Modern Economy and Reconsideration of the Equilibrium Assumption : Is it possible to reconstruct "effective" economics?

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Abstract:

The market autonomously finds an equilibrium where supply and demand meet by using prices as a signal —this is the "invisible hand" of Adam Smith who is often called the "father of economics." However, has the very power of this doctrine, particularly due to its underlying assumption of equilibrium achieved by nominal variables, prevented economists from directly confronting the realities of the modern economy?

This paper reinterprets economic phenomena that traditional models have failed to capture as "dynamic equilibrium," where stability is maintained by the interaction with the internal characteristics of economic agents such as preference structures and external environments like capital transfer. Among the various mathematical expressions derived from the model, perhaps the most crucial is the following:

$$R_t - \rho = n + D_a - \frac{U_{(\theta a)}\theta}{U_c}$$

This means that the discrepancy between the return on assets R_t and the time preference rate ρ (on the left-hand side of the equation) is balanced by two forces on the right-hand side. One is the force of keeping capital within the economy (the marginal utility of assets compared to consumption $\frac{U_{(\theta a)}\theta}{U_c}$) and the other is to promote its diffusion or dilution externally (capital outflow D_a and population growth n). Conventional economics has tended to focus on the left-hand side of this equation to discuss the situation of economies, but this paper argues that it should also be understood from the perspective of its right-hand side.

If the time preference rate is an inherent and entrenched characteristic of economic agents, an economy with a relatively lower time preference rate will have a funds surplus, but a certain portion of this surplus will be balanced by capital outflow or a weak preference for assets, so the decline in the real interest rate will be limited. Conversely, an economy with a relatively higher time preference rate will face a funds deficit, but a certain portion of this deficit will be balanced by capital inflow or a strong preference for assets, so the rise in the real interest rate will be suppressed. The balance between these two forces—the power to generate and retain assets within an economy and the other is that which promotes its diffusion or dilution externally and promote equalization—will generate and maintain differences in asset levels, capital transfer from one economy to another, supply-demand imbalances, and income inequalities even if agents are rational and markets are efficient.

Through these insights, I reinterpret the disequilibrium phenomena facing modern economies as a result of the rational behavior of economic agents and offer clues for more effective macroeconomic policies.

Keywords: dynamic equilibrium; time preference; asset preference; capital flows; global imbalance

1. Reconsideration of the Equilibrium Assumption

1. 1. The "Invisible Hand" Dogma

The market autonomously discovers the point where supply equals demand by using prices as a signal—Adam Smith's famed "invisible hand," and the partial equilibrium that every microeconomics student first learns. This simple, powerful doctrine has captivated aspiring economists for centuries.

Yet precisely because of its power, economists have too often avoided confronting real world economies and instead have been refining more intricate model details, thereby "failing to see the forest for the trees." The consequence of this, in the wake of successive global financial crises, is an intellectual environment recently warned against by sensible economists, with criticisms such as, "spectacularly useless at best, and positively harmful at worst," (1) or "for more than three decades, macroeconomics has gone backwards." (2)

In any era, theory reflects its historical context. In the infancy of economic thought—from Smith's invisible hand, through the marginalist revolution, to Walras's general equilibrium—the scarcity of goods and services reigned. Trade across borders was limited due to the restrictions of transport and communication technology, and financial systems were rudimentary, making transaction scales small.

The crucial question at that time might have been how to meet as many wants as possible under finite resources. This is because under such "goods-short" conditions, "if you build it, they will buy it." With trade geographically constrained, a closed-system view sufficed. With primitive finance, real-side analysis seemed adequate. We can see such a mindset from Say's Law — "supply creates its own demand" —in the early 19th.

However, today's economic society has utterly outgrown those premises. Cross-border trade has become routine, and production chains span continents. Economic blocks have become vast, and and local imbalances are chronic. Advanced economies now suffer from insufficient demand—such as Japanese long-run deflation—rather than goods shortages. Meanwhile, developed financial systems have emancipated asset markets from settlement mechanisms, allowing purely speculative or portfolio flows to sway the real economy.

If "common sense" in the real economy has changed drastically since economics began, the discipline's "common sense" must also evolve. It is time to question afresh the foundational notion of equilibrium.

1. 2. The Historical Evolution of Equilibrium in Economics

Smith's invisible hand was later formalized through the marginalist revolution with its focus on individual agents' optimization behavior and then elevated to "general equilibrium" by Léon Walras. Walras's theory gave equilibrium a rigorous mathematical structure that still exerts profound influence.

However, in the 20th century, circumstances shifted. The Great Depression revealed that these equilibrium theories alone could not explain reality. John Maynard Keynes argued that insufficient effective demand could leave an economy trapped without ever reaching equilibrium, spawning persistent unemployment, and he advocated active fiscal policy to offset the shortfall.

After World War II, economists sought to synthesize Keynesian and pre-Keynesian neoclassical views.

John Hicks's IS–LM model provided a mathematical basis for Keynes's ideas. Paul Samuelson proposed the neoclassical synthesis—Keynesian theory for short-run fluctuations and neoclassical growth theory for the long run—thus shaping postwar policy orthodoxy. Robert Mundell and Marcus Fleming extended the IS–LM model to open economies, thereby deepening the understanding of international macroeconomics.

The 1970s oil shocks, however, produced stagflation—simultaneous inflation and unemployment—that challenged Keynesian prescriptions. Friedman's Monetarism and Lucas's rational-expectations hypothesis revived neoclassical economics and argued that government intervention often destabilizes economies. Building on this, Edward Prescott developed real business-cycle models, explaining short-run fluctuations via supply-side shocks and paving the way for today's dynamic stochastic general-equilibrium models.

In parallel, Joseph Stiglitz and Paul Krugman are known as "New Keynesians." They have incorporated price and wage stickiness, imperfect competition, and information asymmetries—which had been a Keynesian contribution—into micro-founded models while respecting Lucas's critiques.

Yet, the global financial crisis from 2008 delivered a fundamental challenge. Neither neoclassical nor new Keynesian models fully account for the crisis's causes—from chronic global imbalances to protracted deflationary stagnation despite expansionary policies. Aside from them, non-mainstream economists who long emphasized inherent disequilibrium have gained attention, yet no consensus framework has emerged.

Meanwhile, microeconomics has seen a "behavioral turn," escaping the assumption of perfectly rational agents. Fields such as behavioral economics (drawing on cognitive psychology) and game theory (emphasizing interactions among boundedly rational actors) have flourished. Yet so far, their contributions remain confined to specific applications—nudging, auction design—and have not yielded a unified macroeconomic understanding or policy toolkit.

1. 3. Contemporary Economic Issues and Limits of Existing Equilibrium Models

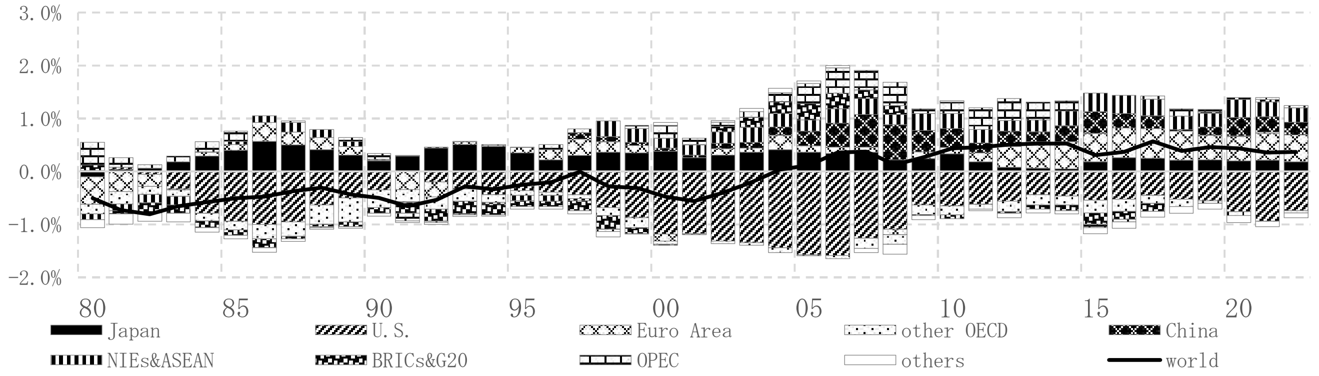
Traditional economics has assumed that a state of force balance—equilibrium—exists, treating disequilibrium as a temporary deviation that autonomously converges to equilibrium by the system. Yet the gulf between economic theory and today's ever more complex reality has widened.

Most mainstream models treat economies as near-closed systems. In physics, closed systems indeed approach equilibrium over time, but real-world economies are open systems, and continuously perturbed by cross-border trade or capital flows. Thus, actual economies never rest at a fixed equilibrium, so any putative equilibrium itself shifts under external shocks or internal structural changes. Market equilibrium becomes nothing more than the statistical average of heterogeneous, unbalanced parts.

This perspective brings into focus how we conceptualize the relationship between part and whole. Since the Lucas's critique in the 1970s, modern models have stressed micro foundations, but even phenomena highlighted in recent decades, such as persistent global imbalances, remain elusive in standard frameworks. Long-run, large-scale creditor-debtor fixed positions, critical to the crisis, should not persist if nominal variables (prices, interest rates, exchange rates) autonomously adjust. As $fig. 1-1$ shows, even years after

2008, the United States continues to record a current-account deficit of 0.5–1.0% of world GDP, while Japan, China, and the post-Euro-crisis Eurozone accumulate surpluses.

fig. 1-1 Current Account Balance of Nations & Regions (% of world GDP)



source) IMF「Workd Economic Outlook」

Also, the long-term deflationary equilibrium of the world economy after the financial crisis or that of the Japanese economy since its financial system crisis in the late 1990s, are phenomena that are difficult to explain with existing models too. Even if everyone rushed to "reduce debt" in the wake of the crisis, the model's automatic adjustment mechanism would have triggered a "self-rebound," but in reality, this was not the case. This is a "fallacy of composition" in which optimal actions for each individual lead to unintended consequences through interactions.

Underlying both puzzles is the relationship between asset markets and the real economy. Since the neoclassical revival, mainstream models have often omitted explicit asset-market dynamics. Yet global imbalances stem in part from unrestricted trust in the U.S. dollar, allowing U.S. external debt to balloon. Japan’s deflation also reflected a chain reaction of credit-market distrust, triggering a demand collapse.

In hindsight, Adam Smith was so prescient that his "invisible-hand" dogma became entrenched for centuries. While the core claim—markets equate supply and demand via price—remains a brilliant organizing principle, it no longer describes all modern phenomena. As long as Smith’s dogma prevails, chronic imbalances and deflationary stagnation driven by expanding asset markets remain beyond our reach.

However, if economists attempt disequilibrium-based theories, they will lose themselves adrift because they have long relied on the anchor of supply–demand equilibrium. Thus, the next better step may be to seek a new anchor that can take the place of supply-demand equilibrium with nominal variables.

Centuries of cumulative economic theory are beautiful, but they are also fragile. The entire edifice may unravel if just a piece shifts. Yet questioning even one piece can open doors to innovative theories.

This paper posits that time preferences—often treated uniformly—are in fact heterogeneous and deeply ingrained in individuals, so they are persisting in various forms at the country or regional level. We hypothesize that the gap between global real interest rates and diverse time preferences underpins many modern disequilibrium phenomena. Chapter 2 surveys prior works on international macroeconomics and asset preferences, laying the groundwork for our approach. Chapter 3 builds and mathematically tests our dynamic model. The final chapter discusses avenues for further research and remaining challenges.

2. View of the Economy under Dynamic Equilibrium

2. 1. Diversity of Time Preferences and Optimal External Balance

Today's global economy is far from the "flat world" derived from standard theory. It features chronic global imbalances, prolonged deflationary equilibria in advanced and emerging economies, and widening disparities in income and wealth. In this section, we trace these phenomena back to diversity and heterogeneity in the time preference rates and sketch the mechanism by which they can produce persistent imbalances.

Time preferences are notoriously hard to measure, so conventional models typically assume a single, uniform discount rate. Yet in reality each individual—and by extension each country or region—possesses a unique, deeply ingrained "personal clock." Recognizing this diversity is the first step toward a model that can explain the puzzles of global imbalance.

In mainstream open-economy macro models, current accounts and net foreign asset positions are often tied to the relationship between a country's time-preference rate ρ and the world real interest rate r. Obstfeld and Rogoff (1994) pioneered this micro-founded intertemporal approach, in which representative households intend to smooth consumption over their lifetimes, and as a result, this will decide the current account balance of each nation and region.

If $\rho > r$, they run persistent current-account surpluses and accumulate claims. If $\rho < r$, they incur deficits and build liabilities. Although short-run imbalances occur, the theory predicts ultimate convergence to zero net-asset positions—countries cannot remain perpetual creditors or debtors. This is because debtors will eventually have to repay their debts, and creditors will eventually have to use their credit to consume in order to gain utility.

However, despite its sophisticated theory, it also has phenomena that are difficult to explain, such as the so-called "consumption correlation puzzle." The model suggests that consumption smoothing would lead to households around the world sharing risks through capital markets and synchronizing their consumption patterns, but the data does not support this prediction.

Similarly, their model is also unable to fully explain the recent chronic global imbalances. Unlike the "short-term imbalances" shown in the model, in reality certain countries have maintained large current account surpluses or deficits over decades, resulting in a fixed external balance.

From such a perspective, the hypothesis emerges that there is an "optimal external balance level" for each economy. This is a perspective that sees the world as an interdependent system in which various economies need each other to offset each other's individual balance shortfalls.

One such a model is Tokushima's model (Tokushima, 2007, 2008). He argues that the external balance is determined as the rate of return on capital converges to the rate of time preference in an open economy. If thinking this way, it seems possible to answer the question, "Why do some countries remain creditor nations and others remain debtor nations?" If the rate of return on capital is higher than a certain uniform rate of time preference, then a country can always be a creditor in the adjustment process, and vice versa.

However, his model also regards external imbalances as a temporary phenomenon in the adjustment process. One of his model's assumptions is "time preference rates are the same around the world," but this paper takes a completely opposite stance. This paper argues that "rather, the time preference rates are diverse, so a divergence occurs with the capital return, which is under pressure to converge due to globalization." If we assume "time preference rates are the same and the capital return converges to it" like Tokushima, the imbalance is merely a temporary phenomenon, but if we take the view "the time preference rates are diverse so capital return converges instead" as this paper, the divergence can be a persistent phenomenon.

In a typical model, capital return and time preference rate will converge under market mechanisms. This is because the interest rate, which is a nominal variable and an opportunity cost, is determined by the degree to which economic agents are willing to spend money on current consumption rather than future.

Yet this paper focuses on the possibility that this assumption does not capture reality. Even if capital return is subject to market arbitrage, the degree of an economic agent's time preference rate is a personal issue. In fact, many previous studies have argued that time preferences are not uniform (e.g., Laibson, 1997; Krusell and Smith, 1998; Harashima, 2022, etc.). Therefore, this paper assumes a mechanism whereby the gap between the rate of return on capital and the rate of time preference can be balanced and maintained through factors of the asset economy, such as the external balance.

2. 2. Asset Preference and Long-Lasting Deflationary Equilibria

Modern economies have been confronted with long-lasting deflationary equilibria—beginning with late-1990s Japan and since observed in advanced economies and some emerging markets. Although recent factors such as the post-COVID recovery and geopolitical turmoil have shifted attention to rising inflation trends and higher bond yields, chronic demand shortfalls continue to depress output, contributing over time to widening income and wealth gaps and fueling social fragmentation.

Such phenomena lie beyond the explanatory reach of conventional frameworks. However, when combined with the “diversity of time preferences” introduced in the previous section, the related notion of “asset preferences” offers a promising line of inquiry.

A long-term deflationary equilibrium is a state in which prices fall persistently, real interest rates remain elevated, and aggregate demand languishes. In Keynesian terms, this is often understood as a liquidity trap. When nominal rates approach zero, the opportunity cost of holding money diminishes, causing households to hoard cash and monetary policy to lose traction. For example, Paul Krugman emphasized in his analysis of Japan’s “Lost Decade” that expectations about the future are crucial once an economy enters a liquidity trap (Krugman, 1998). That perspective has since become standard both domestically and abroad, but these accounts stop short of explaining why agents choose to hoard money in the first place.

Here, I will focus on Yoshiyasu Ono’s concept of the “non-satiation of money utility.” According to him, individuals derive intrinsic utility from holding money, and that utility does not saturate even if money balances rise (Ono, 2007; 2022). Mainstream theory treats money mainly as a measure for acquiring goods, with utility flowing solely from consumption. Even models that grant direct utility from money typically assume diminishing marginal utility and eventual saturation. Ono challenges this assumption, suggesting that the act of continual wealth accumulation itself yields satisfaction. This insight resonates with the fact that people derive joy simply from seeing their bank balances grow.

