on typing space. The total demand is a sum of both public and private demand. In this section, we present the role of the private sector as a demander of public service units to control crime. As a demander, the private sector maximizes the present discounted value of future stream of net benefits, and their present value at time zero is as follows:

$$V(0) = \int_0^\infty \left[U_{pu}\left(n_{pu}\right) - \varsigma_{pu}(c\left(n_{pu}\right))\right] e^{-r_{pu}t} dt, \tag{51}$$

where $U_{pu}\left(n_{pu}\right)$ is the private sector benefit increasing in number of public service units to control crime and concave downward. $\varsigma_{pu}(c\left(n_{pu}\right))$ is the cost to the private sector, the higher the crime control rate, the higher is the cost. The cost curve with respect to crime control rate is concave upward, i.e., increasing in slope.

$r_{pu}$ denotes the discount rate. $n_{pu}(t)$ is the *control variable*,and $c(t)$ is the *state variable*. The maximization problem can be written as

$$\underset{\{n_{pu}(t)\}}{Max}\, V(0) = \int_0^\infty \left[U_{pu}\left(n_{pu}\right) - \varsigma_{pu}(c\left(n_{pu}\right))\right] e^{-r_{pu}t} dt,$$

subject to the constraints that:

$\dot{c}(t) = c'(n_{pu}\left(t\right))\dot{n}_{pu}\left(t\right)$ (state equation, describing how the state variable changes with time),

$c(0) = c_s$ (initial condition),

$c(t) \geq 0$ (non-negativity constraint on state variable),

$c(\infty)$ free (terminal condition).

The current-value Hamiltonian for this case is

$$\widetilde{H} = U_{pu}\left(n_{pu}(t)\right) - \varsigma_{pu}(c\left(n_{pu}\left(t\right)\right)) + \mu_{pu}(t)\; c'(n_{pu}\left(t\right))\dot{n}_{pu}\left(t\right). \tag{52}$$

Now the maximizing conditions are as follows:
$(i)$ $n_{pu}{}^{*}(t)$ maximizes $\widetilde{H}$ for all $t$: $\frac{\partial \widetilde{H}}{\partial n_{pu}} = 0$,
$(ii)$ $\dot{\mu_{pu}} - r_{pu}\mu_{pu} = -\frac{\partial \widetilde{H}}{\partial c}$,
$(iii)$ $\dot{c}^{*} = \frac{\partial \widetilde{H}}{\partial \mu_{pu}}$ (this just gives back the state equation),
$(iv)$ $\lim_{t\to\infty} \mu_{pu}(t)c(t)e^{-r_{pu}t} = 0$ (the transversality condition).
The first two conditions are as follows:

$$\frac{\partial \widetilde{H}}{\partial n_{pu}} = U_{pu}'\left(n_{pu}(t)\right) - \varsigma_{pu}'(c\left(n_{pu}\left(t\right)\right))\ c'\left(n_{pu}\left(t\right)\right) + \mu_{pu}(t)\ c^{''}(n_{pu}\left(t\right))\dot{n}_{pu}\left(t\right) = 0. \tag{53}$$

and

$$\dot{\mu_{pu}} - r_{pu}\mu_{pu} = -\frac{\partial \widetilde{H}}{\partial c} = \varsigma_{pu}'(c\left(n_{pu}\left(t\right)\right)). \tag{54}$$

In equilibrium, $\dot{n}_{pu}\left(t\right) = 0$, and the expression $\frac{\partial \widetilde{H}}{\partial n_{pu}}$ boils down to the following:

$$U_{pu}'\left(n_{pu}(t)\right) - \varsigma_{pu}'(c\left(n_{pu}\left(t\right)\right))\ c'\left(n_{pu}\left(t\right)\right) = 0.$$

If crime control rate goes up, the term $\varsigma_{pu}'(c\left(n_{pu}\left(t\right)\right))$ goes up, and the private sector now faces the following inequality:

$$\frac{\partial \widetilde{H}}{\partial n_{pu}} < 0.$$

The number of public service units demanded by private sector will go down to satisfy the dynamic optimization problem of the private sector. If change in number of public service units demanded by the private sector is proportional to a change in crime control rate, i.e., $C$, or linearization of the demand curve around the steady state equilibrium leads to the following:

$$W_{pu}(t) = K_{pu}\left[\epsilon\left(t\right) - C(t)\right] = -K_{pu}\eta(t), \tag{55}$$

where $\epsilon\left(t\right) = e - e_{s}$; $e$ is a reference crime control rate with respect to which the variation in crime control rate is considered by the private sector for decision making. It is a parameter which may vary over time or remain fixed for a while. $W_{pu}(t)$ is the change in number of public service units with respect to the initial steady state equilibrium value. As it is a deviation variable, i.e., deviation from the steady state, it has a zero initial value. There is a time lag between the change in crime control rate and the change in number of public service units demanded by the private sector, therefore, a dead time element needs to be incorporated in the above expression which results in the following:

$$W_{pu}(t) = -K_{pu}\eta(t - \tau_{d2}). \tag{56}$$

## 6 Solution of the Model-Panel A with a Crime Control Policy

The model is solved for the simplest case when $\tau_{d1} = \tau_{d2} = 0$. For solution of more complex cases, please see appendix. From eq. $(44a)$, (49), and (55), we have the following expressions respectively:

$$\begin{aligned} \frac{dC}{dt} &= -K_c W_A(t), \\ W_{pr}(t) &= K_{pr} C(t), \\ W_{pu}(t) &= K_{pu} \left[ \epsilon(t) - C(t) \right], \\ W_A(t) &= W_1(t) - W_{pu}(t), \\ &= D(t) + W_{pr}(t) - W_{pu}(t). \\ \text{where } D(t) &= W_{Ai}(t) - W_{A0}(t). \end{aligned}$$

In the absence of an exogenous shock in number of public service units, $D(t) = 0$. A policy from panel A must be synchronized with that from panel B, i.e., the supply and demand curves should be moving in the same direction in both panels. In this section, we just present an example on how an optimal policy can be framed from panel A when the demand curve shifts, however, this has to be in line with the policy from panel B as they cannot be treated as independent of each other. Suppose the government adopts a policy (such as a media campaign to create awareness about certain types of crimes, and simultaneously increasing the crime control rate) where demand of public service units gets a shift in the upward direction, i.e.,

$$W_{pu}(t) = K_{pu} \left[ A - C(t) \right],$$

where $A$ is the size of the policy. This implies that

$$\begin{aligned} \frac{dC(t)}{dt} &= -K_c \left[ W_{pr}(s) - W_{pu}(t) \right] \\ &= -K_c \left[ K_{pr} C(t) - K_{pu} A + K_{pu} C(t) \right] \\ &= -K_c \left[ -K_{pu} A + \left( K_{pr} + K_{pu} \right) C(t) \right]. \end{aligned}$$

The above expression can be written as

$$\frac{dC(t)}{dt} + K_c (K_{pr} + K_{pu}) C(t) = K_c K_{pu} A. \quad (57)$$

According to the Routh–Hurwitz stability criterion, the necessary and sufficient condition for stability of the above differential equation is $K_c(K_{pr} + K_{pu}) > 0$, which holds as $K_c$, $K_{pr}$ and $K_{pu}$ are all defined to be positive. This condition ensures that starting from an initial condition away from an initial equilibrium every adjustment mechanism will lead to another equilibrium.

In order to solve the above differential equation, we proceed as follows:
The characteristic function of the differential equation is as follows:

$$x + K_c(K_{pr} + K_{pu}) = 0.$$

The characteristic function has a single root given by:

$$x = -K_c(K_{pr} + K_{pu}).$$

Thus the complementary solution is

$$C_c(t) = C_2 e^{-[K_c(K_{pr}+K_{pu})]t}.$$

The particular solution has the form

$$C_p(t) = C_1.$$

Thus the solution has the form

$$C(t) = C_1 + C_2 e^{-[K_c(K_{pr}+K_{pu})]t}. \quad (58)$$

The constant $C_1$ is determined by substitution into the differential equation as follows:

$$-K_c(K_{pr}+K_{pu})C_2 e^{-[K_c(K_{pr}+K_{pu})]t}+K_c(K_{pr}+K_{pu})C_1+K_c(K_{pr}+K_{pu})C_2 e^{-[K_c(K_{pr}+K_{pu})]t} = K_c K_{pu} A,$$

$$C_1 = \frac{K_{pu}A}{K_{pr} + K_{pu}}.$$

$C_2$ is determined by the initial condition as follows:

$$\begin{aligned} C(0) &= \frac{K_{pu}A}{K_{pr} + K_{pu}} + C_2 = A, \\ C_2 &= A - \frac{K_{pu}A}{K_{pr} + K_{pu}} \\ &= \frac{K_{pr}A}{K_{pr} + K_{pu}}. \end{aligned}$$

Substituting the values of $C_1$ and $C_2$ in eq. (58), we get:

$$C(t) = \frac{K_{pu}A}{K_{pr} + K_{pu}} + \frac{K_{pr}A}{K_{pr} + K_{pu}} e^{-[K_c(K_{pr}+K_{pu})]t}. \quad (59)$$

When $t = 0$, $C(0) = A$ (the initial condition), and when $t = \infty$, $C(\infty) = \frac{K_{pu}A}{K_{pr}+K_{pu}}$ (the final steady state equilibrium value).

