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Abstract

This paper presents a demand-led growth model augmented with induced technical change to address the two Harrod's problems in growth theory. Building on recent developments in the supermultiplier literature, we investigate how both Harrodian instability problems can be resolved through two complementary mechanisms: (1) autonomous, non-capacity-creating demand components growing at an exogenous rate, and (2) endogenous technical change responsive to income distribution. While existing supermultiplier models show how autonomous expenditures stabilize demand-led growth, we integrate induced technical change into the determination of the natural rate of growth. The model achieves twin stabilization through the interplay of two stabilizing mechanisms: the supermultiplier and induced technical change. On the one hand, demand shocks are absorbed via adjustments in the investment share, allowing capital accumulation to align with the exogenously determined growth rate of autonomous expenditures. On the other hand, labor market imbalances trigger productivity adjustments that reconcile natural and warranted growth through changes in the wage share. This dual adjustment mechanism allows the system to sustain normal capacity utilization and stable employment rates, while preserving demand-led growth outcomes. The results suggest that incorporating induced technical change enhances the supermultiplier's capacity to address both of Harrod's instability problems within a unified demand-led framework.

Keywords: Harrodian instability; Supermultiplier model; Induced technical change; Demand-led growth

JEL Codes: E12; E22; O33; O41

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1 Introduction

Roy Harrod's seminal paper *An Essay in Dynamic Theory* (Harrod, 1939) is conventionally considered the beginning of modern growth theory. In his attempt to extend the Keynesian analysis of output determination to the long run, he devised two problems that could affect the process of accumulation in capitalist economies. First, there is no reason to expect that full employment equilibrium growth will emerge. In fact, the 'warranted' growth rate, that is the growth rate allowed by full capacity savings, would equal the rate of growth of the labor force, or 'natural' growth rate, only by accident. Secondly, deviations of the actual growth rate from the warranted growth rate, rather than being self-correcting, would produce cumulative effects and destabilize the economy. This second issue is typically referred to as the 'Harrod instability' problem.

Elaborating on the seminal paper by Serrano (1995), current versions of the supermultiplier growth model (Allain, 2015; Freitas and Serrano, 2015; Lavoie, 2016; Serrano et al., 2019) have shown that the existence of a non capacity generating autonomous component of aggregate demand may tame Harrodian instability. This occurs when firms do not react too strongly to discrepancies between actual and normal growth (or utilization) rates. Within this framework, by assuming that autonomous, i.e. independent of output, expenditures grow at a given rate, both Sraffian and neo-Kaleckian authors produced a demand-led growth theory that features a stable growth rate, exogenous income distribution and normal capacity utilization in the long run. Besides investigating the stability of the growth path, some recent contributions have also begun to address the first Harrod problem, that is the reconciliation of the demand and supply side, in a context where the natural growth rate includes labor productivity growth in addition to exogenous labor supply growth. Palley (2019) and Fazzari et al. (2020) have assumed that the supply side adjusts to the growth rate of demand by making the natural growth rate endogenous. When technical change is a positive function of the employment rate, disequilibrium in the labor market will be self-correcting. Stronger aggregate demand, by driving up employment growth, will also ease supply constrains through labor productivity growth; a symmetric mechanism would work in the opposite direction when demand falls short of supply. Nomaler et al. (2021) model productivity growth, and in turn the natural growth rate, as a function of R&D investment. Aggregate demand and aggregate supply equalize in the long run, but the model loses its demand-led flavor as supply conditions ultimately anchor the balanced growth rate, while demand growth adjusts. Allain (2021) proposes a technical progress function that depends on capital accumulation and the wage share, thus incorporating both the Kaldor-Verdoorn's law and the Classical-Marxian notion of induced technical change. The natural growth rate is thus explained endogenously; but even in this case, the long-run equilibrium that stabilizes the employment rate is determined by supply conditions. Nah and Lavoie (2019) also model labor productivity growth along Kaldor-Verdoorn's lines; in their version of the supermultiplier model, however, labor supply is unconstrained so that the first Harrod problem does not arise.

In this paper, we further explore the potential implications of introducing induced technical in the supermultiplier growth model. The notion that labor productivity growth is a positive function of the wage share relies on the incentive for competitive firms to introduce labor-saving innovations when facing high unit labor costs, that is higher wage shares. It has been formally developed within different analytical frameworks. On the one hand, it follows from the Classical-Marxian analysis of the choice of techniques, which states that new techniques of production are adopted only if they do not decrease the profit rate at the given real wage (Okishio, 1961). In this context, a rise in the wage share requires an increase in labor productivity to avoid a reduction in the rate of profit. On the other hand, neoclassical authors in the 1960s (von Weizsacker, 1962; Kennedy, 1964) proved the same result by assuming that firms choose the profit-maximizing direction of technical change given an innovation possibility frontier, which describes the trade-off between freely available capital- and labor-augmenting innovations.

While the conventional supermultiplier mechanism ensures the long-run equality between the actual and warranted growth rates, the introduction of induced technical change into the supermultiplier model produces a novel way to reconcile the growth rate of demand and supply, thus offering an original solution to the first Harrod problem. The adjustment mechanism works if we assume that income distribution reacts to the dynamics of the employment rate. Divergences between the actual and the natural growth rates yield a disequilibrium in the labor market. If the state of the labor market affects income distribution, there is a direct and stabilizing feedback to the growth rate of labor productivity that may facilitate the equalization of demand and supply growth. When the actual growth rate is higher than the natural growth rate, employment growth will be faster than labor supply growth, and the employment rate will rise. If the tighter labor market puts pressure on real wages relative to labor productivity growth, the wage share increases; at this point, the induced technical chance channel makes labor productivity grow faster, thus stabilizing the employment rate. Symmetrically, a negative shock to demand growth would make the employment rate fall. The wage share would decline and labor productivity growth would slow down, so that the employment rate would settle and the actual and natural growth rates would equalize at a lower level.

In line with the standard conclusions of the supermultiplier literature, the reconciliation between aggregate demand and supply preserves the demand-led flavor of the model as the growth rate of autonomous consumption fully determines long-run growth. The dynamics of the model also ensures that firms operate at the exogenous normal utilization rate in steady state. On the contrary, our analysis departs from the conventional framework regarding income distribution, which loses its exogenous nature. In fact, the wage share becomes the adjusting variable capable of reconciling the natural growth rate with autonomous demand growth, and it becomes therefore endogenous. However, contrary to the growth-distribution trade-offs featured in Kaldor-Robinson growth models with endogenous distribution, the wage share moves in the same direction as the growth rate since higher demand growth fosters faster technical change by increasing employment and the wage share. From this standpoint, our model relates to the wage-led growth literature but with the fundamental difference that the direction of causality between growth and distribution is inverted.

Finally, an additional original feature of our model concerns the employment rate. While the equality between actual and natural growth rate stabilizes the employment rate, the model does not determine its equilibrium value. In fact, the steady-state equilibrium is compatible with a continuum of employment rates. The actual equilibrium employment rate depends on its initial conditions, so that the model exhibits path dependence.