This concept of non-satiation of money utility is a phase of the “asset preferences” we examine. Beyond rational motives tied to future consumption, some agents desire to keep financial assets indefinitely. This idea is rooted in Sidrauski’s Money-in-the-Utility (MIU) model (Sidrauski, 1967), which highlights liquidity services and transaction-cost savings from holding money, but Ono goes further by positing that unbounded asset accumulation can be an optimal choice.

In today’s global economy, especially since the Lehman crisis, demand for safe assets has surged dramatically (Caballero, Farhi, and Gourinchas, 2017). That strong appetite for safety implies more than mere risk aversion. It indicates a population of agents who simply prefer holding financial assets.

The notion of asset preferences calls into question the transversality condition in standard models. The Transversality condition implies that rational agents will not accumulate infinite wealth, since any asset carried into the future could instead be consumed today to raise utility. However, if holding assets yields non-satiating utility, perpetual wealth accumulation may become optimal. After all, many people seem to enjoy seeing their net worth grow, or life’s uncertainty may incentivize them to accrue wealth as long as possible.

Under these preferences, the transversality condition may fail and that failure can constitute an optimal solution for certain agents. This idea echoes the “dynami−cally inefficient over-saving” identified by Diamond (1965) in his overlapping-generations framework.

In addition, although asset preferences and time preference rates are often conflated—since agents with low discount rates also save more—, this paper treats them separately. Asset preferences pertain to the utility function’s response, whereas time preferences govern the discounting of future value. In analyzing the divergence between interest rates and discount rates, distinguishing these channels is crucial.

2. 3. The Asset Economy as a Reaction–Diffusion System

Under the standard model, market equilibrium often yields the conclusion that the time-preference rate (ρ) equals the real interest rate (r). However, this paper holds that it is precisely a state in which $\rho \neq r$ that contain the key to explaining contemporary economic disequilibrium phenomena. Moreover, I regard this not as a temporary state of adjustment processes but as a persistent phenomenon rooted in the preferences of individual economic agents.

What makes this possible is the autonomy of the asset economy. In standard models, financial assets are often treated as a passive instrument for settling net balances of a real economy. Yet in today’s world where daily financial transactions vastly outscale real economic activity, accumulated assets and liabilities operate according to mechanisms distinct from those of the real economy.

As an approach to capture the autonomous dynamics of the asset economy, there is an analogy with the reaction–diffusion equations of physics. Here, “reaction” refers to the endogenous adjustments by agents—with particular time preference and asset preference parameters—who alter their saving and investment in response to current wealth holdings and market interest rates. By contrast, “diffusion” denotes the force by which wealth propagates through economic space driven by differences in capital or asset accumulation.

If only diffusion were at work, wealth would spread uniformly and any spatial heterogeneity would eventually vanish. Yet in a reaction–diffusion system, the reaction effects can counteract this equalizing tendency. For example, in economies dominated by agents with lower time-preference rates and strong asset preferences, assets may be persistently generated. While in economies dominated by agents with higher time-preference rates and weak asset preferences, assets may be persistently consumed. If these local reaction forces outweigh the equalizing power of diffusion, a gradient of asset accumulation can be stably maintained.

This framework offers insights into international economics puzzles like the Feldstein–Horioka paradox or the Lucas puzzle. The former refers to the correlation between national saving and investment despite presumed free capital transfer. The latter asks why capital does not flow into emerging markets, with high marginal productivity due to insufficient capital, at the levels one would expect. Yet, if the reaction term and its interaction with diffusion are taken into account, they admit a coherent explanation.

As organized in this chapter, the model I propose is constructed under the hypothesis that diverse time-preference rates among economic agents generate a gap with the real interest rate, and that this gap is sustained through asset preferences and capital transfer. The next chapter will formalize these ideas in a simple mathematical model and more clearly demonstrate the implications of this hypothesis.

3. Construction and Verification of the Dynamic-Equilibrium Model

3. 1. Formulation of the Model and Optimality Conditions

3. 1. 1. Objectives of the Model

In this chapter, I undertake the development of a theoretical tool capable of explaining contemporary disequilibrium phenomena. In particular, I examine the mechanism by which the gap between diverse time preference rates and the real interest rate ($\rho \neq r$)—a gap that, under standard equilibrium, should converge—can instead be sustained.

While traditional models often assume a uniform time preference rate, this paper regards it as an agent-specific, diverse trait. My analysis shows that its diversity can generate a divergence from the real interest rate, and that this divergence can be sustained by incorporating asset preferences and capital movements.

Moreover, standard macroeconomic theory typically assumes that the representative agent derives utility just from consumption. In the real world, however, individuals also derive utility from holding financial assets. In a modern economy permeated by money, financial assets serve as a “passport to freedom,” expanding life’s range of choices and emancipating individuals from constraints.

My model explicitly introduces utility from financial asset holdings. This extension allows the divergence between time preference and the real interest rate to become persistent, and it supports a dynamic-equilibrium concept characterized by ongoing state changes even if both of them were to coincide.

Furthermore, I recognize that capital transfers between countries or regions are asymmetric. Capital mobility is influenced by various factors, and its direction or friction differ across nations. For example, undergirded by the dollar's reserve-currency status, U.S. Treasuries enjoy persistent and huge demand.

The model integrates these elements by defining the evolution of the state variable—per-capita wealth—as a reaction–diffusion system that combines each economy's internal behavioral choices ("reaction term") and interactions with the external environment ("diffusion term"). These combined forces act to sustain the gap between time preferences and real interest rates.

In this section, I first formalize a general model describing the behavior of a single country or region. I do so without specifying particular functional forms so that the insights are as broadly applicable as possible.

3. 1. 2. Model Foundations

The model is built upon the dynamic optimization framework developed by F. P. Ramsey (1928)—the so-called Ramsey model. This simple, standard model assumes a representative agent with an infinite lifetime who maximizes utility by optimally allocating consumption, so an economy follows a balanced growth path under a given production function and the premise that savings equal investment. The purpose of this paper is to elucidate mechanisms by which global imbalances and long-run deflationary equilibria can arise even under rational optimization behavior. Hence, starting from this framework is appropriate as a theoretical foundation.

Throughout, we will use the following notational conventions unless otherwise noted: $\frac{da(t)}{dt} = \dot{a}$、$\frac{df(x)}{dx} = f'(x)$、$\frac{d^2 f(x)}{dx^2} = f''(x)$、$\frac{\partial F(x,y)}{\partial x} = F_x$、$\frac{\partial}{\partial y}\left(\frac{\partial F(x,y)}{\partial x}\right) = F_{xy}$

3. 1. 3. Economic Agent and Environment

The representative agent employs labor and physical capital to produce a single good. That good is either consumed or invested to form wealth, and the agent derives utility from both consumption and asset holdings.

Production technology is represented by a general function $F(K, L)$, where K is the physical capital stock and L is labor input. Labor grows at a constant rate n and thereafter we analyze all variables in per-capita terms. Denote per-capita capital by $k = \frac{K}{L}$, and per-capita output by $f(k)$, with $f'(k) > 0, f''(k) < 0$.

An economy has real capital markets and financial asset markets. The former is where produced goods are channeled into real investment, and their return is determined by the marginal productivity. The latter is where financial assets are traded, and the real interest rate r is given exogenously. The ratio of financial assets to total assets is defined as $\theta \in [0,1]$, which indicates the degree of financialization of the economy.

Finally, I model the diffusion of assets to the external environment by a function $D\left(a_t, a_{ext,t}\right)$. Here a_t is domestic per-capita wealth, $a_{ext,t}$ is the external reference per-capita wealth, and the diffusion term is driven by their "concentration difference," in analogy with reaction–diffusion systems.

In the remainder of this section, I derive the optimality conditions from these basic settings and examine their economic implications. In subsequent sections, I will specify functional forms, analyze the steady state, and derive the dynamic adjustment paths.

3. 1. 4. Utility Function and Budget Constraint

The agent's per-capita wealth is defined as

$$a_t = k_t + b_t = (1-\theta)a_t + \theta a_t \quad 1.4.1$$

Where a_t is per-capita total assets, k_t is per-capita physical capital, and b_t is per-capita financial assets.

The representative agent maximizes lifetime utility derived from both consumption c_t and financial assets b_t. Using a general utility function $U(c_t, b_t)$, the objective is

$$max \int_0^\infty e^{-\rho t} U(c_t, b_t) dt \quad = \quad max \int_0^\infty e^{-\rho} \; U(c, \theta a_t) dt \qquad 1.4.2$$

Where $\rho(>0)$ is the time preference (discount) rate. We assume $U_c > 0, U_b = U_{(\theta a)} > 0$ and $U_{cc} < 0, U_{bb} = U_{(\theta a)(\theta a)} < 0$, so that utility is increasing and strictly concave in both consumption and asset holdings.

3. 1. 5. Dynamics of Physical and Financial Assets

3. 1. 5. a. Physical-capital dynamics

The change in real capital k_t is expressed as production $f(k_t)$ minus consumption c_t, depreciation of real capital δk_t(δ is the depreciation rate), and dilution due to population growth nk_t(n is the growth rate).

$$\dot{k_t} = f(k_t) - c_t - \delta k_t - nk_t \qquad 1.5.1$$

3. 1. 5. b. Financial-asset dynamics

The change in per capita financial assets b_t is defined as the interest income rb_t minus the dilution nb_t due to population growth and the net outflow of assets to the outside (excess outflow) $D(a_t, a_{ext,t})$.

$$\dot{b_t} = rb_t - nb_t - D(a_t, a_{ext,t}) \qquad 1.5.2$$

3. 1. 6. Dynamics of Total Assets and the Reaction–Diffusion Interpretation

Differentiating the definition of total assets per capita a_t in equation 1.4.1 with respect to time, we get $\dot{a_t} = \dot{k_t} + \dot{b_t}$. Substituting 1.5.1 and 1.5.2 into this equation, we obtain the dynamic equation.

Note that $k_t = (1-\theta)a_t$, $b_t = \theta a_t$, $nk_t + nb_t = n(k_t + b_t) = na_t$.

$$\begin{aligned} \dot{a_t} &= f(k_t) + rb_t - c_t - \delta k_t - nk_t - nb_t - D(a_t, a_{ext,t}) \\ &= f((1-\theta)a_t) + r\theta a_t - c_t - \delta(1-\theta)a_t - na_t - D(a_t, a_{ext,t}) \\ &= f((1-\theta)a_t) - c_t + (r\theta - (1-\theta)\delta - n)a_t - D(a_t, a_{ext,t}) \end{aligned} \qquad 1.6.1$$

This state equation describes how per-capita wealth evolves as income is generated, consumption and saving occur, and assets flow across borders

In models based on the Ramsey model, the real capital stock is often used as the state variable. This is because they are interested in how wealth generated from a given resource and production efficiency should be allocated between consumption and savings. In contrast, this paper aims to depict phenomena such as global imbalances and long-term deflationary equilibrium. In this case, since the important point is the savings-investment balance, I select the total assets of real and financial assets as the state variable.

Moreover, 1.6.1 can be interpreted as a reaction–diffusion system borrowed from physics and biology.

- $\boldsymbol{f((1-\theta)a_t) - c_t + (r\theta - (1-\theta)\delta - n)a_t}$ **: The Reaction Term**

  This shows how assets change as a result of the optimizing behavior. The dynamics of this "response" are determined by internal factors like the production structure or the choices of saving and consumption.

- $-\boldsymbol{D}(\boldsymbol{a}_t, \boldsymbol{a}_{ext,t})$: The Diffusion Term

This describes the mechanism by which assets diffuse into the external environment. The dynamics of this "diffusion" are assumed to be driven by the difference (gradient) in "asset concentration" with the outside world, following the example of a reaction-diffusion system in physics.

3. 1. 7. Optimization Problem and First-Order Conditions

Building on the preceding setup, we formulate the representative agent's lifetime utility-maximization as the following current-value Hamiltonian:

$$H(a_t, c_t, \lambda_t) = U(c_t, \theta a_t) + \lambda_t\left[f\left((1-\theta)a_t\right) - c_t + (r\theta - (1-\theta)\delta - n)a_t - D\left(a_t, a_{ext,t}\right)\right] \qquad 1.7.1$$

Where λ_t is the costate variable representing the shadow price of per-capita asset a_t. The agent chooses consumption c_t and financial assets $b_t = \theta a_t$, starting from an initial wealth a_0.

3. 1. 7. a. First-order condition for consumption

$$\frac{\partial H}{\partial c} = U_c - \lambda_t = 0 \qquad \Rightarrow \qquad \lambda_t = U_c \qquad 1.7.2$$

This condition means that the marginal utility of consumption matches the shadow price of assets, reflecting the trade-off between current consumption and asset accumulation.

3. 1. 7. b. Costate-variable dynamics

The first-order condition on the costate variables is expressed as $\dot{\lambda}_t = \rho\lambda_t - \frac{\partial H}{\partial a_t}$ using the time preference rate and partial derivatives with respect to the state variables.

$$\begin{aligned}\dot{\lambda}_t &= \rho\lambda_t - \frac{\partial}{\partial a_t}\left(U(c_t, \theta a_t) + \lambda_t\left[f\left((1-\theta)a_t\right) - c_t + (r\theta - (1-\theta)\delta - n)a_t - D\left(a_t, a_{ext,t}\right)\right]\right) \\ &= \rho\lambda_t - [U_{(\theta a)}\theta + \lambda_t[f'\left((1-\theta)a_t\right)(1-\theta) + r\theta - (1-\theta)\delta - n - D_a]] \\ &= \rho\lambda_t - [U_{(\theta a)}\theta + \lambda_t[(1-\theta)(f'\left((1-\theta)a_t\right) - \delta) + \theta r - n - D_a]]\end{aligned} \qquad 1.7.3$$

Here, $f'\left((1-\theta)a_t\right) - \delta$ is the net return on real capital minus depreciation. Therefore,

$$R_t = (1-\theta)\left(f'\left((1-\theta)a_t\right) - \delta\right) + \theta r \qquad 1.7.4$$

Then, R_t is the sum of the investment returns from both real capital and financial assets weighted by the allocation ratio θ between them. Furthermore, by dividing both sides of 1.7.3 by the costate variable λ_t and applying 1.7.2 to the right-hand side, the following will be obtained:

$$\frac{\dot{\lambda}_t}{\lambda_t} = \rho - R_t - \frac{U_{(\theta a)}\theta}{U_c} + n + D_a \qquad 1.7.5$$

This equation determines the time path of asset prices. One of the features of this model is that it takes into account not only the time preference rate ρ and investment return R_t, but also the marginal utility of assets with respect to consumption $\frac{U_{(\theta a)}\theta}{U_c}$ and capital outflows to the outside world D_a.

3. 1. 7. c. Interpretation of the transversality condition

Generally, to rule out infinite accumulation, standard dynamic optimization imposes the transversality condition:

$$\lim_{0 \to \infty} e^{-\rho t}\lambda_t a_t = 0 \qquad 1.7.6$$

In this paper's framework, however, because the utility function includes direct utility from financial assets and the state equation features a diffusion term from cross-border capital flows, the usual transversality condition need not hold automatically. In particular, a positive preference for asset accumulation can itself provide a continuing incentive to build wealth indefinitely.

3. 1. 8. Derivation of the Consumption Dynamics Equation

As one of the implications of the optimization, the Euler equation describing per-capita consumption dynamics is derived. It shows how a consumer allocates consumption over different points in time.

First, we differentiate both sides of the first-order condition for consumption 1.7.2 with respect to time:

$$\dot{\lambda}_t = \frac{dU_c}{dt} = U_{cc}\dot{c}_t + U_{c(\theta a)}\theta\dot{a}_t \tag{1.8.1}$$

Next, we divide both sides by $\lambda_t = U_c$.

$$\frac{\dot{\lambda}_t}{\lambda_t} = \frac{U_{cc}}{U_c}\dot{c}_t + \frac{U_{c(\theta a)}}{U_c}\theta\dot{a}_t \tag{1.8.2}$$

Here, to derive the consumption growth rate $\frac{\dot{c}_t}{c_t}$, based on the intertemporal substitution rate of consumption $\sigma = -\frac{U_c}{U_{cc}}c_t$, we substitute $\frac{U_{cc}}{U_c} = -\frac{1}{\sigma c_t}$ into 1.8.2.

$$\frac{\dot{\lambda}_t}{\lambda_t} = -\frac{1}{\sigma}\frac{\dot{c}_t}{c_t} + \frac{U_{c(\theta a)}}{U_c}\theta\dot{a}_t \tag{1.8.3}$$

This 1.8.3 is equal to 1.7.5, so by combining the two the following will be obtained:

$$-\frac{1}{\sigma}\frac{\dot{c}_t}{c_t} + \frac{U_{c(\theta a)}}{U_c}\theta\dot{a}_t = \rho - R_t - \frac{U_{(\theta a)}\theta}{U_c} + n + D_a \tag{1.8.4}$$

Assuming an additively separable utility so that $U_{c(\theta a)} = 0$, we solve for the consumption growth rate.

$$\frac{\dot{c}_t}{c_t} = \sigma\left[(R_t - \rho) + \left(\frac{U_{(\theta a)}\theta}{U_c} - D_a - n\right)\right] \tag{1.8.5}$$

Alternatively, using the coefficient of relative risk aversion $\gamma = \frac{1}{\sigma}$, the following is obtained:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\gamma}\left[(R_t - \rho) + \left(\frac{U_{(\theta a)}\theta}{U_c} - D_a - n\right)\right] \tag{1.8.5$'$}$$

Compared to the usual Keynesian Ramsey rule $\frac{\dot{c}_t}{c_t} = \sigma(r - \rho)$, this equation suggests that capital outflows D_a restrain future consumption, while the marginal utility of assets $\frac{U_{(\theta a)}\theta}{U_c}$ promotes it. The former means that a strong diffusion term of capital flows, $D(a_t, a_{ext,t})$, will push down asset levels and future production levels, thereby slowing consumption. The latter means that a strong utility from financial assets, $b_t = \theta a_t$, will push up asset levels and future production levels, thereby accelerating consumption.