## 7 A Dynamic Optimal Crime Control Policy-Panel A

The social damage due to inadequate/excessive crime control units includes the damage in the initial equilibrium, i.e., before the adoption of a crime control policy, plus the damage during the adjustment process from initial equilibrium to the final. After government adopts a policy to enhance/reduce the number of public service units, it shifts either the supply or the demand curve, e.g., it shifts the demand curve upward by a magnitude depending upon the size of the policy, which is taken as $A$ in the solution of the model with a crime control policy. The crime control rate then adjusts over time to bring the new equilibrium rate which is higher than the previous equilibrium crime control rate and lower than that which existed at the time the policy was implemented depending on the elasticity of supply and demand curves. An excessive number of public service units in society implies that the number is higher on the supply curve than that on the demand curve, and a shortage in their number implies the opposite. When the number on the supply and the demand curve becomes equal, the new equilibrium has arrived. When the number is different on supply and demand curve, that difference is the social damage at that point in time. Furthermore, the number of crime control units in society was lower (in this example) in the previous equilibrium, which is also social damage in equilibrium. If we sum up either the number of excessive units on the supply curve or their number on the demand curve short of supply curve, we get the total social damage in terms of number of units as follows:

$$SD = M_A(t) + \int_{-\infty}^{0} W_{pr}(\infty)dt. \tag{60}$$

From eq. (55), the change in number of crime control units due to change in crime control rate after adoption of crime control policy is as under:

$$W_{pu}(t) = K_{pu}\left[A - C(t)\right],$$
$$\text{or } w_{npu}(t) - w_{ipu}(0) = K_{pu}\left[A - C(t)\right],$$

where $w_{ipu}(0)$ is the initial number of crime control units and $w_{npu}(t)$ is the new number after the implementation of crime control policy as $W_{pu}(t)$ is a deviation variable, i.e., deviation from the initial equilibrium value. An increase in number of crimes controlled per unit time is as follows:

$$INC = A\left[w_{ipu}(0) + K_{pu}\left\{A - C(t)\right\}\right]. \tag{61}$$

If we want to minimize the social damage subject to the constraint that an increase in number of crimes controlled per unit time is greater than or equal to $G_A$ (change in number of crimes controlled per unit time our problem is as follows:

$$\min_{A} SD \quad \text{s.t.} \quad INC \geq G_A \left( = \frac{dM_B}{dt} \right).$$

The choice variable is $A$, i.e., an initial upward jump in the crime control rate chosen by government to shift the demand curve, and the constraint is binding. Lagrangian for the above problem is given below:

$$\mathcal{L} = M_A(t) + \int_{-\infty}^{0} W_{pr}(\infty)dt + \lambda \left[G_A - A\left[w_{ipu}(0) + K_{pu}\left\{A - C(t)\right\}\right]\right].$$

From eq. (44$a$), we have:

$$C(t) = -K_c M_A + E_A.$$

The value of $E_A$ can be found by imposing the initial conditions as follows:

$$\begin{aligned} C(0) &= -K_c M_A(0) + E_A, \\ A &= -K_c K_{pr} C(0) + E_A, \\ E_A &= A\left[1 + K_c K_{pr}\right]. \end{aligned}$$

This implies that

$$M_A(t) = -\frac{1}{K_c}\left[C(t) - A\left\{1 + K_c K_{pr}\right\}\right].$$

Therefore, the Lagrangian can now be written as:

$$\begin{aligned} \mathcal{L} = & -\frac{1}{K_c}\left[C(t) - A\left\{1 + K_c K_{pr}\right\}\right] + \int_{-\infty}^{0} W_{pr}(\infty)dt + \lambda\left[G_A - A\left[w_{ipu}(0) + K_{pu}\left\{A - C(t)\right\}\right]\right] \\ = & -\frac{1}{K_c}\left[\left\{\frac{K_{pu}A}{K_{pr}+K_{pu}} + \frac{K_{pr}A}{K_{pr}+K_{pu}} e^{-[K_c(K_{pr}+K_{pu})]t}\right\} - A\left\{1 + K_c K_{pr}\right\}\right] \\ & + \int_{-\infty}^{0} W_{pr}(\infty)dt + \lambda\left[G_A - A\left[w_{ipu}(0) + K_{pu}\left\{A - \frac{K_{pu}A}{K_{pr}+K_{pu}} - \frac{K_{pr}A}{K_{pr}+K_{pu}} e^{-[K_c(K_{pr}+K_{pu})]t}\right\}\right]\right]. \end{aligned}$$

The first order condition with respect to $A$ is as follows:

$$A = \frac{\lambda w_{ipu}(0) - \frac{1}{K_c}\left[\left\{\frac{-K_{pu}}{K_{pr}+K_{pu}} - \frac{K_{pr}}{K_{pr}+K_{pu}} e^{-[K_c(K_{pr}+K_{pu})]t}\right\} + \left\{1 + K_c K_{pr}\right\}\right]}{-2\lambda K_{pu}\left\{1 - \frac{K_{pu}}{K_{pr}+K_{pu}} - \frac{K_{pr}}{K_{pr}+K_{pu}} e^{-[K_c(K_{pr}+K_{pu})]t}\right\}}. \tag{62}$$

The first order condition of the Lagrangian with respect to $\lambda$ is as follows:

$$G_A - A\left[w_{ipu}(0) + K_{pu}\left\{A - \frac{K_{pu}A}{K_{pr}+K_{pu}} - \frac{K_{pr}A}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}\right] = 0. \tag{63}$$

After substituting the value of $A$ from eq. (62) into (63), the later becomes as follows:

$$\lambda = \frac{J_A}{\sqrt{w_{ipu}^2(0) - 4Q_A G_A}}.$$

$\lambda$ must be positive as the social damage increases with an increase in $G_A$.

$$\text{where } Q_A = -K_{pu}\left\{1 - \frac{K_{pu}}{K_{pr}+K_{pu}} - \frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\},$$
$$J_A = \frac{1}{K_c}\left[\left\{\frac{-K_{pu}}{K_{pr}+K_{pu}} - \frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\} + \{1 + K_c K_{pr}\}\right].$$

Eq. (62) can also be written as

$$A = \frac{\lambda w_{ipu}(0) - J_A}{2\lambda Q_A}. \tag{64}$$

Plugging the value of $\lambda$ into eq. (64) leads to:

$$A = \frac{w_{ipu}(0) - \sqrt{w_{ipu}^2(0) - 4Q_A G_A}}{2Q_A}. \tag{65}$$

$A$ is a policy in a dynamical setting for an optimal number of crime control units. The second order condition for minimization is checked (see appendix). An interesting question would be whether it was possible for the government to follow a policy such that all adjustments could be made instantaneously, such that market clearing was never an issue; and the answer is that it could have been possible only if the government would instantaneously fix the crime control rate and the number of public service units at the new equilibrium values after the policy change. For this, the government would have to know the exact equilibrium point, as well as the exact patterns of supply and demand of number of public service units, which seems impracticable.

## 8 Conclusion

When the government exercises a crime control policy for panel B, government's supply curve shifts downward/upward, which affects the number of crimes controlled and pushes crime control market out of equilibrium. Supply and demand of public service in terms of number of crimes controlled along with the cost adjust over time to lead to the final equilibrium. There are efficiency losses/gains on the dynamic adjustment path as well as in final equilibrium in comparison with the initial equilibrium. The efficiency losses during the adjustment process must also be accounted for while formulating an optimal crime control policy. Eq (33) gives a policy for an optimal number of crimes controlled considering the demand and supply adjustment over time. The expressions are a function of the slopes of demand, supply and cumulative number (a function of supply and demand) curves as well as the initial pre-policy equilibrium number of crimes controlled.

For panel A, crime control rate depends on the parameters $K_c$, $K_{pr}$, $K_{pu}$, $\tau_{d1}$ and $\tau_{d2}$. For given values (estimated through data) of these parameters, we can predict how the crime control rate will change over time, as a result of an exogenous shock resulting in the shift of either the supply, demand or both curves. Figure 3 depicts how a shift in the supply, demand or both curves determines the crime control rate and the number of crime control units in society. An optimal policy (which shifts either the supply, demand or both curves) which minimizes the social damage in terms of inadequate number of crime control/public service units in the initial equilibrium as well as the social loss on dynamic adjustment path (when the number of units is not in equilibrium) subject to a certain increase in number of crimes controlled per unit time can be derived on a case by case basis. In equilibrium, the area under the demand curve is the social benefit in terms of number of crimes controlled per unit time.

# 9 Appendix:

## 9.1 Dynamic Problem of the Tax Policy Maker/Budget Allocator (TPM/BA)

This section discusses the dynamic problem of TPM/BA. Present discounted value of future stream of net social benefits are maximized in a dynamic environment, and the present value at time zero is given below:

$$V(0) = \int_0^\infty \left[U_r(r) - \varsigma_B(m_B(r, e_B))\right] e^{-\sigma t} dt, \tag{66}$$

$\sigma$ denotes the discount rate. $r(t)$ is the *control variable* and $m_B(t)$ the *state variable*. Maximization problem is as follows:

$$\underset{\{r(t)\}}{Max} V(0) = \int_0^\infty \left[U_r(r) - \varsigma_B(m_B(r, e_B))\right] e^{-\sigma t} dt,$$

subject to the constraints that

$\dot{m_B}(t) = m'_{B1}(r(t), e_B(r(t), z_B))\dot{r}(t) + m'_{B2}(r(t), e_B(r(t), z_B))e'_{B1}(r(t), z_B)\dot{r}(t)$ (state equation, describing how the state variable changes with time; $z_B$ are exogenous factors),

$m_B(0) = m_{Bs}$ (initial condition),

$m_B(t) \geq 0$ (non-negativity constraint on state variable),

$m_B(\infty)$ free (terminal condition).

The current-value Hamiltonian is as follows:

$$\widetilde{H} = U_r(r\,(t)) - \varsigma_B(m_B(r(t), e_B(r(t), z_B))) + \mu_B(t)\dot{r}(t)\left[\begin{array}{c} m'_{B1}(r(t), e_B(r(t), z_B)) + m'_{B2}(r(t), e_B(r(t), z_B))* \\ e'_{B1}(r(t), z_B) \end{array}\right]. \tag{67}$$

Now the maximizing conditions are as follows:
$(i)$ $r^*(t)$ maximizes $\widetilde{H}$ for all $t$: $\frac{\partial \widetilde{H}}{\partial r} = 0$,
$(ii)$ $\dot{\mu_B} - \sigma\mu_B = -\frac{\partial \widetilde{H}}{\partial m_B}$,
$(iii)$ $\dot{m_B}^* = \frac{\partial \widetilde{H}}{\partial \mu_B}$ (this just gives back the state equation),
$(iv)$ $\lim\limits_{t\to\infty} \mu_B(t) m_B(t) e^{-\sigma t} = 0$ (the transversality condition).
The first two conditions are as follows:

$$\begin{aligned}\frac{\partial \widetilde{H}}{\partial r} &= U_r'(r\,(t)) - \varsigma_B'(m_B(r(t), e_B(r(t), z_B)))\left\{\begin{array}{c} m'_{B1}(r(t), e_B(r(t), z_B)) + m'_{B2}(r(t), e_B(r(t), z_B))* \\ e'_{B1}(r(t), z_B) \end{array}\right\} \\ &+ \mu_B(t)\dot{r}(t) * \left[\begin{array}{c} m''_{B11}(r(t), e_B(r(t), z_B)) + m''_{B12}(r(t), e_B(r(t), z_B)) e'_{B1}(r(t), z_B) + \\ m''_{B21}(r(t), e_B(r(t), z_B)) e'_{B1}(r(t), z_B) + m''_{B22}(r(t), e_B(r(t), z_B)) e'^2_{B1}(r(t), z_B) + \\ m'_{B2}(r(t), e_B(r(t), z_B)) e''_{B11}(r(t), z_B) \end{array}\right] \\ &= 0, \end{aligned} \tag{68}$$

and

$$\dot{\mu_B} - \sigma\mu_B = -\frac{\partial \widetilde{H}}{\partial m_B} = \varsigma_B'(m_B(r(t), e_B(r(t), z_B))). \tag{69}$$