The paper is divided as follows. Section 2 illustrates the basic levers of the model, while section 3 discusses the model dynamics in the long run. Section 4 presents calibration, simulation results, and numerical stability analysis. Last, section 5 concludes, summarizing our results.

2 Basic elements of the model

2.1 Production and employment

In a closed economy with no public sector, final output Y is produced according to a Leontief production function that uses capital K and labor L in fixed proportions:

$$Y = \min[uK/v, AL], \tag{1}$$

where v denotes the ratio of capital to potential output (Y^p), A is labor productivity, and $u \equiv Y/Y^p$ measures capacity utilization. Cost minimization implies $uK/v = AL$, so that labor demand is $L = uK/(vA)$. We will explore the model under two alternative hypotheses regarding labor supply (N). We will start by assuming that labor is abundant, or infinitely elastic at the ruling wage, so that labor supply instantaneously accommodates labor demand and never constrains capital accumulation and growth. Next, we will assume that labor supply is exogenous and grows at a constant rate n. In this case, we can define the employment rate as $e \equiv L/N = Y/(AN)$. The long-run dynamics of the employment rate could lead to the emergence of the first Harrod problem, for which a solution is presented in subsection 3.2.

2.2 Social classes and income distribution

Capitalists own the means of production, receive profit income Π from ownership of the capital stock, and have a constant propensity to save s. Workers supply labor, earn a real wage w, and do not save. If the wage share is $\omega \equiv wL/Y = w/A$, capitalists' profits are

$$\Pi = (1 - \omega)Y = (1 - \omega)uK/v. \tag{2}$$

2.3 Technical change

As discussed in the Introduction, we assume that labor productivity growth is a positive function of the wage share. This notion is founded both in the Classical-Marxian analysis of the choice of techniques and in the neoclassical induced innovation literature. The logic of the argument is based on the incentive to introduce labor-saving innovations by firms facing rising labor cost. Without dwelling on the microfoundations of the assumption, we simply adopt the reduced form:

$$g_A = f(\omega), \;\; f' > 0, \tag{3}$$

as in Taylor (1991) and Dutt (2013).

2.4 Aggregate demand

We distinguish three sources of aggregate demand. Consumption is either autonomous (Z), that is financed out of financial wealth or through credit, or a function of income: $C = wL+(1-s)\Pi = \omega Y+(1-s)(1-\omega)Y = Y(s\omega+1-s)$. A crucial assumption in the supermultiplier model is that autonomous consumption grows at the exogenous rate g_Z. Investment (I) is modelled along the accelerator principle as a positive function of output: $I = hY$. Notice that h is given at any point in time, but we will assume that it is time dependent as it adjusts to discrepancies between actual and normal utilization rates.

2.5 The short-run equilibrium

In the short run, production adjusts instantaneously to the level of aggregate demand. Equilibrium output solves

$$Y^* = Z + C + I = Z + (s\omega + 1 - s)Y^* + hY^*. \tag{4}$$

Accordingly:

$$Y^* = \frac{Z}{s(1-\omega) - h}, \tag{5}$$

where $1/\left(s(1-\omega) - h\right)$ is the 'supermultiplier'. Both the paradox of saving and the paradox of costs hold as Y^* is a negative function of the saving rate and a positive function of the wage share. Since $Y = uK/v$, the equilibrium utilization rate is $u^* = Y^*v/K = \frac{zv}{s(1-\omega)-h}$, where we defined $z \equiv Z/K$.

The Keynesian tradition emphasizes the dual nature of investment as a source of aggregate demand and productive capacity creation. Since investment builds productive capacity, it defines the law of motion of capital as $\dot{K} = I-\delta K$, where δ is the exogenous depreciation rate.[1] We can then find the growth rate of capital as $g_K = I/K - \delta = hu/v - \delta$, which, in the short-run equilibrium, yields $g_K^* = hu^*/v - \delta = \frac{zh}{s(1-\omega)-h} - \delta$.

[1] We will use the notation $\dot{x}$ and g_x to indicate, respectively, the time derivative and the growth rate of variable x.

3 Long-run analysis

3.1 Abundant labor supply and exogenous income distribution

This first closure of the model mostly coincides with the benchmark Sraffian supermultiplier model, as it features exogenous income distribution ($\omega = \bar{\omega}$) and no labor constraints on growth (as for example in Freitas and Serrano 2015). We simply introduce the explicit dependence of the marginal saving rate on income distribution as in the neo-Kaleckian versions of the supermultiplier model (Allain, 2015; Lavoie, 2016).

Within this setup, the dynamic evolution of the model is described by two differential equations in the state variables z and h. From the definition of z as the autonomous demand-to-capital ratio, it follows that:

$$\dot{z} = z(g_Z - g_K^*) = z\left(g_Z - \left(\frac{zh}{s(1-\bar{\omega}) - h} - \delta\right)\right). \tag{6}$$

The evolution of h, on the other hand, depends on firms' behavioral assumptions. When firms observe an actual degree of capacity utilization higher than the normal one (u_n), they respond by increasing investment as they expect higher future demand; on the contrary, they reduce investment when $u^* < u_n$. This is the source of the so-called Harrodian instability problem and it translates into the following equation:

$$\dot{h} = h\gamma(u^* - u_n) = h\gamma\left(\frac{zv}{s(1-\bar{\omega}) - h} - u_n\right), \tag{7}$$

where $\gamma > 0$ measures the speed of adjustment.

By setting $\dot{z} = \dot{h} = 0$ we can find the (non-trivial) steady-state values z_{ss} and h_{ss} as solutions to the system:

$$\begin{cases} g_Z + \delta = \frac{z_{ss}h_{ss}}{s(1-\bar{\omega})-h_{ss}} \\ \frac{z_{ss}v}{s(1-\bar{\omega})-h_{ss}} = u_n \end{cases}. \tag{8}$$

The equilibrium values are $h_{ss} = (g_Z + \delta)v/u_n$ and $z_{ss} = s(1-\bar{\omega})u_n/v - (g_Z + \delta)$. If z is stationary, capital grows at the rate of autonomous consumption growth g_Z. Positive shocks to g_Z are accommodated by a higher investment share, necessary for faster capital accumulation, and lower autonomous expenditure relative to capital. Since in steady state $Y = u_n K/v$, output also grows at the rate g_Z.

Remembering that, by definition, $Y = AL$, in steady state we have $g_Y = g_A + g_L = f(\bar{\omega}) + g_L = g_Z$. Since labor supply is abundant, there are no constraints on output growth. Income distribution fixes productivity growth, and employment growth emerges residually given that output grows at g_Z. This is analogous to what happens in Nah and Lavoie (2019), though they model labor productivity growth according to the Kaldor-Verdoorn law. We show in the Appendix that the steady state is locally stable if $\gamma < z_{ss}/u_n = s(1-\bar{\omega})/v - (g_Z + \delta)/u_n$, which means that the Harrodian instability effect is not too strong.