3. 1. 9. Steady States of this Model

3. 1. 9. a. Steady State as a Dynamic Equilibrium

Assuming $\dot{c}_t = 0$, then equation 1.8.5 becomes as follows:

$$R_t - \rho = n + D_a - \frac{U_{(\theta a)}\theta}{U_c} \tag{1.9.1}$$

This equation suggests the view that "the divergence between interest rates as the rate of return on assets and the time preference rate (left-hand side) can be maintained by the balance between asset preferences that keep capital domestic and the pressure for capital to diffuse outward (right-hand side)." From this perspective, economics has so far focused on the left-hand side, which is attributable to the real economy, and has not paid enough attention to the right-hand side, which is derived from the asset economy. Below, we will analyze it in three parts.

- Case A: $\boldsymbol{R_t = \rho}$

Since R_t is the weighted average of the return from real investments and financial assets by the financial asset ratio θ, the establishment of $R_t = \rho$ means an exquisite state in which the marginal productivity of capital, the depreciation rate, the rate of return on financial assets, and the time preference rate ρ which is the "personality" of economic agents, happen to coincide through the composition of the asset portfolio based on the financial asset ratio θ. In addition, the right-hand side also needs to be zero, so this requires that multiple dynamic factors such as asset preference $\frac{U_{(\theta a)}\theta}{U_c}$ which tries to keep assets within the economy, population growth n which promotes the dilution of assets, and capital flows D_a which diffuse assets into the external environment, constantly offset each other. It is not impossible that such a state can be established by the choice of savings and consumption, but this means that R_t and $\frac{U_{(\theta a)}\theta}{U_c}$ are simultaneously set to values that make the left-hand side and the right-hand side zero respectively, which is not very realistic.

- Case B: $\boldsymbol{R_t > \rho}$

Since the sign on the left-hand side is positive, the right-hand side must also be positive. Thus, while capital outflow D_a and population growth n tend to be positive, the strength of asset preference $\frac{U_{(\theta a)}\theta}{U_c}$ is unlikely to be large. In an economy with a relatively lower rate of time preference, savings usually increase, creating a funds surplus and falling interest rates. However, in the case of this paper, if there is a capital outflow, dilution due to population growth, or if asset preference is weak, a real interest rate that is relatively high compared to the time preference rate will not decline sufficiently, thus, the divergence of both will be maintained, and external imbalances may be sustained.

- Case C: $\boldsymbol{R_t < \rho}$

Since the sign on the left-hand side is negative, the right-hand side must also be negative. Thus, capital outflows D_a and population growth n tend to be negative, while the strength of asset preference $\frac{U_{(\theta a)}\theta}{U_c}$ tends to be large. In an economy with a relatively higher time preference rate, consumption normally increases, so it faces a funds shortage and rising interest rates. However, in the case of this paper, if there is an increase of assets due to external capital inflows or a declining population or if asset preference is strong, a real interest rate that is relatively lower compared to the time preference rate will be sustained, and the divergence of both and external imbalances may be sustainable.

3. 1. 9. b. Conceptual Interpretation of the Steady State

Here, using the conditions $\dot{c}_t = \dot{a}_t = 0$, we will examine how the control variable c_t and the state variable a_t relate in a steady state. First, under $\dot{c}_t = 0$, equation 1.9.1 can be rearranged as follows:

$$U_c = \frac{U_{(\theta a)}\theta}{n + D_a - (R_t - \rho)} \qquad 1.9.2$$

On the left-hand side, U_c falls as c_t increases by diminishing marginal utility and does not depend on a_t due to additive separability. On the right-hand side, the numerator $U_{(\theta a)}\theta$ decreases with an increase in a_t due to diminishing marginal utility. In the denominator, D_a rises with a_t (more diffusion when domestic assets exceed external assets) and R_t falls with an increase in a_t due to diminishing marginal productivity. Therefore, the denominator increases in a_t, the entire right-hand side decreases in a_t. To restore equality, U_c must fall further, so c_t must rise. Plotting in the $a_t - c_t$ plane yields an upward-sloping curve.

Further, a rise in n, D_a, or ρ increases the denominator, lowers U_c, and thus shifts the $\dot{c_t} = 0$ curve up and to the left. A rise in R_t has the opposite effect.

Next, the $\dot{a_t} = 0$, curve is obtained by setting the left-hand side of the state equation 1.6.1 to zero and solve for c_t:

$$c_t = f\big((1-\theta)a_t\big) + (r\theta - (1-\theta)\delta - n)a_t - D\big(a_t, a_{ext,t}\big) \qquad 1.9.3$$

The right-hand side of this equation can be divided into two parts. One is $f\big((1-\theta)a_t\big) + r\theta a_t$. This is the sum of income from the production function and investment gains. The former increases with a gradual decline (in its rate of increase) while the latter is linear, so it forms an upward-convex curve. The other part is $(-(1-\theta)\delta - n)a_t - D\big(a_t, a_{ext,t}\big)$, which represents dilution of assets due to population growth or depreciation and outflow of capital. To make the relationship of these easier to understand, equation 1.9.3 can be rearranged to give,

$$c_t = \Big(f\big((1-\theta)a_t\big) + r\theta a_t\Big) - \Big(\big((1-\theta)\delta + n\big)a_t + D\big(a_t, a_{ext,t}\big)\Big) \qquad 1.9.3'$$

The part $\Big(\big((1-\theta)\delta + n\big)a_t + D\big(a_t, a_{ext,t}\big)\Big)$ after transformation becomes a monotonically increasing, upward-sloping curve as a_t increases If $D\big(a_t, a_{ext,t}\big)$ is considered to be an increasing function.

Therefore, the formula 1.9.3′ can be said to be income from production and operation minus dilution, depreciation, and outflow. In response to an increase in a_t, it initially increases sharply, then its rate gradually becomes gentler, eventually reaching a peak and starting to decrease. In addition, in terms of the parameters, an increase in δ, n, and $D\big(a_t, a_{ext,t}\big)$ push the curve downward.

According with the above, 1.9.2 when $\dot{c_t} = 0$ and 1.9.3′ when $\dot{a_t} = 0$ are illustrated in $fig. 3-1$.

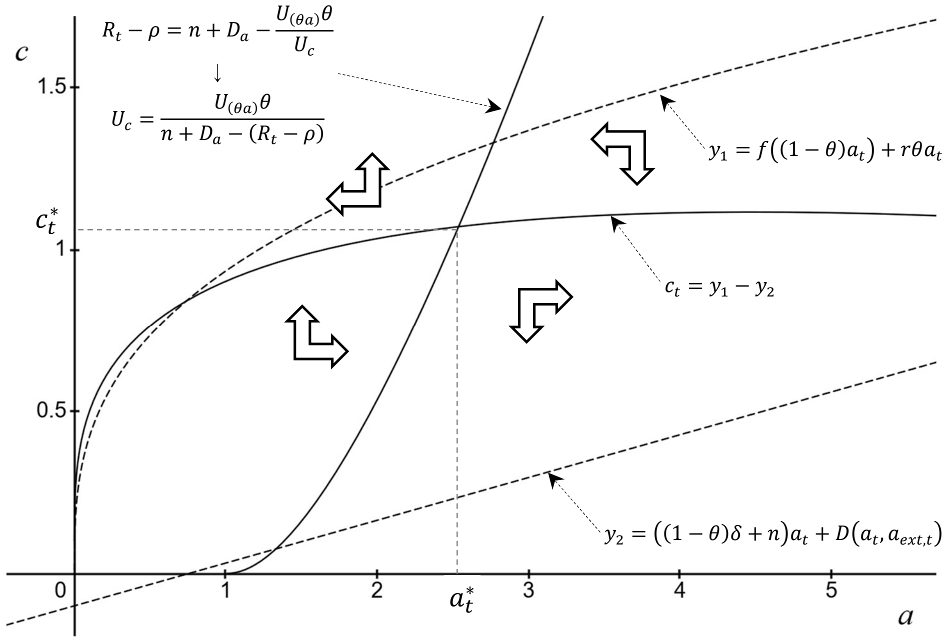
The insights gained from the analysis so far are as follows:

- The starting point of this paper was the view that "time preference rates are not uniform, but are specific and diverse depending on the economic entity, country, and region, so real interest rates do not necessarily converge to them, and the two will rather diverge." From the analysis of the model, there are two factors that generate and maintain this divergence.

- One of these is asset preference. If they prefer assets to consumption, they will have a surplus of funds, and the rise in interest rates will be suppressed. Conversely, if they do not prefer assets, they will have a shortage of funds, and the fall in interest rates will be limited. In other words, these are the forces that keep capital and assets within the economy.

- The other is the mobility of capital and people. If capital outflow occurs or the population increases due to inflow, there will be a shortage of funds, and the fall in interest rates will be limited. Conversely, if capital inflow occurs or the population decreases due to outflow, there will be a surplus of funds, and the rise in interest rates will be suppressed. In other words, this is a force that promotes diffusion and dilution through interaction with the external environment.

fig.3-1 Schematic diagram of the steady state based on this model



- In real economies, even in an economy with a lower time preference rate and a surplus of funds, if there is capital outflow or asset preference is weak, interest rates will not decline sufficiently. Conversely, even in an economy with a higher time preference rate and a shortage of funds, if there is capital inflow or asset preference is strong, interest rates will not rise sufficiently. If these dynamics of asset preference and capital transfer absorb to a certain extent the pressure for real interest rates to converge across countries and regions, then the divergence between time preference rates and real interest rates can be preserved.

3. 2. Mathematical Analysis of the Steady State

3. 2. 1. Specifying Functional Forms

In this section, we apply concrete functional forms to the model developed above, and analyze the existence and stability of the steady state, as well as the impact of parameter changes.

3. 2. 1. a. Production function

The production function for real capital per capita is the standard Cobb-Douglas production function:

$$f(k_t) = Ak_t^{\alpha} \quad (A > 0, \alpha \in [0,1]) \tag{2.1.1}$$

Since $k_t = (1-\theta)a_t$,

$$f(k_t) = f\left((1-\theta)a_t\right) = A\left((1-\theta)a_t\right)^{\alpha} = A(1-\theta)^{\alpha}a_t^{\alpha} \tag{2.1.2}$$

and its marginal product $f'(k_t)$ is as follows:

$$f'(k_t) = A\alpha k^{\alpha-1} = A\alpha\left((1-\theta)a_t\right)^{\alpha-1} = A\alpha(1-\theta)^{\alpha-1}a_t^{\alpha-1} \tag{2.1.3}$$

3. 2. 1. b. Utility function

The utility function $U(c_t, b_t)$ over consumption c_t and financial assets b_t is an additively separable CRRA (Constant Relative Risk Aversion) form, with an additional parameter β capturing asset preference:

$$U(c_t, b_t) = \frac{c_t^{1-\gamma}}{1-\gamma} + \beta \frac{b_t^{1-\psi}}{1-\psi} \qquad (\gamma > 0) \tag{2.1.4}$$

Here, since $b_t = \theta a_t$, the marginal utilities are as follows:

$$U_c = \frac{\partial U(c_t, b_t)}{\partial c_t} = c_t^{-\gamma} \tag{2.1.5}$$

$$U_{(\theta a)} = U_b = \frac{\partial U(c_t, b_t)}{\partial b_t} = \beta b_t^{-\psi} = \beta(\theta a_t)^{-\psi} = \beta \theta^{-\psi} a_t^{-\psi} \tag{2.1.6}$$

Here, we make a certain assumption about β. As is clear from 2.1.4, this parameter indicates the weight of the utility from asset holdings relative to consumption. Usually, it is imagined as a positive constant centered around 1.

However, depending on the economic agent, it may be 0 or a negative value. It means that no utility is gained from asset holdings when it is 0, and it means that disutility is gained from asset holdings when it is negative. For example, when "taking out a loan to buy a car," an agent is actively taking on debt for current consumption. Therefore, this parameter β can take positive or negative values centered around 0.

3. 2. 1. c. Diffusion term about capital

The diffusion term $D(a_t, a_{ext,t})$ represents an outflow that increases as per capita assets a_t are higher than the reference point $a_{ext,t}$. It responds linearly to difference and degree of it can be expressed by ϕ (diffusion coefficient):

$$D(a_t, a_{ext,t}) = \phi(a_t - a_{ext,t}) \qquad (\phi > 0) \tag{2.1.7}$$

Therefore, the partial derivative of this function with respect to a_t is a constant:

$$D_a = \frac{\partial D(a_t, a_{ext,t})}{\partial a_t} = \phi \tag{2.1.8}$$

3. 2. 2. Steady-State Loci

3. 2. 2. a. The $\dot{a}_t = 0$ Locus

Substitute the concrete functional forms into the state equation 1.6.1, set $\dot{a}_t = 0$, and solve for c_t:

$$\dot{a}_t = A(1-\theta)^\alpha a_t^\alpha - c_t + (r\theta - (1-\theta)\delta - n)a_t - \phi(a_t - a_{ext,t})$$
$$\Rightarrow \qquad c_t = (A(1-\theta)^\alpha a_t^\alpha + r\theta a_t) - ((1-\theta)\delta + n + \phi)a_t + \phi a_{ext,t} \tag{2.2.1}$$

As noted above, the right-hand side is total income $A(1-\theta)^\alpha a_t^\alpha + r\theta a_t$ minus dilution and depreciation $((1-\theta)\delta + n + \phi)a_t$. Since the first term is a power function in a_t with exponent $\alpha \in [0,1]$, it increases at a gradually declining rate while the second term is linear. The linear term eventually dominates, creating an inverted-U shape in the $a_t - c_t$ plane.

As a_t increases, c_t initially rises sharply, then peaks, and finally falls. This shape is familiar from the standard Ramsey model, however, we interpret it as the reaction–diffusion balance between endogenous "reaction" (production and return) and exogenous "diffusion" (capital outflow) forces, so our intention is to capture the dynamic pattern formation that occurs in the balance between the two.

3. 2. 2. b. The $\dot{c}_t = 0$ Locus

Substituting a specific function into 1.9.1 which is the steady-state condition for consumption, and solving for c_t. Since the definition of R_t is 1.7.4, using 2.1.3, 2.1.5, 2.1.6, 2.1.8, the following is obtained:

$$(1-\theta)(A\alpha(1-\theta)^{\alpha-1}a_t^{\alpha-1}-\delta)+\theta r-\rho=n+\phi-\frac{(\theta a_t)^{-\psi}}{c_t^{-\gamma}}\beta\theta \qquad 2.2.2$$

The left-hand side has the marginal return on real capital $A\alpha(1-\theta)^{\alpha}a_t^{\alpha-1}$ as its main term, and shows the deviation between the overall return on assets (calculated by subtracting the depreciation rate from this and adding the return on financial assets, taking into account the proportion of financial assets) and the time preference rate ρ which this paper assumes to be "unique and diverse for each economic agent." Furthermore, the entire left-hand side of this equation decreases monotonically with an increase in a_t because the $\alpha-1$ in the exponent is negative, due to the assumption that $\alpha<1$ (the capital distribution rate). It coincides with the model's premise of diminishing marginal productivity.

On the other hand, the right-hand side is composed of the population growth rate, the capital diffusion coefficient, and the ratio of the marginal utility of assets to consumption multiplied by asset preference and the proportion of financial assets. These balance the deviation between the return and time preference rate on the left-hand side. In other words, it is a struggle between forces that keep assets internal (the utility from assets, asset preference, and the degree of financialization) and forces that encourage external diffusion and dilution (the capital diffusion coefficient and population growth).

It is the third term that changes with an increase in a_t, and depending on the sign of asset preference β, this leads to the following different outcomes.

- Case: $\boldsymbol{\beta>0}$ (positive asset preference)

Since asset preference is positive, this term needs to increase for maintaining the equality of the decrease on the left-hand side. Therefore, an increase in a_t causes a decrease in the numerator $a_t^{-\psi}$, so a decrease in the denominator $c_t^{-\gamma}$ (in other words, an increase in consumption) needs to exceed this decrease.