In equilibrium, $\dot{r}(t) = 0$, and the expression $\frac{\partial \widetilde{H}}{\partial r}$ boils down to the following:

$$U_r'(r\,(t)) - \varsigma_B'(m_B(r(t), e_B(r(t), z_B)))\left\{\begin{array}{c} m'_{B1}(r(t), e_B(r(t), z_B)) + m'_{B2}(r(t), e_B(r(t), z_B))* \\ e'_{B1}(r(t), z_B) \end{array}\right\} = 0.$$

If supply curve shifts to the right, then the number of crimes controlled is higher at the existing cost, and the term $\varsigma_B'(m_B(r(t), e_B(r(t), z_B)))$ is higher at the existing cost at that time. The term multiplying $\varsigma_B'(m_B(r(t), e_B(r(t), z_B)))$, i.e., $m'_{B1}(r(t), e_B(r(t), z_B)) + m'_{B2}(r(t), e_B(r(t), z_B)) e'_{B1}(r(t), z_B)$ is a function of cost and has not changed as the cost is the same as before. Therefore, the TPM/BA now faces the following inequality at the existing cost:

$$\begin{aligned}\frac{\partial \widetilde{H}}{\partial r} &= U_r'(r\,(t)) - \varsigma_B'(m_B(r(t), e_B(r(t), z_B)))\left\{\begin{array}{c} m'_{B1}(r(t), e_B(r(t), z_B)) + m'_{B2}(r(t), e_B(r(t), z_B))* \\ e'_{B1}(r(t), z_B) \end{array}\right\} \\ &+ \mu_B(t)\dot{r}(t) * \left[\begin{array}{c} m''_{B11}(r(t), e_B(r(t), z_B)) + m''_{B12}(r(t), e_B(r(t), z_B)) e'_{B1}(r(t), z_B) + \\ m''_{B21}(r(t), e_B(r(t), z_B)) e'_{B1}(r(t), z_B) + m''_{B22}(r(t), e_B(r(t), z_B)) e'^2_{B1}(r(t), z_B) + \\ m'_{B2}(r(t), e_B(r(t), z_B)) e''_{B11}(r(t), z_B) \end{array}\right] \\ &< 0, \end{aligned}$$

The TPM/BA must decrease the cost for satisfying the net social benefit optimization condition after the shock. This implies that there is a negative relationship between the cumulative number of crimes controlled in society and the cost. If the rate of supply of public service in terms of number of crimes controlled is equal to the demand rate, the number of crimes controlled is in equilibrium. If a difference of a finite magnitude comes into force between the supply and demand rates, and the public and the private sector do not react to a change in the cost caused by a difference in the supply and demand rates, the cost will continue changing until the saturation point of society comes. The response of TPM/BA can be depicted by the following formulation:

$$\begin{aligned}
\textit{Cost rate change} &\propto \textit{change in cumulative no. of controlled crime.} \\
R &= \textit{cost rate change.} \\
M_B &= m_B - m_{Bs} = \textit{change in cumulative no. of controlled crime,} \\
m_B &= \textit{cumulative no. of controlled crime at time } t, \\
m_{Bs} &= \textit{cumulative no. of controlled crime in steady state equilibrium.} \\
\textit{Input} - \textit{output} &= \frac{dm_B}{dt} = \frac{d(m_B - m_{Bs})}{dt} = \frac{dM_B}{dt}, \\
\text{or } M_B &= \int (\textit{input} - \textit{output})\, dt. \\
\textit{Cost rate change} &\propto \int (\textit{supply rate} - \textit{demand rate})\, dt, \text{ or} \\
R &= -K_m \int (\textit{supply rate} - \textit{demand rate})\, dt,
\end{aligned}$$

where $K_m$ is the proportionality constant. A negative sign indicates that when (*supply rate* − *demand rate*) is positive, $R$ is negative, i.e., the cost decreases. The above expression can also be written as:

$$\int (\textit{supply rate} - \textit{demand rate})\, dt = -\frac{R}{K_m}, \text{ or}$$

$$\int (w_{Bi} - w_{B0})\, dt = -\frac{R}{K_m}, \tag{70}$$

$$\begin{aligned}
w_{Bi} &= \textit{supply rate}, \\
w_{B0} &= \textit{demand rate}, \\
K_m &= \textit{dimensional constant.}
\end{aligned}$$

When $t = 0$, *supply rate* = *demand rate*, i.e., public service market is in equilibrium and eq. (70) can be expressed as:

$$\int (w_{Bis} - w_{B0s})\, dt = 0. \tag{71}$$

The subscript $s$ denotes steady state equilibrium and $R = 0$ in steady state. Subtracting eq. (71) from eq. (70), we get:

$$\int (w_{Bi} - w_{Bis})\, dt - \int (w_{B0} - w_{B0s})\, dt = -\frac{R}{K_m}, \text{ or}$$

$$\int (W_{Bi} - W_{B0})\, dt = -\frac{R}{K_m}, \tag{72}$$

$$\text{where } w_{Bi} - w_{Bis} = W_{Bi} = \textit{change in supply rate},$$
$$w_{B0} - w_{B0s} = W_{B0} = \textit{change in demand rate}.$$

$R$, $W_{Bi}$ and $W_{B0}$ are deviation variables, i.e., deviation from steady state equilibrium and have zero initial values. Eq. (72) can also be expressed as:

$$R = -K_m \int W_B dt = -K_m M_B, \tag{73}$$

where $W_B = W_{Bi} - W_{B0}$. If $R$ gets a jump as a result of some factor other than a change in cumulative number of crimes controlled, that is another input which can be added to eq. (73) as follows:

$$R = -K_m \int W_B dt + E_B = -K_m M_B + E_B. \tag{73a}$$

There can also be an exogenous shock in cumulative number of crimes controlled other than the feedback of cost.

### 9.2 Solution of the Model-Panel B with a Crime Control Policy

Expressions from eqs. (11$a$), (18),and (23) respectively along with $\tau_{d1} = 0$ are as follows:

$$\frac{dR(t)}{dt} = -K_m W_B(t),$$
$$W_d(t) = -K_d R(t),$$
$$W_m = -K_s (C_c - R),$$

and

$$W_B(t) = W_m(t) - W_d(t),$$

if no exogenous demand or supply shock happens. $W_m(t)$ is number of crimes controlled. Combining the above expressions together, we can write:

$$W_m(t) = -K_{sp} [C_p(t) - R(t)] - K_{sc} [C_c(t) - R(t)], \tag{74}$$

where the $p$ and $c$ subscripts denote the private and the public sector respectively. Now, combining the above expressions together, we can write:

$$\begin{aligned}\frac{dR(t)}{dt} &= -K_m [W_m(t) - W_d(t)] \\ &= -K_m [-K_s \{C_c(t) - R(t)\} + K_d R(t)] \\ &= -K_m [-K_s C_c(t) + (K_s + K_d) R(t)].\end{aligned}$$

Rearranging the above expression gives:

$$\frac{dR(t)}{dt} + K_m(K_s + K_d)R(t) = K_m K_s C_c(t). \tag{75}$$

The Routh-Hurwitz stability criterion (which provides a necessary and sufficient condition for stability of a linear dynamical system) for the above differential equation's stability is $K_m(K_s + K_d) > 0$; and as $K_m$,$K_s$, and $K_d$ are all defined as positive numbers, this criterion holds. This ensures that, away from a given initial equilibrium, every adjustment mechanism will lead to another equilibrium.
Suppose government reduces the per crime controlled cost of the public service provider by $B$, say through provision of some funds to buy an advanced technology for crime control, the above equation can be written as:

$$\frac{dR(t)}{dt} + K_m(K_s + K_d)R(t) = -K_m K_s B. \tag{76}$$

The solution is given by the following expression:

$$R(t) = -\frac{K_s B}{(K_s + K_d)} + \frac{K_s B}{(K_s + K_d)} e^{-[K_m(K_s+K_d)]t}. \tag{77}$$

$R(0) = 0$ (the initial condition), and $R(\infty) = -\frac{K_s B}{(K_s+K_d)}$ (the final steady state equilibrium value). In response to a policy, the per unit crime controlled cost dynamics depends on the parameters $K_s, K_d$, $K_m$ and $B$.

### 9.3 A Dynamic Optimal Crime Control Policy-Panel B

After a crime control policy, there are efficiency gains in post policy equilibrium in comparison with the initial/pre-policy equilibrium. However, there are also some efficiency losses during the adjustment period of crime control market until new equilibrium arrives. As soon as crime control policy is implemented, supply of public service expands, whereas the demand remains the same at the initial per unit cost, pushing the crime control market out of equilibrium. Now the adjustment of per unit cost begins to equalize the supply and demand to bring the new crime control market equilibrium. The post policy equilibrium cost is a function of demand and supply elasticities. From the previous section, change in supply as a result of a crime control policy is as follows:

$$\begin{aligned} &W_m(0) = -K_s\left[C_c(0) - R(0)\right] = K_s B, \\ &\text{as } R(0) = 0. \end{aligned} \tag{78}$$

As a result of crime control policy, supply of public service goes up by $K_s B$. As demand does not change, therefore, cumulative number of crimes controlled also goes up by $K_s B$. Now the market is out of equilibrium, and the market forces push the crime control market toward a new equilibrium through the movement in per unit cost. As the per unit cost changes, the demand and supply of public service also change through feedback. If cumulative number of crime controlled goes up, it indicates a higher supply than demand and vice versa. There is no efficiency loss if crime control market is in equilibrium, and demand and supply are same. If market is out of equilibrium, either supply or demand is excessive at that point in time. Therefore, the total efficiency loss during the adjustment of crime control market is a sum of the differences in supply and demand at all points in time. Total efficiency loss for a crime control policy can be expressed as:

$$\begin{aligned} EL &= \int_{-\infty}^{0} W_m(\infty)dt + \int_{0}^{\infty} \left[W_m(t) - W_d(t)\right] dt \\ &= \int_{-\infty}^{0} W_m(\infty)dt + M_B(t). \end{aligned} \tag{79}$$

Eq. (73$a$) states the following:

$$R(t) = -K_m M_B(t) + E_B.$$

By imposing the initial conditions, we can determine the value of $E_B$ as follows:

$$\begin{aligned} R(0) &= -K_m M_B(0) + E_B, \\ 0 &= -K_m K_s B + E_B, \\ E_B &= K_m K_s B. \end{aligned}$$