3.2 Exogenous labor supply and endogenous income distribution

We now turn to the more innovative contribution of our paper. If we try to keep income distribution exogenous while also adding an exogenous labor supply, steady-state employment growth cannot equal labor supply growth. In fact, $g_L = g_Z - f(\bar{\omega}) \neq n$ unless by a fluke. In Harrod's words, the natural and the warranted growth rates are different, and this creates ever increasing disequilibrium in the labor market. However, if we let income distribution reacts to the disequilibrium in the labor market, the wage share can become the adjusting variable that ensures the stability of employment: $g_L = g_Z - f(\omega_{ss}) = n$.

The adjustment of income distribution in response to labor market dynamics is a mechanism deeply rooted in different economic theories. This relationship draws inspiration from the profit squeeze hypothesis, which posits that tightening labor markets strengthen workers bargaining power, enabling them to secure higher wages and compress profit margins, thereby raising the wage share of income (Goodwin, 1967; Marglin, 1984). Similarly, modern macroeconomic models incorporating a wage-setting curve formalize this link by relating real wages – or the wage share – to employment levels, where lower unemployment increases workers' ability to negotiate higher compensation (Blanchflower and Oswald, 1994; Carlin and Soskice, 2006). These frameworks suggest that the wage share is not exogenously fixed but evolves endogenously depending on labor market conditions. In this spirit, we formalize the wage share as a positive function of the employment rate, introducing a feedback mechanism that enables the convergence between the warranted and the natural rates of growth.

In particular, let us assume that the wage share is an increasing power function of the employment rate: $\omega = ae^{\eta}; a, \eta > 0$. Then, we have $\dot{\omega} = a\eta e^{\eta-1}\dot{e}$ and $\dot{\omega}/\omega = \eta(\dot{e}/e)$, where η is the constant elasticity of the wage share to the employment rate. The wage share becomes an additional state variable and its evolution, given $e = Y/(AN)$, is described by the equation:

$$\dot{\omega} = \omega\eta(\dot{e}/e) = \omega\eta\left(g_Y - f(\omega) - n\right).$$

The time derivative of (5) yields $\dot{Y} = \frac{\dot{Z}}{s(1-\omega)-h} + \frac{(\dot{h}+s\dot{\omega})}{(s(1-\omega)-h)^2}$, so that $g_Y = g_Z + \frac{\dot{h}+s\dot{\omega}}{s(1-\omega)-h}$. If we then plug the output growth rate back into $\dot{\omega}$ and factor the terms depending on $\dot{\omega}$, we find

$$\dot{\omega} = \omega\eta\frac{s(1-\omega)-h}{s(1-\omega)-h-\eta\omega s}\left(g_Z + \frac{\dot{h}}{s(1-\omega)-h} - f(\omega) - n\right). \tag{9}$$

We can couple (9) with (6) and (7) to form a three dimensional dynamical system in (z, h,ω). The introduction of the new equation does not affect the steady-state solutions for h_{ss} and z_{ss}, save for replacing $\bar{\omega}$ with $\omega_{ss} = f^{-1}(g_Z - n)$.

The new steady state features multiple relevant properties. First, it reconciles aggregate demand and aggregate supply while fully retaining the demand-led nature of the model, as long-run growth is anchored by the exogenous growth rate of autonomous consumption g_Z. This adds to contributions such as Fazzari et al. (2020) and Palley (2019), who obtain analogous results while assuming that labor productivity growth is a positive function of the employment rate rather than the wage share. On the other hand, our approach differs from the reconciliation proposed by Allain (2021). He adopts a technical progress function similar to ours, but in his analysis autonomos expenditure eventually depends on the exogenous growth rate of population, so that the natural growth rate of the economy is ultimately supply driven.

Secondly, the model differs from the Sraffian supermultiplier model as income distribution becomes endogenous. However, contrary to the Cambridge growth models criticized by Serrano and Freitas (2017), growth remains fully demand-led in our framework. Even more importantly, the wage share moves in the same direction as the growth rate. Rather than the standard trade-off between growth and distribution featured in the Kaldor-Robinson analyses, we devised the possibility of a win-win result where higher demand growth fosters faster technical change by increasing employment and the wage share.

Finally, notice that even though the employment rate is stable in a steady-state equilibrium, the stationarity conditions of the model do not pin down its equilibrium value. In the Appendix we show that the equilibrium employment rate is:

$$e_{ss} = \frac{e_0[s(1-\omega_0)-h_0]}{s(1-\omega_{ss})-h_{ss}}\exp\left([g_Z - n]t_{ss} - \int_0^{t_{ss}} f(\omega(\tau)d\tau\right) \tag{10}$$

where t_{ss} is the time when the system reaches the steady state; and e_0, ω_0, and h_0 are the starting values of the employment rate, the wage and investment shares. e_{ss} depends on the initial levels of the employment rate, the wage and investment shares and it is thus subject to path dependence. In fact, the model allows for a continuum of equilibrium employment rates compatible with the triple (h_{ss}, z_{ss}, ω_{ss}). In the next section, Figure 3 shows that raising the initial level of the employment rate makes the employment rate converge to a higher long-run value.

4 Simulation results and stability

4.1 Parameter values and initial conditions

To illustrate the model dynamics, we choose a simple linear specification for $g_A = f(\omega)$ (equation 3), assuming that $f(\omega) = \alpha\omega$. Accordingly, α measures the sensitivity of productivity growth to changes in the functional distribution of income. Choosing a functional form of $f(\omega)$ allows to calculate the non-trivial steady-state value of ω_{ss} as:

$$\omega_{ss} = \frac{g_Z - n}{\alpha}. \tag{11}$$

Parameter and exogenous variable values are drawn from the theoretical and empirical literature on the US economy, and are summarized in Table 1.

The model is calibrated on an annual basis, as reflected particularly in the values assigned to the capital-capacity ratio v, the depreciation rate δ, and the growth rate of autonomous demand g_Z. Although the depreciation and growth rates explicitly incorporate a temporal dimension, it is important to emphasize that the capital-capacity ratio also plays a key role in ensuring consistency with an annual calibration. This stems from the fact that v represents a ratio between a stock and a flow variable, as further discussed by Gallo (2022).

The normal rate of capacity utilization is set at $u_n = 82.42\%$, following Setterfield and Budd (2011). This figure aligns closely with empirical estimates for other advanced capitalist economies; for instance, it is comparable to the value of 0.8104 reported by Gallo and Barbieri Góes (2023) for the Euro Area. The capital-capacity ratio is taken from Fazzari et al. (2020), who estimate a long-run capital-output ratio of 1.2 for the U.S. economy using BEA data on non-residential investment and the capital stock; given $u_n = 82.42\%$, this implies a capital-capacity ratio of $v = \frac{K}{Y_p} = \frac{K}{Y_n} \cdot \frac{Y_n}{Y_p} = 1.2 \times 0.8242 = 0.989$, where Y_n is output corresponding to the normal utilization rate. The annual depreciation rate is fixed at $\delta = 0.084$ in line with Fazzari et al. (2020), who provide a value that is consistent with empirical evidence for the U.S. economy, as noted in their Supplementary Appendix. The annual growth rate of autonomous demand, g_Z, is set at 2.5%, consistent with the calibration adopted by Fazzari et al. (2020).