This means that as an increase in capital stock reduces marginal productivity and it no longer matches the time preference rate, allocation to consumption increases.

- Case: $\boldsymbol{\beta<0}$ (negative asset preference)

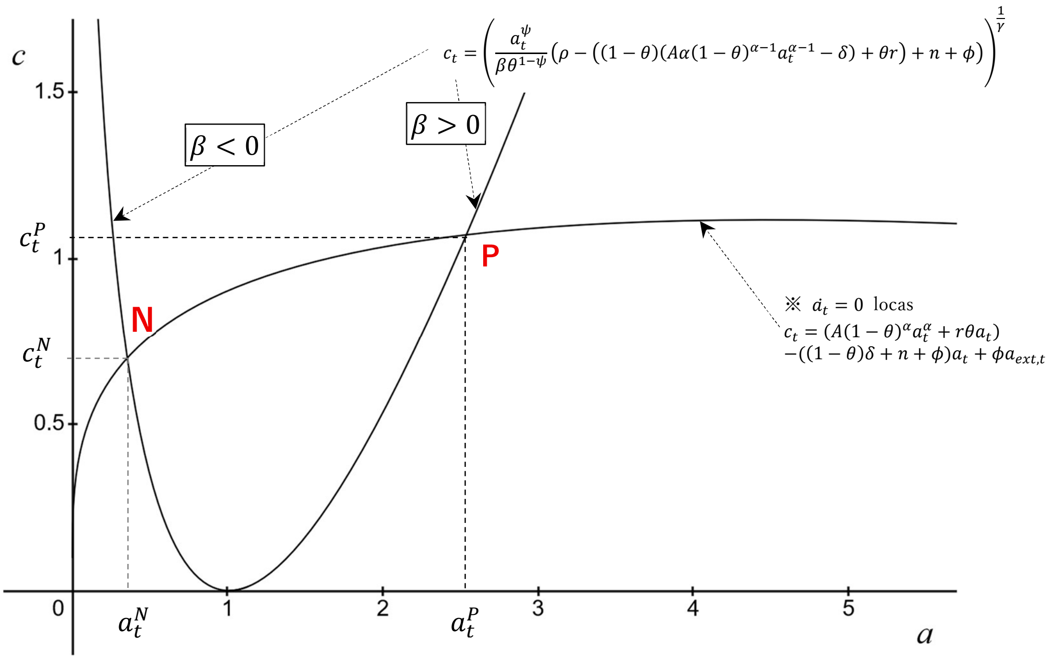
Since asset preference is negative, this term needs to decrease for maintaining the equality of the decrease on the left-hand side. Therefore, an increase in a_t causes a decrease in the numerator $a_t^{-\psi}$, so an increase in the denominator $c_t^{-\gamma}$ (in other words, a decrease in consumption) needs to exceed this decrease.

This means that the allocation to consumption will be increased even if it means reducing assets because an increase in assets only brings disutility.

To trace out the locus defined by equation 2.2.2 in the a_t-c_t plane, we solve it for c_t:

$$c_t=\left(\frac{a_t^{\psi}}{\beta\theta^{1-\psi}}\left(\rho-\left((1-\theta)(A\alpha(1-\theta)^{\alpha-1}a_t^{\alpha-1}-\delta)+\theta r\right)+n+\phi\right)\right)^{\frac{1}{\gamma}} \qquad (\beta\neq 0) \qquad 2.2.3$$

The $(\rho-((1-\theta\ \dots\dots\dots+n+\phi)$ part increases monotonically with an increase in a_t. This is because the main term, marginal return, assumes diminishing marginal productivity, and the term has a negative sign. On the other hand, the outer $\frac{a_t^{\psi}}{\beta\theta^{1-\psi}}$ has the numerator a_t^{ψ} and the denominator β, so just like the balance *fig.3-2 two types of steady state*



【setting of parameters】
ρ : 0.2、β : -0.5 or 0.5、γ : 0.5、ψ : 0.5、A : 1.0、θ : 0.2、α : 0.35、δ : 0.1、r : 0.05、n : 0.002、ϕ : 0.05、$a_{ext,t}$: 2.0

equation above, it depends on the sign of β. With an increase in a_t, there is a monotonous increase if β is positive, and there is a monotonous decrease if it is negative.

Moreover, there is an exponent of $\frac{1}{\gamma}$ on the entire expression in the parenthesis. Of these, γ is the relative risk aversion of consumption, and it is usually considered to be a positive decimal less than 1. As a result of this exponent, even if the value in the parenthesis is negative, c_t can be positive, negative or an imaginary number. Of course, this has no meaning from an economic perspective, so the constraint that the expression in the parenthesis must be positive should be imposed.

Showing in $fig.3-2$, the $\dot{c}_t = 0$ locus is a curve rising upwards from the a_t axis to the right if β is positive, and to the left if negative. These meanings are easier to understand if looking from the a_t axis.

A right-sloping curve with positive asset preference allocates more to consumption because the higher the asset level, the lower the marginal utility of assets, and marginal productivity is also lower, so the rate of return doesn't match the time preference rate. On the other hand, a left-sloping curve with negative asset preference allocates more to consumption even if reducing assets because an increase in assets only brings disutility. As is clear from 2.2.3, this equation cannot be defined when $\beta = 0$ so c_t becomes a straight line extending perpendicularly from the a_t axis as in the usual Ramsey model

3. 2. 3. Steady State and Stability

3. 2. 3. a. Asset preference and two types of steady state

The intersection of the $\dot{a}_t = 0$ locus and the $\dot{c}_t = 0$ locus is the steady state where assets and consumption maintain constant levels. Thus, the economy reaches either a steady state P or N depending on the sign of β as shown in $fig.3-2$. Of these, P is the same as a normal steady state, while N is a unique equilibrium. The lower the asset level, the higher the consumption level desired, so this is called "excess consumption equilibrium" in this paper. Below, we will consider the conditions for its establishment.

First, by rearranging Equation 2.2.2 as follows, it becomes easier to understand how to balance the time preference rate and other factors:

$$\beta\theta^{1-\psi}a_t^{-\psi}c_t^{\gamma} = \left(\rho - \left((1-\theta)(A\alpha(1-\theta)^{\alpha-1}a_t^{\alpha-1} - \delta) + \theta r\right)\right) + n + \phi \qquad 2.3.1$$

The point of the establishment of excess consumption equilibrium is the existence of negative asset preference. From 2.3.1, since the only term that can be negative on the left-hand side is β, it can be seen that its sign is affected by the balance between 1) the difference between the time preference rate and the asset return rate, 2) population growth, and 3) the capital diffusion coefficient on the right side.

Asset preference can be regarded as the force keeping capital and assets within the economy. Therefore, it must become stronger if the asset return rate is lower compared to the time preference rate due to limited effective investment opportunities, or if the degree of capital diffusion outside the country or region is larger. Conversely, negative asset preference is likely to exist when 1) the rate of return is higher compared to the time preference rate because there are a relatively large number of promising investment opportunities, 2) little dilution of per capita assets due to slow or declining population growth, and 3) capital outflows are suppressed or there is a tendency for a surplus inflow of capital.

This is also true for underdeveloped economies that are in the stage of receiving investment, or for the U.S. economy where confidence in the dollar leads to regular large-scale capital inflows and the economy chronically suffers from an investment surplus and current account deficit.

3. 2. 3. b. Two steady states and stability analysis

The steady state, the point where $\dot{a}_t = \dot{c}_t = 0$, is the equilibrium point where the economy settles in the long run. Analyzing its stability is important for understanding the economic implications from the model. Therefore, we will conduct a stability analysis below.

The dynamic equations related to a_t is the following which has already been used in 2.2.1:

$$\dot{a}_t = G_1(a_t, c_t) = A(1-\theta)^{\alpha}a_t^{\alpha} - c_t + (r\theta - (1-\theta)\delta - n)a_t - \phi(a_t - a_{ext,t}) \qquad 2.3.2$$

The one related to c_t is the following which is obtained by applying specific functions to equation 1.8.5′ in the previous section and solving for $\dot{c}_t$:

$$\dot{c}_t = G_2(a_t, c_t) = \frac{c_t}{\gamma}\left[\left((1-\theta)(A\alpha(1-\theta)^{\alpha-1}{a_t}^{\alpha-1} - \delta) + \theta r - \rho\right) + \left(\frac{\beta\theta^{1-\psi}{a_t}^{-\psi}}{c_t^{-\gamma}} - \phi - n\right)\right] \qquad 2.3.3$$

Meanwhile, the linear approximation around the steady state is performed using the Jacobian matrix:

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial G_1}{\partial a_t} & \frac{\partial G_1}{\partial c_t} \\ \frac{\partial G_2}{\partial a_t} & \frac{\partial G_2}{\partial c_t} \end{bmatrix}$$

To determine stability, it is necessary to find the eigenvalue λ of the Jacobian, which can be obtained by solving the characteristic equation $\det(J - \lambda I) = 0$. It can be expressed as $\lambda^2 - Tr(J)\lambda + Det(J) = 0$, where $Tr(J) = J_{11} + J_{22}$ is the sum of the diagonal elements and $Det(J) = J_{11}J_{22} - J_{12}J_{21}$ is the determinant.

The stability of the steady state is classified according to the sign of the real part and the presence or absence of the imaginary part. In this type of model where consumption is a jump variable, an economically meaningful equilibrium is usually a saddle point. The mathematical condition for this is $Det(J) < 0$.

We will derive each element of the Jacobian matrix below. First, J_{11} represents the change in assets due to changes in asset levels. It indicates the extent to which the return from real capital and financial assets exceeds the dilution due to population growth and the diffusion due to capital outflows. This can be an indicator of how strong the motivation for asset accumulation is.

$$J_{11} = \frac{\partial G_1}{\partial a_t} = ((1-\theta)(A\alpha(1-\theta)^{\alpha-1}{a_t}^{\alpha-1} - \delta) + \theta r) - (n+\phi) \tag{2.3.4}$$

Next, J_{12} represents the change in assets due to changes in consumption levels. As is clear from the asset dynamics 2.3.2, it indicates that an increase in consumption reduces asset accumulation by that amount.

$$J_{12} = \frac{\partial G_1}{\partial c_t} = -1 \tag{2.3.5}$$

Furthermore, J_{21} represents the change in consumption due to changes in asset levels. In the following formula 2.3.6, the first term in the brackets [ ] represents the diminishing marginal productivity suppressing consumption, and the second represents the utility or disutility accelerating or decelerating consumption.

$$J_{21} = \frac{\partial G_2}{\partial a_t} = \frac{c^*}{\gamma}\left[A\alpha(\alpha-1)(1-\theta)^{\alpha}{a_t}^{\alpha-2} - \frac{\beta\psi\theta^{1-\psi}{a_*}^{-\psi-1}}{{c_*}^{-\gamma}}\right] \tag{2.3.6}$$

Finally, J_{22} represents the change in consumption due to changes in the consumption level, and this requires some ingenuity. First, for the dynamic formula 2.3.3 for c_t, in the steady state, $\dot{c_t} = 0$ and $c_t \neq 0$ must be true, so the expression in the brackets [ ] should be zero. Then, by applying the product rule of differentiation to this equation, one might expect to get $J_{22} = \frac{\partial G_2}{\partial c_t} = \frac{\partial}{\partial c_t}\left(\frac{c_t}{\gamma}\right)[\ldots] + \frac{c_t}{\gamma}\frac{\partial}{\partial c_t}[\ldots]$, However, since $[\ldots] = 0$, we just need to consider the second term.

$$J_{22} = \frac{\partial G_2}{\partial c_t} = \frac{c^*}{\gamma}\frac{\partial}{\partial c_t}[\ldots] = \frac{c^*}{\gamma}\frac{\partial}{\partial c_t}\left(\frac{\beta\theta^{1-\psi}{a_*}^{-\psi}}{c_*^{-\gamma}}\right) = \frac{c^*}{\gamma}\beta\theta^{1-\psi}{a_*}^{-\psi}\gamma {c_*}^{\gamma-1} = \frac{\beta\theta^{1-\psi}{a_*}^{-\psi}}{c_*^{-\gamma}} \tag{2.3.7}$$

This shows how much the marginal utility of assets is greater than that of consumption. The sign of this term depends solely on β, so if the utility of assets is positive ($\beta > 0$), an increase in the level of consumption will accelerate consumption. Conversely, if negative ($\beta < 0$), the growth rate of consumption will decelerate.

As a result of these factors, the Jacobian matrix can finally be expressed as follows:

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial G_1}{\partial a_t} & \frac{\partial G_1}{\partial c_t} \\ \frac{\partial G_2}{\partial a_t} & \frac{\partial G_2}{\partial c_t} \end{bmatrix} = \begin{bmatrix} ((1-\theta)(A\alpha(1-\theta)^{\alpha-1}{a_t}^{\alpha-1} - \delta) + \theta r) - (n+\phi) & -1 \\ \frac{c^*}{\gamma}\left[A\alpha(\alpha-1)(1-\theta)^{\alpha}{a_t}^{\alpha-2} - \frac{\beta\psi\theta^{1-\psi}{a_*}^{-\psi-1}}{{c_*}^{-\gamma}}\right] & \frac{\beta\theta^{1-\psi}{a_*}^{-\psi}}{c_*^{-\gamma}} \end{bmatrix} \tag{2.3.8}$$

J_{11} and J_{21} are functions with complex parameters, so it is difficult to analytically determine their signs. However, it is possible to gain insight into the dynamics of the model through a combination of parameters.

(a) $\boldsymbol{\beta > 0}$ and $\boldsymbol{J_{11} > 0}$: Case where the utility of assets is positive and high returns can be obtained

Economic agents have a preference for asset accumulation, and the return from capital and assets exceeds their dilution and dissipation. In this case, a strong incentive to accumulate assets is likely to work.

The signs of the elements are expected to be $J_{11} > 0$, $J_{12} < 0$, $J_{21} < 0$, and $J_{22} > 0$. $Det(J)$ is uncertain as it involves subtracting a positive term from a positive term, while the sign of $Tr(J)$ is positive. As a result, it is likely to become an unstable node or focus. This suggests that the economy will diverge due to the synergistic effect of the asset accumulation motive and the accelerating effect on consumption.

(b) $\beta > 0$ and $J_{11} < 0$: Case where the utility of assets is positive but only low returns can be obtained.

Economic agents have a preference for assets, but the return from capital and assets is less than their dilution and dissipation. In this case, as asset accumulation progresses, the rate of return slows and the incentive to accumulate assets weakens due to the balance with each agent's time preference rate.

The signs of the elements are expected to be $J_{11} < 0$, $J_{12} < 0$, $J_{21} < 0$, and $J_{22} > 0$. $Det(J)$ is negative because it involves subtracting a positive term from a negative term, while the sign of $Tr(J)$ is not determined. As a result, the steady state is a saddle point, which is similar to the stability in standard models. This suggests that the decline in return due to asset accumulation acts as moderate feedback, and the economy converges.

(c) $\beta < 0$ and $J_{11} > 0$: Case where the utility of assets is negative but high returns can be obtained

This is a situation where agents have disutility from assets, and their returns exceed dilution and dissipation. Depending on the impact of changes in assets on consumption J_{21}, a result is either of the following:

- **c-1 $J_{21} > 0$ (accelerating consumption effect exceeds suppressing consumption effect)**

The signs of the elements are expected to be $J_{11} > 0$, $J_{12} < 0$, $J_{21} > 0$, and $J_{22} < 0$. $Det(J)$ is uncertain as it involves subtracting a negative term from a negative term, while the sign of $Tr(J)$ is not determined. As a result, there is a high possibility that it will become an unstable node or focus. This suggests that the effect of accelerating consumption due to asset disutility will encourage excessive consumption and cause the economy to diverge.

- **c-2 $J_{21} < 0$ (accelerating consumption effect is less than suppressing consumption effect)**

The signs of the elements are expected to be J_11>0, J_12<0, J_21<0, and J_22<0. Det(J) is always negative because it subtracts positive from negative term, while the sign of Tr(J) is uncertain. Therefore, it becomes a saddle point. This suggests that even under high returns, the suppressing consumption effect due to declining marginal productivity can lead to converge to a steady state.

(d) $\beta < 0$ and $J_{11} < 0$: Case where the utility of assets is negative and only low returns can be obtained

This is a situation where agents have disutility from assets and their return is less than dilution and dissipation. Depending on the impact of changes in assets on consumption J_{21}, a result is either of the following.

- **d-1 $J_{21} > 0$ (accelerating consumption effect exceeds suppressing consumption effect)**

The signs of the elements are expected to be $J_{11} < 0$, $J_{12} < 0$, $J_{21} > 0$, and $J_{22} < 0$. $Det(J)$ is always positive, while Tr(J) is always negative because it is the sum of negative terms. This results in a stable node. This suggests that even if the accelerating effect of asset disutility exceeds the suppressing effect on consumption, the low return on assets may suppress accumulation and promote convergence.