After plugging in the above expression in eq. (73$a$), it transforms to

$$\begin{aligned} R(t) &= -K_m M_B(t) + K_m K_s B, \text{ or} \\ M_B(t) &= -\frac{1}{K_m}\left[R(t) - K_m K_s B\right]. \end{aligned}$$

**With Crime Control Policy Cost Constraint:**
According to eq. (23), public service supply change due to change in per unit cost is:

$$W_m(t) = -K_s \left[C_c(t) - R(t)\right].$$

It can also be written as:

$$w_{nm}(t) - w_{im}(0) = -K_s \left[C_c(t) - R(t)\right],$$

where $w_{im}(0)$ is the initial public service supply, and $w_{nm}(t)$ is the new supply after government exercises crime control policy. $W_m(t) = w_{nm}(t) - w_{im}(0)$, as $W_m(t)$ is a deviation variable, i.e., deviation from the initial steady state equilibrium value. The crime control policy cost ($CCPC$) can be expressed as:

$$CCPC = B\left[w_{im}(0) + K_s \left\{B + R(t)\right\}\right]. \tag{80}$$

Our problem of minimising efficiency loss subject to crime control policy cost constraint is as follows:

$$\min_B EL \quad \text{s.t.} \quad CCPC \leq G_B.$$

$G_B$ is the government's cost for exercising crime control policy. The choice variable is crime control policy, i.e., $B$, and the constraint is binding at $t = 0$. The Lagrangian for the above problem can be written as follows:

$$\begin{aligned}
\mathcal{L} &= \int_{-\infty}^{0} W_m(\infty)dt + M_B(t) + \lambda\left[G_B - B\left[w_{im}(0) + K_s\left\{B + R(t)\right\}\right]\right] \\
&= \int_{-\infty}^{0} \left[K_s B - \frac{K_s^2 B}{(K_s + K_d)}\right] dt \\
&- \frac{1}{K_m}\left[-\frac{K_s B}{(K_s + K_d)} + \frac{K_s B}{(K_s + K_d)} e^{-[K_m(K_s+K_d)]t} - K_m K_s B\right] \\
&+ \lambda\left[G_B - B\left[w_{im}(0) + K_s\left\{B - \frac{K_s B}{(K_s + K_d)} + \frac{K_s B}{(K_s + K_d)} e^{-[K_m(K_s+K_d)]t}\right\}\right]\right] \\
&= \int_{-\infty}^{0} \frac{K_s K_d B}{(K_s + K_d)} dt - \frac{1}{K_m}\left[-\frac{K_s B}{(K_s + K_d)} + \frac{K_s B}{(K_s + K_d)} e^{-[K_m(K_s+K_d)]t} - K_m K_s B\right] \\
&+ \lambda\left[G_B - B\left[w_{im}(0) + K_s\left\{B - \frac{K_s B}{(K_s + K_d)} + \frac{K_s B}{(K_s + K_d)} e^{-[K_m(K_s+K_d)]t}\right\}\right]\right].
\end{aligned}$$

The first order condition with respect to $B$ leads to the following:

$$\begin{aligned}
&\int_{-\infty}^{0} \frac{K_s K_d}{(K_s + K_d)} dt - \frac{1}{K_m}\left[-\frac{K_s}{(K_s + K_d)} + \frac{K_s}{(K_s + K_d)} e^{-[K_m(K_s+K_d)]t} - K_m K_s\right] \\
&- \lambda\left[w_{im}(0) + K_s\left\{B - \frac{K_s B}{(K_s + K_d)} + \frac{K_s B}{(K_s + K_d)} e^{-[K_m(K_s+K_d)]t}\right\}\right] \\
&\qquad -\lambda B K_s\left[1 - \frac{K_s}{(K_s + K_d)} + \frac{K_s}{(K_s + K_d)} e^{-[K_m(K_s+K_d)]t}\right] = 0.
\end{aligned}$$

Rearranging this, we get:

$$\begin{aligned}
&\int_{-\infty}^{0} \frac{K_s K_d}{(K_s + K_d)} dt - \frac{1}{K_m}\left[-\frac{K_s}{(K_s + K_d)} + \frac{K_s}{(K_s + K_d)} e^{-[K_m(K_s+K_d)]t} - K_m K_s\right] \\
&- 2\lambda B K_s\left[1 - \frac{K_s}{(K_s + K_d)} + \frac{K_s}{(K_s + K_d)} e^{-[K_m(K_s+K_d)]t}\right] \\
&\qquad\qquad = \lambda w_{im}(0),
\end{aligned}$$

or

$$B = -\frac{\lambda w_{im}(0) - \left[\int\limits_{-\infty}^{0} \frac{K_s K_d}{(K_s+K_d)} dt - \frac{1}{K_m}\left[-\frac{K_s}{(K_s+K_d)} + \frac{K_s}{(K_s+K_d)} e^{-[K_m(K_s+K_d)]t} - K_m K_s\right]\right]}{2\lambda K_s\left[1 - \frac{K_s}{(K_s+K_d)} + \frac{K_s}{(K_s+K_d)} e^{-[K_m(K_s+K_d)]t}\right]}. \quad (81)$$

The derivative with respect to $\lambda$, is as follows:

$$G_B - B\left[w_{im}(0) + K_s\left\{B - \frac{K_s B}{(K_s+K_d)} + \frac{K_s B}{(K_s+K_d)} e^{-[K_m(K_s+K_d)]t}\right\}\right] = 0. \quad (82)$$

Putting eq. (81) into (82), we get:

$$G_B =$$

$$-w_{im}(0).\frac{\lambda w_{im}(0) - \left[\int\limits_{-\infty}^{0} \frac{K_s K_d}{(K_s+K_d)} dt - \frac{1}{K_m}\left[-\frac{K_s}{(K_s+K_d)} + \frac{K_s}{(K_s+K_d)} e^{-[K_m(K_s+K_d)]t} - K_m K_s\right]\right]}{2\lambda K_s\left[1 - \frac{K_s}{(K_s+K_d)} + \frac{K_s}{(K_s+K_d)} e^{-[K_m(K_s+K_d)]t}\right]}$$

$$+K_s\left\{1 - \frac{K_s}{(K_s+K_d)} + \frac{K_s}{(K_s+K_d)} e^{-[K_m(K_s+K_d)]t}\right\}$$

$$* \left[-\frac{\lambda w_{im}(0) - \left[\int\limits_{-\infty}^{0} \frac{K_s K_d}{(K_s+K_d)} dt - \frac{1}{K_m}\left[-\frac{K_s}{(K_s+K_d)} + \frac{K_s}{(K_s+K_d)} e^{-[K_m(K_s+K_d)]t} - K_m K_s\right]\right]}{2\lambda K_s\left[1 - \frac{K_s}{(K_s+K_d)} + \frac{K_s}{(K_s+K_d)} e^{-[K_m(K_s+K_d)]t}\right]}\right]^2,$$

$$\text{or } 4\lambda^2 Q_B G_B = -2\lambda^2 w_{im}^2(0) + 2\lambda w_{im}(0) J_B + \lambda^2 w_{im}^2(0) + J_B^2 - 2\lambda w_{im}(0) J_B,$$

$$\text{where } Q_B = K_s\left[1 - \frac{K_s}{(K_s+K_d)} + \frac{K_s}{(K_s+K_d)} e^{-[K_m(K_s+K_d)]t}\right],$$

$$J_B = \int\limits_{-\infty}^{0} \frac{K_s K_d}{(K_s+K_d)} dt - \frac{1}{K_m}\left[-\frac{K_s}{(K_s+K_d)} + \frac{K_s}{(K_s+K_d)} e^{-[K_m(K_s+K_d)]t} - K_m K_s\right].$$

or

$$\left\{w_{im}^2(0) + 4Q_B G_B\right\}\lambda^2 - J_B^2 = 0.$$

$$\lambda = \frac{J_B}{\sqrt{w_{im}^2(0) + 4Q_B G_B}}.$$

$\lambda$ is a positive number because when $G_B$ increases, the minimum efficiency loss also increases. From eq. (81):

$$B = -\frac{\lambda w_{im}(0) - J_B}{2\lambda Q_B}. \tag{83}$$

By replacing $\lambda$ with its value in the above expression, we get:

$$B = -\frac{\frac{w_{im}(0)J_B}{\sqrt{w_{im}^2(0)+4Q_B G_B}} - J_B}{\frac{2Q_B J_B}{\sqrt{w_{im}^2(0)+4Q_B G_B}}},$$

$$B = -\frac{w_{im}(0) - \sqrt{w_{im}^2(0) + 4Q_B G_B}}{2Q_B}. \tag{84}$$

The second order condition for minimization can be checked as follows:

$$\mathcal{L} = J_B B + \lambda \left[ G_B - B \left( w_{im}(0) + Q_B B \right) \right].$$

Now we write the Bordered Hessian matrix of the Lagrange function as:

$$BH = \begin{bmatrix} 0 & w_{im}(0) + 2Q_B B \\ w_{im}(0) + 2Q_B B & \frac{-2Q_B J_B}{\sqrt{w_{im}^2(0)+4Q_B G_B}} \end{bmatrix}.$$

The determinant of the above matrix is negative as $-\left(w_{im}(0) + 2Q_B B\right)^2 < 0$, and hence the efficiency loss got minimized.