Population growth is assumed to be 1% per year, consistent with standard calibrations in neoclassical growth models (Barro and Sala-i Martin, 2004) and roughly aligned with the average rate observed in the United States since 1980 (United Nations, 2024).

Given the values assigned to g_Z and n discussed above, we set α to 0.025 to obtain an equilibrium wage share of 60% (via equation 11), in order to be roughly in line with the observed value in many high income countries.

Last, parameter γ is assigned a value of 0.1, consistent with previous simulation exercises featuring accelarator dynamics (Nomaler et al., 2021; Gallo, 2022). The value of the elasticity of the wage share to the employment rate is also set 0.1. Since there is no clear guidance on the values of α, γ, and η, a numerical robustness analysis will be presented in subsection 4.3.

Table 1: Exogenous variables and parameter values

Par.	**Description**	**Value**
s	Capitalists' propensity to save	0.8
g_z	Growth rate of autonomous demand	0.025
u_n	Normal rate of capacity utilization	0.8242
v	Acceleration coefficient	0.989
δ	Depreciation rate	0.084
n	Population growth rate	0.01
α	Sensitivity of productivity growth to distribution	0.025
γ	Sensitivity of the investment share adjustment	0.1
η	Elasticity of the wage share to the employment rate	0.1

Initial conditions are summarized in Table 2. We start from the steady-state positions for the autonomous demand-to-capital ratio ($z_0 = z_{ss}$) and the investment share ($h_0 = h_{ss}$), equal in the baseline to 15.75% and to 14.29%, respectively.[2] To generate the model dynamics, we introduce an exogenous distributive shock by setting the initial wage share 5% above its steady-

[2]It ought to be noted that these values are compatible with the empirical evidence (see e.g. Barbieri Góes 2024).

state value of 60%, i.e. $\omega_0 = 1.05 \cdot \omega_{ss} = 0.63$.

Table 2: Initial Conditions

Variable	Description	Value
z_0	Autonomous demand-capital ratio	0.1575
h_0	Investment share	0.1429
ω_0	Wage share	0.63

4.2 Simulation results

Given the parameter space and initial conditions discussed in the previous subsection, we can now discuss simulation results.[3] The full model we simulate is composed of equations (6), (7), and (9) with $f(\omega) = \alpha\omega$, as illustrated above:

$$\begin{cases} \dot{z} &= z\left[g_Z - \left(\dfrac{zh}{s(1-\omega)-h} - \delta\right)\right] \\ \dot{h} &= h\gamma\left(\dfrac{zv}{s(1-\omega)-h} - u_n\right) \\ \dot{\omega} &= \omega\eta\,\dfrac{s(1-\omega)-h}{s(1-\omega)-h-\eta\omega s}\left(g_Z + \dfrac{\dot{h}}{s(1-\omega)-h} - \alpha\omega - n\right). \end{cases} \tag{12}$$

Figure 1 presents the 3D phase plane of the system. Starting from a positive wage share shock, the model converges through dumpened fluctuations in the three endogenous variables to its long-run position.

To get a better sense of the out-of-equilibrium dynamics, we plot the trajectories of the state variables over time in Figure 2. The simulation illustrates how a positive distributive shock to the wage share propagates through the system, generating cyclical adjustments with dampening oscillations across all three endogenous state variables.

Over the traverse path, the initial positive shock to ω generates a demand increase through the multiplier, yielding a surge in the short-run capacity utilization rate and in the short-run growth rate. This produces an initial increase in h and a simultaneous reduction in z during the early periods. Last, the distributive adjustment mechanism activates, with ω further increasing in the early phase of the traverse (as the actual growth rate surpasses the natural rate), then

[3] The model is simulated in Python using the `solve_ivp` function from the `scipy.integrate` library. The Runge-Kutta 45 (RK45) method is employed as the numerical integrator, which is an explicit, adaptive step-size solver based on the Dormand-Prince formulation. This method is well-suited for smooth, non-stiff systems of ordinary differential equations and allows for efficient and accurate computation of the models dynamic trajectories over the chosen time horizon.

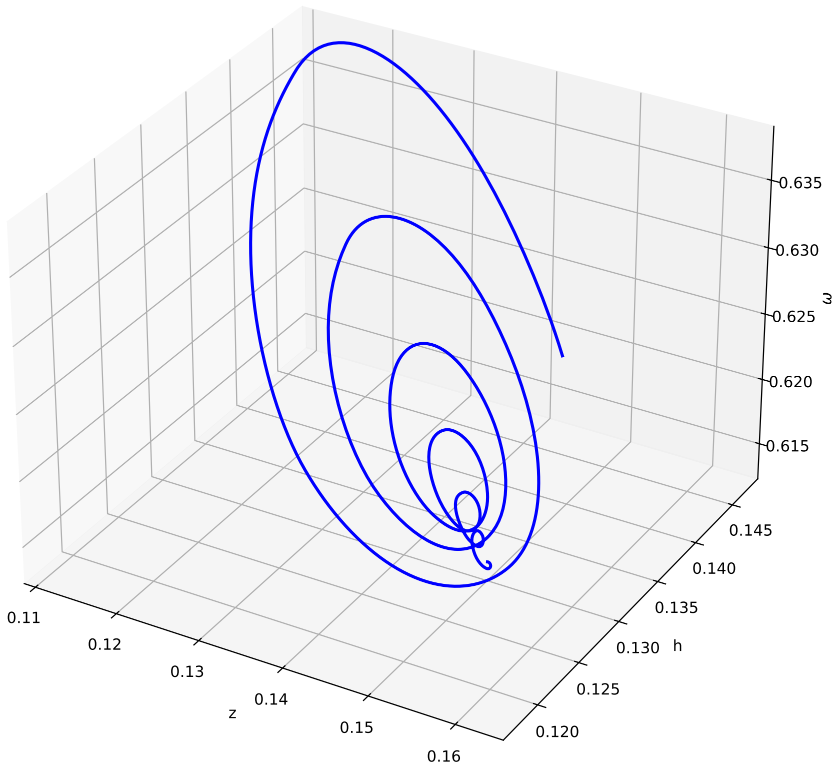

Figure 1: 3D phase plane

triggering a wage share reduction due to the feedback effect on labor productivity growth, which initiates a reversal of the adjustment process. The progressive decline in the wage share then helps stabilize the dynamics of the investment share and the autonomous demand-capital ratio.