- **d-2 $J_{21} < 0$ (accelerating consumption effect is less than suppressing consumption effect)**

The signs of the elements are expected to be $J_{11} < 0$, $J_{12} < 0$, $J_{21} < 0$J, and $J_{22} < 0$. $Det(J)$ is uncertain, while $Tr(J)$ is always negative because it is the sum of negative terms. This results a mixture of stable nodes or focus, and saddle points. This indicates that the effect of declining marginal productivity in suppressing consumption, combined with the low returns from assets strongly suppresses consumption.

In the standard case where asset preference is positive ($\beta > 0$), the condition for a saddle point is that the gradual decline in marginal productivity, combined with dilution and dissipation to the external environment, results in low returns from assets, which leads to negative feedback ($J_{11} < 0$). This is related to point P.

In the non-standard case ($\beta < 0$), the condition is more complicated. One way to reach a saddle point is when the consumption-suppressing effect of diminishing marginal productivity exceeds the consumption-accelerating effect ($J_{21} < 0$). Another way is when the consumption-accelerating effect of asset disutility exceeds the consumption-suppressing effect of diminishing marginal productivity ($J_{21} > 0$), it can still become a stable node or focus under low returns ($J_{11} < 0$).

These findings suggest that this "excess consumption equilibrium" is not merely theoretical, but can actually exist in reality, driven by interactions by interactions with the external environment and negative asset preferences.

3. 2. 4. Changes in parameters and their effects on the steady state

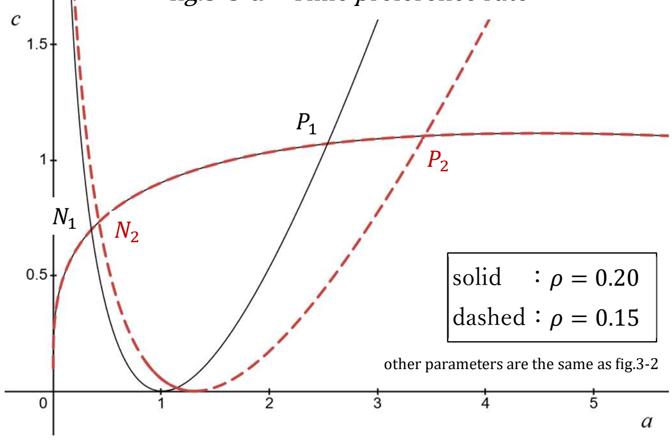
This section considers the effects of changes in parameters on the steady state. Since it is difficult to find an analytical solution, graph drawing software is used.

(a) Time preference rate

When the time preference rate ρ decreases, the $\dot{c}_t = 0$ locus shifts to the right. As a result:

- $\beta > 0$: P shifts from P1 to P2, leading to an increase in assets and a slight increase in consumption. This occurs because a decreased time preference rate emphasizes future utility, which encourages asset accumulation and a slight rise in the production level.
- $\beta < 0$: Point N shifts from N1 to N2, resulting in slight increases in both assets and consumption. Although asset accumulation progresses, this increase is limited because an increase in assets brings only disutility in this case.

fig.3-3-a Time preference rate

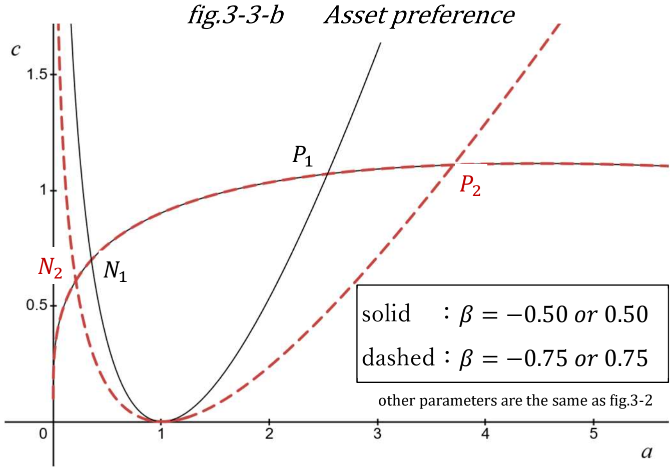


(b) Asset preference

When the asset preference β strengthens, the $\dot{c}_t = 0$ locus is compressed downward along the c_t axis (vertical axis). As a result:

- $\beta > 0$: Point P shifts from P1 to P2, leading to an increase in assets and a slight increase in consumption. This is because positive asset preference increases allocation to assets, which now bring greater utility than before, and consequently, production levels rise slightly.
- $\beta < 0$: Point N shifts from N1 to N2, resulting in decreases in both assets and consumption. This is because negative asset preference leads to decreased allocation to assets that bring greater disutility than before, resulting in a decline in production levels.

fig.3-3-b Asset preference



(c) Relative risk aversion of assets

When the relative risk aversion of assets ψ increases (become more risk averse), the slope of the $\dot{c}_t = 0$ locus becomes gentler. As a result:

- $\beta > 0$: Point P shifts from P1 to P2, assets increase while consumption remains flat. This occurs because, with positive asset preference, asset holdings increase due to a decrease in marginal utility, but their impact on production is limited.
- $\beta < 0$: Point N shifts from N1 to N2, both assets and consumption decrease. This is because, with negative asset preference, assets are reduced to avoid disutility, and as a result, consumption also decreases due to a fall in the production level.

(d) Financial asset ratio

When the financial asset ratio θ increases, first the $\dot{a}_t = 0$ locus shifts downward since real capital decreases and production levels fall. In addition:

- $\beta > 0$: The slope of the $\dot{c}_t = 0$ locus becomes gentler, point P shifts from P1 to P2. Assets increase while consumption remains flat because the increase in assets is driven by financial assets, but its impact is limited by the decline in the production level.
- $\beta < 0$: The $\dot{c}_t = 0$ locus shifts to the left, point N shifts from N1 to N2. Both assets and consumption decrease. This is because assets are reduced to avoid disutility, and as a result, production levels also decline.

(e) Capital diffusion coefficient

When the diffusion coefficient ϕ decreases, the peak of the $\dot{a}_t = 0$ locus shifts to the right. This occurs because a higher asset concentration leads to lower capital outflow. In addition:

- $\beta > 0$: The $\dot{c}_t = 0$ locus shifts to the right and point P shifts from P1 to P2. Both assets and consumption increase. This is because, at higher asset levels, capital outflows are more suppressed, leading to increases in production levels and consumption.
- $\beta < 0$: The $\dot{c}_t = 0$ locus shifts slightly to the right, point N shifts from N1 to N2. Assets increase, and consumption decreases slightly. This is because, since the asset level is low, the reduction in capital inflows is greater than the outflows, so the impact on production levels, income, and consumption is limited.

fig.3-3-c Relative risk aversion of assets

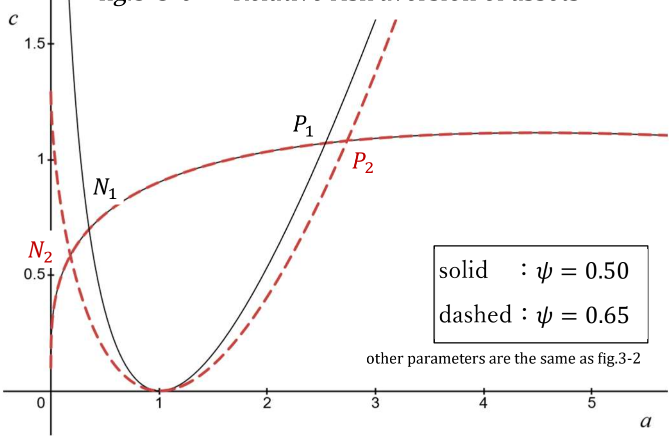


fig.3-3-d Financial asset ratio

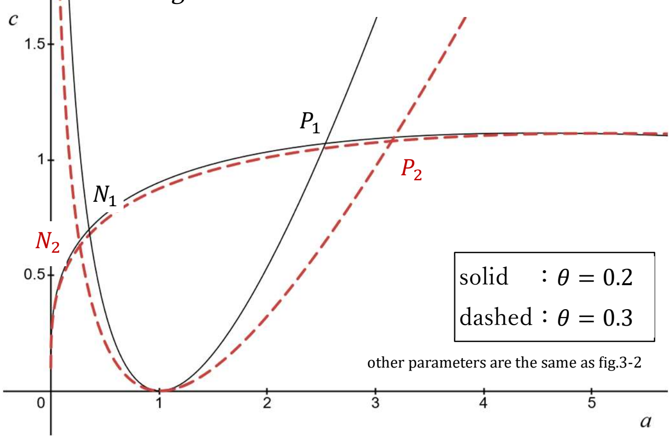


fig.3-3-e Capital diffusion coefficient

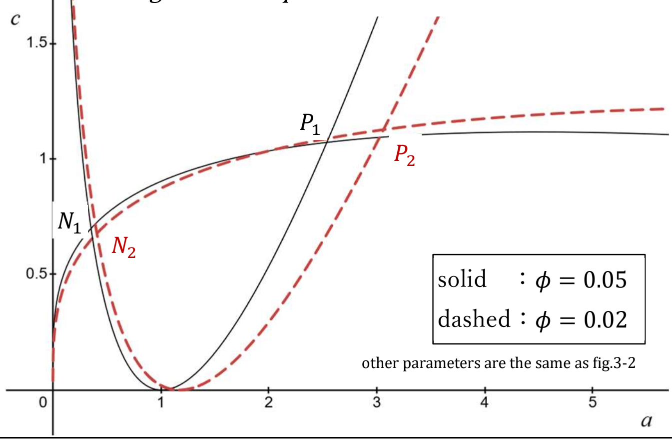


Table.3-1 parameters and stability analysis of single economy model

		$\beta>0$ (positive asset preference)						$\beta<0$ (negative asset preference)					
		P1	P2					N1	P2				
			ρ	β	ψ	θ	ϕ		ρ	β	ψ	θ	ϕ
parameters	ρ	0.200	**0.150**	0.200	0.200	0.200	0.200	0.200	**0.150**	0.200	0.200	0.200	0.200
	β	0.500	0.500	**0.750**	0.500	0.500	0.500	▲0.500	▲0.500	**▲0.750**	▲0.500	▲0.500	▲0.500
	ψ	0.500	0.500	0.500	**0.650**	0.500	0.500	0.500	0.500	0.500	**0.650**	0.500	0.500
	γ	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
	A	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	θ	0.200	0.200	0.200	0.200	**0.300**	0.200	0.200	0.200	0.200	0.200	**0.300**	0.200
	α	0.350	0.350	0.350	0.350	0.350	0.350	0.350	0.350	0.350	0.350	0.350	0.350
	δ	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100
	r	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050
	n	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002
	ϕ	0.050	0.050	0.050	0.050	0.050	**0.020**	0.050	0.050	0.050	0.050	0.050	**0.020**
	a_{ext}	2.000	2.000	2.000	2.000	2.000	2.000	2.000	2.000	2.000	2.000	2.000	2.000
a* : asset		2.538 (0.0%)	3.435 (35.3%)	3.701 (45.8%)	2.738 (7.9%)	3.151 (24.1%)	3.063 (20.7%)	0.353 (0.0%)	0.423 (19.7%)	0.209 (▲40.7%)	0.180 (▲48.9%)	0.269 (▲23.9%)	0.407 (15.3%)
c* : consumption		1.072 (0.0%)	1.105 (3.1%)	1.111 (3.6%)	1.082 (0.9%)	1.082 (0.9%)	1.127 (5.1%)	0.699 (0.0%)	0.733 (4.8%)	0.610 (▲12.8%)	0.586 (▲16.2%)	0.628 (▲10.1%)	0.678 (▲3.1%)
f((1-θ)a) : production		1.281 (0.0%)	1.424 (11.2%)	1.462 (14.1%)	1.316 (2.7%)	1.319 (2.9%)	1.368 (6.8%)	0.643 (0.0%)	0.684 (6.5%)	0.535 (▲16.7%)	0.508 (▲20.9%)	0.557 (▲13.3%)	0.675 (5.1%)
f'((1-θ)a) : productivity		0.221 (0.0%)	0.181 (▲17.9%)	0.173 (▲21.7%)	0.210 (▲4.8%)	0.209 (▲5.2%)	0.195 (▲11.5%)	0.796 (0.0%)	0.708 (▲11.0%)	1.118 (40.5%)	1.231 (54.7%)	1.037 (30.3%)	0.726 (▲8.8%)
Rt		0.107	0.075	0.068	0.098	0.092	0.086	0.567	0.496	0.824	0.915	0.671	0.511
asset preference term of eq.		0.145	0.127	0.184	0.154	0.160	0.136	▲0.315	▲0.294	▲0.572	▲0.663	▲0.419	▲0.289
BOP		0.027	0.072	0.085	0.037	0.058	0.021	▲0.082	▲0.079	▲0.090	▲0.091	▲0.087	▲0.032
Jacobian elements	J11	0.035	0.003	▲0.004	0.026	0.010	0.044	0.495	0.424	0.752	0.843	0.589	0.469
	J12	▲1.000	▲1.000	▲1.000	▲1.000	▲1.000	▲1.000	▲1.000	▲1.000	▲1.000	▲1.000	▲1.000	▲1.000
	J21	▲0.158	▲0.102	▲0.109	▲0.165	▲0.121	▲0.125	▲1.016	▲0.766	▲1.718	▲1.359	▲1.227	▲0.776
	J22	0.145	0.127	0.184	0.154	0.160	0.136	▲0.315	▲0.294	▲0.572	▲0.663	▲0.419	▲0.289
Tr(J)		0.180	0.130	0.180	0.180	0.170	0.180	0.180	0.130	0.180	0.180	0.170	0.180
Det(J)		▲0.153	▲0.101	▲0.110	▲0.161	▲0.119	▲0.119	▲1.171	▲0.890	▲2.148	▲1.918	▲1.474	▲0.911
D		0.646	0.422	0.471	0.678	0.505	0.507	4.718	3.579	8.625	7.705	5.925	3.677
judgement		saddle point	saddle point	saddle point	saddle point	saddle point	saddle point	saddle point	saddle point	saddle point	saddle point	saddle point	saddle point

Note: Light texts are the same as in the standard case. The lower ( ) indicates the rate of increase or decrease compared to the standard case.

Furthermore, based on $Table.3-1$, we will now discuss the balance equations (1.9.1 and 2.2.2), particularly for the case where $\beta>0$. First, in these five cases, a decline in the time preference rate ρ and capital diffusion coefficient ϕ, or an increase in asset preference β, relative risk aversion of assets ψ, and financial asset ratio θ, all lead to increases in asset levels, consumption levels, and production levels. Simultaneously, the rate of return from capital and assets declines due to a decrease in marginal productivity.

Moreover, in the case of a decline in the time preference rate on the far left, the force that keeps assets internal such as asset preference on the right-hand side of the balance equation declines due to a decrease in the marginal utility of assets. However, this is balanced by a decline in the time preference rate itself. Next, in the three subsequent cases to the right of that (an increase in asset preference, relative risk aversion, and financial asset ratio), these are all components of the force that keeps assets internal on the right-hand side and the increase itself compensates for the decline in the rate of return of capital and assets. Finally, in the case of a decline in the diffusion coefficient on the far right, the decline in the decrease in the pressure for capital to diffuse itself on the right-hand side is balanced by the decline in the rate of return.

In this way, by viewing the divergence between the rate of return on assets and the time preference rate as being maintainable in balance between the forces that generate and retain assets within the economy and the forces that diffuse them in the external environment, we can paint a more diverse and realistic picture of economies. In the next section, we will extend this to a bilateral model and consider the interactions between multiple economies and their convergence to a steady state and stability.

3. 3. Expansion to a Bilateral Model

3. 3. 1. Basic Settings and Formulation of the Model

In this section, we will expand the single-country model constructed in the previous section to a bilateral model. In particular, I will examine what kind of economic phenomena differences in time preference rate, asset preference, and capital diffusion coefficient lead to in multiple heterogeneous economies. In the model presented here, the world is composed of Country H, Country L, and the rest of the world. The time preference rate ρ, asset preference β, and capital diffusion coefficient ϕ differ across countries, but other parameters are assumed to be the same. The state variable a_t and the associated real capital stock $k_t = (1-\theta)a_t$ and financial assets $b_t = \theta a_t$, as well as the control variable c_t, are also distinguished between Country H and Country L.

In the following, when it is not necessary to distinguish between the two countries, subscripts $i, j = H, L$ (where $i \neq j$) are used for parameters and variables by country. In addition, although the variables depend on time, the subscript t is omitted unless specifically required.