### 9.4 A Dynamic Optimal Crime Control Policy-Panel A

The social damage due to inadequate/excessive crime control units includes the damage in the initial equilibrium, i.e., before the adoption of a crime control policy, plus the damage during the adjustment process from initial equilibrium to the final. After government adopts a policy to enhance/reduce the number of public service units, it shifts either the supply or the demand curve, e.g., it shifts the demand curve upward by a magnitude depending upon the size of the policy, which is taken as $A$ in the solution of the model with a crime control policy. The crime control rate then adjusts over time to bring the new equilibrium rate which is higher than the previous equilibrium crime control rate and lower than that which existed at the time the policy was implemented depending on the elasticity of supply and demand curves. An excessive number of public service units in society implies that the number is higher on the supply curve than that on the demand curve, and a shortage in their number implies the opposite. When the number on the supply and the demand curve becomes equal, the new equilibrium has arrived. When the number is different on supply and demand curve, that difference is the social damage at that point in time. Furthermore, the number of crime control units in society was lower (in this example) in the previous equilibrium, which is also social damage in equilibrium. If we sum up either the number of excessive units on the supply curve or their number on the demand curve short of supply curve, we get the total social damage in terms of number of units as follows:

$$SD = M_A(t) + \int_{-\infty}^{0} W_{pr}(\infty)dt. \tag{85}$$

From eq. (**??**), the change in number of crime control units due to change in crime control rate after adoption of crime control policy is as under:

$$\begin{aligned} W_{pu}(t) &= K_{pu}\left[A - C(t)\right], \\ \text{or } w_{npu}(t) - w_{ipu}(0) &= K_{pu}\left[A - C(t)\right], \end{aligned}$$

where $w_{ipu}(0)$ is the initial number of crime control units and $w_{npu}(t)$ is the new number after the implementation of crime control policy as $W_{pu}(t)$ is a deviation variable, i.e., deviation from the initial equilibrium value. An increase in number of crime control units per unit time is as follows:

$$INC = A\left[w_{ipu}(0) + K_{pu}\left\{A - C(t)\right\}\right]. \tag{86}$$

If we want to minimize the social damage subject to the constraint that an increase in number of crimes controlled per unit time is greater than or equal to $G_A$ $\Big($change in number of crimes controlled per unit time our problem is as follows:

$$\min_{A} SD \quad \text{s.t.} \quad INC \geq G_A\left(= \frac{dM_B}{dt}\right).$$

The choice variable is $A$, i.e., an initial upward jump in the crime control rate chosen by government to shift the demand curve, and the constraint is binding. Lagrangian for the above problem is given below:

$$\mathcal{L} = M_A(t) + \int_{-\infty}^{0} W_{pr}(\infty)dt + \lambda\left[G_A - A\left[w_{ipu}(0) + K_{pu}\left\{A - C(t)\right\}\right]\right].$$

From eq. (44$a$), we have:

$$C(t) = -K_c M_A + E_A.$$

The value of $E_A$ can be found by imposing the initial conditions as follows:

$$\begin{aligned} C(0) &= -K_c M_A(0) + E_A, \\ A &= -K_c K_{pr} C(0) + E_A, \\ E_A &= A\left[1 + K_c K_{pr}\right]. \end{aligned}$$

This implies that

$$M_A(t) = -\frac{1}{K_c}\left[C(t) - A\left\{1 + K_c K_{pr}\right\}\right].$$

Therefore, the Lagrangian can now be written as:

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{K_c}\left[C(t) - A\left\{1 + K_c K_{pr}\right\}\right] + \int_{-\infty}^{0} W_{pr}(\infty)dt + \lambda\left[G_A - A\left[w_{ipu}(0) + K_{pu}\left\{A - C(t)\right\}\right]\right] \\
&= -\frac{1}{K_c}\left[\left\{\frac{K_{pu}A}{K_{pr}+K_{pu}} + \frac{K_{pr}A}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\} - A\left\{1 + K_c K_{pr}\right\}\right] \\
&+ \int_{-\infty}^{0} W_{pr}(\infty)dt + \lambda\left[G_A - A\left[w_{ipu}(0) + K_{pu}\left\{A - \frac{K_{pu}A}{K_{pr}+K_{pu}} - \frac{K_{pr}A}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}\right]\right].
\end{aligned}$$

The first order condition with respect to $A$ is as follows:

$$\begin{aligned}
&-\frac{1}{K_c}\left[\left\{\frac{K_{pu}}{K_{pr}+K_{pu}} + \frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\} - \left\{1 + K_c K_{pr}\right\}\right] \\
&-\lambda\left[w_{ipu}(0) + K_{pu}\left\{A - \frac{K_{pu}A}{K_{pr}+K_{pu}} - \frac{K_{pr}A}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}\right] \\
&-\lambda A K_{pu}\left\{1 - \frac{K_{pu}}{K_{pr}+K_{pu}} - \frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\},
\end{aligned}$$

which implies that

$$\begin{aligned}
&-\frac{1}{K_c}\left[\left\{\frac{K_{pu}}{K_{pr}+K_{pu}} + \frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\} - \left\{1 + K_c K_{pr}\right\}\right] \\
&-2\lambda A K_{pu}\left\{1 - \frac{K_{pu}}{K_{pr}+K_{pu}} - \frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\} \\
&= \lambda w_{ipu}(0).
\end{aligned}$$

or

$$A = \frac{\lambda w_{ipu}(0) - \frac{1}{K_c}\left[\left\{\frac{-K_{pu}}{K_{pr}+K_{pu}} - \frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\} + \left\{1 + K_c K_{pr}\right\}\right]}{-2\lambda K_{pu}\left\{1 - \frac{K_{pu}}{K_{pr}+K_{pu}} - \frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}}. \tag{87}$$

The first order condition of the Lagrangian with respect to $\lambda$ is as follows:

$$G_A - A\left[w_{ipu}(0) + K_{pu}\left\{A - \frac{K_{pu}A}{K_{pr}+K_{pu}} - \frac{K_{pr}A}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}\right] = 0. \tag{88}$$

After substituting the value of $A$ from eq. (87) into (88), the later becomes as follows:

$$G_A = w_{ipu}(0).\frac{\lambda w_{ipu}(0) - \frac{1}{K_c}\left[\left\{\frac{-K_{pu}}{K_{pr}+K_{pu}} - \frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\} + \{1 + K_c K_{pr}\}\right]}{-2\lambda K_{pu}\left\{1 - \frac{K_{pu}}{K_{pr}+K_{pu}} - \frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}}$$

$$+ K_{pu}\left\{1 - \frac{K_{pu}}{K_{pr}+K_{pu}} - \frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}$$

$$* \left[\frac{\lambda w_{ipu}(0) - \frac{1}{K_c}\left[\left\{\frac{-K_{pu}}{K_{pr}+K_{pu}} - \frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\} + \{1 + K_c K_{pr}\}\right]}{-2\lambda K_{pu}\left\{1 - \frac{K_{pu}}{K_{pr}+K_{pu}} - \frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}}\right]^2.$$

$$\text{or } 4\lambda^2 Q_A G_A = 2\lambda^2 w_{ipu}^2(0) - 2\lambda w_{ipu}(0) J_A - \lambda^2 w_{ipu}^2(0) - J_A^2 + 2\lambda w_{ipu}(0) J_A,$$

$$\text{where } Q_A = -K_{pu}\left\{1 - \frac{K_{pu}}{K_{pr}+K_{pu}} - \frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\},$$

$$J_A = \frac{1}{K_c}\left[\left\{\frac{-K_{pu}}{K_{pr}+K_{pu}} - \frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\} + \{1 + K_c K_{pr}\}\right].$$

This implies that

$$\left\{w_{ipu}^2(0) - 4Q_A G_A\right\}\lambda^2 - J_A^2 = 0,$$

$$\lambda = \frac{J_A}{\sqrt{w_{ipu}^2(0) - 4Q_A G_A}}.$$

$\lambda$ must be positive as the social damage increases with an increase in $G_A$.
Eq. (87) can also be written as

$$A = \frac{\lambda w_{ipu}(0) - J_A}{2\lambda Q_A}. \tag{89}$$

Plugging the value of $\lambda$ into eq. (89) leads to:

$$A = \frac{\frac{w_{ipu}(0)J_A}{\sqrt{w_{ipu}^2(0) - 4Q_A G_A}} - J_A}{\frac{2Q_A J_A}{\sqrt{w_{ipu}^2(0) - 4Q_A G_A}}},$$

$$A = \frac{w_{ipu}(0) - \sqrt{w_{ipu}^2(0) - 4Q_A G_A}}{2Q_A}. \tag{90}$$

The second order condition for minimization is checked as follows:

$$\mathcal{L} = J_A A + \int_{-\infty}^{0} W_{pr}(\infty)dt + \lambda\left[G_A - A\left[w_{ipu}(0) - Q_A A\right]\right].$$

The Bordered Hessian matrix of the Lagrange function is as follows:

$$BH = \begin{bmatrix} 0 & w_{ipu}(0) - 2Q_A A \\ w_{ipu}(0) - 2Q_A A & \frac{2Q_A J_A}{\sqrt{w_{ipu}^2(0) - 4Q_A G_A}} \end{bmatrix},$$

which has a negative determinant as $-(w_{ipu}(0) - 2Q_A A)^2 < 0$, therefore the social damage is minimized.

## 10 General Solution of Model-Panel A

Figure 4 depicts a dynamic crime control model after joining together the blocks of inputs and outputs for various agents. Laplace transform is a convenient tool for solving differential equations. After Laplace transform, Figure 4 gets transformed to Figure 5. Let us first evaluate the transfer function relating $C(s)$ to $W_1(s)$ in Figure 5 (the part marked as $A$) as follows:
We have the following equations relating inputs to outputs for various blocks in $A$ assuming that $\epsilon(s) = 0$:

$$\begin{aligned} C(s) &= -\frac{K_c}{s}.W(s), \\ W_{pu}(s) &= -K_{pu}e^{-s\tau_{d2}}C(s), \\ W_B(s) &= W_1(s) - W_{pu}(s). \end{aligned}$$

We can solve the above equations simultaneously for $C(s)$ in terms of $W_1(s)$ as follows:

$$\begin{aligned} C(s) &= -\frac{K_c}{s}\left[W_1(s) - W_{pu}(s)\right], \\ C(s) &= -\frac{K_c}{s}\left[W_1(s) + K_{pu}e^{-s\tau_{d2}}C(s)\right], \\ C(s)\left[1 + \frac{K_c K_{pu} e^{-s\tau_{d2}}}{s}\right] &= -\frac{K_c}{s}W_1(s), \\ \frac{C(s)}{W_1(s)} &= \frac{-K_c}{s + K_c K_{pu} e^{-s\tau_{d2}}}. \end{aligned}$$

Using the above expression to reduce part $A$ in Figure 5 to one block and shifting $W_0(s)$ in backward direction, results in Figure 6, from which we can find the overall transfer function for $D(s)$. We have the following equations:

$$\begin{aligned} C(s) &= \frac{-K_c}{s + K_c K_{pu} e^{-s\tau_{d2}}}\left[D(s) + W_{pr}(s)\right], \\ \text{where } D(s) &= W_i(s) - W_0(s), \\ W_{pr}(s) &= K_{pr}e^{-s\tau_{d1}}C(s). \end{aligned}$$