The dynamic trajectories exhibit oscillatory patterns with varying convergence speeds. As autonomous demand growth falls short of capacity growth, z initially plunges below its steady-state value (z_{ss}). Then, it rebounds sharply, overshooting the equilibrium, and continuing to oscillate with gradually diminishing amplitude. Even after 400 periods, z has not fully stabilized, though the oscillations have substantially dampened. Similarly, h displays pronounced cyclical behavior, initially surging above its steady-state value (h_{ss}) in the early phase of the traverse (when $u^* > u_n$), before falling below it. The subsequent oscillations in h demonstrate relatively faster convergence compared to the other variables, with minimal deviations from equilibrium observed in the later periods of the simulation. Last, the wage share exhibits the most persistent disequilibrium dynamics. Following its initial positive shock, ω increases further before beginning a series of dampened oscillations around a gradually declining trend.

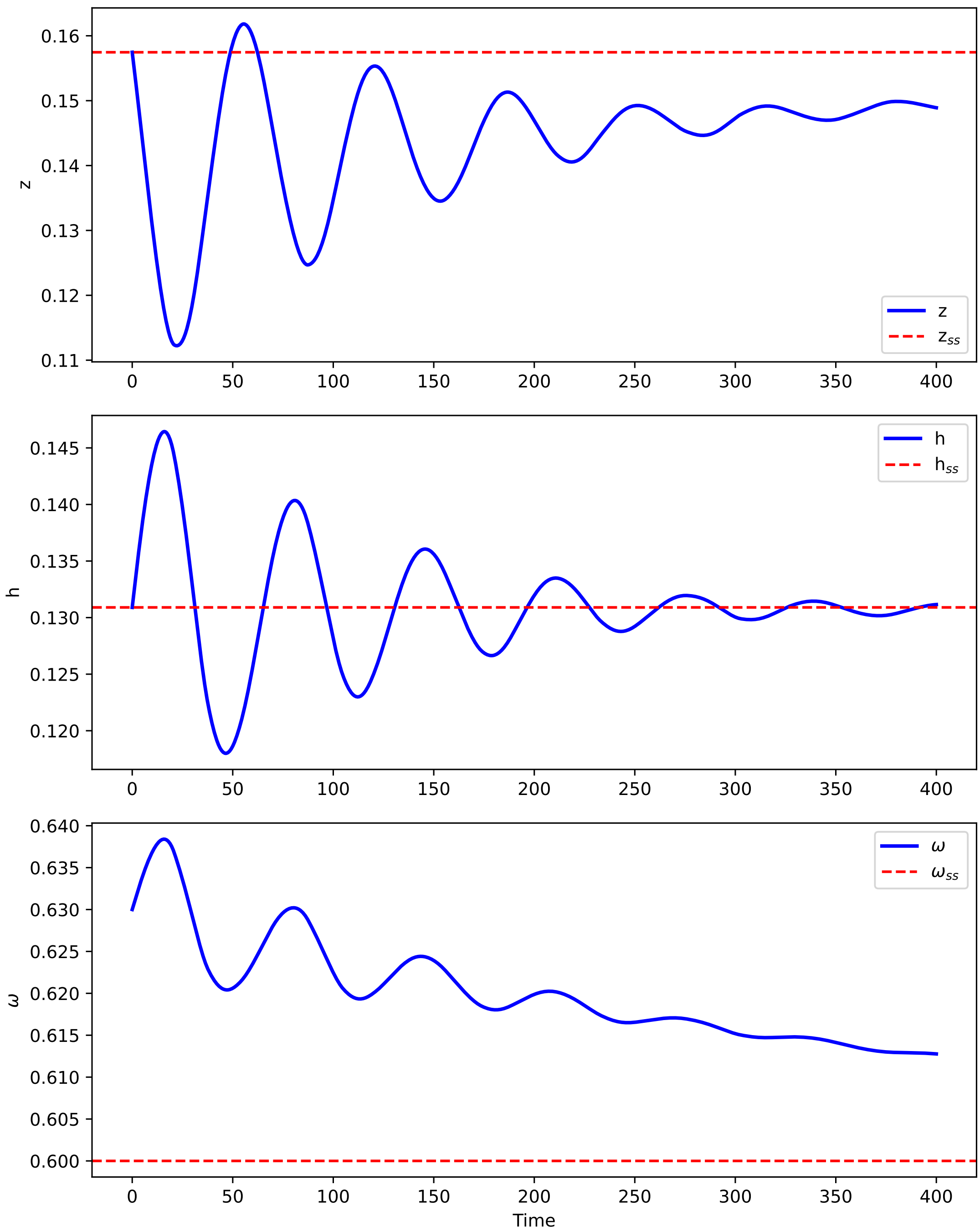

Figure 2: Simulation results

This cyclical adjustment process illustrates the feedback mechanisms between distribution and growth in the model. The simulation demonstrates that while the system tends toward its equilibrium configuration, the adjustment process requires substantial time, particularly for the distributive variables.

On the other hand, the dynamics of the employment rate (figure 3) shows no convergence to a predetermined equilibrium value, as it is characterized by path dependence. As illustrated above (equation 10) and in the Appendix, the equilibrium employment rate chiefly depends on its initial condition. History thus affects employment dynamics: starting from a higher employment rate ($e_0 = 0.68$) allows to maintain a higher utilization of labor at any time step both in the transitional dynamics and in equilibrium compared to a lower initial state ($e_0 = 0.58$).

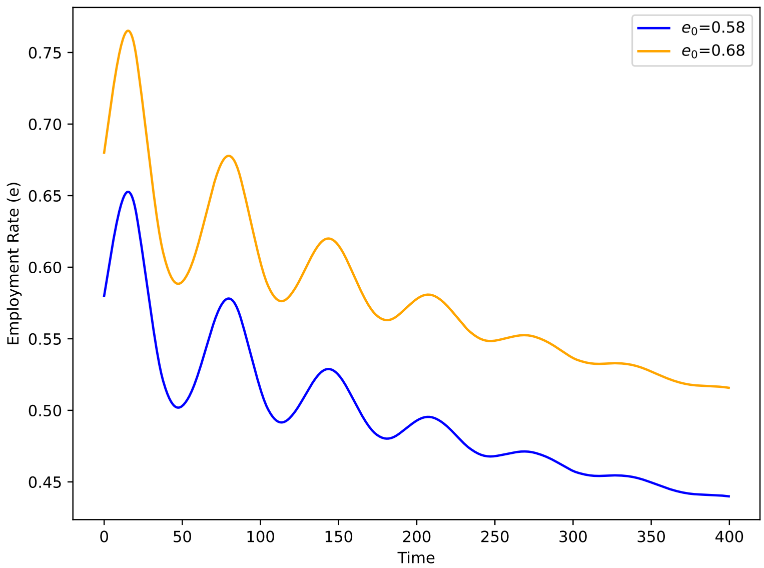

Figure 3: The trajectory of the employment rate with different e_0

4.3 Robustness of the stability analysis

Following the methodology adopted by Nomaler et al. (2021), we examine the robustness of the local stability properties of the system by performing a grid search over the parameter space. Conducting such an analysis across the full set of parameters would be both computationally intensive and of limited analytical value. We therefore restrict our focus to a subset of parameters for which no clear guidance is available from either the theoretical or empirical literature – namely, the adjustment speed of the investment share to deviations in the capacity utilization rate (γ), the elasticity of the wage share to the employment rate (η), and the sensitivity of productivity growth to changes in the functional distribution of income (α). Parameters that are well-anchored in prior research are held constant at their baseline values, as discussed above. Specifically, we vary γ in the interval $[0, 0.4]$, η in the range $[0, 0.5]$, and α between $[0, 0.1]$,[4] using 100 evenly spaced points along each dimension. For each combination of parameters, we numerically compute the eigenvalues of the Jacobian matrix evaluated at the steady state.