In this section, we use a logarithmic utility function where the relative risk aversion is set to 1 (i.e., $\gamma = \psi = 1$ from the previous section). Therefore, taking into account equations 2.1.4, 2.1.5, and 2.1.6, the following is obtained:

$$U(c_i, b_i) = \ln c_i + \beta_i \ln b_i \qquad 3.1.1$$

$$U_c = \frac{\partial U(c_i, b_i)}{\partial c_i} = c_i^{-1} = \frac{1}{c_i} \qquad 3.1.2$$

$$U_{(\theta a)} = U_b = \frac{\partial U(c_i, b_i)}{\partial b_i} = \beta_i b_i^{-1} = \beta_i \frac{1}{b_i} = \frac{\beta_i}{\theta a_i} \qquad 3.1.3$$

Also, the dynamic equation of assets is as follows from equation 2.2.1:

$$\dot{a}_i = A(1-\theta)^\alpha a_i^\alpha - c_i + (\theta r - (1-\theta)\delta - n)a_i - \phi_i(a_i - a_j) \qquad 3.1.4$$

In a typical bilateral model, it is assumed that the current account balance is offset between only two countries. However, if this is the case, there would be no difference between the two countries in terms of ϕ which is an important parameter in this paper. Therefore, by adding the "remainder of the world" (whose current account balance is denoted by Res.), the following relationship is assumed to hold:

$$\phi_i(a_i - a_j) = -\phi_j(a_j - a_i) + Res. \qquad 3.1.5$$

3. 3. 2. Dynamic Equations for Assets and Consumption

The $\dot{a}_i = 0$ locus is as follows, which is obtained by setting $\dot{a}_i = 0$ in equation 3.1.4 and solving for c_i:

$$\begin{aligned} c_i &= A(1-\theta)^\alpha a_i^\alpha + (\theta r - (1-\theta)\delta - n)a_i - \phi_i(a_i - a_j) \\ &= (A(1-\theta)^\alpha a_i^\alpha + \theta r a_i) - ((1-\theta)\delta + n + \phi_i)a_i + \phi_i a_j \end{aligned} \qquad 3.2.1$$

Next, the dynamic equations for consumption are as follows, based on equations 3.1.2, 3.1.3, and 2.3.3. (Note that $\theta^{1-1} = \theta^0 = 1$, $1/c_i = c_i^{-1}$, $1/a_i = a_i^{-1}$.)

$$\dot{c}_i = c_i\left[\left((1-\theta)(A\alpha(1-\theta)^{\alpha-1}{a_i}^{\alpha-1}-\delta)+\theta r-\rho_i\right)+\left(\frac{\beta_i {a_i}^{-1}}{{c_i}^{-1}}-n-\phi\right)\right] \quad 3.2.2$$

Therefore, the $\dot{c}_i = 0$ locus is obtained by setting $\dot{c}_i = 0$ in equation 3.2.2 and solving for c_i as follows:

$$c_i = \frac{a_i}{\beta_i}\left(\rho_i - \left((1-\theta)(A\alpha(1-\theta)^{\alpha-1}{a_i}^{\alpha-1}-\delta)+\theta r\right)+n+\phi_i\right) \quad 3.2.3$$

Incidentally, the important balance equation in this paper, which correspond to 1.9.1 in the first section and 2.2.2 in the previous section, can be derived in this section by setting $\dot{c}_i = 0$ in 3.2.2 as follows:

$$\left((1-\theta)(A\alpha(1-\theta)^{\alpha-1}{a_i}^{\alpha-1}-\delta)+\theta r\right)-\rho_i = n+\phi_i-\frac{\beta_i {a_i}^{-1}}{{c_i}^{-1}} \quad 3.2.4$$

3. 3. 3. Deriving the Steady State and Its Economic Implications

3. 3. 3. a. Deriving the steady state

We will analyze the stability of the steady state based on the equations derived earlier. However, in the model of this section, the steady state implies $\dot{a}_H = \dot{a}_L = \dot{c}_H = \dot{c}_L = 0$ which would typically require dealing with a 4x4 Jacobian matrix. To reduce the dimensionality, we will substitute the equation for c_i from the $\dot{c}_i = 0$ locus (equation 3.2.3) into the dynamic equation for $\dot{a}_i$ (equation 3.1.4), so reduce the dimensions to two. In other words, this means finding the equation for assets under the consumption being in a steady state.

Substituting equation 3.2.3 into equation 3.1.4 and rearranging it, the following is derived:

$$\begin{aligned}
\dot{a}_i &= A(1-\theta)^\alpha a_i^\alpha - \left(\frac{a_i}{\beta_i}\left(\rho_i - \left((1-\theta)(A\alpha(1-\theta)^{\alpha-1}{a_i}^{\alpha-1}-\delta)+\theta r\right)+n+\phi_i\right)\right)+(\theta r-(1-\theta)\delta-n)a_i-\phi_i(a_i-a_j) \\
&= A(1-\theta)^\alpha a_i^\alpha - \frac{a_i}{\beta_i}\left(\rho_i - \left((1-\theta)(A\alpha(1-\theta)^{\alpha-1}{a_i}^{\alpha-1}-\delta)+\theta r\right)+n+\phi_i\right)+\theta r a_i-(1-\theta)\delta a_i-na_i-\phi_i a_i+\phi_i a_j \\
&= A(1-\theta)^\alpha a_i^\alpha - \frac{a_i}{\beta_i}\rho_i + \frac{a_i}{\beta_i}(1-\theta)A\alpha(1-\theta)^{\alpha-1}{a_i}^{\alpha-1} - \frac{a_i}{\beta_i}(1-\theta)\delta + \frac{a_i}{\beta_i}\theta r - \frac{a_i}{\beta_i}n - \frac{a_i}{\beta_i}\phi_i + \theta r a_i - (1-\theta)\delta a_i - na_i - \phi_i a_i + \phi_i a_j \\
&= (1-\theta)a_i A(1-\theta)^{\alpha-1}a_i^{\alpha-1} + \frac{\alpha}{\beta_i}(1-\theta)a_i A(1-\theta)^{\alpha-1}{a_i}^{\alpha-1} - (1-\theta)a_i\delta - \frac{a_i}{\beta_i}(1-\theta)\delta + \theta r a_i + \frac{a_i}{\beta_i}\theta r - \frac{a_i}{\beta_i}\rho_i - na_i - \frac{a_i}{\beta_i}n - \phi_i a_i - \frac{a_i}{\beta_i}\phi_i + \phi_i a_j \\
&= \left(1+\frac{\alpha}{\beta_i}\right)(1-\theta)a_i A(1-\theta)^{\alpha-1}a_i^{\alpha-1} - \left(1+\frac{1}{\beta_i}\right)(1-\theta)a_i\delta + \left(1+\frac{1}{\beta_i}\right)\theta r a_i - \frac{a_i}{\beta_i}\rho_i - \left(1+\frac{1}{\beta_i}\right)na_i - \left(1+\frac{1}{\beta_i}\right)\phi_i a_i + \phi_i a_j \\
&= \left((1-\theta)a_i\left(\left(1+\frac{\alpha}{\beta_i}\right)A(1-\theta)^{\alpha-1}a_i^{\alpha-1} - \left(1+\frac{1}{\beta_i}\right)\delta\right) + \left(1+\frac{1}{\beta_i}\right)\theta a_i r\right) - \frac{1}{\beta_i}\rho_i a_i - \left(1+\frac{1}{\beta_i}\right)(na_i+\phi_i a_i)+\phi_i a_j
\end{aligned} \quad 3.3.1$$

Then, since it is a steady state, we set $\dot{a}_i = 0$, and divide the entire equation by a_i to get the following:

$$\left((1-\theta)\left(\left(1+\frac{\alpha}{\beta_i}\right)A(1-\theta)^{\alpha-1}a_i^{\alpha-1} - \left(1+\frac{1}{\beta_i}\right)\delta\right) + \left(1+\frac{1}{\beta_i}\right)\theta r\right) - \frac{1}{\beta_i}\rho_i - \left(1+\frac{1}{\beta_i}\right)(n+\phi_i)+\phi_i\frac{a_j}{a_i} = 0 \quad 3.3.2$$

$$\Rightarrow \quad \left((1-\theta)\left(\left(1+\frac{\alpha}{\beta_i}\right)A(1-\theta)^{\alpha-1}a_i^{\alpha-1} - \left(1+\frac{1}{\beta_i}\right)\delta\right) + \left(1+\frac{1}{\beta_i}\right)\theta r\right) - \frac{1}{\beta_i}\rho_i = \left(1+\frac{1}{\beta_i}\right)(n+\phi_i)-\phi_i\frac{a_j}{a_i} \quad 3.3.3$$

At first glance, it looks complicated, but its essence is the same as the balance equations 1.9.1, 2.2.2 and 3.2.4. The term corresponding to $(1-\theta)$ includes the first derivative of the production function $A(1-\theta)^{\alpha-1}a_i^{\alpha-1}$ and the depreciation rate δ, so its essence is close to the net return of production. Next, θr represents the return rate on financial assets multiplied by the holding ratio. Therefore, these two parts are approximate the overall return rate on assets in this economy and related to R_t. Moreover, ρ_i is the time preference rate, so the left-hand side represents the deviation between this rate and the interest rate, similar to the balance equation.

On the other hand, the right-hand side also includes the population growth rate n and the capital diffusion coefficient ϕ_i which represent the forces that diffuse assets to the outside as in the balance equation. If these forces are strong, the right-hand side will be large. In such cases, even if the time preference rate is low and there is a financial surplus, the decline in the real interest rate will be suppressed, and the deviation will be maintained.

The last term in this equation differs from the previous balance equations, where the strength of asset preference was expressed. Here, this term represents the asset level of the other country compared to the home country, multiplied by the diffusion coefficient. This means that when consumption is already in a steady state, the tendency for capital to flow is one factor that compensates for the deviation between the interest rate and the time preference rate.

Each term is multiplied by what could be called an "adjustment term," such as $1+\frac{\alpha}{\beta_i}$, $1+\frac{1}{\beta_i}$, or $\frac{1}{\beta_i}$. These terms become smaller as asset preference β_i becomes larger. What is important to note is that the last term $\phi_i \frac{a_j}{a_i}$ is not multiplied by this "adjustment term." As a result, the larger the asset preference, the greater the influence of the diffusion term and external interactions rather than the mechanisms internal to the economy.

In addition, only the term $A(1-\theta)^{\alpha-1}a_i^{\alpha-1}$ of capital's marginal productivity has a capital share α in the numerator instead of 1, so the larger the capital share, the greater the impact on production.

Note that as a result of these "adjustment terms", in the case of negative asset preference, it becomes difficult to define a_i in the positive range. For this reason, we will limit our discussion to the case where $\beta_i > 0$.

3. 3. 3. b. Stability of the steady state

Since the focus of this section is on the interaction between two countries, we will consider the system as a set of simultaneous nonlinear equations that determines the steady-state a_H^* and a_L^*, based on the equation 3.3.2.

First, the steady-state conditions for Country H and Country L are as follows. The asset levels of each country can be found by solving these two equations, but this is generally difficult to do analytically.

$$\left((1-\theta)\left(\left(1+\frac{\alpha}{\beta_H}\right)A(1-\theta)^{\alpha-1}a_H^{\alpha-1}-\left(1+\frac{1}{\beta_H}\right)\delta\right)+\left(1+\frac{1}{\beta_H}\right)\theta r\right)-\frac{1}{\beta_H}\rho_H-\left(1+\frac{1}{\beta_H}\right)(n+\phi_H)+\phi_H\frac{a_L}{a_H}=0 \qquad 3.3.4H$$

$$\left((1-\theta)\left(\left(1+\frac{\alpha}{\beta_L}\right)A(1-\theta)^{\alpha-1}a_L^{\alpha-1}-\left(1+\frac{1}{\beta_L}\right)\delta\right)+\left(1+\frac{1}{\beta_L}\right)\theta r\right)-\frac{1}{\beta_L}\rho_L-\left(1+\frac{1}{\beta_L}\right)(n+\phi_L)+\phi_L\frac{a_H}{a_L}=0 \qquad 3.3.4L$$

The dynamic equations for each country are given by equations 3.3.1. We will analyze the stability based on these.

$$\dot{a}_H=\left((1-\theta)a_H\left(\left(1+\frac{\alpha}{\beta_H}\right)A(1-\theta)^{\alpha-1}a_H^{\alpha-1}-\left(1+\frac{1}{\beta_H}\right)\delta\right)+\left(1+\frac{1}{\beta_H}\right)\theta a_H r\right)-\frac{1}{\beta_H}\rho_H a_H-\left(1+\frac{1}{\beta_H}\right)(na_H+\phi_H a_H)+\phi_H a_L \qquad 3.3.5H$$

$$\dot{a}_L=\left((1-\theta)a_L\left(\left(1+\frac{\alpha}{\beta_L}\right)A(1-\theta)^{\alpha-1}a_L^{\alpha-1}-\left(1+\frac{1}{\beta_L}\right)\delta\right)+\left(1+\frac{1}{\beta_L}\right)\theta a_L r\right)-\frac{1}{\beta_L}\rho_L a_L-\left(1+\frac{1}{\beta_L}\right)(na_L+\phi_L a_L)+\phi_L a_H \qquad 3.3.5L$$

(a) Deriving the Jacobian matrix and Jacobian elements

As in the previous section, we derive the Jacobian matrix and its elements. There are two state variables a_H and a_L, so the Jacobian matrix is as follows:

$$J=\begin{bmatrix}J_{11} & J_{12}\\ J_{21} & J_{22}\end{bmatrix}=\begin{bmatrix}\frac{\partial \dot{a}_H}{\partial a_H} & \frac{\partial \dot{a}_H}{\partial a_L}\\ \frac{\partial \dot{a}_L}{\partial a_H} & \frac{\partial \dot{a}_L}{\partial a_L}\end{bmatrix}$$

The equations for $\dot{a}_H$ and $\dot{a}_L$ have a symmetric structure, so their derivatives are derived in two parts.

＜Diagonal elements＞

The diagonal elements J_11 and J_22 of the matrix indicate the effect of the domestic asset level on the domestic asset accumulation speed. Their structure is similar to the steady-state condition equations for $\dot{a}_i = 0$.

$$J_{11} = \frac{\partial \dot{a}_H}{\partial a_H} = \frac{\partial}{\partial a_H}\left(\left((1-\theta)a_H\left(\left(1+\frac{\alpha}{\beta_H}\right)A(1-\theta)^{\alpha-1}a_H^{\alpha-1} - \left(1+\frac{1}{\beta_H}\right)\delta\right) + \left(1+\frac{1}{\beta_H}\right)\theta a_H r\right) - \frac{1}{\beta_H}\rho_H a_H - \left(1+\frac{1}{\beta_H}\right)(na_H + \phi_H a_H) + \phi_H a_L\right)$$

$$= \frac{\partial}{\partial a_H}\left(\left(1+\frac{\alpha}{\beta_H}\right)A(1-\theta)^{\alpha}a_H^{\alpha} - \left(1+\frac{1}{\beta_H}\right)(1-\theta)\delta a_H + \left(1+\frac{1}{\beta_H}\right)\theta a_H r - \frac{1}{\beta_H}\rho_H a_H - \left(1+\frac{1}{\beta_H}\right)(na_H + \phi_H a_H) + \phi_H a_L\right)$$

$$= \left(1+\frac{\alpha}{\beta_H}\right)A\alpha(1-\theta)^{\alpha}a_H^{\alpha-1} - \left(1+\frac{1}{\beta_H}\right)(1-\theta)\delta + \left(1+\frac{1}{\beta_H}\right)\theta r - \frac{1}{\beta_H}\rho_H - \left(1+\frac{1}{\beta_H}\right)(n+\phi_H)$$

$$= \left(1+\frac{\alpha}{\beta_H}\right)A\alpha(1-\theta)^{\alpha}a_H^{\alpha-1} + \left(1+\frac{1}{\beta_H}\right)(\theta r - (1-\theta)\delta - n - \phi_H) - \frac{1}{\beta_H}\rho_H \qquad 3.3.6H$$

$$J_{22} = \frac{\partial \dot{a}_L}{\partial a_L} = \left(1+\frac{\alpha}{\beta_L}\right)A\alpha(1-\theta)^{\alpha}a_L^{\alpha-1} + \left(1+\frac{1}{\beta_L}\right)(\theta r - (1-\theta)\delta - n - \phi_L) - \frac{1}{\beta_L}\rho_L \qquad 3.3.6L$$

If rearranging this as in $3.3.6H$ and $3.3.6L$, the terms including $a_i^{\alpha-1}$ derived from the marginal productivity will become smaller as asset levels rise both mathematically and according to economic assumptions. On the other hand, the other terms are not affected by asset levels. Since all terms except θr have negative signs, they are likely to eventually outweigh the positive term, causing the entire expression to become negative. In other words, negative feedback is at work where assets increase.

Also, like the steady-state condition equation, this equation is multiplied by an "adjustment term" such as $1+\frac{\alpha}{\beta_i}$, $1+\frac{1}{\beta_i}$, or $\frac{1}{\beta_i}$. This term becomes smaller as asset preference β_i. Thus, the stronger the asset preference, the slower the adjustment will be, or in other words, the more stable the economy.