We can solve for $C(s)$ in terms of $D(s)$ as follows:

$$C(s) = \frac{-K_c}{s + K_c K_{pu} e^{-s\tau_{d2}}} \left[ D(s) + W_{pr}(s) \right],$$

$$C(s) = \frac{-K_c}{s + K_c K_{pu} e^{-s\tau_{d2}}} \left[ D(s) + K_{pr} e^{-s\tau_{d1}} C(s) \right],$$

$$C(s) \left[ 1 + \frac{K_c K_{pr} e^{-s\tau_{d1}}}{s + K_c K_{pu} e^{-s\tau_{d2}}} \right] = \frac{-K_c}{s + K_c K_{pu} e^{-s\tau_{d2}}} D(s),$$

$$\frac{C(s)}{D(s)} = \frac{\frac{-K_c}{s + K_c K_{pu} e^{-s\tau_{d2}}}}{1 + \frac{K_c K_{pr} e^{-s\tau_{d1}}}{s + K_c K_{pu} e^{-s\tau_{d2}}}},$$

$$\frac{C(s)}{D(s)} = \frac{-K_c}{s + K_c K_{pu} e^{-s\tau_{d2}} + K_c K_{pr} e^{-s\tau_{d1}}}. \tag{91}$$

$K_c$, $K_{pu}$, $K_{pr}$, $\tau_{d1}$ and $\tau_{d2}$ are all positive numbers and the crime control rate depends on these five empirical parameters. Useful results and conclusions can be drawn by inversion and solution of eq. (91). If inversion of eq. (91) is to be done by partial fractions, then the following approximation has to be made:

$$e^{-\tau s} \approx 1 - \tau s. \tag{92}$$

Second better approximation is:

$$e^{-\tau s} \approx \frac{1 - (\tau/2)s}{1 + (\tau/2)s}. \tag{93}$$

A third approximation (better than the above two) is as follows:

$$e^{-\tau s} \approx \frac{1 - \tau s/2 + \tau^2 s^2/12}{1 + \tau s/2 + \tau^2 s^2/12}. \tag{94}$$

Eq. (92) gives a crude approximation. One could possibly choose either eq. (93) (which is simpler) or (94) (which is laborious but more accurate). If $D(t) = A$, a step input, i.e., an exogenous shift in either the number of public service units demanded, and/or a shift in supply, then after Laplace transform

$$D(s) = \frac{A}{s}.$$

Using the final value theorem of Laplace transform we get:

$$C(\infty) = \frac{-A}{K_{pu} + K_{pr}}, \tag{95}$$

$$C(\infty) = C(t) \mid_{t=\infty}.$$

Eq. (93) can be rewritten as:

$$e^{-\tau s} \approx \frac{2 - \tau s}{2 + \tau s}.$$

Using this approximation, eq. (91) can be written as:

$$\frac{C(s)}{D(s)} = \frac{-K_c}{s + K_c K_{pu}\left(\frac{2-s\tau_{d2}}{2+s\tau_{d2}}\right) + K_c K_{pr}\left(\frac{2-s\tau_{d1}}{2+s\tau_{d1}}\right)},$$

$$\frac{C(s)}{D(s)} = \frac{-K_c\left(2+s\tau_{d1}\right)\left(2+s\tau_{d2}\right)}{s\left(2+s\tau_{d1}\right)\left(2+s\tau_{d2}\right) + K_c K_{pu}\left(2+s\tau_{d1}\right)\left(2-s\tau_{d2}\right) + K_c K_{pr}\left(2-s\tau_{d1}\right)\left(2+s\tau_{d2}\right)},$$

$$= \frac{-K_c\left\{\tau_{d1}\tau_{d2}s^2 + 2\left(\tau_{d1}+\tau_{d2}\right)s + 4\right\}}{\left[\begin{array}{c}\tau_{d1}\tau_{d2}s^3 + 2\left(\tau_{d1}+\tau_{d2}\right)s^2 + 4s + K_c K_{pu}\left\{-\tau_{d1}\tau_{d2}s^2 + 2\left(\tau_{d1}-\tau_{d2}\right)s + 4\right\}\\ +K_c K_{pr}\left\{-\tau_{d1}\tau_{d2}s^2 + 2\left(\tau_{d2}-\tau_{d1}\right)s + 4\right\}\end{array}\right]},$$

$$= \frac{-K_c\left\{\tau_{d1}\tau_{d2}s^2 + 2\left(\tau_{d1}+\tau_{d2}\right)s + 4\right\}}{\left[\begin{array}{c}\tau_{d1}\tau_{d2}s^3 + \left[2\left(\tau_{d1}+\tau_{d2}\right) - K_c K_{pu}\tau_{d1}\tau_{d2} - K_c K_{pr}\tau_{d1}\tau_{d2}\right]s^2 + \\ \left[2K_c K_{pu}\left(\tau_{d1}-\tau_{d2}\right) + 2K_c K_{pr}\left(\tau_{d2}-\tau_{d1}\right) + 4\right]s + 4K_c K_{pu} + 4K_c K_{pr}\end{array}\right]}.$$

The denominator of the above expression can be written as:

$$as^3 + bs^2 + cs + d,$$

where

$$\begin{aligned}
a &= \tau_{d1}\tau_{d2},\\
b &= 2\left(\tau_{d1}+\tau_{d2}\right) - K_c\tau_{d1}\tau_{d2}(K_{pu}+K_{pr}),\\
c &= 2\left[K_c\left(\tau_{d1}-\tau_{d2}\right)\left(K_{pu}-K_{pr}\right) + 2\right],\\
d &= 4K_c(K_{pu}+K_{pr}).
\end{aligned}$$

This implies that

$$\frac{C(s)}{D(s)} = \frac{-K_c\left\{as^2 + 2\left(\tau_{d1}+\tau_{d2}\right)s + 4\right\}}{as^3 + bs^2 + cs + d}. \tag{96}$$

The roots of the denominator of eq. (96) depict the qualitative response of the crime control rate, therefore it will be convenient (for future reference) to write it as follows:

$$as^3 + bs^2 + cs + d = 0. \tag{97}$$

Now let us discuss the dimensions of the parameters involved. $\tau_{d1}$ and $\tau_{d2}$ and have the dimensions of time.

$$\begin{aligned}
Dimensions\ of\ K_c &= (Dimensions\ of\ C)/(time \times Dimensions\ of\ W_A) \\
&= \frac{Number\ of\ new\ crimes\ controlled}{time \times No.\ of\ new\ public\ service\ units} \\
Dimensions\ of\ K_{pu} &= (Dimensions\ of\ W_{pu})/(Dimensions\ of\ C) \\
&= \frac{No.\ of\ new\ public\ service\ units\ demanded}{Number\ of\ new\ crimes\ controlled}, \\
Dimensions\ of\ K_{pr} &= (Dimensions\ of\ W_{pr})/(Dimensions\ of\ C) \\
&= \frac{No.\ of\ new\ public\ service\ units\ supplied}{Number\ of\ new\ crimes\ controlled}.
\end{aligned}$$

Therefore $K_cK_{pu}$ and $K_cK_{pr}$ have dimensions of $1/time$. Using these facts, we can write: $a$ has dimensions of $time^2$; $b$ has dimensions of $time$; $c$ is dimensionless and $d$ has dimensions of $1/time$. We can see that eq. (97) is dimensionally consistent (as $s$ has dimensions of $1/time$).

**Method to Solve eq. (96):**

Let a step input of magnitude $A$ is given to $D$, then

$$D(s) = \frac{A}{s}. \tag{98}$$

Putting this in eq. (96), we get:

$$C(s) = \frac{-AK_c\left\{as^2 + 2\left(\tau_{d1} + \tau_{d2}\right)s + 4\right\}}{s(as^3 + bs^2 + cs + d)}. \tag{99}$$

The parameters $K_c$, $K_{pr}$, $K_{pu}$, $\tau_{d1}$ and $\tau_{d2}$ are to be estimated empirically. This gives the values of $a$, $b$, $c$ and $d$. Find roots of eq. (97) and invert eq. (99) to time function of $C$ by using partial fractions and table of Laplace transform. Using the Final Value Theorem of Laplace transform on eq. (99), we get:

$$C(\infty) = -AK_c \times \frac{4}{d}. \tag{100}$$

Using the value of $d = 4K_c(K_{pu} + K_{pr})$, we get:

$$C(\infty) = \frac{-A}{K_{pu} + K_{pr}}. \tag{101}$$

We get the same $C(\infty)$ from eq. (99) as that from eq. (91). Similarly using the Initial Value Theorem of Laplace transform on eq. (99), we get:

$$C(0) = 0. \tag{102}$$

The qualitative nature of the solution $C(t)$ is dependent on location of roots of the denominator of $C(s)$ in the complex plane. Please look at Figure 7 in which several roots are located. Table 1 gives the form of the terms in the expression for $C(t)$ corresponding to these roots. $X1, X2, \ldots, Y1, Y2, \ldots$.. are all positive.

An optimal policy minimizing the social damage in terms of excessive/inadequate number of public service units in initial equilibrium, as well as the social loss in terms of excessive or inadequate number on dynamic adjustment path (when number of public service units demanded is not equal to supply) before arriving at final equilibrium, subject to a certain increase in number of crimes controlled per unit time can be derived on a case by case basis.

In equilibrium, the area under the demand curve is the social benefit in terms of number of crimes controlled per unit time. For estimating an optimal policy, the parameters $K_c$, $K_{pr}$, $K_{pu}$, $\tau_{d1}$ and $\tau_{d2}$ need to be estimated. The values of $K's$ can be estimated in the same manner as demand and supply elasticities. Time lags $\tau_{d1}$ and $\tau_{d2}$ can also be estimated through various techniques. As the optimal policy is a function of these parameters, Delta method can be used for confidence interval.

## References


Agnew, R.: (1999), A general strain theory of community differences in crime rates, *Journal of research in crime and delinquency* **36**(2), 123–155.

Antonaccio, O., Smith, W. R. and Gostjev, F. A.: (2015), Anomic strain and external constraints: A reassessment of mertonŠs anomie/strain theory using data from ukraine, *International Journal of Offender Therapy and Comparative Criminology* **59**(10), 1079–1103.

Bacher-Hicks, A., Billings, S. B. and Deming, D. J.: (2019), The school to prison pipeline: Long-run impacts of school suspensions on adult crime, *Technical report*, National Bureau of Economic Research.

Benson, B. L., Kim, I. and Rasmussen, D. W.: (1994), Estimating deterrence effects: a public choice perspective on the economics of crime literature, *Southern Economic Journal* pp. 161–168.

Bowles, R. and Garoupa, N.: (1997), Casual police corruption and the economics of crime, *International review of Law and Economics* **17**(1), 75–87.

Burton Jr, V. S. and Cullen, F. T.: (1992), The empirical status of strain theory, *Journal of Crime and Justice* **15**(2), 1–30.

Chassang, S., Del Carpio, L., Kapon, S. et al.: (2020), Making the most of limited government capacity: Theory and experiment, *Technical report*, National Bureau of Economic Research, Inc.

Danziger, S. and Wheeler, D.: (1975), The economics of crime: Punishment or income redistribution, *Review of social economy* **33**(2), 113–131.