The resulting stability maps are shown in the 2D plane in Figure 4, and in 3D in Figure 5. This sampling of the parameter space allows us to systematically evaluate how different combinations of (γ, η, α) influence the system stability. We restrict our analysis to economically meaningful steady states, excluding from consideration all combinations of parameters for which: (1) the Keynesian stability condition is violated ($s(1 - \omega_{ss}) - h_{ss} < 0$), (2) z_{ss}, h_{ss}, ω_{ss} are negative, or (3) $\omega_{ss} > 1$. In such cases, the Jacobian matrix is not computed, and these parameter constellations are represented by white areas in Figure 4.

[4] Although the range for α may appear narrow, this restriction is justified by the crucial role the parameter plays in determining the steady-state wage share ω_{ss} (see equation 11). Specifically, for higher values of α – given plausible values of g_Z and n – the wage share in the steady state converges toward zero, i.e., $\omega_{ss} \to 0$. To avoid economically implausible outcomes, we cap α at 0.1, which already corresponds to a very low wage share of 0.125.

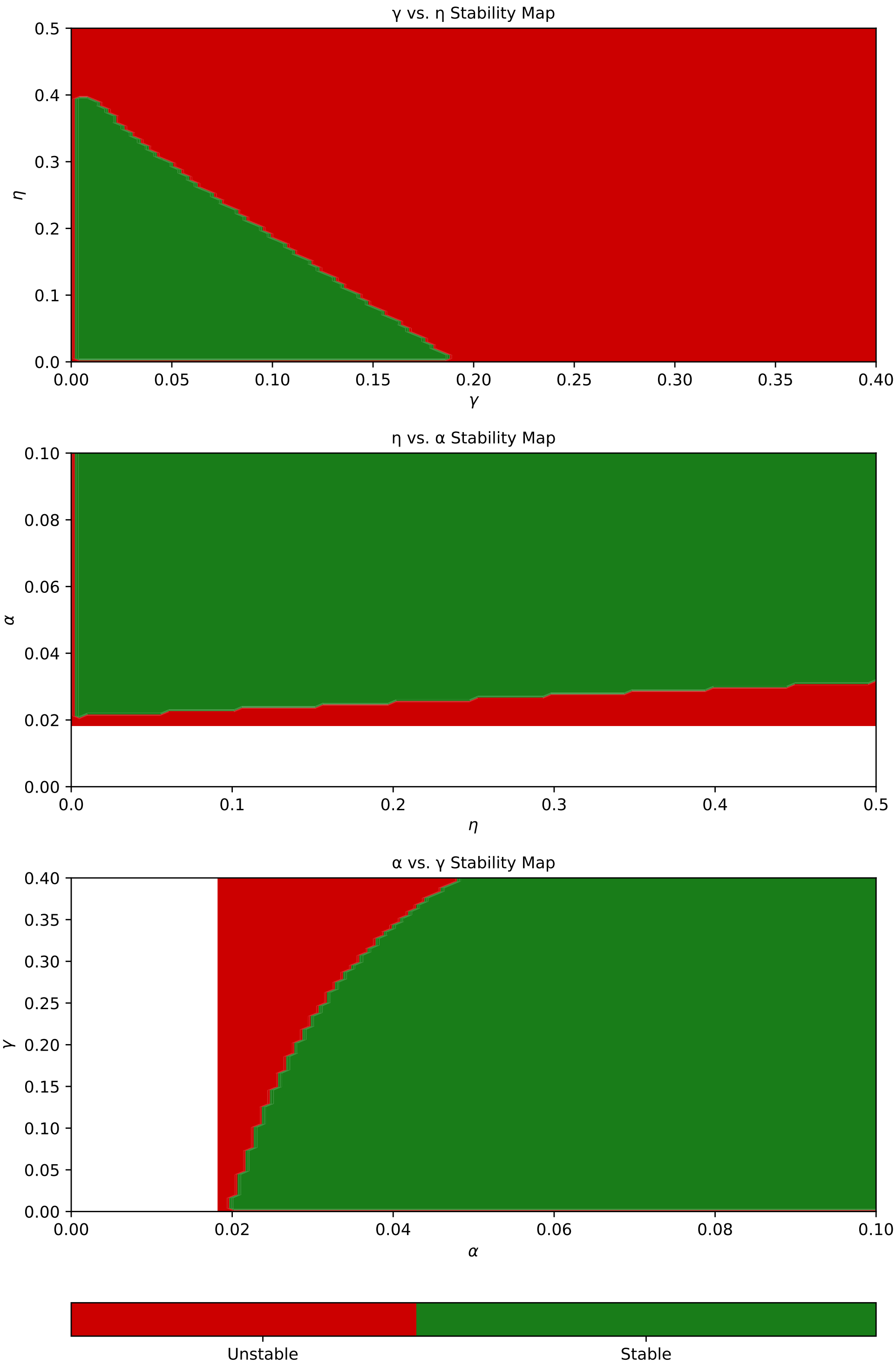

Figure 4: Stability maps for different values of α, γ, and η

The stability maps reveal that system stability is particularly sensitive to (γ, η), with instability arising when both parameters take relatively high values. This underscores that one key assumption regarding baseline supermultiplier models holds true even within our extended framework: Harrodian instability must not be too strong. In other terms, γ must remain fairly low to ensure that induced investment does not adjust capacity to demand too quickly outside its fully adjusted position (Freitas and Serrano, 2015; Serrano and Freitas, 2017; Gallo, 2022). Regarding η, higher values tend to destabilize the model, amplifying the reaction of the wage share to changes in the employment rate. In contrast, α appears to play a stabilizing role across most of parameter space; instability emerges only when it becomes very small, as illustrated in both (η, α)- and (α, γ)-planes. Therefore, unlike the destabilizing role of γ and η, the strength of the induced innovation mechanism plays a key role in taming Harrodian instability.