＜Off-diagonal elements＞

The off-diagonal elements J_{12} and J_{21} indicate the impact that fluctuations in assets in other countries have on the asset accumulation speed in one's own country. These elements directly represent the respective diffusion coefficients.

$$J_{12} = \frac{\partial \dot{a}_H}{\partial a_L} = \frac{\partial}{\partial a_L}(((1-\theta)a_H \dots\dots\dots + \phi_H a_L) = \phi_H \qquad 3.3.7H$$

$$J_{21} = \frac{\partial \dot{a}_L}{\partial a_H} = \frac{\partial}{\partial a_H}(((1-\theta)a_L \dots\dots\dots + \phi_L a_H) = \phi_L \qquad 3.3.7L$$

(b) Considerations on stability

As mentioned previously, stability analysis using the Jacobian matrix is performed by examining the sign conditions of the sum of the diagonal elements ($Tr(J) = J_{11} + J_{22}$) and the determinant ($Det(J) = J_{11}J_{22} - J_{12}J_{21}$). If J_{11} and J_{22} are negative, then $Tr(J) < 0$, indicating that an autoregressive stabilization mechanism is at work. Under the assumption of diminishing marginal productivity, J_{11} and J_{22} are likely to become negative as asset levels increase, suggesting that $Tr(J) < 0$ is likely to occur at sufficiently high asset levels.

Next, if the diffusion coefficients (ϕ_H and ϕ_L) are sufficiently large, the possibility of $Det(J) < 0$ increases. In this case, the steady state becomes a saddle point. Of course, a large diffusion coefficient also implies a strong ability to self-adjust imbalances among countries. However, if there are countries with heterogeneous preferences and diffusion coefficients, and a saddle point is formed in a pattern of imbalances, it suggests that these imbalances may persist along a stable manifold.

3. 3. 4. Parameter Effects and Convergence Paths

In this final section, we will use the bilateral model derived above to examine how changes in key parameters affect the steady state and whether the dynamic path converges. For the former, we will utilize graph-drawing software, as in the previous section. For the latter, we will simulate the convergence of various arbitrary initial values through sequential calculations using R (the R script is provided at the end of this paper).

The state variables are a_H and a_L. Therefore, it is necessary to identify the steady state on the $a_H - a_L$ plane and calculate the convergence path to that point. To do this, we must first rewrite the steady state conditions $3.3.4H$ and $3.3.4L$ in the forms of $a_L = g^L(a_H)$ and $a_H = g^H(a_L)$, respectively, and then plot them. The derivations are as follows:

$$\left((1-\theta)\left(\left(1+\frac{\alpha}{\beta_H}\right)A(1-\theta)^{\alpha-1}a_H^{\alpha-1}-\left(1+\frac{1}{\beta_H}\right)\delta\right)+\left(1+\frac{1}{\beta_H}\right)\theta r\right)-\frac{1}{\beta_H}\rho_H-\left(1+\frac{1}{\beta_H}\right)(n+\phi_H)+\phi_H\frac{a_L}{a_H}=0$$

$$\left(1+\frac{\alpha}{\beta_H}\right)A(1-\theta)^{\alpha}a_H^{\alpha-1}-\left(1+\frac{1}{\beta_H}\right)(1-\theta)\delta+\left(1+\frac{1}{\beta_H}\right)\theta r-\frac{1}{\beta_H}\rho_H-\left(1+\frac{1}{\beta_H}\right)(n+\phi_H)=-\phi_H\frac{a_L}{a_H}$$

$$\left(1+\frac{\alpha}{\beta_H}\right)A(1-\theta)^{\alpha}a_H^{\alpha-1}+\left(1+\frac{1}{\beta_H}\right)\Big(\theta r-\big((1-\theta)\delta+n+\phi_H\big)\Big)-\frac{1}{\beta_H}\rho_H=-\phi_H\frac{a_L}{a_H}$$

$$\phi_H\frac{a_L}{a_H}=\frac{1}{\beta_H}\rho_H-\left(1+\frac{\alpha}{\beta_H}\right)A(1-\theta)^{\alpha}a_H^{\alpha-1}-\left(1+\frac{1}{\beta_H}\right)\Big(\theta r-\big((1-\theta)\delta+n+\phi_H\big)\Big)$$

$$a_L=\frac{a_H}{\phi_H}\left(\frac{1}{\beta_H}\rho_H-\left(1+\frac{\alpha}{\beta_H}\right)A(1-\theta)^{\alpha}a_H^{\alpha-1}-\left(1+\frac{1}{\beta_H}\right)\Big(\theta r-\big((1-\theta)\delta+n+\phi_H\big)\Big)\right) \quad 3.3.8H$$

$$a_H=\frac{a_L}{\phi_L}\left(\frac{1}{\beta_L}\rho_L-\left(1+\frac{\alpha}{\beta_L}\right)A(1-\theta)^{\alpha}a_L^{\alpha-1}-\left(1+\frac{1}{\beta_L}\right)\Big(\theta r-\big((1-\theta)\delta+n+\phi_L\big)\Big)\right) \quad 3.3.8L$$

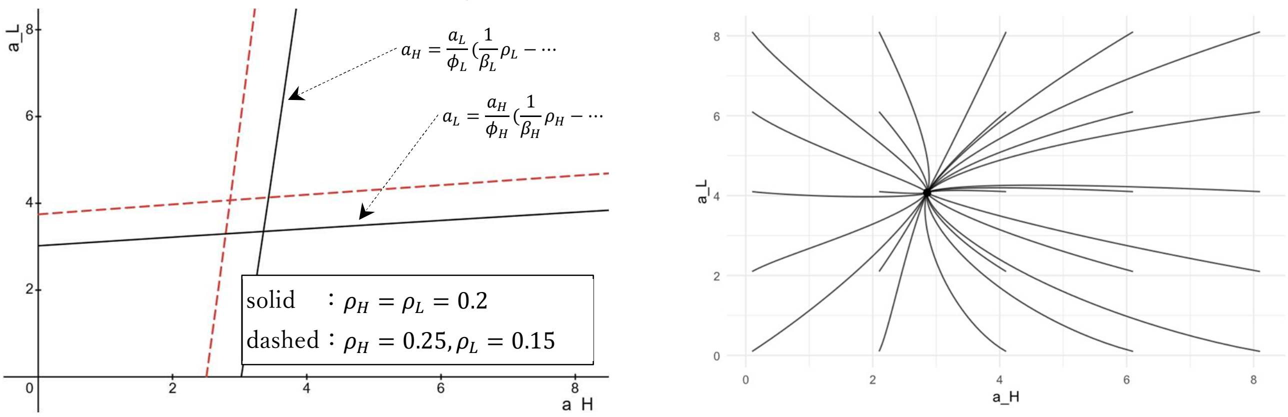
From this point, we will analyze the model's behavior for each case. For comparison, we have intentionally adjusted the changes in asset levels due to the initial time preference rate to be the standard, and the subsequent changes due to asset preferences and diffusion coefficients to be set at the same levels.

(a) Time preference rate

If the time preference rates differ, the asset level will be higher in L (economy with a lower rate) and lower in H (with a higher rate). This is because a greater weight placed on future utility leads to more allocation to assets.

As shown in $Table.3-2$, the consumption level is slightly higher in H (with a higher time preference rate). Furthermore, although L, with its low time preference rate, exhibits a high production level, its marginal productivity is low. This disparity in asset levels leads to capital outflows from L, resulting in a current account surplus.

fig.3-4-a Time preference rate

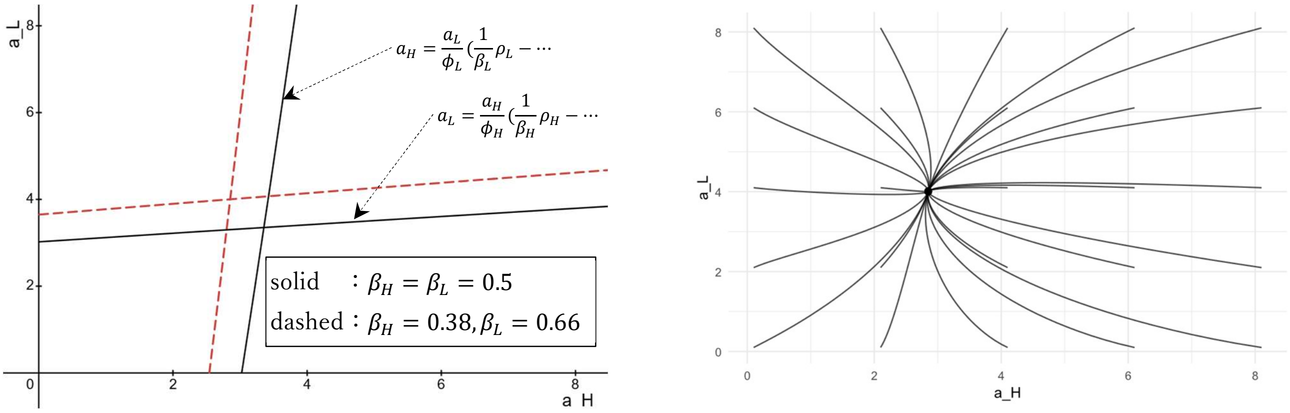


(b) Asset Preference

If the degree of asset preference differs, the asset level will be higher in L (economy with a strong asset preference). This is because placing greater importance on the utility derived from assets leads to a larger allocation to them.

As shown in $Table.3-2$, the consumption level will be slightly higher in H with a weak asset preference. Furthermore, while L, with its strong asset preference, exhibits a high production level, its marginal productivity is low. This disparity in asset levels leads to capital outflows from L, resulting in a current account surplus.

fig.3-4-b Asset Preference



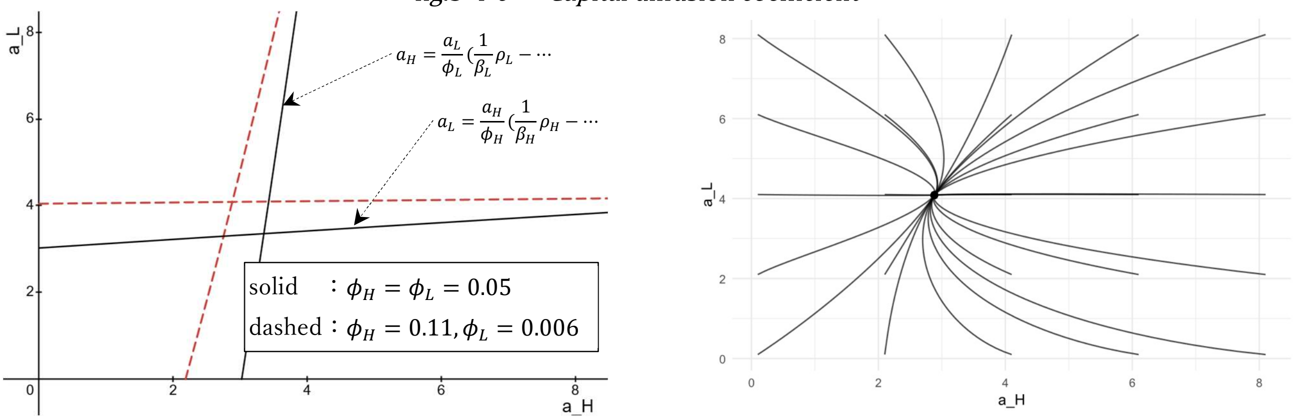
(c) Capital diffusion coefficient

If the capital diffusion coefficients differ, the asset level will be higher in L (the economy with a smaller diffusion coefficient). This is because less capital diffusing outside leads to greater asset accumulation within the economy.

As shown in $Table.3-2$, the consumption level is high in H (with a large diffusion coefficient), but L (with a small coefficient) also shows a consumption level higher than its original state, and the increase is greater than that observed with similar changes in asset levels caused by other parameters. This is because, in the case of the diffusion coefficient, the production level is raised not only through the $\dot{c}_t = 0$ locus but also through the $\dot{a}_t = 0$ locus.

Furthermore, although L, with a small diffusion coefficient, exhibits a high production level, its marginal productivity is low. While the disparity in asset levels still leads to capital outflows from L, resulting in a current account surplus, the magnitude of this surplus is smaller than in other cases, attributed to the smallness of the diffusion coefficient. On the other hand, H, with a large diffusion coefficient, experiences a substantial current account deficit.

fig.3-4-c Capital diffusion coefficient

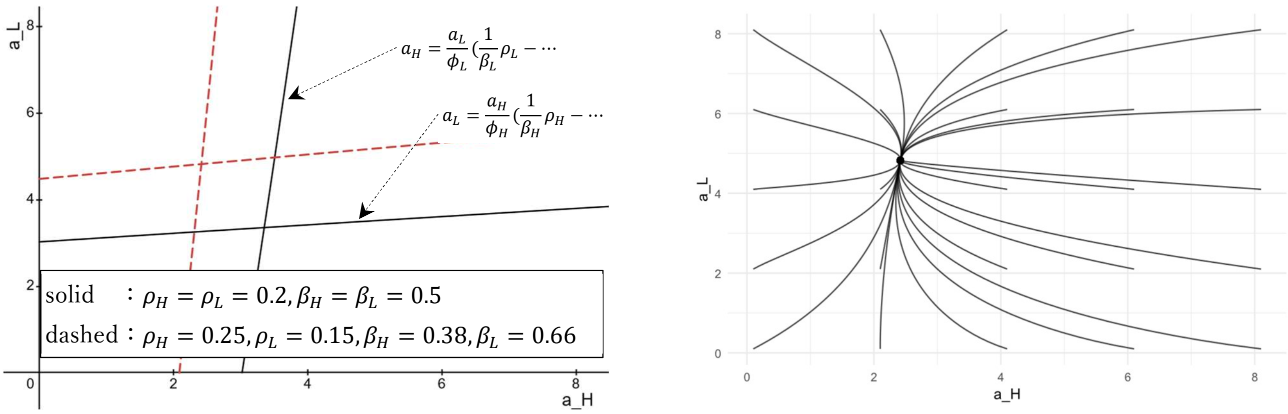


(d) Time preference rate and asset preference

The asset level is higher in L (with a low time preference rate and strong asset preference). The magnitude of this increase is somewhat greater than the sum of the results observed when these parameters are changed individually.

As shown in $Table.3-2$ the consumption level is higher in H (with a high time preference rate and weak asset preference). Furthermore, while L, with its low time preference rate and strong asset preference, exhibits a high production level, its productivity is low. This leads to capital outflows from L, resulting in a current account surplus.

fig.3-4-d Time preference rate and asset preference



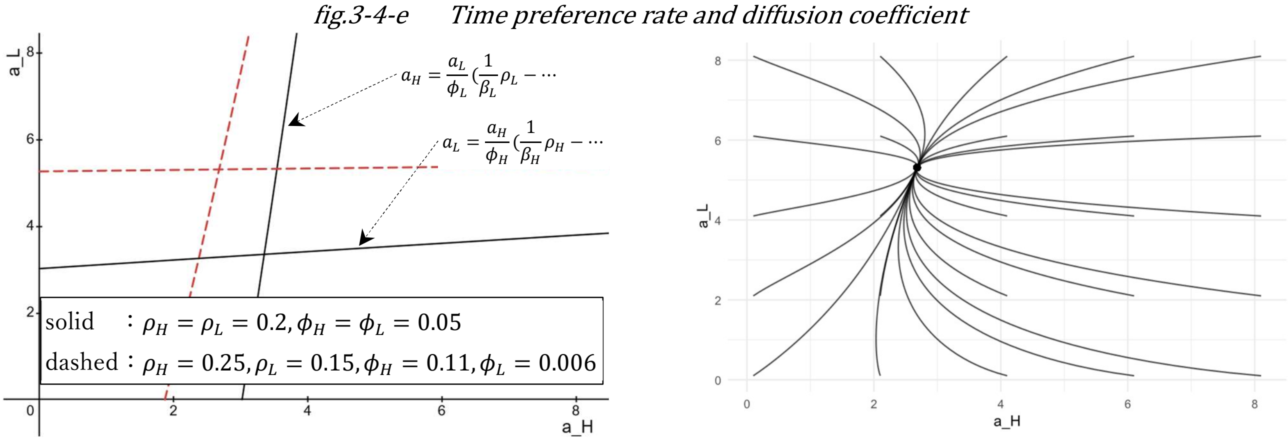
(e) Time preference rate and diffusion coefficient

The asset level is higher in L (with a low time preference rate and a small diffusion coefficient). The magnitude of this increase is much greater than the combined total when these parameters are changed individually. This is because, in the case of the diffusion coefficient, the production level itself is raised through the $\dot{a}_t = 0$ locus.

As shown in $Table.3-2$, the consumption level is higher in H (with a high time preference rate and a large diffusion coefficient), but L (with a low time preference rate and a small coefficient) also shows a consumption level higher than its original level. This is attributed to the production level itself being raised in the case of the diffusion coefficient.