Devi, T. and Fryer Jr, R. G.: (2020), Policing the police: The impact of" pattern-or-practice" investigations on crime, *Technical report*, National Bureau of Economic Research.

Dippel, C. and Poyker, M.: (2019), Do private prisons affect criminal sentencing?, *Technical report*, National Bureau of Economic Research.

Lehrer, S. F. and Lepage, L.-P.: (2020), How do nypd officers respond to terror threats?, *Economica* **87**(347), 638–661.

Rubin, P. H.: (1978), The economics of crime, *Atlanta Economic Review* **28**(4), 38–43.

Van Winden, F. A. and Ash, E.: (2012), On the behavioral economics of crime, *Review of Law & Economics* **8**(1), 181–213.

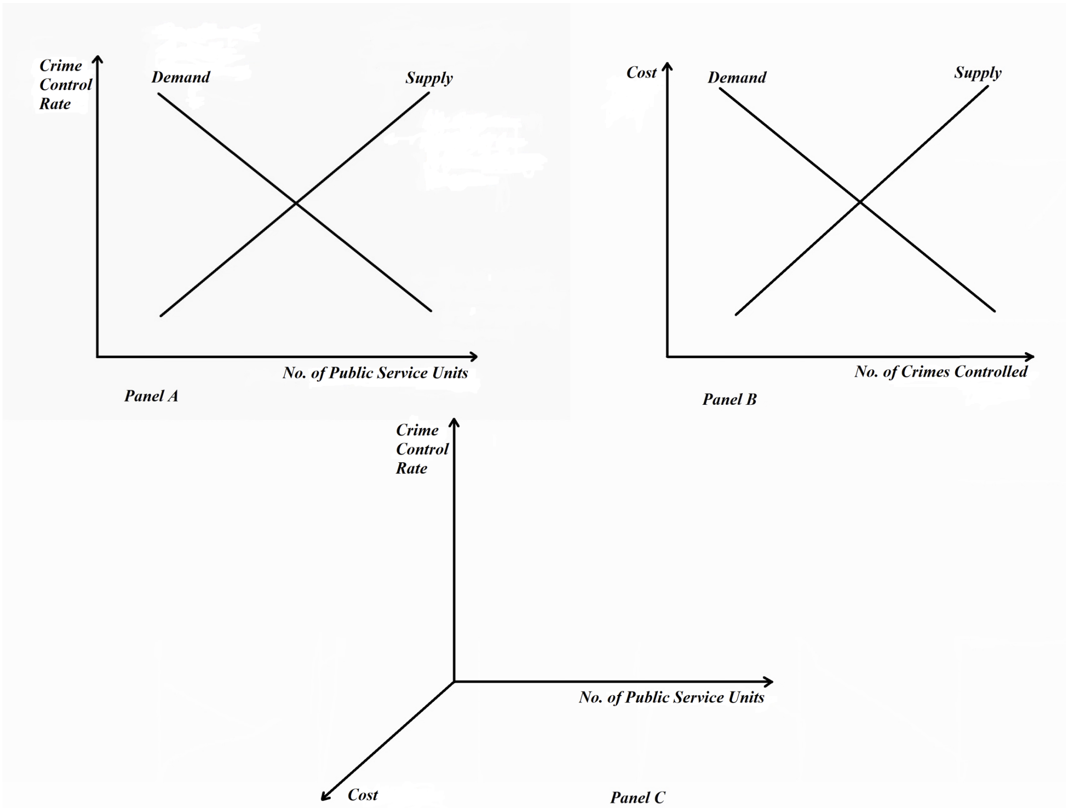


Figure 1: Theoretical concept of crime control model.

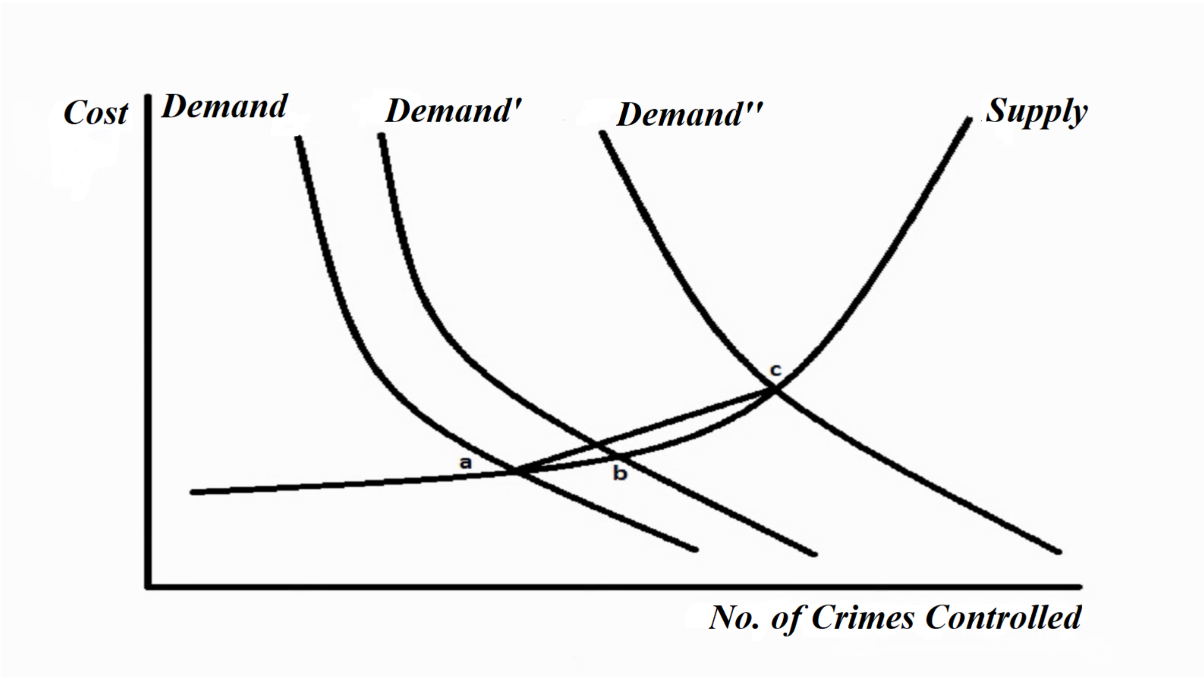


Figure 2: When is linearity a reasonable assumption?

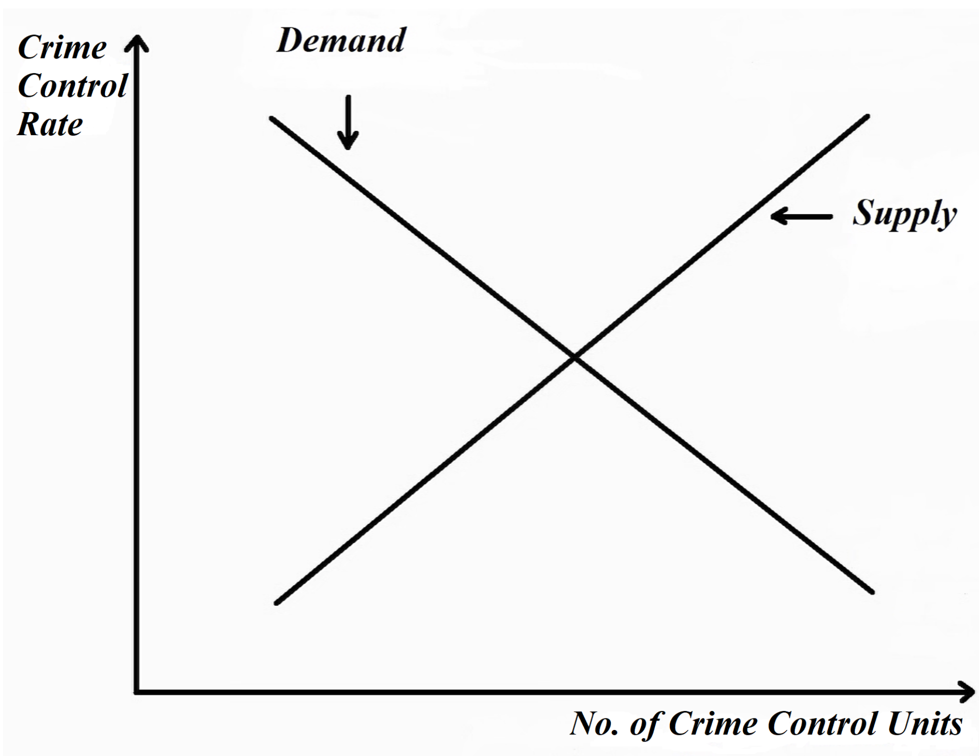


Figure 3: Theoretical concept of crime control model in panel A.

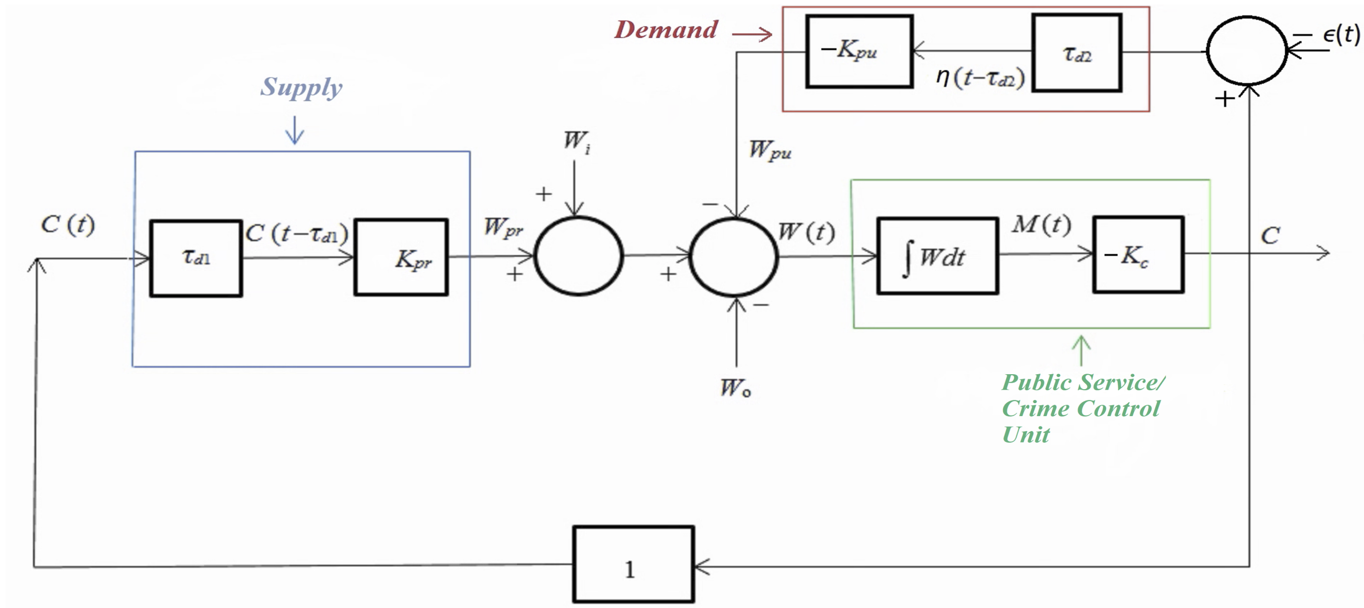


Figure 4: A dynamic optimal crime control model for panel A.