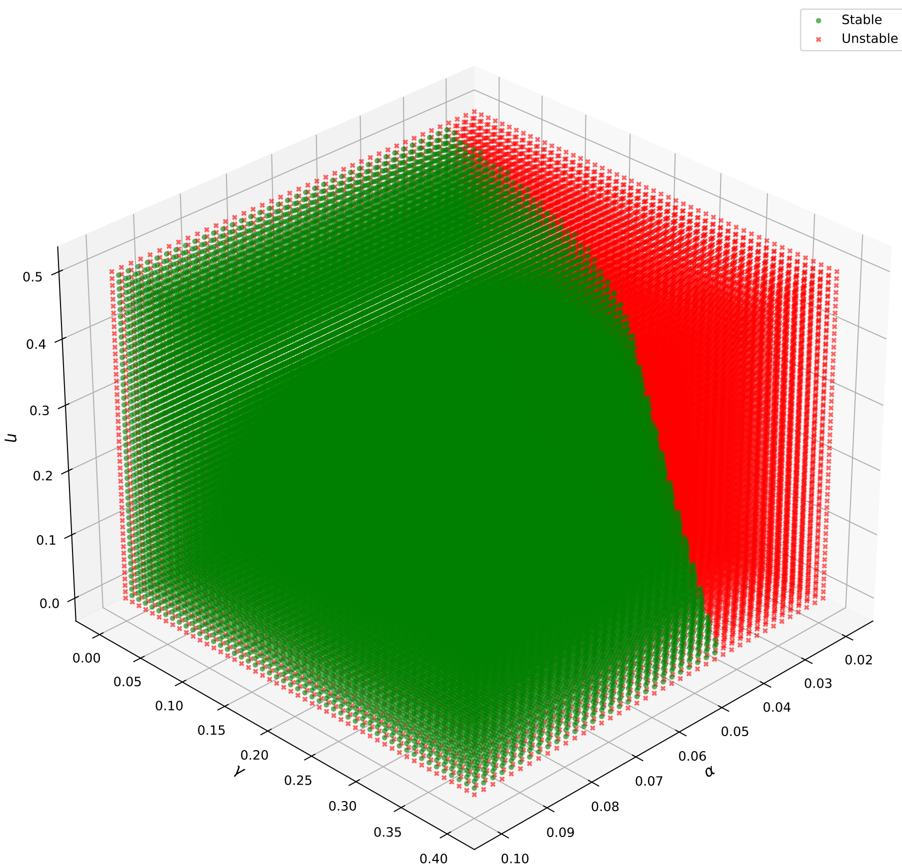

Figure 5: 3D Stability map for different values of α, γ, and η

The 3D stability map (Figure 5) provides a more comprehensive view of how these three parameters interact to define stable regions, highlighting that achieving stability requires balancing all three parameters simultaneously. Moderate values of η and low-to-intermediate values of γ are conducive to stability when coupled with sufficiently high values of α.

5 Conclusion

This paper has developed a supermultiplier growth model augmented with induced technical change to address both of Harrod's problems in growth theory. By integrating these two complementary mechanisms, we have shown how a demand-led growth framework can simultaneously resolve Harrodian instability and reconcile demand-led growth with supply constraints.

The model achieves twin stabilization. The interplay between the supermultiplier and the induced technical change mechanisms simultaneously ensures that: i) demand shocks are absorbed through changes in the investment share, thus allowing capital accumulation to adjust to the exogenously given growth rate of autonomous expenditures; ii) productivity growth responds to distributive pressures arising from labor market imbalances, thereby aligning the natural rate of growth with the warranted rate. In the long run, actual, warranted and natural growth rates are equalized so that both Harrod's problems are addressed.

Our analysis demonstrates that the model can sustain normal capacity utilization and stable employment rates without surrendering to supply-side determinism. Growth remains fully demand-led in our framework, with the dynamics of autonomous non-capacity creating components of aggregate demand anchoring the long-run growth rate. Notably, our model presents an important departure from traditional growth-distribution trade-offs, as higher demand growth can foster faster technical change by increasing employment and the wage share, creating a potential win-win scenario. Furthermore, a feature of our model is that while the employment rate stabilizes in the steady state, its equilibrium value exhibits path dependence, being determined by the initial conditions of the system. This generates indeterminacy of the long-run employment equilibrium.

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References

Allain, O. (2015). Tackling the instability of growth: a Kaleckian-Harrodian model with an autonomous expenditure component. *Cambridge Journal of Economics*, 39(5):1351–1371.

Allain, O. (2021). A supermultiplier model of the natural rate of growth. *Metroeconomica*, 72(3):612–634.

Barbieri Góes, M. C. (2024). An analysis of the patterns of economic growth in the US. *Review of Political Economy*, pages 1–36. 20/8/2024.

Barro, R. J. and Sala-i Martin, X. (2004). *Economic Growth*. MIT Press, Cambridge, MA, 2 edition.

Blanchflower, D. G. and Oswald, A. J. (1994). *The Wage Curve*. MIT Press, Cambridge, MA.

Carlin, W. and Soskice, D. (2006). *Macroeconomics: Imperfections, Institutions, and Policies*. Oxford University Press, Oxford, UK.

Dutt, A. (2013). Endogenous technological change in classical-Marxian models of growth and distribution. In Michl, T. and Rezai, A., editors, *Social Fairness and Economics: Essays in the Spirit of Duncan Foley*. Routledge, New York, NY.

Fazzari, S. M., Ferri, P. E., and Variato, A. M. (2020). Demand-led growth and accommodating supply. *Cambridge Journal of Economics*, 44(3):583–605.

Freitas, F. and Serrano, F. (2015). Growth rate and level effects, the stability of the adjustment of capacity to demand and the Sraffian supermultiplier. *Review of Political Economy*, 27(3):258–281.

Gallo, E. (2022). When is the long run? – Historical time and adjustment periods in demand-led growth models. *Metroeconomica*, 73(4):1155–1178.

Gallo, E. and Barbieri Góes, M. C. (2023). Investment, autonomous demand and long-run capacity utilization: an empirical test for the Euro Area. *Economia Politica*, 40(1):225–255.

Goodwin, R. M. (1967). A growth cycle. In Feinstein, C. H., editor, *Socialism, Capitalism and Economic Growth: Essays Presented to Maurice Dobb*, pages 54–58. Cambridge University Press, Cambridge, UK.

Harrod, R. F. (1939). An essay in dynamic theory. *Economic Journal*, 49:14–33.

Kennedy, C. (1964). Induced bias in innovation and the theory of distribution. *Economic Journal*, 74(295):541–547.

Lavoie, M. (2016). Convergence towards the normal rate of capacity utilization in neo-Kaleckian models: The role of non-capacity creating autonomous expenditures. *Metroeconomica*, 67(1):171–201.

Marglin, S. A. (1984). Growth, distribution, and inflation: A centennial synthesis. *Cambridge Journal of Economics*, 8(2):115–144.

Nah, W. J. and Lavoie, M. (2019). Convergence in a neo-Kaleckian model with endogenous technical progress and autonomous demand growth. *Review of Keynesian Economics*, 7(3):275–291.

Nomaler, Ö., Spinola, D., and Verspagen, B. (2021). R&D-based economic growth in a supermultiplier model. *Structural Change and Economic Dynamics*, 59:1–19.