Furthermore, although L, with a low time preference rate and a small diffusion coefficient, exhibits a high production level, its productivity is low. While the disparity in asset levels leads to capital outflows from L, resulting in a current account surplus, its magnitude is smaller than in other cases due to the small diffusion coefficient. On the other hand, H has a substantial current account deficit, which is larger than when each parameter changes individually.

fig.3-4-e Time preference rate and diffusion coefficient



As a result of the stability analysis, all of the aforementioned steady states are found to be stable nodes (not saddle points), as illustrated by the convergence path and $Table.3-2$. This outcome is achieved because consumption, a jump variable, is initially treated as a function of assets based on the steady-state condition equation in the bilateral model of this section and then substituted into the dynamic equation for assets to reduce the system's dimension.

However, it is important to note that even when employing such a method, the steady state does not necessarily converge to a stable node or focus as observed in this particular case. The analysis results presented here demonstrate that, under the model constructed in this paper, the rational optimization behavior of each economic agent—who possesses different time preference rates, asset preferences, capital transfer characteristics, etc.—does not converge to an average or center as depicted in standard models. Instead, it reaches a distinct and sustainable state that reflects the "individuality" of each economy.

Table.3-2 Effects by parameters and stability of the bilateral model

		bench mark		changes of parameters											
				ρ		β		ϕ		ρ、β		ρ、ϕ		combined	
		H	L	H	L	H	L	H	L	H	L	H	L	H	L
parameters	ρ	0.200	0.200	**0.250**	**0.150**	0.200	0.200	0.200	0.200	**0.250**	**0.150**	**0.250**	**0.150**	**0.250**	**0.150**
	β	0.500	0.500	0.500	0.500	**0.380**	**0.660**	0.500	0.500	**0.380**	**0.660**	0.500	0.500	**0.550**	**0.450**
	A	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	θ	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	**0.250**	**0.150**
	α	0.350	0.350	0.350	0.350	0.350	0.350	0.350	0.350	0.350	0.350	0.350	0.350	0.350	0.350
	δ	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100
	r	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050
	n	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002
	ϕ	0.050	0.050	0.050	0.050	0.050	0.050	**0.110**	**0.006**	0.050	0.050	**0.110**	**0.006**	**0.075**	**0.025**
a* : asset		3.354 (0.0%)	3.354 (0.0%)	2.857 (▲14.8%)	4.074 (21.5%)	2.852 (▲15.0%)	4.006 (19.4%)	2.883 (▲14.1%)	4.086 (21.8%)	2.417 (▲27.9%)	4.822 (43.8%)	2.682 (▲20.0%)	5.313 (58.4%)	2.917 (▲13.0%)	4.277 (27.5%)
c* : consumption		1.171 (0.0%)	1.171 (0.0%)	1.191 (1.7%)	1.158 (▲1.1%)	1.187 (1.4%)	1.157 (▲1.2%)	1.265 (8.0%)	1.212 (3.5%)	1.206 (3.0%)	1.137 (▲3.0%)	1.403 (19.8%)	1.261 (7.7%)	1.229 (4.9%)	1.197 (2.2%)
f((1-θ)a) : production		1.413 (0.0%)	1.413 (0.0%)	1.336 (▲5.5%)	1.512 (7.0%)	1.335 (▲5.5%)	1.503 (6.4%)	1.340 (▲5.2%)	1.514 (7.2%)	1.260 (▲10.8%)	1.604 (13.5%)	1.306 (▲7.5%)	1.659 (17.5%)	1.315 (▲6.9%)	1.571 (11.2%)
f'((1-θ)a) : productivity		0.184 (0.0%)	0.184 (0.0%)	0.205 (11.0%)	0.162 (▲11.9%)	0.205 (11.1%)	0.164 (▲10.9%)	0.203 (10.3%)	0.162 (▲12.0%)	0.228 (23.7%)	0.146 (▲21.0%)	0.213 (15.7%)	0.137 (▲25.8%)	0.210 (14.2%)	0.151 (▲17.9%)
Rt		0.077	0.077	0.094	0.060	0.094	0.061	0.093	0.060	0.112	0.046	0.100	0.039	0.095	0.051
asset preference term of eq.		0.175	0.175	0.208	0.142	0.158	0.191	0.219	0.148	0.190	0.156	0.262	0.119	0.232	0.126
BOP		0.000	0.000	▲0.061	0.061	▲0.058	0.058	▲0.132	0.007	▲0.120	0.120	▲0.289	0.016	▲0.102	0.034
Jacobian elements	J11		▲0.515		▲0.588		▲0.655		▲0.669		▲0.751		▲0.756		▲0.589
	J12		0.050		0.050		0.050		0.110		0.050		0.110		0.075
	J21		0.050		0.050		0.050		0.006		0.050		0.006		0.025
	J22		▲0.515		▲0.445		▲0.409		▲0.414		▲0.356		▲0.348		▲0.442
Tr(J)			▲1.031		▲1.033		▲1.064		▲1.083		▲1.106		▲1.104		▲1.031
Det(J)			0.263		0.259		0.265		0.276		0.265		0.263		0.258
D			0.010		0.030		0.070		0.068		0.166		0.169		0.029
judgement		stable node		stable node		stable node		stable node		stable node		stable node		stable node	

Note: Light texts are the same as in the standard case. The lower ( ) indicates the rate of increase or decrease compared to the standard case.

According to $Table.3-2$, in the case where both time preference and asset preference are altered (columns " ρ, β"), L, which has a low time preference rate and a strong asset preference, exhibits high asset and production levels. However, it also shows low productivity and profitability, along with a weak ability to retain assets internally (as indicated by the asset preference term in the balance equation). Conversely, H, with a high time preference rate and a weak asset preference, demonstrates lower asset and production levels but higher productivity and profitability, coupled with a strong ability to retain assets internally. As a result, capital inflows from L to H are sustained, and the current account imbalance persists. Regarding consumption levels, H is higher than L. This observation appears to be related to the recent economic stagnation experienced by some developed and emerging countries, such as Japan, prompting questions as to why a nation with well-established capital accumulation and a robust production base might fall into a deflationary equilibrium.

In contrast, the rightmost case (column "Combined") illustrates a scenario where H possesses higher time preference rates, asset preferences, financial asset ratios, and capital diffusion coefficients. In this combined scenario, H exhibits lower asset and production levels but higher productivity and consumption levels. Despite the high diffusion coefficient, H's strong ability to hold onto assets ensures that capital inflows to H are maintained, and its current account deficit persists. This analysis may contribute to our understanding of the U.S. economy's mechanism, which functions as the "market of the world" and is a primary source of global imbalances.

4. Summary and Future Issues and Prospects

Here is a summary of the key findings derived from this paper:

- Assuming that the time preference rate is specific to each economic entity and thus varies by country and region, the deviation between this rate and the real interest rate typically generates a surplus or shortage of funds. In standard models, this deviation is adjusted by the interest rate through market mechanisms, however, this is not necessarily the case in the model presented in this paper.

  The underlying causes are asset preference, which aims to retain capital and assets within the economy, and capital flows, which promote their diffusion outside the economy. As illustrated in the initial balance equation, if a certain level of utility is derived from assets and the system is open to external interactions, a steady state can be reached where the deviation between the interest rate and the time preference rate can be sustained by these two forces, as a result of the optimizing behavior of economic entities. This state is referred to as a "dynamic equilibrium" in this paper.

- The asset level in the steady state is higher when the time preference rate is low and asset preference is strong. In such cases, the consumption level depends on the prevailing steady-state situation, often being higher when the time preference rate is high and asset preference is weak. The asset level is also high when the capital diffusion coefficient is low, in this scenario, the production level itself is raised due to the retention of capital and assets within the economy, which frequently leads to a high consumption level.

  However, if economic agents exhibit a negative asset preference, meaning they derive disutility from holding assets, it is possible to reach an "excess consumption equilibrium." This equilibrium differs from the normal steady state, and its behavior deviates from the standard one.

- Although these findings are theoretical, the behaviors that differ from the standard model can converge to a saddle point in a single-country model or a stable node in a bilateral model, as demonstrated by stability analysis or simulation. This convergence is achieved through a balance between two forces: one is the internal dynamics that generate and retain assets, and the other is the interaction with the external environment that tends to diffuse them.

  Depending on these factors, a steady state can be realized where capital flows from one economy to another while maintaining distinct asset and consumption levels. This also contributes to an understanding of current account imbalances and deflationary equilibrium.

Considering the analysis from this perspective, even if economic agents are rational and markets are efficient, we can gain insights into persistent questions such as:

- Why has the relationship between creditor nations and debtor nations remained fixed for such a long time?
- Why have many developed and emerging countries fallen into a long-term deflationary equilibrium?
- Why has a divide between the "haves" and the "have-nots" become widespread and entrenched?

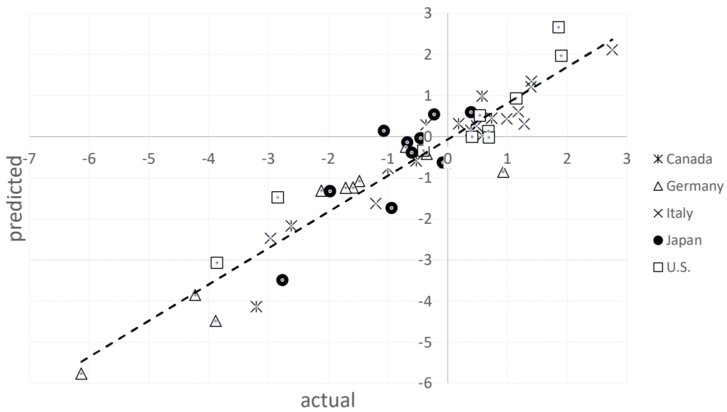
Of course, this paper has thus far only explored theoretical possibilities and is still in its nascent stages, leaving various issues to be addressed. One such issue is the "excess consumption equilibrium" due to negative asset preference, as pointed out in Section 3(2). However, this paper merely highlights this possibility. As the simulation results suggest, this concept is crucial for deepening our understanding of the challenges facing the modern economy.

Next, while Sections 3(2) and 3(3) primarily focused on examining the time preference rate, asset preference, and capital diffusion coefficient, there are other parameters that warrant discussion. One such parameter is the financial asset ratio, which represents an important aspect of the modern economy. Another is population growth, a crucial factor, especially when considering developing countries or inter-regional relations within a country. Additionally, all of these are components of the balance equations (1.9.1, 2.2.2, and 3.2.4).

Furthermore, a critical question is whether the theory presented in this paper can accurately capture the real economy. Particularly, from the perspective of "not compromising theory in the name of measurability," this paper utilizes parameters that are inherently difficult to measure, such as the time preference rate and asset preference, as the core components of the model.

However, this does not imply a complete lack of empirical possibilities. For instance, fig.4-1 displays a regression analysis of long-term real interest rates over the past 10 years for five of G7 countries, treated as panel data. If actual interest rates are indeed influenced (pulled up or down) by time preference rates as suggested by the model, then we can estimate the extent to which each country's deviations from the annual average level

fig.4-1 actual and predected long-term real interest rate



【result of regression】

- ○ estimate equation：
  $r_{it} = Const. + \alpha\, n_{it} + \beta\, BOP_{it} + \gamma\, EXR_{it} + \delta\, S_{it} + \epsilon\, C_{it} + \mu_t$
  - r_{it}：real interest rate(long-term bond yield - deflator)
  - n_{it}：population growth
  - BOP_{it}：current account balance compared to nominal GDP
  - EXR_{it}：changes of exchange rate to U.S. dollars
  - S_{it}：net savings ratio compared to disposal income
  - C_{it}：consumption compared to nominal GDP
  - μ_t：dummy terms of year
- ○ estimate period：2015～2024
- ○ number of data：50（panel data of 5 countries × 10 years）
  - ※ except U.K. and France only gross savings ratio
  - ※ add dummy terms to top and bottom 5% data
- ○ Adjusted R-squared：0.82495
- ○ Coefficient and t value（Significant at * is 10%、** is 5%、*** is 1%）

variables	coefficient	t value
（Constant）	-7.17741	-2.60866 **
population growth	0.58274	1.94603 *
current account balance	0.16799	1.93685 *
changes of exchange rate	-0.02889	-0.96866
net savings ratio	-0.19607	-3.65302 ***
propensity to consume	0.13838	3.09377 ***

※ source）OECD Stat.

(which is theoretically absorbed by year dummies) can be explained by the elements included in the balance equation. While the limited number of data points, target regions, and period means it is far from being empirically conclusive, we observe that the signs for the population growth rate, propensity to consume, and current account balance (which is synonymous with capital account deficit/capital outflow) are positive, and the net savings rate is negative. These signs are consistent with the theoretical hypothesis of this paper. Further data collection and analysis are required for robust verification, but this preliminary analysis may offer some valuable clues.

This paper attempts to reconsider the assumptions of conventional economic models, motivated by the awareness that they are unable to fully capture modern economic trends such as global imbalances and secular deflationary equilibrium. By utilizing the concept of "dynamic equilibrium," which incorporates the time preference specific to each economic agent, the utility arising from asset holdings, and the interaction with the external environment through capital transfer, we have clarified the mechanism by which a divergence continues to exist between the real interest rate and the time preference rate.

The mathematical expression presented in this paper represents one approach to comprehensively capturing the fluctuations of both the real economy and the asset economy. At this point, it remains purely theoretical, but it is hoped that it will contribute to economics building a framework that leads to a deeper understanding of the real economy.

Footnotes

(1) From Krugman's speech at the London School of Economics in 2016

(2) From Romer's 2016 Omicron Delta Epsilon Society Commons Memorial Lecture

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Note: R source code for the bilateral model simulation

```
library(deSolve)
library(ggplot2)

# parameters setting
params <- list(
  # common parameters
  A = 1,     # total factor productivity
  alpha = 0.35,  # capital share
  theta = 0.2,   # ratio of financial assets
  r = 0.05,    # rete of return of financial assets
  delta = 0.1,   # depreciation rate of real capital stock
  n = 0.002,    # population growth

  # no-common parameters
  rho_H = 0.2, # time preference rate of H
  rho_L = 0.2, # time preference rate of L
  beta_H = 0.5,  # asset preference of H
  beta_L = 0.5,  # asset preference of L
  phi_H = 0.05, # diffusion coefficient of H
  phi_L = 0.05  # diffusion coefficient of L
)

# Definition of Differential Equation
# equation 3.3.5H and 3.3.5L to R function
```

```
# a is a vector of state variable (a[1] = a_H, a[2] = a_L)
ramsey_model <- function(t, a, params) {
  # get out parameters from list
  with(as.list(c(a, params)), {

    # dynamic equation 3.3.5H
    dadH <- (1 + alpha/beta_H) * (1-theta) * a[1] * A * (1-theta)^(alpha-1) * a[1]^(alpha-1) -
      (1 + 1/beta_H) * (1-theta) * delta * a[1] +
      (1 + 1/beta_H) * theta * r * a[1] -
      (1/beta_H) * rho_H * a[1] -
      (1 + 1/beta_H) * (n * a[1] + phi_H * a[1]) +
      phi_H * a[2]

    # dynamic equation 3.3.5L
    dadL <- (1 + alpha/beta_L) * (1-theta) * a[2] * A * (1-theta)^(alpha-1) * a[2]^(alpha-1) -
      (1 + 1/beta_L) * (1-theta) * delta * a[2] +
      (1 + 1/beta_L) * theta * r * a[2] -
      (1/beta_L) * rho_L * a[2] -
      (1 + 1/beta_L) * (n * a[2] + phi_L * a[2]) +
      phi_L * a[1]
    # return time derivative as vector
    list(c(dadH, dadL))
  })
}

# create grid of initial condition
# Set multiple points around the equilibrium point
a_grid_H <- seq(0.1, 10, by = 2)
a_grid_L <- seq(0.1, 10, by = 2)
# time points setting
time_points <- seq(0, 100, by = 0.1)
# A data frame storing all simulation results
results_df <- data.frame()

# Run a simulation for each combination of initial values
for (a_H_init in a_grid_H) {
  for (a_L_init in a_grid_L) {
    a_init <- c(a_H = a_H_init, a_L = a_L_init)
    # Solve numerically with deSolve::ode()
    solution <- ode(
      y = a_init,
      times = time_points,
      func = ramsey_model,
      parms = params
    )
    # add result to the data frame
    solution_df <- as.data.frame(solution)
    solution_df$a_H_init <- a_H_init
    solution_df$a_L_init <- a_L_init
    results_df <- rbind(results_df, solution_df)
  }
}

# Drawing a topological space diagram
ggplot(results_df, aes(x = a_H, y = a_L, group = interaction(a_H_init, a_L_init))) +
  geom_path(alpha = 0.7, color = "dodgerblue") +
  geom_point(data = results_df[results_df$time == max(results_df$time), ],
             aes(x = a_H, y = a_L), color = "red", size = 2) +
  labs(
    x = "a_H",
    y = "a_L"
  ) +
  theme_minimal()
```