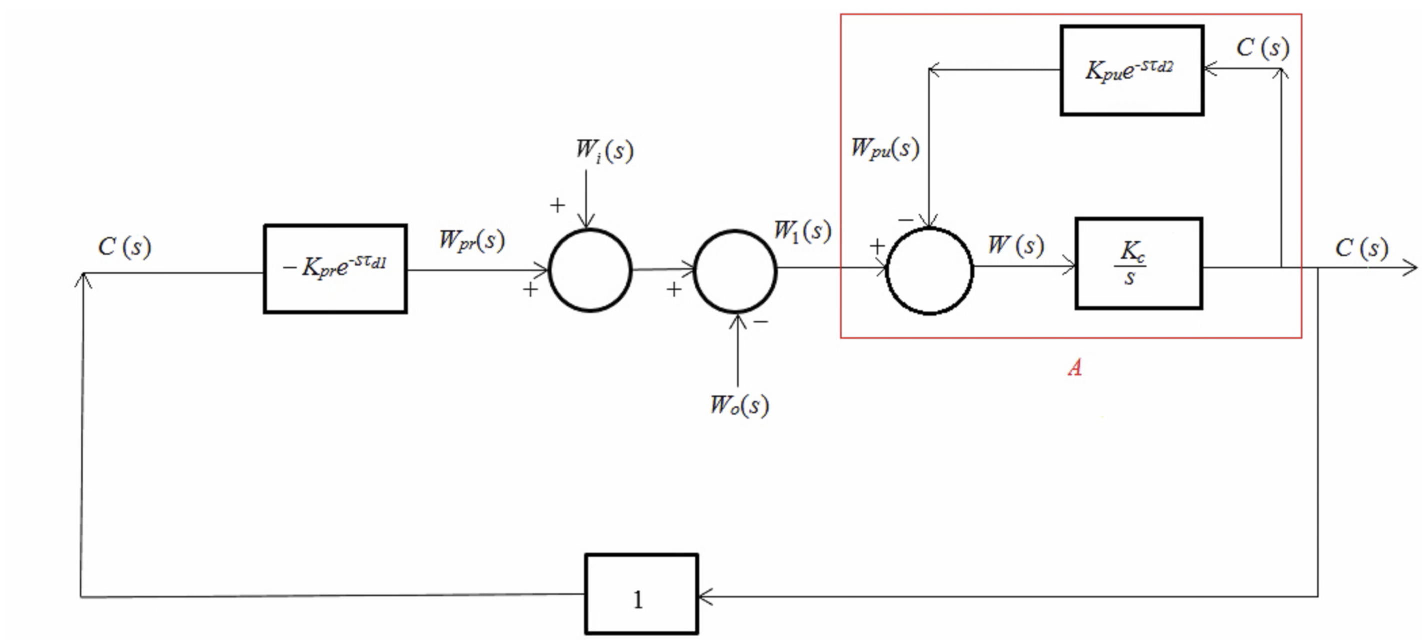


Figure 5: A dynamic crime control model for panel A after Laplace Transform.

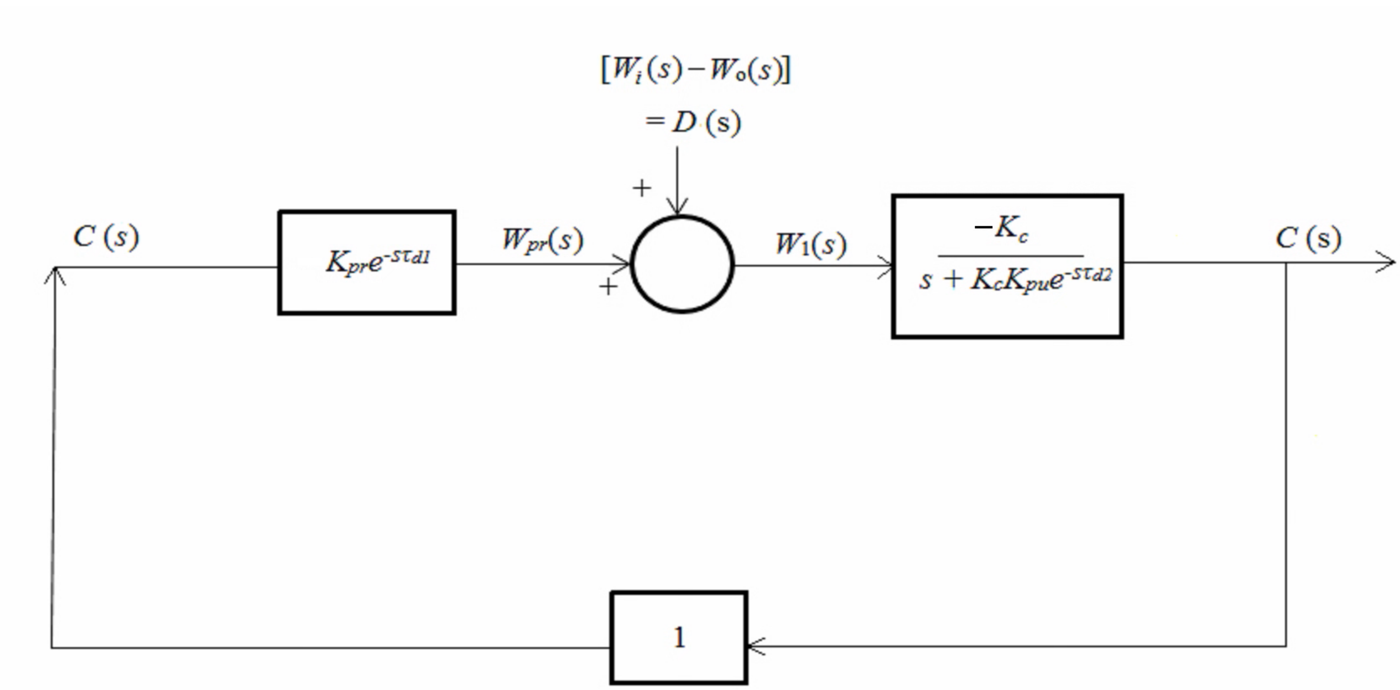


Figure 6: Crime control model in panel A after solution of block A in figure 5.

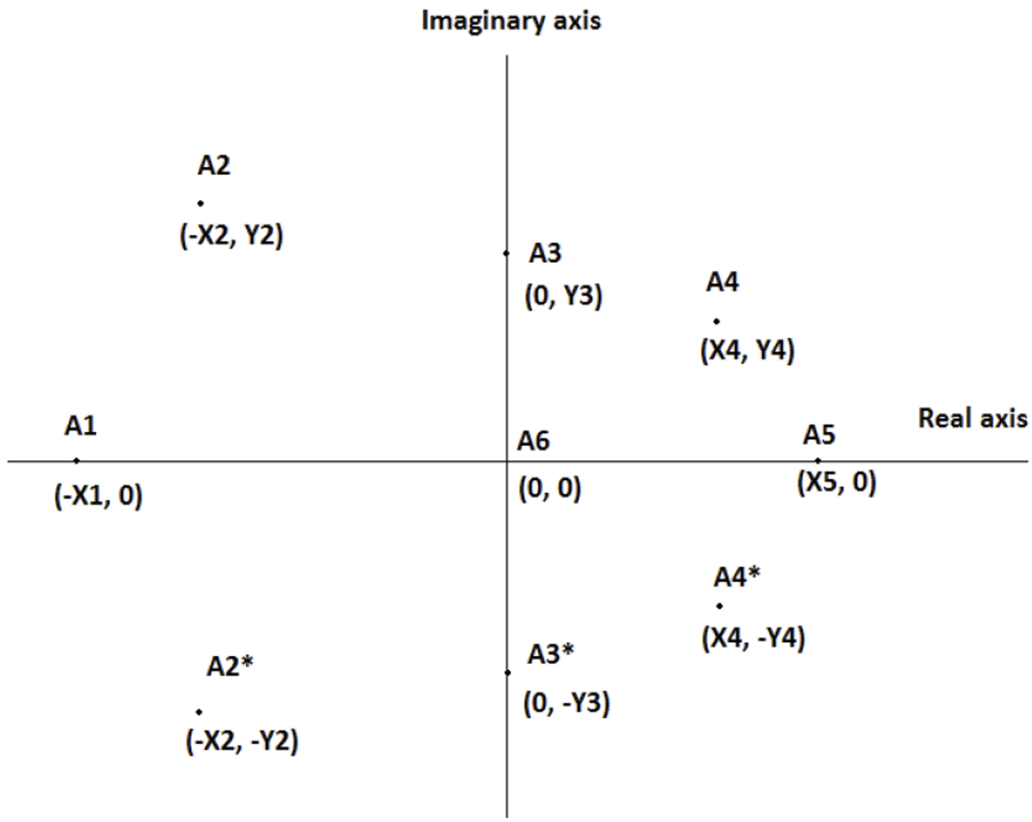


Figure 7: Location of roots in a complex plane corresponding to Table 1.

**Table 1:**

| S. No | Roots | Terms in P(t) for t > 0 | Description of Response |
|---|---|---|---|
| 1 | A1 | $C_1 e^{-X_1 t}$ | Bounded non-cyclic |
| 2 | A2, A2* | $e^{-X_2 t}(C_1 \cos Y_2 t + C_2 \sin Y_2 t)$ | Bounded cyclic |
| 3 | A3, A3* | $C_1 \cos Y_3 t + C_2 \sin Y_3 t$ | Cyclic (constant amplitude) |
| 4 | A4, A4* | $e^{X_4 t}(C_1 \cos Y_4 t + C_2 \sin Y_4 t)$ | Unbounded cyclic |
| 5 | A5 | $C_1 e^{X_5 t}$ | Unbounded non-cyclic |
| 6 | A6 | $C_1$ | Constant |