Okishio, N. (1961). Technical changes and the rate of profit. *Kobe University Economic Review*, 7:86–99.

Palley, T. (2019). The economics of the super-multiplier: A comprehensive treatment with labor markets. *Metroeconomica*, 70(2):325–340.

Serrano, F. (1995). Long Period Effective Demand and the Sraffian Supermultiplier. *Contributions to Political Economy*, 14:67–90.

Serrano, F. and Freitas, F. (2017). The Sraffian supermultiplier as an alternative closure for heterodox growth theory. *European Journal of Economics and Economic Policies: Intervention*, 14(1):70–91.

Serrano, F., Freitas, F., and Bhering, G. (2019). The Trouble with Harrod: the fundamental instability of the warranted rate in the light of the Sraffian Supermultiplier. *Metroeconomica*, 70(2):263–287.

Setterfield, M. and Budd, A. (2011). A Keynes-Kalecki model of cyclical growth with agent-based features. In Arestis, P., editor, *Microeconomics, Macroeconomics and Economic Policy: Essays in Honour of Malcolm Sawyer*, pages 228–250. Springer, New York, NY.

Taylor, L. (1991). *Income distribution, inflation, and growth*. MIT Press, Cambridge, MA.

United Nations (2024). *World Population Prospects 2024: The 2024 Revision*. Department of Economic and Social Affairs, Population Division, United Nations, New York, NY.

von Weizsacker, C. (1962). A New Technical Progress Function. *German Economic Review*, 11(3):248–265. Reprinted 2019.

A Appendix

A.1 Stability Analysis: 2D case

Linearization of the dynamical system formed by equations (6) and (7) around its steady state yields the following Jacobian matrix:

$$J(z_{ss}, h_{ss}) = \begin{bmatrix} \dot{z}_z & \dot{z}_h \\ \dot{h}_z & \dot{h}_h \end{bmatrix}_{ss} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix},$$

where $J_{11} = -(g_Z + \delta)$, $J_{12} = -s(1-\omega)\left(\frac{u_n}{v}\right)^2$, $J_{21} = \gamma v(g_Z + \delta)/z_{ss}$, $J_{22} = \gamma u_n(g_Z + \delta)/z_{ss}$. We can calculate the determinant of J as: $DetJ = J_{11}J_{22} - J_{21}J_{12} = \gamma u_n(g_Z + \delta)/z_{ss}\left[s(1-\omega)\frac{u_n}{v} - (g_Z+\delta)\right] = \gamma u_n(g_Z + \delta) > 0$. On the other hand, the trace of J is $TrJ = J_{11} + J_{22} = (g_Z + \delta)\left[\gamma u_n/z_{ss} - 1\right]$. Hence, $\gamma < z_{ss}/u_n = s(1-\omega)/v - (g_Z+\delta)/u_n$ ensures that the trace of Jacobian matrix is negative and the system is locally stable.

A.2 Stability Analysis: 3D Case

Linearization of the dynamical system formed by equations (6), (7) and (9) around its steady-state position yields the following Jacobian matrix

$$J(z_{ss}, h_{ss}, \omega_{ss}) = \begin{bmatrix} \dot{z}_z & \dot{z}_h & \dot{z}_\omega \\ \dot{h}_z & \dot{h}_h & \dot{h}_\omega \\ \dot{\omega}_z & \dot{\omega}_h & \dot{\omega}_\omega \end{bmatrix}_{ss} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix},$$

where $J_{11} = -(g_Z + \delta)$, $J_{12} = -s(1-\omega)\left(\frac{u_n}{v}\right)^2$, $J_{13} = -s(g_Z + \delta)u_n/v$, $J_{21} = \gamma v(g_Z + \delta)/z_{ss}$, $J_{22} = \gamma u_n(g_Z + \delta)/z_{ss}$, $J_{23} = s\gamma u_n(g_Z + \delta)/z_{ss}$, $J_{31} = \frac{\omega\eta}{s(1-\omega)-h-\eta\omega s}J_{21}$, $J_{32} = \frac{\omega\eta}{s(1-\omega)-h-\eta\omega s}J_{22}$, $J_{33} = \frac{\omega\eta}{s(1-\omega)-h-\eta\omega s}\left(J_{23} - f'(\omega)\left(s(1-\omega)-h\right)\right)$.

The Routh-Hurwitz necessary and sufficient conditions for stability cannot be established purely analytically. We have proceeded to calibrate and simulate the model to investigate its stability properties.

A.3 Time path of the employment rate

Let us solve the differential equation $\dot{e} = e\left(g_Z + \frac{\dot{h}+s\dot{\omega}}{s(1-\omega)-h} - f(\omega) - n\right)$. We start by setting $D = s(1-\omega) - h$, so that the term $\frac{\dot{h}+s\dot{\omega}}{s(1-\omega)-h}$ can be rewritten as $\frac{\dot{h}+s\dot{\omega}}{D} = -\frac{\dot{D}}{D} = -\frac{d}{dt}\ln D$. Substituting this back into the original equation, we get: $\dot{e}/e = \left(g_Z - \frac{d}{dt}\ln D - f(\omega) - n\right)$.

Rewriting the left-hand side as the logarithmic derivative: $\frac{d}{dt}\ln e = g_Z - \frac{d}{dt}\ln D - f(\omega) - n$. Integrating both sides with respect to time : $\ln e + k_0 = -\ln D + k_1 + \int [g_Z - f(\omega(t)) + n]dt$; hence, $\ln e + \ln D = \ln(eD) = k_1 - k_0 + [g_Z - n]t - \int f(\omega(t)dt \equiv k_2 + [g_Z - n]t - \int f(\omega(t)dt$. Exponentiating the second and the last element in the equality chain and making the time dependence explicit: $e(t)D(t) = \exp\left(k_2 + [g_Z - n]t - \int f(\omega(t)dt\right) = k_3 \exp\left([g_Z - n]t - \int f(\omega(t)dt\right)$, where $k_3 \equiv e^{k_2}$. Since $[g_Z - n]0 - \int_0^0 f(\omega(t)dt = 0$, we can calculate

$$e(0)D(0) = k_3 \exp\left([g_Z - n]0 - \int_0^0 f(\omega(t)dt,\right) = k_3.$$

Hence, substituting $D(t) = s(1 - \omega(t)) - h(t)$, the final solution is:

$$e(t) = \frac{e(0)[s(1-\omega(0)) - h(0)]}{s(1-\omega(t)) - h(t)} \exp\left([g_Z - n]t - \int_0^t f(\omega(\tau))d\tau\right).$$

If we call t_{ss} the moment the system reaches the steady state, the equilibrium employment rate will be

$$e(t_{ss}) = \frac{e(0)[s(1-\omega(0)) - h(0)]}{s(1-\omega_{ss}) - h_{ss}} \exp\left([g_Z - n]t_{ss} - \int_0^{t_{ss}} f(\omega(\tau)d\tau\right),$$

and it will thus depend on the initial conditions of the system.